General Integral Representation of the Holomorphic Functions on the Analytic Subvariety

By

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§1. Introduction

Henkin[1] and Ramirez[2] obtained an integral representation of holomorphic functions for strictly pseudoconvex domains in $\mathbb{C}^n$. Range and Siu[3] gave a generalization of Henkin-Ramirez's formula to the domains in $\mathbb{C}^n$ with piecewise smooth strictly pseudoconvex boundaries Sommer[4] proved an integral formula of Weil type for analytic polyhedra in $\mathbb{C}^n$. Sergeev and Henkin[5] also obtained an integral representation for the strictly pseudoconvex polyhedra. Stout[6] and Hatziafratis[7] have respectively proved integral formulas for strictly pseudoconvex domains in codimension-one and codimension-$m$ complex submanifolds of $\mathbb{C}^n$. The formula which was given by Stout is valid not only for nonsingular hypersurfaces, but also for certain subvarieties which may possess sufficiently restricted singular points. Hatziafratis' work is based on the results of Stout.

In this paper we derive integral formulas which include all the above ([1]–[7]) integral formulas for holomorphic functions. The papers of Stout[6], Hatziafratis[7] and the author[8] are most relevant references to this work.

§2. Definitions, Symbols and Terms

Definition 1 (Polyhedral domain[8]) Let $\Omega$ be a domain of holomorphy in $\mathbb{C}^n$. An open set $D \subset \subset \Omega$ is called a polyhedral domain if there is a neighbourhood $U_{\bar{D}}$ of $\bar{D}$ and holomorphic mappings:
and $D_\alpha \subset \subset \mathcal{C}^{m_\alpha}$, $\alpha = 1,2,\ldots, N$, such that
\[
D = X_1^{-1}(D_1) \cap \cdots \cap X_N^{-1}(D_N).
\]

If $P_1,\ldots,P_N$ are differentiable functions in the neighbourhoods $\theta_1,\ldots,\theta_N$ of $\partial D_1,\ldots,\partial D_N$ respectively, and
\[
D_\alpha \cap \theta_\alpha = \{ z \in \theta_\alpha; P_\alpha(z) < 0 \}, \quad \alpha = 1, 2, \ldots, N,
\]
then $\partial D \equiv X_1^{-1}(\theta_1) \cup \cdots \cup X_N^{-1}(\theta_N)$ and a point $z \in X_1^{-1}(\theta_1) \cup \cdots \cup X_N^{-1}(\theta_N)$ belongs to $D$ if and only if $z \in X_\alpha^{-1}(\theta_\alpha)$ and $P[\alpha(z)] < 0$ for some $\alpha: 1 \leq \alpha \leq N$. $D$ is called a non-degenerate polyhedral domain, if we can choose the functions $X_\alpha$ and $P_\alpha$ so that
\[
d(P_{z_1} \cdot X_{z_1})(z) \wedge \cdots \wedge d(P_{z_i} \cdot Z_{z_i})(z) \neq 0.
\]

Whenever $P_{z_i}[X_{z_i}(z)] = \cdots = P_{z_i}[X_{z_i}(z)] = 0$, for all $1 \leq z_1 < \cdots < z_i \leq N$.

In this paper we only consider non-degenerate polyhedral domains.

A nondegenerate polyhedral domain will be called a strictly pseudoconvex polyhedron if $P_\alpha(\alpha = 1,2,\ldots,N)$ are strictly plurisubharmonic functions; and called a holomorphic polyhedron (including Weil polyhedron) if the mapping $P_\alpha(\alpha = 1,2,\ldots,N)$ are pluriharmonic functions (or usual harmonic functions when $n = 1$), i.e. $P_\alpha(\alpha = 1,2,\ldots,N)$ are twice continuously differentiable and
\[
\partial^2 P_\alpha / \partial z_j \partial \bar{z}_k = 0, \quad j,k = 1,2,\ldots,n.
\]

There exist continuously differentiable support functions for the nondegenerate polyhedral domains, support functions holomorphic in $z$ for strictly pseudoconvex polyhedrons, and holomorphic support functions for the holomorphic polyhedron.

**Definition 2** (Space with slits\[19\]) A compact metric space $R$ is called a slit space or a space with slit if $S$ is a nonempty closed subset of $R$ each point of which is an accumulation point of $R-S$ and $R-S$ is homeomorphic to a topological product $X \times Y$, where $X$ is a connected $\tilde{m}$-dimensional differential manifold of class $C^2$, called the base space, and $Y$ is a compact set, called the side space. The homeomorphism $\varphi: X \times Y \rightarrow R-S$ is called the coordinate function.

**Example.** The closure $R$ of any bounded domain in a $\tilde{n}$-dimensional
Euclidean space \( E^n \) can be considered a slit space with the boundary as its slit. \( Y \) is then a set consisting of a single point.

**Definition 3.** A sequence of spaces \( R_1 \supset R_2 \supset \cdots \supset R_k \) is called a chain of slit spaces, if each \( R_v \) is a slit space with \( R_{v+1} \) as its slit \((v = 1, \ldots, k - 1)\).

Firstly, we consider the following two types of bounded domains \( D \subset \Omega \) in \( \mathbb{C}^n \):

1. Its boundary \( \partial D \) consists of a chain of slit spaces, and this chain can be written as:

\[
\partial D = \sigma^{(1)} \supset \sigma^{(2)} \supset \cdots \supset \sigma^{(\beta)} = \sigma_1^{(\beta - 1)} \supset \sigma_1^{(\beta - 1)} \supset \cdots \supset \sigma_k^{(\beta - 1)}
\]

where \( \sigma_{v+1}^{(\beta - 1)} \) is the slit of \( \sigma_v^{(\beta - 1)} \), \( \sigma_{v+1}^{(\beta - 1)} \) is the slit of \( \sigma_v^{(\beta - 1)} = \bigcup_{j_1 \cdots j_v} \sigma_{j_1 \cdots j_v}^{(\beta - 1)} \), \( \sigma_{j_1 \cdots j_v}^{(\beta - 1)} \) is of real dimension \( 2n - \beta - v + 1 \); \( \sigma^{(0)} = \bigcup_{k_1 \cdots k_i} \sigma_{k_1 \cdots k_i}^{(0)} \), and \( \sigma_{k_1 \cdots k_i}^{(0)} \) is of real dimension \( 2n - i \). \( \sigma_k^{(\beta - 1)} \) is called the distinguished boundary of \( D \).

**Example.** The closed bicylinder \( R = \{(z_1, z_2) : |z_1| \leq 1, |z_2| \leq 1\} \) can be considered a space with the boundary \( R_1 = \{(z_1, z_2) : |z_1| = 1, |z_2| \leq 1 \text{ and } |z_1| \leq 1, |z_2| = 1\} \) as slit. Moreover the boundary \( R_1 \) can also be considered a space with slit \( R_2 = \{(z_1, z_2) : |z_1| = 1, |z_2| = 1\} \).

2. Its boundary \( \partial D \) consists of a chain of slit spaces, and this chain can be written as:

\[
\partial D = \sigma^{(1)} \supset \cdots \supset \sigma^{(n)} \supset \cdots \supset \sigma^{(n)} = \sigma_1^{(\beta - 1)} \supset \sigma_1^{(\beta - 1)} \supset \cdots \supset \sigma_k^{(\beta - 1)},
\]

where \( \sigma^{(\beta)} \) is slit of \( \sigma^{(n)} \), and the dimensions of \( \sigma^{(n)} \) may be at least one dimension greater than the dimensions of \( \sigma^{(\beta)} \). \( \sigma_k^{(\beta - 1)} \) is also called the distinguished boundary of \( D \).

**Example.** The closure of all invariant subspaces of the classical domain \(^9\) consists of a chain of slit spaces mentioned above.

Secondly, if \( F_1, \ldots, F_m \) are holomorphic functions in the neighbourhood \( U_{\bar{D}} \) of \( \bar{D} \), and set

\[
Z(F_1, \ldots, F_m) = \{z \in U_{\bar{D}} : F_1(z) = \cdots = F_m(z) = 0\}.
\]

We assume that \( Z(F_1, \ldots, F_m) \) meet \( \partial D \) transversally. We set \( \bar{D} = Z(F_1, \ldots, F_m) \cap D \), and consider
\[ 
\langle I \rangle \quad \partial \mathcal{D} = \sigma^{(1)} \circ \sigma^{(2)} \circ \cdots \circ \sigma^{(\beta)} = \sigma^{(\beta-1)}_1 \circ \sigma^{(\beta-1)}_2 \circ \cdots \circ \sigma^{(\beta-1)}_k, 
\]
\[ 
\langle II \rangle \quad \partial \mathcal{D} = \sigma^{(1)} \circ \cdots \circ \sigma^{(\alpha)} \circ \cdots \circ \sigma^{(\beta)} = \sigma^{(\beta-1)}_1 \circ \cdots \circ \sigma^{(\beta-1)}_k, 
\]

where \( \sigma^{(\theta)} = Z(F_1, \cdots, F_m) \cap \sigma^{(\theta)} \) and \( \sigma^{(\beta-1)}_j = Z(F_1, \cdots, F_m) \cap \sigma^{(\beta-1)}_k. \)

When \( m = 0 \), \( \langle I \rangle \) and \( \langle II \rangle \) coincide with \( \langle I \rangle \) and \( \langle II \rangle \) respectively.

According to Hefer's theorem, we have

\[ F_l(\zeta) - F_l(z) = \sum_{j=1}^n (\zeta_j - z_j) h_{ij}(\zeta, z), l = 1, 2, \cdots, m, \]

where \( h_{ij} \) are holomorphic functions on a neighbourhood of \( \mathcal{D} \times \mathcal{D}. \)

\section{3. Some Lemmas}

In what follows let \( D \) be a nondegenerate polyhedral domain.

**Lemma 1.** Let \( M_1 \) be a continuously differentiable support function for \( \mathcal{D} \), then we have

\[ 
\bar{\partial}_s \det_{(n)} \left( \frac{N_1}{M_1}, h_1, \cdots, h_k, \bar{\partial}_t \left( \frac{N_1}{M_1} \right) \cdots, \bar{\partial}_t \left( \frac{N_1}{M_1} \right) \right) 
\]

\[ = F_{k+1}(\zeta) \det_{(n)} \left( \frac{N_1}{M_1}, h_1, \cdots, h_k, \bar{\partial}_t \left( \frac{N_1}{M_1} \right) \cdots, \bar{\partial}_t \left( \frac{N_1}{M_1} \right) \right) 
\]

on \( \mathcal{D} - Z(F_{k+1}). \)

**Proof.** Since \( M_1 = M_1(\zeta, z) = \sum_{j=1}^n (\zeta_j - z_j) N_{1j}(\zeta, z) \), i.e. \( \sum_{j=1}^n (\zeta_j - z_j) \frac{N_{1j}}{M_1} = 1 \),

we obtain \( \sum_{j=1}^n (\zeta_j - z_j) \bar{\partial}_s(N_{1j}/M_1) = 0. \) Thus we have the following determinant of \( (n+1) \times (n+1) \):
on $Z(F_1,\ldots,F_k) - Z(F_{k+1})$. Taking it into account that $F_1,\ldots,F_m$ are the holomorphic functions and $\bar{\partial}_\zeta\bar{\partial}_\zeta(N_j/M_1) = 0$, by (2) we have

$$(-1)^{k+1} F_{k+1}(\zeta) \det_{(o)}\left(\frac{N_1}{M_1}, h_1, \ldots, h_k, \bar{\partial}_\zeta\left(\frac{N_1}{M_1}\right), \ldots, \bar{\partial}_\zeta\left(\frac{N_1}{M_1}\right)\right)$$

$$= \det_{(o)}\left(h_1, \ldots, h_k, h_{k+1}, \bar{\partial}_\zeta\left(\frac{N_1}{M_1}\right), \ldots, \bar{\partial}_\zeta\left(\frac{N_1}{M_1}\right)\right)$$

$$=(-1)^{k+1} \det_{(o)}\left(\bar{\partial}_\zeta\left(\frac{N_1}{M_1}\right), h_1, \ldots, h_{k+1}, \bar{\partial}_\zeta\left(\frac{N_1}{M_1}\right), \ldots, \bar{\partial}_\zeta\left(\frac{N_1}{M_1}\right)\right)$$

$$=(-1)^{k+1} \bar{\partial}_\zeta\det_{(o)}\left(\frac{N_1}{M_1}, h_1, \ldots, h_{k+1}, \bar{\partial}_\zeta\left(\frac{N_1}{M_1}\right), \ldots, \bar{\partial}_\zeta\left(\frac{N_1}{M_1}\right)\right).$$

Thus we obtain (1).

Since

$$\bar{\partial}_\zeta\left(\frac{N_{1j}}{M_1}\right) = \bar{\partial}_\zeta N_{1j} - \frac{N_{1j}}{M_1} \cdot \bar{\partial}_\zeta M_1,$$

we can apply the properties of the determinant and write (1) as:

$$\frac{1}{M_1^{n-k-1}} \det_{(o)} (N_1, h_1, \ldots, h_k, h_{k+1}, \bar{\partial}_\zeta N_1, \ldots, \bar{\partial}_\zeta N_1)$$

$$= \frac{F_{k+1}(\zeta)}{M_1^{n-k}} \det_{(o)} (N_1, h_1, \ldots, h_k, \bar{\partial}_\zeta N_1, \ldots, \bar{\partial}_\zeta N_1).$$

Especially when $k=0$, we have
\[ \tilde{\partial}_\zeta \det_\phi \left( \frac{N_1}{M_1}, \frac{\tilde{\partial}_\zeta N_1}{M_1}, \cdots, \frac{\tilde{\partial}_\zeta N_1}{M_1} \right) = \frac{F_1(\zeta)}{M_1^n} \det_\phi (N_1, \tilde{\partial}_\zeta N_1, \cdots, \tilde{\partial}_\zeta N_1) \]

on \( D - Z(F_1) \).

**Lemma 2.** If

\[ B_{k+1}^F(\zeta) = \frac{(n-k-1)!}{|V_{k+1}^F(\zeta)|^2} \sum_{1 \leq j_1 < \cdots < j_{k+1} \leq n} \left( -1 \right)^{j_1 + \cdots + j_{k+1}} \frac{\partial (F_1, \cdots, F_{k+1})}{\partial (\zeta_{j_1}, \cdots, \zeta_{j_{k+1}})} \wedge \frac{d\zeta_{j_1}}{1 \neq j_1, \cdots, j_{k+1}} \]

where

\[ |V_{k+1}^F(\zeta)|^2 = \sum_{1 \leq j_1 < \cdots < j_{k+1} \leq n} \frac{\partial (F_1, \cdots, F_{k+1})}{\partial (\zeta_{j_1}, \cdots, \zeta_{j_{k+1}})}^2 \neq 0 \]

(especially \( B_0^F(\zeta) = n! \omega(\zeta), \omega(\zeta) = d\zeta_1 \wedge \cdots \wedge d\zeta_n \)), then we have

\[ B_{k+1}^F(\zeta) \wedge dF_{k+1} = \frac{(-1)^{n+k}}{n-k} B_k^F(\zeta) \]

on \( D \).

**Remark.** When \( k = 0 \), as \( |V_{k+1}^F(\zeta)|^2 = \sum_{j=1}^n \left| \frac{\partial F_1}{\partial \zeta_j} \right|^2 \), so (5) may be written as

\[ B_1^F(\zeta) \wedge dF_1 = \frac{(n-1)!}{|V_1^F(\zeta)|^2} \sum_{j=1}^n (-1)^j \frac{\partial F_1}{\partial \zeta_j} d\zeta_1 \wedge \cdots \wedge [d\zeta_j] \wedge \cdots \wedge d\zeta_n \wedge \]

\[ \sum_{j=1}^n \frac{\partial F_1}{\partial \zeta_j} d\zeta_j \]

\[ = (-1)^n (n-1)! d\zeta_1 \wedge \cdots \wedge d\zeta_n = \frac{(-1)^n}{n} B_0^F(\zeta). \]

**The proof of Lemma 2**. First of all, Notice that \( |V_{k+1}^F(\zeta)| \neq 0 \) implies \( |V_k^F(\zeta)| \neq 0 \). Since we can assume that
where $\zeta_0 \in Z(F_1, \ldots, F_k)$, according to the implicit function theorem, restricted to $Z(F_1, \ldots, F_k)$ locally at a point $\zeta_0$, we have
\[ \zeta_{n-k+j} = \bar{F}_j(\zeta^{(k)}), \quad j = 1, \ldots, k; \zeta^{(k)} = (\zeta_1, \ldots, \zeta_{n-k}) \]
such that
\[ F_j(\zeta^{(k)}), \quad \bar{F}_i(\zeta^{(k)}), \ldots, \bar{F}_k(\zeta^{(k)}) \equiv 0, \quad j = 1, \ldots, k, \]
therefore, the following equations are true,
\[ (*) \quad \frac{\partial F_j}{\partial \zeta_{n-k+1}} \frac{\partial \bar{F}_1}{\partial \zeta_1} + \ldots + \frac{\partial F_j}{\partial \zeta_n} \frac{\partial \bar{F}_k}{\partial \zeta_k} = - \frac{\partial F_j}{\partial \zeta_i}, \quad i = 1, \ldots, n-k. \]

For a fixed sequence $1 \leq j_1 < \cdots < j_k \leq n$, let us assume that
\[ 1 \leq j_1 < \cdots < j_k < n - k + 1 \leq j_{k+1} < \cdots < j_k \leq n. \]

Then it follows from (*) that
\[ \prod_{1 \leq i \leq n-k} d\zeta_i = d\zeta_{j_1} \cdots d\zeta_{j_k} \cdots d\zeta_{n-k} \cdots d\zeta_{n-k+1} \cdots d\zeta_{n-1} \cdots d\zeta_n \]
where
\[ \delta_1 = (-1)^{n(n-1)}(-1)^{j_1+\cdots+j_k}(-1)^{(k-1)(k-2)/2}. \]

So we have
\[ \prod_{1 \leq i \leq n-k} d\zeta_i = \delta_2 \frac{\partial(\bar{F}_1, \ldots, \bar{F}_{n-k+1}, \ldots, \bar{F}_k)}{\partial(\zeta_{j_1}, \ldots, \zeta_{j_k})} \prod_{1 \leq i \leq n-k} d\zeta_i, \]
where
\[ \delta_2 = (-1)^{n(n-1)}(-1)^{j_1+\cdots+j_k}(-1)^{(k-1)(k-2)/2}. \]

From the above we obtain
where \( \delta = (-1)^{k(n-k)}(-1)^{k(k+1)/2} \).

It follows from (**) that

\[
(***) \quad B_k^\delta(\zeta) = \frac{(n-k)\delta}{|V_k^F(\zeta)|^2} \sum_{j_1 < \ldots < j_k \leq n} \frac{\partial(F_1, \ldots, F_k)}{\partial(\zeta_{j_1}, \ldots, \zeta_{j_k})} \cdot \frac{\partial(F_1, \ldots, F_k)}{\partial(\zeta_{n-k+1}, \ldots, \zeta_n)} \left( \frac{\partial(F_1, \ldots, F_k)}{\partial(\zeta_{n-k+1}, \ldots, \zeta_n)} \right)^{-1} \wedge d\zeta_i.
\]

Using the above expression, we obtain

\[
B_{k+1}^\delta \wedge DF_{k+1} = \frac{(n-k-1)!(-1)^{n+k}}{|V_{k+1}^F(\zeta)|^2} \sum_{i=1}^{n} (-1)^{i_1 + \ldots + i_k} \left[ \sum_{i=1}^{n} \frac{\partial F_{k+1}}{\partial \zeta_i} \wedge \frac{\partial (F_1, \ldots, F_k, F_{k+1})}{\partial (\zeta_{j_1}, \ldots, \zeta_{j_k}, \zeta_i)} \right] \wedge d\zeta_i \wedge \left( \sum_{i=1}^{n} \frac{\partial F_{k+1}}{\partial \zeta_i} \wedge \frac{\partial (F_1, \ldots, F_k, F_{k+1})}{\partial (\zeta_{n-k+1}, \ldots, \zeta_n)} \right)^{-1} \left( \frac{\partial (F_1, \ldots, F_k)}{\partial (\zeta_{n-k+1}, \ldots, \zeta_n)} \right)^{-1} \wedge d\zeta_i.
\]

**Lemma 3.** Let \( D_k = Z(F_1, \ldots, F_k) \cap D \), \( D_0 = D \), \( D_n = \overline{D} \) and \( M_d \) a continuously differentiable support function. If \( f(z) \) is a holomorphic function on \( \overline{D} \) and \( |V_{k+1}^F(\zeta)| \neq 0 \), then
\[ c_{k+1} \int_{\partial D_{k+1}} \frac{f(\zeta)}{M_{1}^{n-k+1}} \det_{(0)}(N_{1}, h_{1}, \ldots, h_{k+1}, \vec{\partial}_{\zeta} N_{1}, \ldots, \vec{\partial}_{\zeta} N_{1}) \wedge B_{k+1}^{F}(\zeta) \]

\[ = c_{k} \int_{\partial D_{k}} \frac{f(\zeta)}{M_{1}^{n-k}} \det_{(0)}(N_{1}, h_{1}, \ldots, h_{k}, \vec{\partial}_{\zeta} N_{1}, \ldots, \vec{\partial}_{\zeta} N_{1}) \wedge B_{k}^{F}(\zeta) \]  

(6)

where \( c_{k} = (-1)^{k(n+1)}(-1)^{k(n+1)/2}/(n-k)! (2\pi i)^{n-k} \).

**Proof.** Let \((\partial D_{k})_{\varepsilon} = \{\zeta \in \partial D_{k} : |F_{k+1}(\zeta)| > \varepsilon\}\). By lemma 1 and lemma 2, taking account that \( c_{k+1} = (-1)^{n+k}(n-k)2\pi i c_{k} \), we obtain

\[ c_{k} \int_{(\partial D_{k})_{\varepsilon}} \frac{f(\zeta)}{M_{1}^{n-k+1}} \det_{(0)}(N_{1}, h_{1}, \ldots, h_{k}, \vec{\partial}_{\zeta} N_{1}, \ldots, \vec{\partial}_{\zeta} N_{1}) \wedge B_{k}^{F}(\zeta) \]

\[ = \int_{(\partial D_{k})_{\varepsilon}} d_{\zeta} \left[ \frac{f(\zeta)}{M_{1}^{n-k-1}} \det_{(0)}(N_{1}, h_{1}, \ldots, h_{k}, h_{k+1}, \vec{\partial}_{\zeta} N_{1}, \ldots, \vec{\partial}_{\zeta} N_{1}) \right] \]

\[ \wedge (-1)^{n+k}(n-k)B_{k+1}^{F}(\zeta) \wedge \frac{dF_{k+1}}{F_{k+1}} \]

\[ = (n-k)(-1)^{n+k}c_{k} \int_{(\partial D_{k})_{\varepsilon}} \frac{f(\zeta)}{M_{1}^{n-k-1}} \det_{(0)}(N_{1}, h_{1}, \ldots, h_{k+1}, \vec{\partial}_{\zeta} N_{1}, \ldots, \vec{\partial}_{\zeta} N_{1}) \]

\[ \wedge B_{k+1}^{F}(\zeta) \wedge \frac{dF_{k+1}}{F_{k+1}} \]

\[ = \frac{c_{k+1}}{2\pi i} \int_{|\tau| = \varepsilon} \left[ \int_{(\partial D_{k}: F_{k+1}(\zeta) = \tau_{0})} \frac{f(\zeta)}{M_{1}^{n-k-1}} \det_{(0)}(N_{1}, h_{1}, \ldots, h_{k+1}, \vec{\partial}_{\zeta} N_{1}, \ldots, \vec{\partial}_{\zeta} N_{1}) \right] dt \]

\[ \wedge B_{k+1}^{F}(\zeta) \wedge \frac{dF_{k+1}}{F_{k+1}} \]  

(7)

Let \( s_{0} = \partial D_{k+1} = \{\zeta \in \partial D_{k} : F_{k+1}(\zeta) = 0\}\) and \( v(\zeta) \) be the normal direction at \( \zeta \in s_{0} \). We consider the smooth mapping \( f_{\zeta}(\zeta, \tau) : \zeta + \tau v(\zeta) \rightarrow V_{0} \equiv \{\zeta \in \partial D_{k} : F_{k+1}(\zeta) = \tau\} \). Since \( s_{0} \) is compact and the Jacobian \( Jf(\zeta, \tau) \neq 0 \) for every \((\zeta, 0) \) in \( R^{2n-2k-1} \), there is the inverse \( f^{-1} \). Here \( \varepsilon \) is chosen to be sufficiently small. From the above, we conclude that \( \{\zeta \in \partial D_{k} : F_{k+1}(\zeta) = \tau\} \) is (for \( \tau \in C, |\tau| < \varepsilon \) and \( \varepsilon \) a small positive number) diffeomorphic to \( \partial D_{k+1} \).

When \( \varepsilon \rightarrow 0 \), the left side and right side of (7) tend to the right side.
and left side of (6) respectively.

**Corollary 1.** Let \( f(z) \) be a holomorphic function on \( D_1 \), then when \( z \in D_1 \), and \( |\nabla^F_1(\zeta)| \neq 0 \) on \( \partial D_1 \), we have

\[
f(z) = c_1 \int_{\partial D_1} \frac{f(\zeta)}{M^1_t} \det_{(n)}(N_1, h_1, \bar{\partial}_\zeta N_1, \cdots, \bar{\partial}_\zeta N_1) \land B^1_t(\zeta).
\]

**(Proof.)** Applying remark for lemma 2 and (6), we have

\[
c_1 \int_{\partial D_1} \frac{f(\zeta)}{M^1_t} \det_{(n)}(N_1, h_1, \bar{\partial}_\zeta N_1, \cdots, \bar{\partial}_\zeta N_1) \land B^1_t(\zeta)
= c_0 \int_{\partial D_0} \frac{f(\zeta)}{M^1_t} \det_{(n)}(N_1, \bar{\partial}_\zeta N_1, \cdots, \bar{\partial}_\zeta N_1) \land n! \omega(\zeta)
= \frac{1}{(2\pi)^n} \int_{\partial D} \frac{f(\zeta)}{M^1_t} \det_{(n)}(N_1, \bar{\partial}_\zeta N_1, \cdots, \bar{\partial}_\zeta N_1) \land \omega(\zeta).
\]

By Cauchy-Fantappie formula, the right-hand side of (9) equals to \( f(z) \), and (8) is obtained.

**Remark.** In fact, representation (8) is more evident than that in [6].

**Corollary 2.** Let \( f(z) \) be a holomorphic function on \( \tilde{D} \), then when \( z \in \tilde{D} \) and \( |\nabla^m_1(\zeta)| \neq 0 \) on \( \partial \tilde{D} \), we have

\[
f(z) = c_m \int_{\partial \tilde{D}} \frac{f(\zeta)}{M^m_1} \det_{(n)}(N_1, h_1, \cdots, h_m, \bar{\partial}_\zeta N_1, \cdots, \bar{\partial}_\zeta N_1) \land B^m_m(\zeta).
\]

**(Proof.)** By lemma 3 and its corollary 1, we obtain (10).

**Lemma 4.** Let \( T^0_0 = \sum_{j=1}^n (\zeta_j - z_j)S^0_0(\zeta, z) \neq 0(\zeta \neq z) \) for some continuous functions \( S^0_0(i=1,2) \) on \( D \), and let \( T_1 = \sum_{j=1}^n (\zeta_j - z_j)S_{ij}(\zeta, z) \) be continuously differentiable support functions for \( \tilde{D} \). Then we have

\[
\det_{(n)} \left( \frac{S^0_0(T^0_0)}{h_1, \cdots, h_k, \bar{\partial}_\zeta S_1}{T^1_1}, \cdots, \bar{\partial}_\zeta S_{n-k-1}^T \right)
\]
\[ = \det\left( \frac{S^{(2)}_0}{T^{(2)}_0} h_1, \ldots, h_k \bar{\partial}_z \left( \frac{S_1}{T_1} \right), \ldots, \bar{\partial}_z \left( \frac{S_{n-k-1}}{T_{n-k-1}} \right) \right) \]  

(11)

on \( D_k \).

**Proof.** Since \( \sum_{j=1}^{n} (\zeta_j - z_j) S_{ij} / T_i = 1 \), then

\[ \sum_{j=1}^{n} (\zeta_j - z_j) \bar{\partial}_z (S_{ij} / T_i) = 0, \quad l = 1, \ldots, n - k - 1. \]  

(12)

Since

\[ \sum_{j=1}^{n} (\zeta_j - z_j) \left( \frac{S^{(1)}_0}{T^{(1)}_0} - \frac{S^{(2)}_0}{T^{(2)}_0} \right) = 0, \]  

(13)

and on \( D_k \)

\[ 0 = F_l(\zeta) - F_l(z) = \sum_{j=1}^{n} (\zeta_j - z_j) h_{ij}(\zeta, z), l = 1, \ldots, k, \]  

(14)

thus by (12)–(14), we obtain

\[ \det\left( \frac{S^{(1)}_0}{T^{(1)}_0} - \frac{S^{(2)}_0}{T^{(2)}_0} h_1, \ldots, h_k \bar{\partial}_z \left( \frac{S_1}{T_1} \right), \ldots, \bar{\partial}_z \left( \frac{S_{n-k-1}}{T_{n-k-1}} \right) \right) = 0, \]

on \( D_k \), i.e. we have (11) on \( D_k \).

**Lemma 5.** Let \( T_l = \sum_{j=1}^{n} (\zeta_j - z_j) S_{ij}(\zeta, z), l = 0, 1, 2, \ldots, n - k - 1, \) be continuous differentiable support functions for \( \bar{D} \), then

\[ \det\left( \frac{S_0}{T_0} h_1, \ldots, h_k \bar{\partial}_z \left( \frac{S_1}{T_1} \right), \ldots, \bar{\partial}_z \left( \frac{S_{n-k-1}}{T_{n-k-1}} \right) \right) \]

and

\[ -\det\left( \frac{S_0}{T_0} h_1, \ldots, h_k \bar{\partial}_z \left( \frac{S_1}{T_1} \right), \ldots, \bar{\partial}_z \left( \frac{S_{n-k-1}}{T_{n-k-1}} \right) \right) \]  

(15)
is the exact differential form of \( \bar{\partial} \) on \( D_k \). Here \( \bar{S}_t \) and \( \bar{T}_t \) are of the same properties as \( S_t \) and \( T_t \) respectively.

**Proof.** Since

\[
\bar{\partial} \det_{(0)} \left( S_1, h_1, \ldots, h_k; \frac{S_1}{T_1}, \ldots, \frac{S_{n-k-1}}{T_{n-k-1}} \right)
\]

\[
= \det_{(0)} \left( \frac{\bar{\partial}_t(S_1)}{T_1}, h_1, \ldots, h_k; \frac{\bar{\partial}_t(S_1)}{T_1}, \ldots, \frac{\bar{\partial}_t(S_{n-k-1})}{T_{n-k-1}} \right)
\]

\[
+ (-1)^{l-1} \det_{(0)} \left( S_1, h_1, \ldots, h_k; \frac{\bar{S}_1}{T_1}, \ldots, \frac{\bar{S}_l}{T_l}, \ldots, \frac{\bar{S}_{n-k-1}}{T_{n-k-1}} \right)
\]

\[
= (-1)^{l-1} \det_{(0)} \left( \frac{\bar{S}_1}{T_1}, h_1, \ldots, h_k; \frac{\bar{S}_1}{T_1}, \ldots, \frac{\bar{S}_l}{T_l}, \ldots, \frac{\bar{S}_{n-k-1}}{T_{n-k-1}} \right)
\]

\[
+ (-1)^{l-1} \det_{(0)} \left( S_1, h_1, \ldots, h_k; \frac{\bar{\partial}_t(S_1)}{T_1}, \ldots, \frac{\bar{\partial}_t(S_{n-k-1})}{T_{n-k-1}} \right),
\]

by lemma 4, replacing \( S_t'/T_l \) and \( S_t/T_l \) by \( S_0/T_0 \) in two determinants of the right-hand side of (16), we can conclude that (15) is an exact differential form of \( \bar{\partial} \).

**Corollary.** With the identical assumptions of lemma 5,

\[
\det_{(0)} \left( \frac{S_0}{T_0}, h_1, \ldots, h_k; \frac{\bar{\partial}_t(S_1)}{T_1}, \ldots, \frac{\bar{\partial}_t(S_{n-k-1})}{T_{n-k-1}} \right)
\]

\[
- \det_{(0)} \left( \frac{S_0}{T_0}, h_1, \ldots, h_k; \frac{\bar{\partial}_t(\bar{S}_1)}{T_1}, \ldots, \frac{\bar{\partial}_t(\bar{S}_{n-k-1})}{T_{n-k-1}} \right)
\]

is an exact differential form of \( \bar{\partial} \). Here \( \bar{S}_1, \ldots, \bar{S}_{n-k-1} \) and \( \bar{T}_1, \ldots, \bar{T}_{n-k-1} \) are of the same properties as \( S_1, \ldots, S_{n-k-1} \) and \( T_1, \ldots, T_{n-k-1} \) respectively.

**§4. Main Theorems**

Let \( \{U_j\}_{j=1}^N \) be a finite open covering of an open neighbourhood \( U \) of
\[ \partial D, \text{ and let } X_j: U_j \to \mathbb{R} (1 \leq j \leq N) \text{ be } C^1 \text{ functions, such that} \]

(i) \( D \cap U_{\partial D} = \{ z \in U: \text{ for } 1 \leq j \leq N \text{ either } z \notin U_j \text{ or } X_j(z) < 0 \}, \)

(ii) for \( 1 \leq i_1 < \cdots < i_l \leq N \), \( dX_{i_1}, \ldots, dX_{i_l} \) are linearly independent over \( \mathbb{R} \) at every point of \( \bigcap_{r=1}^l U_{i_r}. \)

For every ordered subset \( \{ j_1, \ldots, j_\theta \} \) of \( \{ 1, \ldots, N \} \), define

\[ \sigma^{(\theta)}_{j_1 \cdots j_\theta} = \{ z \in \partial D, U_{\partial D} : X_{j_1}(z) = \cdots = X_{j_\theta}(z) = 0 \} \]

and choose the orientation on \( \sigma^{(\theta)}_{j_1 \cdots j_\theta} \) such that the orientation is skew symmetric in \( (j_1, \ldots, j_\theta) \) and the following equations hold when \( D \) is given the natural orientation:

\[ \partial D = \bigcup_{j=1}^N \sigma^{(j)}_{j_1 \cdots j_\theta} = \bigcup_{j_1 < \cdots < j_\theta} \sigma^{(j_1 \cdots j_\theta)} \partial \sigma^{(j_1 \cdots j_\theta)} = \bigcup_{j=1}^N \sigma^{(j_1 \cdots j_\theta)} \]

\( \partial^{(\theta)}_{j_1 \cdots j_\theta}, \partial D, \sigma^{(\theta)}, \partial \sigma^{(j_1 \cdots j_\theta)} \) is defined as above, and it is easy to verify the following (cf. [3]):

\[ \partial(\sum_{\Theta} (-1)^\Theta \sigma^{(\Theta)} \times \Delta^{(\Theta)}_0) = \sum_{\Theta} \sigma^{(\Theta)} \times \Delta^{(\Theta-1)}_0 \times \Delta^{(\Theta)}_0, \quad (18) \]

where \( \Theta = \{ j_1, \ldots, j_\theta \} \) is an ordered subset of \( \{ 1, \ldots, N \} \), \( j_1 < \cdots < j_\theta \),

\[ \Delta = \{ \mu = (\mu_0, \mu_1, \ldots, \mu_N) \in \mathbb{R}^{N+1}, \mu_j \geq 0, \sum_{j=0}^N \mu_j = 1 \}, \quad (19) \]

\[ \Delta^{(\theta)}_{j_1 \cdots j_\theta} = \{ \mu \in \Delta: \mu_0 + \mu_{j_1} + \cdots + \mu_{j_\theta} = 1, \Delta_0^{(\theta)} = (1, 0, \ldots, 0) \}, \quad (20) \]

\[ \Delta_0^{(\theta)} = \bigcup_{j_1 < \cdots < j_\theta} \Delta^{(\theta)}_{j_1 \cdots j_\theta} \Delta^{(\theta-1)}_0 = \bigcup_{j_1 < \cdots < j_\theta} \Delta^{(\theta-1)}_{j_1 \cdots j_\theta}. \]

**Theorem 1.** Let \( D \) be a nondegenesate polyhedral domain in \( C^n \) whose boundary can be written as a chain of slit spaces;

\[ \sigma^{(1)} \supset \sigma^{(2)} \supset \cdots \supset \sigma^{(\theta)}. \]

(22)
Assume that $Z(F_1, \ldots, F_m)$ meets $\partial D$ transversally yielding a chain of slit spaces;

$$\partial \tilde{D} = \phi^{(1)} \supset \phi^{(2)} \supset \cdots \supset \phi^{(n)},$$

and that $|\nabla_{\bar{Z}}(\zeta)| \neq 0$ on $\partial \tilde{D}$. Then

$$f(z) = e_m \sum_{\Phi} \int_{\partial^{(n)} \times \Delta^{(n-1)}} f(\zeta) \det_{(n)}(Q, h_1, \ldots, h_m, \partial_{\zeta} Q, \ldots, \partial_{\zeta} Q) \wedge B_m^p(\zeta) \text{ if } \beta > 1$$

and

$$f(z) = e_m \int f(\zeta) \det_{(n)}(N^0, h_1, \ldots, h_m, \partial_{\zeta} N^0, \ldots, \partial_{\zeta} N^0) \wedge B_m^p(\zeta) \text{ if } \beta = 1,$$

for any holomorphic function on $\bar{D}$ and $z \in \bar{D}$. Here

$$Q = (Q_1, \ldots, Q_n)^t, \quad Q_p = \mu_0 \frac{N^0_p(\zeta, z)}{M_0} + \sum_{j \in \Omega} \frac{N_j(\zeta, z)}{M_j} \text{ on } \Delta^{(n)}_0,$$

where $M_0 = \sum_{p=1}^n (\zeta_p - z_p) N^0_p(\zeta, z) \neq 0$ (when $\zeta \neq z$), $M_j = \sum_{p=1}^n (\zeta_p - z_p) N_{jp}(\zeta, z) \neq 0$ (when $\zeta \neq z$), i.e. $M_0, M_j$ are the continuously differentiable support functions; and $Q_p = \sum_{j \in \Omega} \mu_j N_{jp}(\zeta, z)/M_j$ on $\Delta^{(n-1)}_0$.

**Proof.** Since $\sum_{j=1}^n (\zeta_j - z_j) Q_j(\zeta, z, \mu) \equiv 1$ on $\Delta^{(n)}_0$, then $\sum_{j=1}^n (\zeta_j - z_j) \partial_{\zeta} Q_j = 0$. According to Hefer's theorem, we have $0 = F_1(\zeta) - F_1(z) = \sum_{j=1}^n (\zeta_j - z_j) h_j(\zeta, z)$ on $\partial \bar{D} \times \bar{D}$, $l = 1, \ldots, m$. By $(***)$ we have $\partial_{\zeta} B_m^p(\zeta) = 0$. As a result we obtain

$$d[\det_{(n)}(Q, h_1, \ldots, h_m, \partial_{\zeta} Q, \ldots, \partial_{\zeta} Q) \wedge B_m^p(\zeta)] = \det_{(n)}(\partial_{\zeta} Q, h_1, \ldots, h_m, \partial_{\zeta} Q, \ldots, \partial_{\zeta} Q) \wedge B_m^p(\zeta) = 0$$

on $\phi^{(n)} \times \Delta^{(n)}_0$. Using Stokes' theorem and taking account of (18) and (26), we obtain
Applying corollary 2 of lemma 3 to the right-hand side of (27), we obtain (24).

**Remark.** Obviously, (24) includes the generalizations of Range and Siu's formula\(^\text{[3]}\), and of Sergeev and Henkin's formula\(^\text{[5]}\) on the analytic subvariety.

**Theorem 2.** Let \( f(z) \) be a holomorphic function on \( \bar{D} \), then, for \( z \in \bar{D} \), we have

\[
f(z) = \sum_{\partial D} f(\zeta) \det_{(n)}(Q, h_1, \ldots, h_m, \partial_Q Q, \ldots, \partial_Q Q) \land B^F_m(\zeta). (28)
\]

**Remark 1.** When \( m = 0 \), \( \epsilon_m = 1/n! \) \((2\pi i)^n B^E_0(\zeta) = n! \omega(\zeta) \) in (28), and (28) can be rewritten as:

\[
f(z) = \frac{1}{(2\pi i)^n} \int_{\partial D} f(\zeta) \det_{(n)}(S_0, h_1, \ldots, h_m, \bar{\partial}_\zeta(S_1), \ldots, \bar{\partial}_\zeta(S_n-m-1)) \land \omega(\zeta). (29)
\]

This is the generalized Cauchy-Fantappie formula for the bounded domains in \( \mathbb{C}^n \). In fact, let \( S_1 = \cdots = S_{n-m-1} = S_0 \), (29) is the Cauchy-Fantappie formula.

**Remark 2.** If \( S_1 = \cdots = S_{n-m-1} = S_0 \), then \( T_1 = \cdots = T_{n-m-1} = T_0 \). Thus (28) is the generalization of the generalized Cauchy-Fantappie formula on analytic subvarieties.

**Remark 3.** For fixed \( z \in \bar{D} \), we consider the following surface in \( \mathbb{C}^{2n-2m} \)

\[
M_z = \{ (\zeta, w) : \sum_{j=1}^{n} (\zeta_j - z_j)w_j = 1, \ \zeta \in \partial D \}.
\]
We make the following assumptions: \( w_j = w_j(\zeta, z) \), \( (\zeta, z) \in \partial \bar{D} \times \bar{D}, j = 1, \ldots, n \), which belong to \( C^1(\partial \bar{D}) \) in \( \zeta \), and the condition \( \sum_{j=1}^{n} (\zeta_j - z_j)w_j = 1 \) is fulfilled. \( \mathcal{C}_0 \) denotes a cycle on \( M_z \) and cycle \( \mathcal{C}_0 \) represents a homology class \( \lambda \in \mathcal{H}_{2n-2m-1}(M_z) \). Then, for any holomorphic function \( f(z) \) in \( \bar{D} \) and any cycle \( \mathcal{C} \in \lambda \), we have

\[
f(z) = c_m \int_{\mathcal{C}} f(\zeta) \det(m) \left( \frac{S_0}{T_0}, h_1, \ldots, h_m, d\left( \frac{S_1}{T_1} \right), \ldots, d\left( \frac{S_{n-m-1}}{T_{n-m-1}} \right) \right) \wedge B_m^F(\zeta). \tag{30}
\]

**Proof of theorem 2.** According to the corollary of lemma 5, we have

\[
\int_{\partial \bar{D}} f(\zeta) \det(m) \left( \frac{S_0}{T_0}, h_1, \ldots, h_m, \bar{\partial}_\zeta \left( \frac{S_1}{T_1} \right), \ldots, \bar{\partial}_\zeta \left( \frac{S_{n-m-1}}{T_{n-m-1}} \right) \right) \wedge B_m^F(\zeta)
\]

\[
= \int_{\partial \bar{D}} f(\zeta) \det(m) \left( \frac{S_0}{T_0}, h_1, \ldots, h_m, \bar{\partial}_\zeta \left( \frac{S_0}{T_0} \right), \ldots, \bar{\partial}_\zeta \left( \frac{S_0}{T_0} \right) \right) \wedge B_m^F(\zeta)
\]

\[
= \int_{\partial \bar{D}} f(\zeta) \det(m) \left( S_0, h_1, \ldots, h_m, \bar{\partial}_\zeta S_0, \ldots, \bar{\partial}_\zeta S_0 \right) \wedge B_m^F(\zeta)
\]

By further applying corollary 2 of lemma 3, we obtain (28).

**Theorem 3.** Let \( D \) be a bounded domain with piecewise smooth boundaries in \( C^n \), the boundary \( \partial D \) of \( D \) consisting of a chain of slit spaces

\[
\partial D = \sigma^{(1)} \supset \cdots \supset \sigma^{(n)} \supset \sigma^{(n)} \supset \cdots \supset \sigma^{(n)} \supset \sigma^{(\beta)}.
\]

Assume that \( \sigma^{(\beta)} \) be a \( 2n-\beta \) dimensional boundary chain, i.e. there is a \( 2n-\beta+1 \) dimensional chain \( \tau_0 \), such that \( \partial \tau_0 = \sigma^{(\beta)} \). Correspondingly

\[
\partial D = \sigma^{(1)} \supset \cdots \supset \sigma^{(n)} \supset \sigma^{(n)} \supset \cdots \supset \sigma^{(n)} \supset \sigma^{(\beta)}.
\]

\[
\partial \tau_m = \sigma^{(\beta)}, \text{ and when } \zeta \in \partial \bar{D}, |\nabla_f(\zeta)| \neq 0 \text{ and}
\]

\[
\operatorname{rank} \frac{\partial (\mathcal{N}_1^\beta, \ldots, \mathcal{N}_n^\beta)}{\partial (\mathcal{F}_1, \ldots, \mathcal{F}_n)} \leq n - m - \beta. \tag{31}
\]
Then, for a holomorphic function \( f(z) \) on \( \bar{D} \) we have

\[
f(z) = c_m \int_{\partial D \times \Delta^{(\beta - 1)}} f(\zeta) \det_{(n)}(Q, h_1, \ldots, h_m, \partial_{\bar{u}Q}, \ldots, \partial_{\bar{v}Q}) \wedge B_m^F(\zeta), \text{ for } z \in \bar{D}.
\]

(32)

**Proof.** Let

\[
c = \partial(\tau_0 \times \Delta^{(\beta - 1)}) = \sigma^{(\beta)} \times \Delta^{(\beta - 1)} + \varepsilon_0 \tau_0 \times \partial \Delta^{(\beta - 1)},
\]

\[
\tilde{c} = \partial(\tau_m \times \Delta^{(\beta - 1)}) = \tilde{\sigma}^{(\beta)} \times \Delta^{(\beta - 1)} + \varepsilon_m \tau_m \times \partial \Delta^{(\beta - 1)},
\]

where \( \varepsilon_0, \varepsilon = \pm 1 \). Thus on \( \Delta^{(\beta - 1)} \), we have

\[
\det_{(n)}(Q, h_1, \ldots, h_m, \partial_{\bar{u}Q}, \ldots, \partial_{\bar{v}Q})
\]

\[
= \chi_0(\zeta, \mu) + \chi_1(\zeta, \mu) + \cdots + \chi_{\beta - 1}(\zeta, \mu),
\]

(33)

where \( \chi_r(\zeta, \mu) \) are differential forms, the degrees of \( d\mu_\theta \) and \( \overline{\partial}_{\bar{u}j} \) are \( r \) and \( n - m - r - 1 \) respectively. By (31), \( \chi_r(\zeta, \mu) = 0 \), if \( r < \beta - 1 \), and by the degree reasons, we have

\[
\int_{\tau_m \times \partial \Delta^{(\beta - 1)}} f(\zeta) \chi_{\beta - 1}(\zeta, \mu) \wedge B_m^F(\zeta) = 0.
\]

Thus

\[
\int_{\tau_m \times \partial \Delta^{(\beta - 1)}} f(\zeta) \det_{(n)}(Q, h_1, \ldots, h_m, \partial_{\bar{u}Q}, \ldots, \partial_{\bar{v}Q}) \wedge B_m^F(\zeta) = 0.
\]

Therefore we obtain

\[
\int_{\partial D \times \Delta^{(\beta - 1)}} f(\zeta) \det_{(n)}(Q, h_1, \ldots, h_m, \partial_{\bar{u}Q}, \ldots, \partial_{\bar{v}Q}) \wedge B_m^F(\zeta)
\]

\[
= \int_{\bar{C}} f(\zeta) \det_{(n)}(Q, h_1, \ldots, h_m, \partial_{\bar{u}Q}, \ldots, \partial_{\bar{v}Q}) \wedge B_m^F(\zeta).
\]

(34)

On the other hand, let \( \bar{C}_1 = \partial(\bar{D} \times \Delta_0^{(0)}) = \partial \bar{D} \times \Delta_0^{(0)}. \) Since \( \bar{C} \) and \( \bar{C}_1 \) are the cycles of real dimension \( 2n - 2m - 1 \), and
on $\partial D = \sigma^{(1)}$, then

$$
\int_{\mathcal{C}} f(\zeta) \det_{(n)}(Q, h_1, \ldots, h_m, \partial_{\xi_1} Q, \ldots, \partial_{\xi_n} Q) \wedge B^F_m(\zeta)
$$

Applying (34), (35) and (25), we obtain (32).

**Theorem 4.** Let $D$ be a nondegenerate polyhedral domain in $\mathbb{C}^n$, such that its boundary $\partial D$ consists of a chain of slit spaces

$$
\partial D = \sigma_1^{(0)} \supset \sigma_2^{(0)} \supset \cdots \supset \sigma_k^{(0)} \supset \sigma_{k+1}^{(0)} \supset \cdots \supset \sigma_{\beta}^{(0)},
$$

where $\sigma_{k+1}^{(0)}, \ldots, \sigma_{\beta}^{(0)}$ are the boundary surfaces of polyhedral type which is defined by pluriharmonic functions. Let $\Phi_{j,\zeta}(z) = \sum_{p=1}^{n} (z_p - z_p) \varphi_{j_\beta}(z)$ be the holomorphic support functions on $\sigma_{k+1}^{(0)}$ and $H_{j\lambda} = \varphi_{j_\lambda}/\Phi_{j\zeta}$. Let

$$
\partial \tilde{D} = \sigma_1^{(0)} \supset \sigma_2^{(0)} \supset \cdots \supset \sigma_k^{(0)} \supset \sigma_{k+1}^{(0)} \supset \sigma_{k+2}^{(0)} \supset \cdots \supset \sigma_{\beta}^{(0)}
$$

be the corresponding chain of slits of $\partial \tilde{D}$ and assume that $|\nabla^F_m(\zeta)| \neq 0$ on $\partial \tilde{D}$. Then for a holomorphic function $f(z)$ in $\tilde{D}$ we have

$$
f(z) = c_0^m \sum_{(k+1 \leq \rho \leq n-m)} \int_{\sigma_1^{(0)} \times \delta f_{\rho-k}}^{k+1} f(\zeta) \sum_{p=1}^{k+1} (-1)^{k+1-\rho} \det_{(n)}(h_1, \ldots, h_m, H_{j_\rho}, \ldots, H_{j_{n-m}} \partial_{\xi_1} Q, \ldots, \partial_{\xi_n} Q) \wedge B^F_m(\zeta), \text{ for } z \in \tilde{D}
$$
where $C^0_m = (1)^{m(n-1)} \frac{m(m+1) \cdots (n-m)(n-m-1)}{2\cdots (n-m-n)}$, $J_\theta = \{j_1, \ldots, j_\theta\} \subseteq \{1, \ldots, N\}$, $j_1 < \cdots < j_\theta$, $J_{\theta-k} = \{j_1, \ldots, j_{\theta-k}\} \subseteq \{1, \ldots, N\}$, $j_1 < \cdots < j_{\theta-k}$.

**Proof.** It is easy to verify the following

$$\partial \left( \sum_{J_\theta} (-1)^n \delta^{(0)} \times \Delta_{J_\theta - k} \right) = \sum_{J_\theta} \delta^{(0)} \times \Delta_{J_\theta - k}$$

$$\sum_{j_1 < \cdots < j_{\theta+1}} \delta^{(0)}_{j_1 \cdots j_{\theta+1}} \times \Delta_0.$$  \hspace{1cm} (37)

Since

$$\begin{vmatrix} 1 & 0 & \cdots & 0 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ Q_1 & h_{11} & \cdots & h_{1m} & H_{j_11} & \cdots & H_{j_{\theta+1}1} & \partial_{\zeta} Q_1 & \cdots & \partial_{\zeta} Q_1 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ Q_n & h_{1n} & \cdots & h_{mn} & H_{j_{\theta+1}n} & \cdots & H_{j_{\theta+1}n} & \partial_{\zeta} Q_n & \cdots & \partial_{\zeta} Q_n \end{vmatrix}$$

on $\delta^{(0)} \times \Delta_{J_\theta - k} (k+1 \leq \theta \leq N)$, then

$$\sum_{\rho=1}^{k+1} (-1)^{k+1-\rho} \det_{(n)}(h_1, \ldots, h_m, \ldots, [H_{j_{\rho+1}}], \ldots, \partial_{\zeta} Q, \ldots, \partial_{\zeta} Q)$$

$$= \det_{(n)}(h_1, \ldots, h_m, H_{j_1}, \ldots, H_{j_{\theta+1}}, \partial_{\zeta} Q, \ldots, \partial_{\zeta} Q).$$

Then

$$d\left( \sum_{\rho=1}^{k+1} (-1)^{k+1-\rho} \det_{(n)}(h_1, \ldots, h_m, H_{j_1}, \ldots, H_{j_{\theta+1}}, Q, \partial_{\zeta} Q, \ldots, \partial_{\zeta} Q) \right)$$

$$\wedge B^F_m(\zeta) = \partial_{\zeta} \det_{(n)}(h_1, \ldots, h_m, H_{j_1}, \ldots, H_{j_{\theta+1}}, \partial_{\zeta} Q, \ldots, \partial_{\zeta} Q) \wedge B^F_m(\zeta) = 0.$$  \hspace{1cm} (38)

It is from (37), (38) and Stokes' theorem, that

$$C^0_m \sum_{J_\theta} \int_{\Delta_{J_\theta - k}} f(\zeta) \sum_{\rho=1}^{k+1} (-1)^{k+1-\rho} \det_{(n)}(h_1, \ldots, h_m, H_{j_1}, \ldots, [H_{j_{\rho+1}}], Q, \partial_{\zeta} Q, \ldots, \partial_{\zeta} Q) \wedge B^F_m(\zeta)$$

$$= C^0_m \sum_{j_1 < \cdots < j_{\theta+1}} \int_{\Delta_{J_\theta - k}} f(\zeta) \sum_{\rho=1}^{k+1} (-1)^{k+1-\rho} \det_{(n)}(h_1, \ldots, h_m, H_{j_1}, \ldots, [H_{j_{\rho+1}}], Q, \partial_{\zeta} Q, \ldots, \partial_{\zeta} Q) \wedge B^F_m(\zeta).$$
\[ [H_{j_1}, \ldots, H_{j_{k+1}}, \partial_{\text{g}^{\Omega}Q}, \ldots, \partial_{\text{g}^{\Omega}Q}) \wedge B_m^{\text{g}^{\Omega}(\zeta)} = C_m^0 \sum_{j_1 < \ldots < j_{k+1}} \int_{\rho_0}^{f(\zeta)} \frac{k+1}{M_0^{n-m-1}} \sum_{\rho = 1}^{k+1} (-1)^{k+1-\rho} \det_{(0)}(h_1, \ldots, h_m, H_{j_1}, \ldots, H_{j_{k+1}}, \partial_{\text{g}^{\Omega}N^0}, \ldots, \partial_{\text{g}^{\Omega}N^0}) \wedge B_m^{\text{g}^{\Omega}(\zeta)}. \] 

(39)

Since

\[
\begin{vmatrix}
1 & 0 & \ldots & 0 & 1 & \ldots & 1 & G & \ldots & G \\
N_0^0 & h_{11} & \ldots & h_{m1} & H_{j1} & \ldots & H_{jk} & \partial_{\text{g}^{\Omega}N^0} & \ldots & \partial_{\text{g}^{\Omega}N^0} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
N_n^0 & h_{1n} & \ldots & h_{mn} & H_{jn} & \ldots & H_{jk} & \partial_{\text{g}^{\Omega}N^0} & \ldots & \partial_{\text{g}^{\Omega}N^0} \\
\end{vmatrix}
= 0
\]

(40)

where \( G = \sum_{j=1}^n (\zeta_j - z_j) \partial_{\text{g}^{\Omega}N^0_j} \), then

\[
\frac{1}{M_0^{n-m-k}} \det_{(0)}(h_1, \ldots, h_m, H_{j1}, \ldots, h_{jk}, \partial_{\text{g}^{\Omega}N^0}, \ldots, \partial_{\text{g}^{\Omega}N^0}) - \frac{n-m-k}{M_0^{n-m-k+1}}
\]

\[
G \wedge \det_{(0)}(h_1, \ldots, h_m, H_{j1}, \ldots, H_{jk}, N^0, \partial_{\text{g}^{\Omega}N^0}, \ldots, \partial_{\text{g}^{\Omega}N^0})
\]

\[
= \sum_{\rho = 1}^k (-1)^{k-\rho} \det_{(0)}(h_1, \ldots, h_m, H_{j1}, \ldots, [H_{j1}], \ldots, H_{jk})
\]

\[
= \frac{N^0}{M_0^{n-m-k+1}} \partial_{\text{g}^{\Omega}N^0}, \ldots, \partial_{\text{g}^{\Omega}N^0}). \quad (41)
\]

On the other hand, we have

\[
\partial_{\zeta} \det_{(0)}(h_1, \ldots, h_m, H_{j1}, \ldots, [H_{j1}], \ldots, H_{jk}, \partial_{\zeta}N^0, \ldots, \partial_{\zeta}N^0)
\]

\[
= \det_{(0)}(h_1, \ldots, h_m, H_{j1}, \ldots, [H_{j1}], \ldots, H_{jk}, \partial_{\zeta}N^0, \ldots, \partial_{\zeta}N^0)
\]

\[
= \frac{1}{M_0^{n-m-k}} \det_{(0)}(h_1, \ldots, h_m, H_{j1}, \ldots, [H_{j1}], \ldots, H_{jk}, \partial_{\zeta}N^0, \ldots, \partial_{\zeta}N^0) \]
\[ \frac{n-m-k}{M_0^{m-k+1}} G \wedge \det_{(\rho)}(h_1, \ldots, h_m, H_{j_1}, \ldots, H_{j_k}, N^0, \partial_\zeta N^0, \ldots, \partial_\zeta N^0) \]

(42)

on \( \partial D \).

Moreover, since \( \partial \bar{\sigma}^{(0)}_{j_1 \cdots j_k+1} = \bigcup_{j_p} (-1)^{k+1-\rho} \partial \bar{\sigma}^{(0)}_{j_1 \cdots j_k+1} \), and

\[ \sum_{j_1 < \cdots < j_k+1} \frac{1}{(k+1)!} \sum_{j_1 < \cdots < j_k+1} \]

then by Stokes' formula and (41), (42), we obtain

\[ C_m \sum_{j_1 < \cdots < j_k+1} \int_{\mathcal{D}(\rho)} f(\xi) \sum_{\rho = 1}^{k+1} \int_{\mathcal{D}(\rho)} (-1)^{k+1-\rho} \det_{(\rho)}(h_1, \ldots, h_m, H_{j_1}, \ldots, H_{j_k+1}, N^0, \partial_\zeta N^0, \ldots, \partial_\zeta N^0) \wedge B^F_m(\zeta) \]

\[ = C_m \frac{1}{(k+1)!} \sum_{\rho = 1}^{k+1} \sum_{j_1 < \cdots < j_k+1} \int_{\mathcal{D}(\rho)} (-1)^{k+1-\rho} f(\xi) \det_{(\rho)}(h_1, \ldots, h_m, H_{j_1}, \ldots, H_{j_k+1}, N^0, \partial_\zeta N^0, \ldots, \partial_\zeta N^0) \wedge B^F_m(\zeta) \]

\[ = C_m \frac{1}{(k+1)!} \sum_{\rho = 1}^{k+1} \sum_{j_1 < \cdots < j_k+1} \int_{\mathcal{D}(\rho)} d[f(\xi) \det_{(\rho)}(h_1, \ldots, h_m, H_{j_1}, \ldots, H_{j_k+1}, N^0, \partial_\zeta N^0, \ldots, \partial_\zeta N^0) \wedge B^F_m(\zeta)] \]

\[ = \frac{1}{k!} \sum_{j_1 < \cdots < j_k} \int_{\mathcal{D}(\rho)} f(\xi) \frac{1}{M_0^{m-k}} \det_{(\rho)}(h_1, \ldots, h_m, H_{j_1}, \ldots, H_{j_k}, N^0, \partial_\zeta N^0, \partial_\zeta N^0) \wedge B^F_m(\zeta) - \frac{n-m-k}{M_0^{m-k+1}} G \wedge \det_{(\rho)}(h_1, \ldots, h_m, H_{j_1}, \ldots, H_{j_k}, N^0, \partial_\zeta N^0, \partial_\zeta N^0) \wedge B^F_m(\zeta) \]

\( H_{j_k}, N^0, \partial_\zeta N^0, \ldots, \partial_\zeta N^0 \wedge B^F_m(\zeta) \)
Using (43) repeatedly we obtain

\[ C_m^0 \sum_{j_1<\ldots<j_{k+1}} \int_{\Omega^{(0)}} f(\zeta) \sum_{\rho = 1}^{k+1} (-1)^{k+1-\rho} \det_{(\rho)}(h_1,\ldots,h_m,H_{j_1},\ldots,H_{j_{k+1}}) \]

\[ = C_m^0 \sum_{j_1<\ldots<j_{k+1}} \int_{\Omega^{(0)}} f(\zeta) \sum_{\rho = 1}^{k+1} (-1)^{k+1-\rho} \det_{(\rho)}(h_1,\ldots,h_m,H_{j_1},\ldots,H_{j_{k+1}}) \]

\[ \frac{N^0}{M_0^{n-m-k+1}} \partial_{\zeta} N^0,\ldots,\partial_{\zeta} N^0,\partial_{\zeta} N^0) \land B_m^E(\zeta). \]  

Using (43) repeatedly we obtain

\[ C_m^0 \sum_{j_1<\ldots<j_{k+1}} \int_{\Omega^{(0)}} f(\zeta) \sum_{\rho = 1}^{k+1} (-1)^{k+1-\rho} \det_{(\rho)}(h_1,\ldots,h_m,H_{j_1},\ldots,H_{j_{k+1}}) \]

\[ = C_m^0 \sum_{j_1<\ldots<j_{k+1}} \int_{\Omega^{(0)}} f(\zeta) \det_{(\rho)}(h_1,\ldots,h_m) \frac{N^0}{M_0^{n-m-k}} \partial_{\zeta} N^0,\ldots,\partial_{\zeta} N^0) \land B_m^E(\zeta) \]

\[ = C_m^0 \int_{\partial B} \frac{f(\zeta)}{M_0^{n-m}} \det_{(\rho)}(N^0, h_1,\ldots,h_m,\partial_{\zeta} N^0,\ldots,\partial_{\zeta} N^0) \land B_m^E(\zeta). \]  

Applying (25) to the right-hand side of (44), we have

\[ f(z) = C_m^0 \sum_{j_1<\ldots<j_{k+1}} \int_{\Omega^{(0)}} f(\zeta) \sum_{\rho = 1}^{k+1} (-1)^{k+1-\rho} \det_{(\rho)}(h_1,\ldots,h_m) \]

\[ H_{j_1},\ldots,[H_{j_{k+1}}],\ldots,\partial_{\zeta} N^0,\partial_{\zeta} N^0) \land B_m^E(\zeta). \]  

(39) and (45) imply (36).

**Remark.** When \( k = n - m - 1 \), (45) can be rewritten as:

\[ f(z) = C_m^0 \sum_{j_1<\ldots<j_{n-m}} \int_{\Omega^{(0)}} f(\zeta) \sum_{\rho = 1}^{n-m} (-1)^{n-m-\rho} \det_{(\rho)}(h_1,\ldots,h_m,H_{j_1},\ldots,H_{j_{n-m}}) \]

\[ [H_{j_{n-m}},N^0) \land B_m^E(\zeta) \]

\[ = C_m^0 \sum_{j_1<\ldots<j_{n-m}} \int_{\Omega^{(0)}} f(\zeta) \det_{(\rho)}(h_1,\ldots,h_m, \]

\[ H_{j_1},\ldots,H_{j_{n-m}}) \land B_m^E(\zeta). \]  

(46)
Let $D$ be a holomorphic polyhedron. Then (45) and (46) are generalizations\(^8\) of the integral representation formulas of holomorphic functions for analytic polyhedrons\(^4\) in analytic subvarieties (the generalization of Weil's integral representation in analytic subvarieties\(^7\) is also included).

References


