Operators Characterized by Certain Cauchy-Schwarz Type Inequalities

By

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Abstract

A Hilbert space operator $T$ satisfying either (**) \( \langle T\xi, \eta \rangle \leq |\langle T \xi, \xi \rangle| \langle T \eta, \eta \rangle \) for all $\xi, \eta \in \mathcal{H}$, or (*) \( |\langle T \xi, \xi \rangle| \leq \langle |T|\xi, \xi \rangle \) for all $\xi \in \mathcal{H}$ is studied. The condition (**) defines a slightly larger class than the hyponormality, and for compact operators (**) is equivalent to the normality. The condition (*) is characterized by using an operator whose numerical radius is less than 1, and among other things we show that (*) and the normality are equivalent for matrices. Moreover, we show that (*) and the normality are equivalent for trace class operators in Appendix.

§ 0. Introduction

The purpose of the present paper is to study operators $T$ satisfying either

\[ (**): \quad |\langle T\xi, \eta \rangle| \leq |\langle T \xi, \xi \rangle| |\langle T \eta, \eta \rangle| \quad \text{for all } \xi, \eta \in \mathcal{H}, \]

or

\[ (*) : \quad |\langle T\xi, \xi \rangle| \leq |\langle |T|\xi, \xi \rangle| \quad \text{for all } \xi \in \mathcal{H}. \]

Here, $T$ is an operator on a Hilbert space $\mathcal{H}$ with absolute value $|T| = (T^*T)^{1/2}$.

It is obvious that (**) implies (*). Based on the Cauchy-Schwarz inequality, one can show that (**) is equivalent to the operator inequality $|T^*| \leq |T|$ (Theorem 1.1). In particular, if $T$ is normal (i.e., $TT^* = T^*T$), then (**) holds. In the operator theory, several extensions of the notion of the normality are known (see, for example, [8]). One of the most important and most widely studied classes among them is the hyponormality (i.e., $TT^* \leq T^*T$) (see, for example, [5]). Since the square root function $t^{1/2}$ $(t \geq 0)$ preserves the (natural) order among positive operators ([7]), a hyponormal operator $T$ actually satisfies $|T^*| \leq |T|$ (and hence (**)). Therefore, we are looking at a slightly (and strictly ... see the end of §1) larger class than the hyponormality.

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In § 1, we will identify the class of operators satisfying (**). We will also show that for compact operators the validity of (**) is equivalent to the normality based on the following result due to T. Ando ([1]): A compact hyponormal operator is automatically normal (see also [4] and [9]).

In § 2, we will consider the condition (*). Firstly we will characterize (*) by making use of an operator \( X \) whose numerical radius \( w(X) \) satisfies \( w(X) \leq 1 \) (Theorem 2.1). Secondly we will also show that for (finite) matrices \( \dim(\mathcal{H}) = \infty \) the condition (***) actually implies the normality (Theorem 2.3). Consequently, (*), (**), and the normality are all equivalent for matrices.

A beautiful characterization of an operator \( X \) with \( w(X) = 1 \) was obtained by T. Ando ([2]). Based on this characterization and Theorem 2.1, in § 3 we will show that the class of operators satisfying (*) is strictly larger than the class of operators satisfying (***) (when \( \dim(\mathcal{H}) = \infty \)).

Finally, in Appendix, we will extend to the result obtained in § 2 to trace class operators. Based on T. Ando’s factorization of a numerical contraction \( X \) (i.e., \( w(X) = 1 \) ([2])), we will show that a numerical contraction and its adjoint have the same invariant vectors, and that for trace class operators the validity of (*) is equivalent to the normality.

The results in Appendix were suggested by the referee, and the author would like to thank the referee for the suggestion.

### § 1. Inequality (**)

In this section, we consider the following inequality for an operator \( T \in \mathcal{B}(\mathcal{H}) \):

\[
(**) \quad \langle (T\xi, \eta) \rangle^2 \leq \langle |T| \xi, \xi \rangle \langle |T| \eta, \eta \rangle \quad \text{for all } \xi, \eta \in \mathcal{H}.
\]

**Theorem 1.1.** For an operator \( T \in \mathcal{B}(\mathcal{H}) \), (**) holds for all \( \xi, \eta \in \mathcal{H} \) if and only if \( |T^*| \leq |T| \).

**Proof.** Let \( T = U |T| \) be the polar decomposition of \( T \). Then, since \( |T^*| = U |T| U^* \),

\[
\langle (T\xi, \eta) \rangle^2 = \langle (U |T|^{1/2} |T|^{1/2} \xi, \eta) \rangle^2
\]

\[
= \langle (|T|^{1/2} \xi, |T|^{1/2} U^* \eta) \rangle^2
\]

\[
\leq |||T|^{1/2} \xi||^2 |||T|^{1/2} U^* \eta||^2
\]

\[
= \langle |T| \xi, \xi \rangle (|T| \eta, \eta)
\]

for all \( \xi, \eta \in \mathcal{H} \). Therefore, if \( |T^*| \leq |T| \), then we get (**).

Conversely when (**) is valid, by replacing \( \xi, \eta \) by \( U^* \xi, \xi \), we get
Hence, we conclude $|T^*| \leq |T|$.

From the above theorem, we easily see that the normality of $T$ implies (**) But, in general, the inequality (**) does not imply that $T$ is normal (for example, an isometry). However, when $T$ is compact, we obtain:

**Theorem 1.2.** Let $T \in \mathcal{B}(\mathcal{H})$ be compact. Then (**) holds for all $\xi, \eta \in \mathcal{H}$ if and only if $T$ is normal.

To prove Theorem 1.2, we need the following fact due to T. Ando ([1] ••• see also [4] and [9]):

**Proposition 1.3.** A compact hyponormal operator in $\mathcal{B}(\mathcal{H})$ is normal.

**Proof of Theorem 1.2.** Let $T=U|T|$ be the polar decomposition of a compact operator $T$. We must show that $|T^*| \leq |T|$ implies the normality of $T$. By setting $S=U|T|^{1/2}$, we observe that

$$SS^*=U|T|U^*=|T^*|$$

$$\leq |T| = |T|^{1/2}U^*U|T|^{1/2} = S^*S,$$

i.e., $S$ is hyponormal. Since $T$ is compact, $S$ is also compact. Thus $S$ is actually normal by Proposition 1.3. On the other hand, since $S=U|T|^{1/2}$ is the polar decomposition of $S$, the normality of $S$ implies $UU^*=U^*U$ and $U|T|^{1/2}=|T|^{1/2}U$. Thus, $U|T|=|T|U$, and hence $T$ is normal. q.e.d.

The function $t^{1/2}$ ($t \geq 0$) is operator monotone ([7]). Therefore, the hyponormality (i.e., $TT^* \leq T^*T$) implies $|T^*| \leq |T|$. But $|T^*| \leq |T|$ does not necessarily imply the hyponormality of $T$. For example, consider the $2 \times 2$-matrices

$$r = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad s = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$ 

Note that $r \leq s$ and $r^2 \leq s^2$. We set
Then we compute
\[
TT^* = \begin{bmatrix}
0 & r^2 \\
0 & s^2 \\
s^2 & \ddots
\end{bmatrix}
\quad \text{and} \quad
T^*T = \begin{bmatrix}
r^2 & \ddots \\
s^2 & \ddots \\
& \ddots & \ddots
\end{bmatrix}.
\]

Therefore \(|T^*| \leq |T|\), but \(T\) is not hyponormal (because of \(r^2 \leq s^2\)).

§ 2. Inequality (*)

In this section, we consider the following inequality for an operator \(T \in \mathcal{B}(\mathcal{H})\):

\[
(*) \quad |(T\xi, \xi)| \leq |T| |\xi, \xi| \quad \text{for all } \xi \in \mathcal{H}.
\]

Obviously the inequality (**) implies (*), and hence the normality of \(T\) implies (*).

For an operator \(T \in \mathcal{B}(\mathcal{H})\), \(\sup \{|(T\xi, \xi)| : \xi \in \mathcal{H}, \|\xi\| = 1\}\) is called the numerical radius of \(T\) and denoted by \(w(T)\). Then the following inequality is standard ([6]):

\[
(1/2)\|T\| \leq w(T) \leq \|T\|.
\]

**Theorem 2.1.** For an operator \(T \in \mathcal{B}(\mathcal{H})\), (*) holds for all \(\xi \in \mathcal{H}\) if and only if

\[
U \|T\|^{1/2} = \|T\|^{1/2}X
\]

for some \(X \in \mathcal{B}(\mathcal{H})\) with \(w(X) \leq 1\), where \(T = U \|T\|\) is the polar decomposition of \(T\).

Related results were obtained in [3].

**Proof.** Suppose that (*) holds for all \(\xi \in \mathcal{H}\). For each positive integer \(n \in \mathbb{N}\), we define \(X_n \in \mathcal{B}(\mathcal{H})\) by

\[
X_n = \left\{ \|T\| + (1/n)I \right\}^{-1/2}U \left\{ \|T\| + (1/n)I \right\}^{1/2}.
\]

Then, for all \(\xi \in \mathcal{H}\),
(\(X_n\xi, \xi\)) := (U |T| + (1/\(n\)I)|^{1/2} \xi, \|T| + (1/\(n\)I)|^{-1/2} \xi) \\
= (U |T| + (1/\(n\)I)|^{1/2} \xi, \|T| + (1/\(n\)I)|^{-1/2} \xi) \\
= (T |T| + (1/\(n\)I)|^{-1/2} \xi, \|T| + (1/\(n\)I)|^{-1/2} \xi) \\
+ (1/\(n\))(U |T| + (1/\(n\)I)|^{-1/2} \xi, \|T| + (1/\(n\)I)|^{-1/2} \xi).

Thus, by (*) \((X_n\xi, \xi)\) is majorized by

\(|T| |T| + (1/\(n\)I)|^{-1/2} \xi, \|T| + (1/\(n\)I)|^{-1/2} \xi) \\
+ (1/\(n\))\|T| + (1/\(n\)I)|^{-1/2} \xi\|^2 \\
= (\{|T| + (1/\(n\)I)|^{1/2} \xi, \|T| + (1/\(n\)I)|^{-1/2} \xi\}) \\
= (\xi, \xi).

Therefore, we get

\(w(X_n) \leq 1\) and \(\|X_n\| \leq 2\).

Thus, by the Alaoglu theorem, we can construct a subnet \(\{X_{n_j}\}_{j \in J}\) converging weakly to some \(X \in \mathcal{B}(\mathcal{H})\) with \(\|X\| \leq 2\) from the sequence \(\{X_n\}_{n \in N}\). Then, we have \(w(X) \leq 1\) since

\((X\xi, \xi) = \lim_j (X_{n_j}\xi, \xi) \leq (\xi, \xi)\).

Now, from the definition of \(\{X_j\}_{j \in J}\), we get

\[(1) \quad U |T| + (1/F(j))|^{1/2} \Rightarrow \{T| + (1/F(j))|^{1/2}X_{F(j)}\}

for some mapping \(F: J \rightarrow N\) (in fact, \(X_j = X_{F(j)}\)). Hence, we conclude

\[U |T|^{1/2} = |T|^{1/2}X\]

by taking weak limits of both sides of (1) (see [7]).

Conversely, assume that \(U |T|^{1/2} = |T|^{1/2}X\) for some \(X \in \mathcal{B}(\mathcal{H})\) with \(w(X) \leq 1\). Then, for all \(\xi \in \mathcal{H}\),

\[|(T\xi, \xi)| = |(U |T|^{1/2} |T|^{1/2} \xi, \xi)| \\
= |(|T|^{1/2} |T|^{1/2} \xi, \xi)| \\
= |(|X| |T|^{1/2} \xi, |T|^{1/2} \xi)| \\
\leq |(|T|^{1/2} \xi, |T|^{1/2} \xi)| \\
= (|T| \xi, \xi)\text{ q.e.d.}

From the above theorem, we easily obtain:

**Corollary 2.2.** When \(T \in \mathcal{B}(\mathcal{H})\) satisfies (*) for all \(\xi \in \mathcal{H}\), we have
and

$$|T^*| \leq 4 |T|$$

$$UU^* \leq U^*U.$$  

For matrices (i.e., dim $\mathcal{H} < \infty$), we obtain the following characterization:

**Theorem 2.3.** Let $\mathcal{H}$ be a finite-dimensional Hilbert space. Then for $T \in \mathcal{B}(\mathcal{H})$, (*) holds for all $\xi \in \mathcal{H}$ if and only if $T$ is normal.

**Proof.** We may assume that $\mathcal{H} = \mathbb{C}^n$. Then $\mathcal{B}(\mathcal{H}) = M_n(\mathbb{C})$ (:= \{complex $n \times n$-matrix\}). Thanks to the obvious unitary invariance, we may and do assume that $T$ is of the form

$$
\begin{bmatrix}
  a_{11} & \cdots & a_{1n} \\
  \vdots & \ddots & \vdots \\
  0 & \cdots & a_{nn}
\end{bmatrix}
$$

(i.e., $T$ is an upper triangular matrix).

We will show that

$$a_{ij} = 0 \quad \text{if } i < j$$

by induction on the size $n$ of a matrix.

For $n = 1$ this result is trivial. Let

$$T = \begin{bmatrix} a & \beta^* \\ 0 & B \end{bmatrix} \quad \text{and} \quad |T| = \begin{bmatrix} x & \zeta^* \\ \zeta & Z \end{bmatrix}.$$  

Here, $a = a_{11}$, $x$ is a non-negative number, $\beta$ and $\zeta$ are (column) vectors in $\mathbb{C}^n$, $B$ is an upper triangular matrix in $M_n(\mathbb{C})$, and $Z$ is a positive matrix in $M_n(\mathbb{C})$. Then, since $T^*T = |T|^2$, we have

$$|a|^2 = x^2 + \zeta^*\zeta$$

by comparing the 1–1 components. Therefore $|a| \geq x$. On the other hand, with $\xi := (1, 0)$ in $\mathbb{C}^{n+1}$, we have $|a| \leq x$ by the assumption (*). Hence

$$x = |a| \quad \text{and} \quad \zeta = 0.$$  

Furthermore, since $T^*T = |T|^2$, we have

$$a^* \beta = 0$$

by comparing the 1–2 components.

We will show $\beta = 0$ by the contradiction argument. Then, the result follows from the induction hypothesis.

Assume $\beta \neq 0$, and hence $x = |a| = 0$ from (2). We choose and fix a (column) vector $\xi' (=\beta)$ in $\mathbb{C}^n$ such that
Let \( \xi=\langle p, \xi' \rangle \) in \( C^{n+1} \) \((p>0)\). Since \( a=0 \) and \( \xi=0 \), straightforward computations show

\[
(T\xi, \xi) = kp + (B\xi', \xi')
\]

and

\[
(|T|\xi, \xi) = (Z\xi', \xi').
\]

Therefore (*) does not hold for \( p \) sufficiently large, a contradiction. \ q. e. d.

\section*{§ 3. Relation of (*) and (**)}

From Theorem 1.2 and 2.3, for a finite-dimensional Hilbert space \( \mathcal{H} \), (*) is equivalent to (**) (and to the normality of \( T \)). Recall that (**) implies (*). But, in general, (*) does not imply (**) (i.e., \( |T^*| \leq |T| \) by Theorem 1.1).

We will consider an operator \( T \) of the form

\[
T = \begin{bmatrix}
0 & & \\
\alpha_1 & 0 & \\
& \alpha_2 & 0 \\
& & \ddots
\end{bmatrix}
\]

to explain this phenomenon. Here, \( \alpha_n \)'s are positive numbers to be fixed later.

We note that

\[
|T^*| = \begin{bmatrix}
0 & & \\
\alpha_1 & & \\
& \alpha_2 & \\
& & \ddots
\end{bmatrix}
\quad \text{and} \quad
|T| = \begin{bmatrix}
\alpha_1 & & \\
& \alpha_2 & \\
& & \ddots
\end{bmatrix}.
\]

Therefore, if the sequence \( \{\alpha_n\} \) is strictly decreasing, then \( |T^*| \leq |T| \) (i.e., (**)) does not hold. On the other hand, Corollary 2.3 indicates that, if \( \{\alpha_n\} \) decreases too rapidly, then (*) does not hold either. Thus, we are forced to choose a slowly decreasing sequence \( \{\alpha_n\} \) so that \( T \) does not satisfy (**) but (*).

We set

\[
e_n^{-1} = 3 \cdot 2^n - 4 \quad (n \geq 1).
\]

By using this sequence \( \{e_n\} \) (of positive numbers converging to 0), we set

\[
\alpha_1 = 1 \quad \text{and} \quad \alpha_{n+1} = \alpha_n (1 + e_n^{-1})^{-1} \quad (n \geq 1).
\]

Then \( \{\alpha_n\} \) is obviously decreasing, and it remains to show that (*) holds. For this purpose, we need the following result due to T. Ando ([2]): \( w(Y) \leq 1 \) if and only if \( Y = (I + A)^{1/2}B(I - A)^{1/2} \) with \(-I \leq A \leq I\) and \( \|B\| \leq 1 \) (we actually need just the easier half).

We define the sequence \( \{a_n\} \subset [-1, 1] \) (in fact, \( a_n \in [-1, 0) \)) by

\[
a_n^{-1} = -3 \cdot 2^{n-1} + 2 \quad (n \geq 1),
\]
and we set

\[
X = \begin{bmatrix}
\sqrt{1+a_1} & 0 \\
0 & \sqrt{1+a_2} \\
\sqrt{1+a_3} & \ddots & \ddots & \ddots
\end{bmatrix}
\begin{bmatrix}
\sqrt{1-a_1} \\
0 \\
\sqrt{1-a_2} \\
\vdots
\end{bmatrix}
\]

with

\[
U = \begin{bmatrix}
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & \ddots
\end{bmatrix}.
\]

In fact, the above \(U\) is the partial isometry appearing in the polar decomposition of \(T\). Then, by T. Ando's result, we have \(w(X) \leq 1\). It is straightforward to see

\[
X = \begin{bmatrix}
0 & 0 & 0 & \sqrt{1+e_1} \\
0 & \sqrt{1+e_2} & 0 & 0 \\
\sqrt{1+e_3} & \ddots & \ddots & \ddots
\end{bmatrix},
\]

thanks to

\[
1+e_n = (1+a_{n+1})(1-a_n).
\]

It is also easy to see

\[
U|T|^{1/2} = |T|^{1/2}X.
\]

Therefore, we see that \(T\) satisfies (*) by Theorem 2.1.

Since \(\sum_{n=1}^{\infty} e_n\) is convergent, so is

\[
\alpha_1 \cdot \alpha_n^{-1} = \prod_{\ell=1}^{n-1} (1+e_\ell).
\]

Therefore, \(\lim_{n \to \infty} \alpha_n = 0\), and the above \(T\) is not compact.

The author does not know whether the condition (*) and the normality are different for compact operators (in fact, we can confine ourselves to the case \(T\) is compact quasi-nilpotent according to the way used in Theorem 2.3), and this problem seems to deserve further investigation.

**Appendix**

Theorem 2.3 is extended to trace class operators, that is, for them the condition (*) is equivalent to the normality. We show this by the a method different from that of Theorem 2.3.

**Lemma.** Let \(X \in \mathcal{B}(\mathcal{H})\) be a numerical contraction, i.e., \(w(X) \leq 1\). Then \(X\xi = \xi\) implies \(X^*\xi = \xi\).
Proof. By T. Ando’s factorization of a numerical contraction ([2]), there exist a self-adjoint contraction \( A \in \mathcal{B}(\mathcal{H}) \) and a contraction \( B \in \mathcal{B}(\mathcal{H}) \) such that
\[
X = (I + A)^{1/2} B (I - A)^{1/2}
\]
and \( B \) is isometric on the range of \( I - A \). Since
\[
(\xi, \xi) = (X \xi, \xi) = (I + A)^{1/2} B (I - A)^{1/2} \xi, \xi
\]
\[
\leq \|B(I - A)^{1/2} \xi\| \cdot \|B(I + A)^{1/2} \xi\|
\]
\[
\leq (1/2) \| (I - A)^{1/2} \xi \|^2 + \| (I + A)^{1/2} \xi \|^2
\]
\[
= (\xi, \xi),
\]
we have
\[
(3) \quad B(I - A)^{1/2} \xi = c(I + A)^{1/2} \xi
\]
for some scalar \( c \) and
\[
(4) \quad ((I - A) \xi, \xi) = ((I + A) \xi, \xi).
\]
From (4), we have
\[
(5) \quad (A \xi, \xi) = 0.
\]
Since \( B \) is isometric on the range of \( (I - A) \), we have from (3)
\[
(I - A) \xi = (I - A)^{1/2} B^* B (I - A)^{1/2} \xi
\]
\[
= c(I - A)^{1/2} B^* (I + A)^{1/2} \xi
\]
\[
= c X^* \xi
\]
by the polarization identity. Therefore
\[
(\xi, \xi) = ((I - A) \xi, \xi) = c(X^* \xi, \xi)
\]
\[
= c(\xi, \xi)
\]
and hence \( c = 1 \) and \( X^* \xi = \xi - A \xi \). But since
\[
\xi = X^* \xi = (I + A)^{1/2} B (I - A)^{1/2} \xi
\]
\[
= (I + A) \xi,
\]
we obtain \( X^* \xi = \xi \). q.e.d.

Theorem. Let \( T \in \mathcal{B}(\mathcal{H}) \) be of trace class. Then (*) holds for all \( \xi \in \mathcal{H} \) if and only if \( T \) is normal.

Proof. By Theorem 2.1, there exists a numerical contraction \( X \in \mathcal{B}(\mathcal{H}) \) such that
\[
U |T|^{1/2} = |T|^{1/2} X,
\]
where \( T = U | T | \) is the polar decomposition of \( T \). The space \( \mathcal{C}_d(H) \) of Hilbert-Schmidt class operators becomes a Hilbert space with the inner product \( \langle K, L \rangle = Tr(L^*K) \) for \( K, L \in \mathcal{C}_d(H) \). We define the operator \( \Phi \) on \( \mathcal{C}_d(H) \) by \( \Phi(K) = U^* K X \). Then \( \Phi(|T|^{1/2}) = U^* |T|^{1/2} X = |T|^{1/2} \).

Now, by T. Ando’s factorization, we are led to the representation
\[
X = (I + A)^{1/2} B (I - A)^{1/2}
\]
with a self-adjoint contraction \( A \in \mathcal{B}(H) \) and a contraction \( B \in \mathcal{B}(H) \). Then
\[
\Phi = L_{U^* X} R_X = L_{U^* X} R_{(I - A)^{1/2}} R_B R_{(I + A)^{1/2}} = (I - R_A)^{1/2} (L_{U^* X} R_B)^* (I + R_A)^{1/2}.
\]
Here, \( L_D \) and \( R_D \) are the left- and right-multiplication operator on \( \mathcal{C}_d(H) \) induced by \( D \in \mathcal{B}(H) \) respectively. Again by T. Ando’s result, we get \( \omega(\Phi) \leq 1 \). By virtue of the above lemma, we have \( \Phi(|T|^{1/2}) = U |T|^{1/2} X^* = |T|^{1/2} \) and hence
\[
U^* |T|^{1/2} = |T|^{1/2} X^*.
\]
Therefore, we get \( \text{Re}(U^* T |T|^{1/2}) = |T|^{1/2} \text{Re}(X) \) and \( \text{Im}(U^* T |T|^{1/2}) = |T|^{1/2} \text{Im}(X) \).

Let \( \{E_{\text{Re}(U)}(S) : S \text{ is a Borel subset of } \mathbb{R}\} \) and \( \{E_{\text{Re}(X)}(S) : S \text{ is a Borel subset of } \mathbb{R}\} \) be the spectral projections of \( \text{Re}(U) \) and \( \text{Re}(X) \) respectively. Then, since \( \text{Re}(U^* T |T|^{1/2}) = |T|^{1/2} \text{Re}(X) \), we have \( E_{\text{Re}(U)}(S) |T|^{1/2} = |T|^{1/2} E_{\text{Re}(X)}(S) \). This implies
\[
E_{\text{Re}(U)}(S) T |T|^{1/2} E_{\text{Re}(U)}(S) \leq |T|.
\]
But \( E_{\text{Re}(U)}(S) T |T|^{1/2} E_{\text{Re}(U)}(S) \leq |T| \) is possible only when \( E_{\text{Re}(U)}(S) \) commutes with \( |T| \). Therefore, we are led to the commutativity of \( \text{Re}(U) \) and \( |T| \). In a similar fashion, \( \text{Im}(U) \) commutes with \( |T| \). Hence, we obtain
\[
U^* |T|^{1/2} = |T| U^*,
\]
i.e., \( T \) is quasi-normal. Since \( T \) is of trace class, \( T \) is normal by Proposition 1.3.

q.e.d.

References

[5] Conway, J.B. and Szymanski, W., Linear combinations of hyponormal operators,


