On Orthogonality of Two Subfactors Constructed from Factor Maps

By

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Abstract

Two subfactors constructed from two factor maps are considered. A characterization is given for the angles between two subfactors to become trivial.

§1. Introduction

Motivated by the Jones Index theory ([4]), in [6] Watatani and the author introduced the notion of angles between two subfactors. On the other hand, subfactors constructed via the ergodic theory were studied in [2], [3]. In this paper we consider two (finite to one) factor maps of a (single) ergodic transformation and investigate relative positions between the resulting subfactors. We at first find a condition guaranteeing that the intersection of two subfactors becomes a subfactor of finite index. We then determine when the set of the angles is trivial (i.e., this set reduces to the singleton $\{\frac{\pi}{2}\}$). This is equivalent to the "commuting square" condition, see [6].) Our characterization is given in terms of two kinds of fibres obtained from factor maps.

After collecting basic definition in §2, we state and prove our main results in §3. Some examples (based on sofic systems) are explained in §4.

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§ 2. Preliminaries

2.1. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{M}, \mathcal{N}$ two subspaces with orthogonal projections $p$ and $q$. When $p$ and $q$ do not commute, there exist a Hilbert space $\mathcal{K}$ and positive contractions $s$ and $c$ on $\mathcal{K}$ with null kernels and $s^2 + c^2 = 1$ such that the two projections $p$ and $q$ are unitarily equivalent to

$$I_{(p \wedge q)\mathcal{K}} \oplus \begin{pmatrix} I_{\mathcal{K}} & 0 \\ 0 & 0 \end{pmatrix} \oplus I_{(p \wedge q)\mathcal{K}} \oplus 0_{(p \wedge q)\mathcal{K}} \oplus 0,$$

$$I_{(p \wedge q)\mathcal{K}} \oplus \begin{pmatrix} c^2 & cs \\ cs & s^2 \end{pmatrix} \oplus 0_{(p \wedge q)\mathcal{K}} \oplus I_{(p \wedge q)\mathcal{K}} \oplus 0,$$

respectively. Let $\Theta$ be the operator uniquely determined by $c = \cos \Theta$ and $s = \sin \Theta$ ($0 \leq \Theta \leq \frac{\pi}{2}$).

**Definition 1** ([6]). The angles between two subspaces denoted by $\text{Ang}(p, q)$ is the set in $[0, \frac{\pi}{2}]$ defined as

$$\text{Ang}(p, q) = \begin{cases} sp \Theta, & \text{if } pq \neq qp, \\ \frac{\pi}{2}, & \text{if } pq = qp. \end{cases}$$

2.2. Let us consider a finite factor $L$ containing two subfactors $\mathcal{M}$ and $\mathcal{N}$. Let $\xi$ be a trace vector in $L^2(L)$ and we consider the two subspaces $\mathcal{M}\xi$ and $\mathcal{N}\xi$. The corresponding projections $e_\mathcal{M}$ and $e_\mathcal{N}$ are related to the conditional expectations $E_\mathcal{M}$ and $E_\mathcal{N}$ ([4]) via

$$e_\mathcal{M}(x\xi) = E_\mathcal{M}(x)\xi, \ e_\mathcal{N}(x\xi) = E_\mathcal{N}(x)\xi, \ \text{for } x \in L.$$

**Definition 2.** The angles between two subfactors denoted by $\text{Ang}(\mathcal{M}, \mathcal{N})$ is defined as $\text{Ang}(e_\mathcal{M}, e_\mathcal{N})$.

(See [6] for related results.)
2.3. An (ergodic) dynamical system \((X, \mathcal{F}, \mu, T)\) consists of a probability Lebesgue space \((X, \mathcal{F}, \mu)\) and a nonsingular (ergodic) transformation \(T\). A dynamical system \((Y, \mathcal{G}, \nu, S)\) is a factor of \((X, \mathcal{F}, \mu, T)\) if there exists a measurable (projection) map \(\varphi : X \to Y\) such that \(\mu \circ \varphi^{-1} = \nu\) and \(\varphi \circ T = S \circ \varphi\). Then there exists a (unique) disintegration of \((X, \mathcal{F}, \mu)\) over \((Y, \mathcal{G}, \nu)\), that is, there exists a family of measures \(\{m_y\}_{y \in Y}\) satisfying

\[
m_y(\varphi^{-1}(y)) = 1 \quad \text{for} \quad y \in Y,
\]

for each \(B \in \mathcal{F}\), the map \(y \mapsto m_y(\varphi^{-1}(y) \cap B)\) is \((\mathcal{G})\)-measurable,

\[
\mu(\cdot) = \int_Y m_y(\cdot) \, d\nu(y).
\]

The existence of a finite invariant measure is not necessary in what follows. But for simplicity in the rest we assume the existence of such a measure. It is also possible to deal with a countable ergodic equivalence relation ([1]) instead of an ergodic transformation. However such generalizations are straightforward and we will deal with just the above mentioned case.

§3. Main Theorem

Let \((Y_i, \mathcal{F}_i, \mu_i, S_i)\) \((i=1, 2)\) be a factor of an ergodic dynamical system \((X, \mathcal{F}, \mu_X, T)\) with the factor map \(\pi_i\). We assume that \(\pi_i\) is a finite (= \(m_i\)) to one map and \(\mu_X\) is an invariant measure for \(T\). We further assume the existence of a natural number \(m\) satisfying

\[
(\pi_1^{-1} \pi_2^{-1} \pi_2)^m(x) = (\pi_1^{-1} \pi_2^{-1} \pi_2)^{m+1}(x) \quad \text{a.e.} \quad x \in X.
\]

(The meaning of this condition in subfactor set-up will be clarified later.)

**Proposition 3.** The partition \(\xi = \{(\pi_1^{-1} \pi_2^{-1} \pi_2)^m(x); x \in X\}\) is measurable under the above assumption (1).

A partition \(\xi_0\) is said to be measurable when there exist countable measurable sets \(\{A_n\}_{n \in \mathbb{N}}\) (called a basis for \(\xi_0\)) such that \(\xi_0 = \{\bigcap_{n=1}^{\infty} B_n; B_n = A_n, A_{n+1}\}\). The proposition guarantees that the quotient space of \(X\) by \(\xi\) is Lebesgue (see [5] for instance).

**Proof.** Let \(\{D_n\}_n\) be a basis for the Lebesgue space \(Y_1\). Set
$B_1 = Y_1, B_{2n+1} = D_n^*$ and $B_{2n} = D_n^*(n \geq 1)$. The assumption implies the invariance of the $(\pi_1^{-1}\pi_2^{-1}\pi_2^{-1})^m(x)$ under $\pi_1^{-1}\pi_1$ and $\pi_2^{-1}\pi_2$ and the existence of a natural number $p$ satisfying $|\pi_1(\pi_1^{-1}\pi_2^{-1}\pi_2^{-1})^m(x)| \leq p$. Let us consider a family of countable measurable sets $\{A_{(n_1,n_2,\ldots,n_p)}; (n_1, n_2, \ldots, n_p) \in \mathbb{N}^p\} \subseteq Y_1$ defined by

$$A_{(n_1,n_2,\ldots,n_p)} = B_{n_1} \cap B_{n_2} \cap \cdots \cap B_{n_p}.$$  

We will show that $\{(\pi_1^{-1}\pi_1\pi_2^{-1}\pi_2^{-1})^m(\pi_1^{-1}(A_{(n_1,n_2,\ldots,n_p)})); (n_1, n_2, \ldots, n_p) \in \mathbb{N}^p\}$ is a basis for the partition $\xi$. Remark that $(\pi_1^{-1}\pi_1\pi_2^{-1}\pi_2^{-1})^m(\pi_1^{-1}(A_{(n_1,n_2,\ldots,n_p)}))$ is measurable because $\pi_1$ and $\pi_2$ are finite to one ([5]). Take an element $C(= (n_1, n_1\ldots, n_1)) \in \xi$ and consider a disjoint one $C'$ with $C$. And let us write that $\pi_1(C') = \{y_1, \ldots, y_q\} (q \leq p)$. Since the basis $\{D_n\}$ separates two distinct points, we can choose a set of natural numbers $\{k_j; 1 \leq j \leq p\}$ such that $\pi_1(x) \in B_{k_j}, y_j \notin B_{k_j} (1 \leq j \leq q)$. (For $j(q+1 \leq j \leq p), k_j = 1$.) Then $\{(\pi_1^{-1}\pi_1\pi_2^{-1}\pi_2^{-1})^m(\pi_1^{-1}(A_{(k_1,k_2,\ldots,k_p)}))\}$ contains the element $C$ but does not include $C'$. 

**Lemma 4.** The function

$$x \mapsto |\pi_1^{-1}\pi_1(x) \cap \pi_2^{-1}\pi_2(x)| (=: c(x))$$

is measurable and hence constant by the ergodicity.

**Proof.** We may assume $X \cong Y_1 \times \{1, 2, \ldots, m_i\} (i=1, 2)$. Define the map $\varphi_i$ by $\varphi_i(y, s) = (y, s+1) \pmod{m_i}$ in $\{1, 2, \ldots, m_i\}$. Consider the measurable sets

$$M_k = \{x \in X; (\varphi_1)^l(x) = (\varphi_2)^l(x) \text{ for some } l\}.$$  

Since

$$\{x; c(x) \geq m\} = \bigcup_{0=n_0<\ldots<n_{m-1}} \bigcap_{k=n_0,n_1,\ldots,n_{m-1}} M_k,$$

c(x) is a measurable function. 

We call this constant the crossing number (denoted by $c(\pi_1, \pi_2)$ for $\pi_1$ and $\pi_2$). The discussion so far allows us to give another factor system $(X/\xi, \mathcal{F}_3, \mu_3, S_3)$ with the finite (=$m_3$) to one factor map $\pi_3$ over the Lebesgue space $X/\xi$. 

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Let $M$ be the Krieger factor corresponding to $(X, \mathcal{F}, \mu_X, T)$. As in [3], the above three factor maps give rise to the three subfactors in $M$. For the readers convenience we briefly recall the construction described in [1], [2], and [3]:

Let $\mathcal{R}_T$ be the (countable) ergodic equivalence relation (on $X$) generated by $T$: $x \sim y$ if $y = T^n x$ for some $n \in \mathbb{Z}$.

The left counting measure $(\mu_X)_l$ on $\mathcal{R}_T(d(\mu_X)_l(u, v) = d(\mu_X)(v))$ gives us the Hilbert space

$$\mathcal{H} = L^2(\mathcal{R}_T; (\mu_X)_l) = \{\xi; \text{ a measurable function on } \mathcal{R}_T \text{ and}$$

$$\|\xi\|^2 = \int_{\mathcal{R}_T} |\xi(u, v)|^2 d(\mu_X)_l = \int_X \sum_{u \sim v} |\xi(u, v)|^2 d(\mu_X)(v) < \infty\}.$$

The factor $M = W^*(\mathcal{R}_T)$ is generated by the convolution operators $L_f$ of "nice" functions $f$ on $\mathcal{R}_T$ such as $(L_f \xi)(u, v) = \sum_{w \sim v} f(u, w) \xi(w, v)$.

Let $N_1, N_2,$ and $N_3$ be the von Neumann subalgebras on $\mathcal{H}$ defined by

$$N_i = \{L_f \in M; f(u, v) = f(u', v') \text{ for } \pi_i(u) = \pi_i(u'), \pi_i(v) = \pi_i(v')\}.$$

(Note that $[M; N_i] = m_i$ (i=1, 2) ([3]).) From the construction, we have $N_1 \cap N_2 = N_3$. Actually $N_i$ is a factor since it is isomorphic to $W^*(\mathcal{R}_T)$. 

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**Diagram:**

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Fig. 1

\[
\begin{array}{ccc}
(X, T) & \xrightarrow{\pi_1} & (Y_1, S_1) \\
\phantom{X} & \downarrow & \phantom{X} \\
\pi_2 & \xrightarrow{\phantom{X}} & \pi_3 \\
(Y_2, S_2) & \phantom{X} & (X/\xi, S_3)
\end{array}
\]
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Remark that our previous assumption (1) is equivalent to that the index $[M: N_3]$ is finite ($= m_3$). The corresponding orthogonal projection $e_{N_i}$ is given by

$$
(e_{N_i}, \xi)(u, v) = \frac{1}{m_i} \sum_{\begin{array}{c} u' \sim v' \\ \pi_i(u) = \pi_i(u') \\ \pi_i(v) = \pi_i(v') \end{array}} \xi(u', v').
$$

Similarly the measurable partition \{\pi_1^{-1} \pi_1(x) \cap \pi_2^{-1} \pi_2(x); x \in X\} gives us the subfactor $N_0$ containing $N_1$ and $N_2$. By Lemma 3.2. (2) ([6]), we have $\text{Ang}_M(N_1, N_2) = \text{Ang}_{N_0}(N_1, N_2)$. Hence we may restrict our attention to the case $c(\pi_1, \pi_2) = 1$ by starting with $N_0$ instead of $M$.

**Definition 5.** A pair of factor maps $\pi_1$ and $\pi_2$ is said to be exclusive if there exists a measurable set $A$ with $\mu_X(A) > 0$ satisfying: For any $x \in A$, we have $x_1, x_2 \in X$ such that $\pi_1(x) = \pi_1(x_1)$, $\pi_2(x) = \pi_2(x_2)$ and $\pi_1^{-1} \pi_1(x_2) \cap \pi_2^{-1} \pi_2(x_1) = \emptyset$.

We are ready to state our main theorem.

**Theorem 6.** Let $M$ be the factor and $N_1$ and $N_2$ the subfactors constructed above. Then the following are equivalent.

1. $e_{N_1} e_{N_2} = e_{N_2} e_{N_1}$ (i.e., $\text{Ang}(N_1, N_2) = (\frac{\pi_1}{2})$).
2. The pair of $\pi_1$ and $\pi_2$ is not exclusive.
3. $(\pi_1^{-1} \pi_1 \pi_2^{-1} \pi_2)(x) = (\pi_2^{-1} \pi_2 \pi_1^{-1} \pi_1)(x)$ a.e.x.

**Proof.** As was pointed out above, we may and do assume $c(\pi_1, \pi_2) = 1$.

(2)$\Rightarrow$(3); The assumption means that $\pi_1^{-1} \pi_1(x_2) \cap \pi_2^{-1} \pi_2(x_1) \neq \emptyset$ a.e.x for any $x_1$ and $x_2$ satisfying $\pi_1(x) = \pi_1(x_1)$ and $\pi_2(x) = \pi_2(x_2)$. Hence we have (3).

(3)$\Rightarrow$(1); Obvious.

(3)$\Rightarrow$(2); Assume that the pair of $\pi_1$ and $\pi_2$ is exclusive. Then there exists a measurable set $A$ such that for any $x \in A$, we have $x_1, x_2 \in X$ such that $\pi_1(x) = \pi_1(x_1)$, $\pi_2(x) = \pi_2(x_2)$ and $\pi_1^{-1} \pi_1(x_2) \cap \pi_2^{-1} \pi_2(x_1) = \emptyset$. Hence, there exists at least one point in $\pi_2^{-1} \pi_2(x_1)$ which is never contained in $(\pi_1^{-1} \pi_1 \pi_2^{-1} \pi_2)(x)$.

(1)$\Rightarrow$(3); Suppose that there exists a measurable set $A$ with $\mu_X(A) > 0$
satisfying \((\pi_1^{-1}\pi_1\pi_2^{-1}\pi_2)(x) \neq (\pi_2^{-1}\pi_2\pi_1^{-1}\pi_1)(x)\) for \(x \in A\). Choose a cross section \(A_0(\subseteq \mathcal{A})(\mu_3(A_0) > 0)\) over the factor system \((X/\xi, \mu_3)\) with the factor map \(\pi_3\) ([5]). Set \(B = (\pi_1^{-1}\pi_1\pi_2^{-1}\pi_2)(A_0)/(\pi_2^{-1}\pi_2\pi_1^{-1}\pi_1)(A_0) = \bigcup_{x \in A_0}(\pi_1^{-1}\pi_1\pi_2^{-1}\pi_2)(x)/(\pi_2^{-1}\pi_2\pi_1^{-1}\pi_1)(x)\). Then we have

\[
\mu_X(B) = \int_{x/\xi} \chi_{\pi_3(A_0)}(z)m_x(B)d\mu_3(z)
\]

\[
= \int_{x/\xi} \chi_{\pi_3(A_0)}(z) \sum_{y \in C(z)} m_y(\{y\})d\mu_3(z)
\]

\[
\geq \frac{1}{m_3}\mu_3(\pi_3(A_0)) > 0,
\]

where \(C(z)\) is the set \((\pi_1^{-1}\pi_1\pi_2^{-1}\pi_2)(A_0 \cap \pi_3^{-1}(z))/(\pi_2^{-1}\pi_2\pi_1^{-1}\pi_1)(A_0 \cap \pi_3^{-1}(z))\). Let \(\xi\) be the function \(\xi(u, v) = \chi_D(u, v)\chi_B(u)\), where \(D = \{(u, u); u \in X\}\). For any \(u \in A_0\), we have

\[
(e_{N_1}e_{N_2}\xi)(u, u) = \sum_{v \in (\pi_1^{-1}\pi_2^{-1}\pi_1)(u)} \xi(v, v) = 0,
\]

\[
(e_{N_2}e_{N_1}\xi)(u, u) = \sum_{v \in (\pi_1^{-1}\pi_2^{-1}\pi_1)(u)} \xi(v, v) > 0.
\]

Therefore, \(e_{N_1}e_{N_2} \neq e_{N_2}e_{N_1}\).

\(\square\)

\section*{Examples}

A labeled graph means a directed graph \(\Gamma = (V, A, i, t)\) together with label maps \(\lambda: A \to \mathcal{A}\) and \(\mu: V \to \mathcal{B}\), where \(V\) (resp. \(A\)) is the set of all vertices (resp. arcs). Here \(\mathcal{A}\), \(\mathcal{B}\) are sets of labels and \(i, t\) are initial and terminal vertex maps (see the example presented below).

We will construct two factor maps from two labeled graphs \(\Gamma_1\) and \(\Gamma_2\) with the same directed graph \((V, A, i, t)\) but different label maps \((\lambda, \mu_1)\) and \((\lambda, \mu_2); \lambda: A \to \mathcal{A}, \mu_j: V \to \mathcal{B}_j\) \((j = 1, 2)\). Set \(X = \{(x_n); x_n \in A, t(x_n) = i(x_{n+1})\}, Y_j = \{(y_n); \beta; \text{there exists } x_n \in X \text{ satisfying } y_n = \lambda(x_n) \text{ and } \beta = \mu_j(i(x_0))\}\) and define the shifts \(T: X \to X\) and \(S_j: Y_j \to Y_j\) by

\[
T((x_n)) = (x_{n+1}),
\]

\[
S_j((\lambda(x_n)), \mu_j(i(x_0))) = ((\lambda(x_{n+1})), \mu_j(i(x_1))).
\]
In this setting the map $\pi_j: X \to Y_j(\pi_j(x_n) = (\lambda(x_n)_i, \mu_j(i(x_n)))$ is a factor map of $(X, T)$ over $(Y_j, S_j)$. Set $Y = \{(y_n)_{n}'; \text{ there exists a } (x_n)_{n} \in X \text{ satisfying } y_n = \lambda(x_n), S((y_n)_{n}) = (y_{n+1})_{n}, \text{ and } \pi((x_n)_{n}) = (\lambda(x_n))_{n}$. Notice that $(Y, S)$ is a common factor of $(Y_j, S_j)$.

To construct concrete $\Gamma_1$ and $\Gamma_2$, we now consider a finite group $G(=\{g_1, g_2, \cdots, g_n\})$ with $\Gamma = \{\gamma_1, \gamma_2, \cdots, \gamma_m\}$ as generators. Set $V = \{1, 2, \cdots, n\}$. Two vertices $i, j$ are connected by a (unique) oriented arc $\alpha$ if $\gamma_k \in H_j$ for some (unique) $\gamma_k \in \Gamma$. We define the label map $\lambda: A = \{\alpha's\} \to \mathcal{A} = \Gamma$ by $\lambda(\alpha) = \gamma_k$. To construct two different labeled graphs (two different $\mu_j$'s), we choose two subgroups $H_1$ and $H_2$. Choose a right coset representative $\mathcal{B}_j = \{g^j_s\}; G = \cup_s g^j_s H_j (j = 1, 2)$. We define the label map $\mu_j: V \to \mathcal{B}_j$ as follows; $\mu_j(k) = g^j_s$ if $g_k H_j = g^j_s H_j$. For example, let $G$ be the symmetric group $S_3$ on $\{1, 2, 3\}$ with the subgroup $H_1 = S_2$ on $\{1, 2\}$. Set $g_1 = e, g_2 = (12) = \gamma_1, g_3 = (23) = \gamma_2, g_4 = (13), g_5 = (123), g_6 = (132) (H_1 = \{g_1, g_2\})$. The corresponding directed graph is as follows:

And the labeled graph $\Gamma_1$ is drawn simply by Figure 3. Similarly the different subgroup $H_2 = \{g_1, g_3\}$ gives rise to the labeled graph $\Gamma_2$ described by Figure 4;
We have constructed two labeled graphs with \((\lambda, \mu_1)\) and \((\lambda, \mu_2)\) from two subgroups, and as was explained earlier we have \((X, T), (Y_1, S_1), (Y_2, \ldots\)
\[ S_2 \), and \((Y, S)\). \((X, T)\) (resp. \((Y, S)\)) have the unique ergodic probability measure \(\mu_x(\text{resp.} \mu_y)\) with maximal entropy \(\mu_y = \mu_x \circ \pi_3^{-1}\) (see [3] for details). Hence, we have two factor systems \((Y_1, \mu_1, S_1)\) and \((Y_2, \mu_2, S_2)\) of the ergodic dynamical system \((X, \mu_x, T)\) (\(\mu_1\) and \(\mu_2\) are the image measures of \(\mu_x\)). Following the same procedure explained in the previous section, we have two subfactors. Here, we apply the theorem to, for instance, the pair of the two subfactors \(N_1\) and \(N_2\) constructed from the subgroups \(H_1 = \{g_1, g_2\}, H_2 = \{g_1, g_3\}(\subseteq G)\). It is straightforward to show that the conditions (2) and (3) of the theorem do not hold. Therefore, \(\text{Ang}(N_1, N_2) \neq \{\pi_i\}_{i=1}^2\).

Remark 7. For simplicity, let us assume that \(G = H_1 \vee H_2\) and \(H_1 \cap H_2 = \{e\}.\) (The latter is equivalent to \(c(\pi_1, \pi_2) = 1\).) Then we may identify \(X\) with \(Y \times G\) and \(T, \pi_j\) look like
\[
T((y_n, g)) = ((y_{n+1}, y_0 g), \pi_j(y, gH_j) \in Y \times G/H_j \cong Y \times \beta_j)
\]
If we consider the automorphisms \(\{\alpha_g\}_{g \in G} \subseteq N[\beta_T]\), the normalizers, on \(Y \times G\) defined by
\[
\alpha_g(y, h) = (y, g^{-1}h),
\]
then they induce an outer action of \(G\) on \(M = \text{W}^*(\beta_T)\). It is easy to check that the fixed point algebra by \(H_i\) is precisely \(N_i(i=1,2)\). Therefore the quadrilateral \((M, N_1, N_2, N_3)\) can be identified with \((M, M^{H_1}, M^{H_2}, M^G)\). Then, by Proposition 7.9 ([6]), we have the trivial angle iff \(|G| = |H_1| \times |H_2|\)\). Notice that this condition precisely means that two factor maps are not exclusive (because of the crossing number = 1).

More generally, \(\text{Ang}(N_1, N_2)\) might be calculated by looking at how two kinds of fibres meet. This seems to deserve further investigation.

References

