Unitarily Invariant Norms under Which the Map $A \to |A|$ Is Lipschitz Continuous

By

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Abstract

We will characterize the unitarily invariant norms (for compact operators) under which the map $A \to \|A\| = (A^*A)^{1/2}$ is Lipschitz-continuous. Although the map is not Lipschitz-continuous for the trace class norm, we will obtain a certain Lipschitz-type estimate by making use of the Macaev ideal.

§0. Introduction

In [9] E. B. Davies showed the following Lipschitz-type estimates in the Schatten $p$-norm ($1 < p < +\infty$):

\[
\begin{align*}
\left\| \|A\|X - X|A| \right\|_p & \leq \text{Const}. \|AX - XA\|_p ; & A = A^* \in C_p , \\
\left\| |A| - |B| \right\|_p & \leq \text{Const}. \|A - B\|_p ; & A, B \in C_p .
\end{align*}
\]

Related results can be found in [1], [2], [3], [14] and [15]. For $p = 1$ and $+\infty$ (where $\|\cdot\|_\infty = \|\cdot\|$, the usual operator norm) the above Lipschitz-type estimates are known to fail. Instead some weaker estimates have been investigated by several authors ([7], [13], [16], [18]). See also [6] for some recent results.

An obvious next problem is to characterize unitarily invariant norms (of compact operators) under which the map $A \to |A| = (A^*A)^{1/2}$ is Lipschitz-continuous. In the present article we will obtain quite a complete solution to this problem based on very powerful analysis in [9] and Arazy’s result, [4].

One of the difficulties of dealing with the map $A \to |A|$ is its non-linearity. A very clever trick in [9] is to reduce the desired Lipschitz-continuity to the boundedness of a certain linear operator (Schur-Hadamard multiplier, etc.). Therefore, interpolation (for linear operators) is at our disposal, and in §2 all

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the Lipschitz-type estimates in [9] are shown to remain valid for a symmetri-

cally normed ideal (see §1 for its belief explanation) which is an interpolation

space between some $C_{p_1}$ and $C_{p_2}$, $1 < p_1 < p_2 < +\infty$.

What is probably more interesting is that the converse is also true. For

example, we can show that the Lipschitz-continuity of the map $A \to |A|$ implies

the boundedness (relative to the relevant norm) of the triangle projection ([17]).

Therefore, we can use Arazy's theorem, [4], stating that a symmetrically normed

ideal possesses the above-mentioned interpolation property if and only if the

triangle projection is bounded. This converse result will be proved in §3.

Although

$$\| |A| - |B| \|_p \leq \text{Const.} \| A - B \|_p$$

is not valid for $p = 1, +\infty$, we will obtain Lipschitz-type estimates involving

these norms in §4. For example, if the above left side is replaced by the

norm of the Macaev ideal (see [11]), the result remains valid for $p = 1$. The

dual version can be also obtained by using the "predual" of the Macaev

ideal. The Macaev ideal plays important roles in analysis on compact opera-
tors ([11], [12]). Its importance is also emphasized in the recent book [8],

where relationship between this ideal and the Dixmier (non-normal) trace is

discussed.

§1. Symmetrically Normed Ideal ([11], [19])

In this section we collect basic facts on symmetrically normed ideals (of

compact operators on a Hilbert space), and details on this subject matter can

be found in [11], [19].

Let $f$ be the space of the sequences with finitely many non-zero terms. A

norm $\Phi(\cdot)$ on $f$ (with normalization $\Phi(1, 0, 0, \ldots) = 1$) is called a symmetric

norm if $\Phi(\xi_1, \xi_2, \ldots)$ is invariant under the permutations (of terms) and

$$\Phi(\xi_1, \xi_2, \ldots) = \Phi(|\xi_1|, |\xi_2|, \ldots).$$

Let $S_\Phi$ be the Banach space of sequences $a = \{a_n\}_{n=1, 2, \ldots}$ satisfying

$$\sup_m \Phi(a_1, a_2, \ldots, a_m, 0, 0, \ldots) \leq \Phi(a),$$

and let $S_\Phi^{(0)}$ be the closure of $f$ relative to the norm $\Phi$.

Throughout let $H$ be a separable Hilbert space. For a compact operator

$A$ on $H$ let $s_n(A)$ ($n = 1, 2, \ldots$) be the $n$-th singular number of $A$, that is, the
$n$-th largest (with multiplicities counted) eigenvalue of $|A|$. We now introduce

two Banach spaces $I(S_\Phi)$, $I(S_\Phi^{(0)})$ consisting of compact operators. A compact

operator $A$ belongs to $I(S_\Phi)$ if the associated sequence $s(A) = \{s_n(A)\}_{n=1, 2, \ldots}$ lies

in $S_\Phi$. The space $I(S_\Phi)$ is a Banach space under the norm

$$\| A \|_{I(S_\Phi)} = \Phi(s(A)).$$
The second Banach space $I(S_\phi^{(0)})$ is defined as the closure (in $I(S_\phi)$) of the space of the finite rank operators. The space $I(S_\phi)$ may or may not be a separable Banach space while $I(S_\phi^{(0)})$ is always separable. The both spaces are two sided ideals in $B(H)$, the bounded operators, and $\| \cdot \|_{I(S_\phi)}$ is symmetric in the sense that

$$\| XAY \|_{I(S_\phi)} \leq \| X \| \| A \|_{I(S_\phi)} \| Y \|,$$

where $\| \cdot \|$ denotes the usual operator norm (throughout the article). In particular (and actually equivalently) we get the unitary invariance

$$\| UAV \|_{I(S_\phi)} = \| A \|_{I(S_\phi)}$$

for unitaries $U, V$.

Basic properties of these Banach spaces are:

1. $I(S_\phi)$ is separable if and only if $I(S_\phi^{(0)}) = I(S_\phi^{(0)\|})$. (This can be checked by just looking at $\Phi$, i.e., mononormalizing in [11] or regular in [19].)

2. Any separable symmetrically normed ideal is of the form $(I(S_\phi^{(0)}), \| \cdot \|_{I(S_\phi)})$ for some symmetric norm $\Phi$. For a given $\Phi$ (not equivalent to $\Phi^\infty$ defined later) we define the dual norm $\Phi'$ by

$$\Phi'(\zeta) = \sup \left\{ \sum_{i=1}^{\infty} \xi_i \zeta_i : \zeta = \{\zeta_i\} \in f \text{ and } \Phi(\zeta) \leq 1 \right\}.$$

Then the dual space $I(S_\phi^{(0)})^*$ can be identified with $I(S_\phi)$. Here the duality is given by the bilinear form

$$(A, B) \in I(S_\phi^{(0)}) \times (I(S_\phi)) \mapsto \text{Tr}(AB) \in \mathbb{C}.$$

If we set $\Phi_p(\zeta) = \left( \sum_{i=1}^{\infty} |\xi_i|^p \right)^{1/p}, 1 \leq p \leq +\infty$ (with the usual convention for $p = +\infty$), we get $I(S_\phi_p) = I(S_\phi^{(0)p}) = C_p$, the Schatten $C_p$-ideal, and $I(S_\phi^{(0)}) = C_\infty$, the compact operators. In the rest of the article, we will deal with either $I(S_\phi)$ or $I(S_\phi^{(0)})$ which is strictly smaller than $C_\infty$. Whenever there is no possibility of confusion, the norm $\| \cdot \|_{I(S_\phi)}$ will be denoted by $\| \|$. For later use we list some properties of symmetrically normed ideals.

**Lemma 1.** We have

$$\| \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \| = \| \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix} \| = \| \begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix} \| = \| \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix} \| = \| A \|.$$

**Proof.** The result follows from the obvious facts:

- $s_n \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} = s_n \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix} = s_n(A)$,
- $s_n \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix} = s_n \begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix}$.
Lemma 2. We have

\[
\|\begin{bmatrix} A & B \\ C & D \end{bmatrix}\| \leq \|A\| + \|B\| + \|C\| + \|D\| \leq 4 \|\begin{bmatrix} A & B \\ C & D \end{bmatrix}\|.
\]

Proof. The first inequality follows from the triangle inequality and Lemma 1. To show the second, notice

\[
\|\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\| = \|\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\|
\leq \|\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\|\|\begin{bmatrix} A & B \\ C & D \end{bmatrix}\|\|\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\|
\leq \|\begin{bmatrix} A & B \\ C & D \end{bmatrix}\|.
\]

Therefore, Lemma 1 shows

\[
\|A\| \leq \|\begin{bmatrix} A & B \\ C & D \end{bmatrix}\|.
\]

We similarly show that \(\|B\|, \|C\|, \text{ and } \|D\|\) are majorized by the same quantity. Q.E.D.

The next two results are Theorems 5.1 and 6.3 in Chap. III, [11], respectively.

Lemma 3. Let \(X\) be a bounded operator. If there is a sequence \(\{X_n\}\) in \(I(S_\phi)\) converging to \(X\) in the weak operator topology and \(\sup_n\|X_n\| < +\infty\), then \(X\) belongs to \(I(S_\phi)\) and \(\|X\| \leq \sup_n\|X_n\|\).

Lemma 4. Assume \(A \in I(S_\phi^{(p)})\). If a sequence \(\{X_n\}\) of self-adjoint operators converges to \(X\) in the strong operator topology, then \(X_nA \to AX, AX_n \to AX, \text{ and } X_nAX_n \to XAX\) in the norm \(\|\cdot\|\).

§ 2. Lipschitz-Type Estimates for Commutators

In this section we show that certain Lipschitz-type estimates are valid in a symmetrically normed ideal \((I(S_\phi), \|\cdot\|)\) which is an interpolation space between \(C_{p_1}\) and \(C_{p_2}\), \(1 < p_1 < p_2 < +\infty\). The reader can find details on the general interpolation theory in [5]. (Information on interpolation spaces be-
between symmetrically normed ideals can be found in [4], [10]. In our set-up (since \( C_{p_1} \subseteq C_{p_2} \)) the assumption means the following: We must have \( C_{p_1} \subseteq I(S_\phi) \subseteq C_{p_2} \) with continuous inclusion operators. Let \( T \) be a linear mapping from \( C_{p_2} \) into itself. Whenever \( T(C_{p_1}) \subseteq C_{p_1} \), and \( T \) is bounded relative to \( \| \cdot \|_{p_2} \) and \( \| \cdot \|_{p_1} \), we must have \( T(I(S_\phi)) \subseteq I(S_\phi) \) and \( T \) has to be bounded relative to \( \| \cdot \| \).

The central core for analysis in [9] was the next result based on the theory of Volterra operators ([12]).

**Lemma 5.** (Corollary 5 and Corollary 6 in [9]) There is a constant \( \gamma_p \), \( 1 < p < \infty \), satisfying the following:

(i) For any \( \lambda_i, \mu_i > 0, i = 1, 2, \ldots, n \) and any \( n \times n \)-matrix \( A = [A_{ij}] \), the \( n \times n \)-matrix \( B = [B_{ij}] \), \( B_{ij} = (\lambda_i - \mu_i) \times (\lambda_i + \mu_i)^{-1} \times A_{ij} \), satisfies \( \| B \|_p \leq \gamma_p \| A \|_p \).

(ii) For any \( \lambda_i, \mu_i \geq 0, i = 1, 2, \ldots, n \), and any \( n \times n \)-matrix \( C = [C_{ij}] \), we have

\[
\| [(\lambda_i - \mu_j)C_{ij}]_{ij} \|_p \leq \gamma_p \| [(\lambda_i + \mu_j)C_{ij}]_{ij} \|_p.
\]

Obviously (i) and (ii) are equivalent. Let us emphasize that \( \gamma_p \) is an absolute constant which does not depend on \( n, A, \) and the choice of \( \lambda_i \)'s and \( \mu_i \)'s. As was shown in [9], this lemma is based on the facts

\[
\begin{align*}
\| P_{\pi/4}(A) \|_p &\leq \frac{1}{4} \gamma_p \| A \|_p \\
P_\theta(A) &= U_\theta P_{\pi/4}(U_\theta^* A) \\
B &= - A - \int_0^{\pi/2} P_\theta(A) g'(\theta) d\theta
\end{align*}
\]

(see p. 150, 151 for the definitions of \( P_\theta, U_\theta, g(\theta), \) etc.). Our interpolation assumption (applied to the linear operator \( P_{\pi/4} \)) immediately implies \( \| P_{\pi/4}(A) \| \leq \text{Const.} \| A \| \). Therefore, by repeating the arguments in [9], we conclude that Lemma 5 remains valid for \( \| \cdot \| \).

**Theorem 6.** Assume that a symmetrically normed ideal \((I(S_\phi), \| \cdot \|)\) is an interpolation space between \( C_{p_1} \) and \( C_{p_2} \), \( 1 < p_1 < p_2 < +\infty \). For a bounded operator \( X \) we have

(i) \( \| \| X - X \| \| \| B \| \| \leq \text{Const.} \| AX - XB \| \); \( A = A^*, B = B^* \in I(S_\phi) \),

(ii) \( \| \| X - X \| \| \| A \| \| \leq \text{Const.} \| AX - XA \| \); \( A \in I(S_\phi), X = X^* \),

(iii) \( \| A_1 X - X A_2 \| \leq \text{Const.} \| A_1 X + X A_2 \| ; A_1, A_2 \in I(S_\phi) \).

**Proof.** (i) Since Lemma 5 is valid for \( \| \cdot \| \), the identical arguments as in the proof of Theorem 7, [9], together with Lemma 2 show

\[
\| \| a x - x a \| \| \leq \text{Const.} \| a x - x a \| , \quad a = a^*,
\]
for $n \times n$-matrices $a, x$. The operator $A \in I(S_{n\phi})$ being compact, we can find an increasing sequence $\{p_n\}$ of projections with $\dim p_n H = n$, $p_n \to I$ strongly and $[p_n, A] = 0$. Setting $a_n = p_n A p_n$ and $x_n = p_n X p_n$, we see $[a_n, x_n] = p_n [A, X] p_n$ and $[|a_n|, x_n] = p_n [|A|, x] p_n$. The above inequality for matrices then implies

$$
\| p_n [|A|, X] p_n \| \leq \text{Const} \cdot \| p_n [A, X] p_n \|
$$

$$
\leq \text{Const} \cdot \| p_n \| \| [A, X] \| \| p_n \|
$$

$$
\leq \text{Const} \cdot \| [A, X] \| .
$$

Since $p_n [|A|, X] p_n \to [|A|, X]$ strongly (hence weakly), Lemma 3 says

$$
\| [A, X] \| \leq \text{Const} \cdot \| [A, X] \| ,
$$

which is exactly (i) with $A = B$. The general case can be obtained by applying this special case to

$$
\hat{A} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} (= \hat{A}^*), \quad \hat{X} = \begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix}.
$$

In fact, we get

$$
\begin{bmatrix} 0 & A |X - X|B| \\ |B|X^* - X^*|A| & 0 \end{bmatrix} \leq \text{Const} \cdot \begin{bmatrix} 0 & AX - XB \\ BX^* - X^*A & 0 \end{bmatrix} .
$$

Since $(|B|X^* - X^*|A|)^* = -(|A|X - X|B|)$, we have

$$
\| A |X - X|B| \| = \frac{1}{2} \{ \| A |X - X|B| \| + \| B|X^* - X^*|A| \| \}
$$

$$
\leq \text{the above left side} \quad \text{(by Lemma 2)} .
$$

We similarly get

$$
\text{the above right side} \leq \| AX - XB \| + \| BX^* - X^*A \| \quad \text{(by Lemma 2)}
$$

$$
= 2 \| AX - XB \| .
$$

(ii) For $A, B \in I(S_{n\phi})$, we set

$$
a = \begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix} (= a^*), \quad b = \begin{bmatrix} 0 & B^* \\ B & 0 \end{bmatrix} (= b^*), \quad x = \begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix} .
$$

Notice that

$$
|a| = \begin{bmatrix} |A^*| & 0 \\ 0 & |A| \end{bmatrix} \quad \text{and} \quad |b| = \begin{bmatrix} |B| & 0 \\ 0 & |B^*| \end{bmatrix} .
$$

Hence (i) applied to $a, b, x$ implies
The estimate and Lemma 2 show

\[ \| A^*X - X|B^*| \| \leq \text{Const.} \| AX - XB \| \leq \text{Const.} \| A*X - X*B^* \| . \]

When \( A = B \) and \( X = X^* \), (since \( (A*X - XA^*)^* = -(AX - XA) \)) (2) reduces to (ii).

(iii) This can be obtained by applying (i) (with \( A = B \)) to

\[ a = \begin{bmatrix} A_1 & 0 \\ 0 & -A_2 \end{bmatrix} (= a^*), \quad x = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} . \]

Q.E.D.

The theorem remains valid for a (separable) symmetrically normed ideal \( I(S_{\sigma}) \), the norm of \( I(S_{\sigma}) \) being just the restriction of \( \| \cdot \| = \| \cdot \|_{I(S_{\sigma})} \).

Corollary 7. Let \( (I(S_{\sigma}), \| \cdot \|) \) be as in Theorem 6 and \( A, B \in B(H) \). If \( A - B \) belongs to \( I(S_{\sigma}) \), then so does \( |A| - |B| \) and

\[ \| |A| - |B| | \| \leq \text{Const.} \| A - B \| . \]

Proof. When \( A, B \in I(S_{\phi}) \), by setting \( X = 1 \) in (2) we get

\[ \| |A| - |B| | \| \leq \text{Const.} \{ \| |A| - |B| | \| + \| A^* - B^* | \| \} \]

\[ = \text{2 Const.} \| |A| - |B| | \| . \]

To deal with the general case, we choose an increasing sequence \( \{ p_n \}_{n=1,2,...} \) of finite rank projections tending to 1 strongly. Since \( p_nA p_n, p_nB p_n \) are finite rank operators (\( \subseteq I(S_{\sigma}) \)), the above estimate implies

\[ \| | p_nA p_n| - | p_nB p_n| | \| \leq \text{Const.} \| p_n(A - B)p_n | \| \]

\[ \leq \text{Const.} \| |A - B| | \| (< + \infty \text{ by the assumption}) . \]

Since \( | p_nA p_n| - | p_nB p_n| \to |A| - |B| \) strongly, Lemma 3 guarantees \( |A| - |B| \in I(S_{\sigma}) \) as well as the desired inequality. Q.E.D.

This perturbation result fails for the trace class ideal \( C_1 \) and for \( C_\infty \). However, different perturbation results will be obtained in §4.

§3. The Converse of Theorem 6

Let \( I \) be either \( I(S_{\phi}) \) or \( I(S_{\sigma}) \).

Proposition 8. For a symmetrically normed ideal \( I, \| \cdot \| \) the following seven conditions are equivalent:
(i) There exists a constant $K$ such that for each $n \in \mathbb{N}_+$, $n \times n$-matrix $C = [C_{ij}]$, and $\lambda_i, \mu_i \geq 0$ ($i = 1, 2, \ldots, n$) we have
$$||[(\lambda_i - \mu_i)C_{ij}]|| \leq K ||[(\lambda_i + \mu_i)C_{ij}]||.$$ 

(ii) $||A|X - X|A|| \leq \text{Const.} \cdot ||AX - XA||$ for $A = A^* \in I$, $X \in B(H)$.

(iii) $||A|X - X|B|| \leq \text{Const.} \cdot ||AX - XB||$ for $A = A^*, B = B^* \in I$, $X \in B(H)$.

(iv) $||A|X - X|B|| \leq \text{Const.} \cdot \{||AX - XB|| + ||A^*X - XB^*||\}$ for $A, B \in I$, $X \in B(H)$.

(v) $||A|X - X|A|| \leq \text{Const.} \cdot ||AX - XA||$ for $A \in I$, $X = X^* \in B(H)$.

(vi) $||A_1X - XA_2|| \leq \text{Const.} \cdot ||A_1X + XA_2||$ for $A_i \in I_+$, $X \in B(H)$.

(vii) $||A| - |B|| \leq \text{Const.} \cdot ||A - B||$ for $A, B \in I$.

Here the six constants in (ii) ~ (vii) do not depend on involved operators.

**Proof.** In §2 we actually showed the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v), (ii) $\Rightarrow$ (vi), and (iv) $\Rightarrow$ (vii). Since (vi) $\Rightarrow$ (i) is obvious, it suffices to prove (v) $\Rightarrow$ (ii) and (vii) $\Rightarrow$ (ii).

(v) $\Rightarrow$ (ii). For $A = A^* \in I$ and $X \in B(H)$ we set
$$a = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}, \quad x = \begin{bmatrix} 0 & X^* \\ X & 0 \end{bmatrix} (= x^*) .$$

Applying (v) to $a, x$, we get
$$\left\| \begin{bmatrix} 0 & |A|X - X|A| \\ |A^*X - X^*A| & 0 \end{bmatrix} \right\| \leq \text{Const.} \left\| \begin{bmatrix} 0 & AX - XA \\ AX^* - X^*A & 0 \end{bmatrix} \right\| .$$

Since $(|A|X^* - X^*|A|)^* = -(|A|X - X|A|)$ and $(AX^* - X^*A)^* = -(AX - XA)^* = -(AX - XA)$, as before we obtain (ii) by using Lemma 2.

(vii) $\Rightarrow$ (ii). The "semi-group theory trick" in the proof of Theorem 1, [1], shows (ii) with the additional assumption $X = X^*$. Then by using the same trick as in (v) $\Rightarrow$ (ii) we can drop the self-adjointness of $X$.

Q.E.D.

**Theorem 9.** Assume that for a separable symmetrically normed ideal $(I = I(S_0^0), ||\cdot||)$ one (hence all) of the seven equivalent estimates in Proposition 8 is satisfied. Then $I$ is an interpolation space between $C_{p_1}$ and $C_{p_2}$, $1 < p_1 < p_2 < +\infty$.

To prove the theorem, we prepare the following lemma:

**Lemma 10.** Under the same assumption as in the above theorem, there is a constant $\tilde{K}$ such that for each $n \in \mathbb{N}_+$ and $n \times n$-matrix $C = [C_{ij}]$ we have
$$\left\| \begin{bmatrix} C_{11} & \cdots & C_{1n} \\ C_{22} & \ddots & \vdots \\ 0 & \cdots & C_{nn} \end{bmatrix} \right\| \leq \tilde{K} ||C|| .$$
Proof. For each \( k \in \mathbb{N}_+ \), we set
\[
D_k(= D_k(C)) = \left[ \begin{array}{c}
\frac{k^i - k^j}{k^i + k^j} C_{ij}
\end{array} \right].
\]
Proposition 8, (i), implies \( \|D_k\| \leq K\|C\| \), where \( K \) does not depend upon \( k \) (and \( n \)). Letting \( k \to \infty \), we have
\[
\left\| \begin{array}{c}
0 & -C_{ij} \\
C_{ij} & 0
\end{array} \right\| \leq K\|C\|.
\]
Notice that
\[
\begin{bmatrix}
C_{11} & \cdots & C_{ij} \\
\vdots & \ddots & \vdots \\
0 & \cdots & C_{nn}
\end{bmatrix} = \frac{1}{2} \left\{ C \left[ \begin{array}{c}
0 & -C_{ij} \\
C_{ij} & 0
\end{array} \right] + \left[ \begin{array}{c}
C_{11} & 0 \\
0 & C_{nn}
\end{array} \right] \right\}.
\]
Since
\[
\left\| \begin{array}{c}
C_{11} & 0 \\
0 & C_{nn}
\end{array} \right\| \leq \|C\| \quad \text{(for any } \| \cdot \|)
\]
is known, \( \tilde{K} = 2^{-1}(1 + K + 1) \) does the job. Q.E.D.

Proof of Theorem 9. Let us identify \( H \) with \( l^2(\mathbb{N}_+) \). By using the canonical basis \( \{e_i\}_{i=1,2,\ldots} \), one can represent an operator as an (infinite) matrix. For an infinite matrix \( C = [C_{ij}] \), we set
\[
T(C) = \left[ \begin{array}{c}
C_{11} & C_{ij} \\
C_{22} & \ddots \\
\vdots & \ddots & \ddots \\
0 & \cdots & \cdots & \ddots \\
1 & \cdots & \cdots & \cdots & 1 \\
0 & \cdots & \cdots & \cdots & 0
\end{array} \right].
\]
We also set
\[
p_n = \left[ \begin{array}{c}
1 & \cdots & n \\
\vdots & \ddots & \vdots \\
1 & \cdots & 1 \\
0 & \cdots & 0
\end{array} \right] \quad \text{(n-dimensional projection)}.
\]
For each \( C \in I = I(S_0^{(0)}) \), Lemma 4 guarantees that \( \{p_nCp_n\} \) is Cauchy in \( \| \cdot \| \). Since the constant \( \tilde{K} \) in Lemma 10 does not depend on \( n \), \( \{T(p_nCp_n)\} \) is also Cauchy in \( I \) and there is an element \( Y \) in \( I \) such that \( \lim_{n \to \infty} \|T(p_nCp_n) - Y\| = 0 \).
Take a vector \( \zeta \) in \( p_mH \) (\( m \in \mathbb{N}_+ \)). Since \( \| \cdot \| \leq \| \cdot \|_H \), we have
\[
\|T(p_nCp_n)\zeta - Y\zeta\|_H \leq \|T(p_nCp_n) - Y\|\|\zeta\|_H \to 0.
\]
as \( n \to \infty \). For \( n \geq m \), \( T(p_n C p_n)\xi \) obviously does not depend upon \( n \) so that we conclude

\[
T(p_n C p_n)\xi = Y\xi \quad \text{for} \quad n \geq m \quad (\xi \in p_m H).
\]

For each \( i, j \), by choosing \( n \geq i, j \) we get

\[
y_{ij} = (Ye_i, e_j) = (T(p_n C p_n)e_i, e_j)
= \begin{cases} 
C_{ij} & \text{if } i \leq j, \\
0 & \text{otherwise}.
\end{cases}
\]

Therefore, we conclude \( Y = T(C) \) (and hence \( T(C) \in I \)) and \( \| T(C) \| = \lim_{n \to \infty} \| T(p_n C p_n) \| \). Since

\[
\| T(p_n C p_n) \| \leq \bar{K} \| p_n C p_n \|
\leq \bar{K} \| C \|
\]

we conclude that \( \| T(C) \| \leq \bar{K} \| C \| \), that is, \( T: I \to I \) is bounded. Thanks to Arazy's characterization (Corollaries 3.4, 4.12, [4]) the ideal \( I \) is an interpolation space between \( C_{p_1} \) and \( C_{p_2} \), \( 1 < p_1 < p_2 < +\infty \). Q.E.D.

Many other characterizations for \( I(S_{\phi}^{(0)}) \) to be such an interpolation space are given in [4]. Also as remarked in p. 458, [4], these characterizations are valid for a (not necessarily separable) ideal \( I(S_{\phi}) \) by the simple duality argument.

We remark that Theorem 9 also remains valid for \( I = I(S_{\phi}) \) by the duality. In fact, let us assume that \( \| \cdot \|_{I(S_{\phi})} \) satisfies the inequality

\[
\left\| \begin{bmatrix} \lambda_i - \mu_j \\ \lambda_i + \mu_j \end{bmatrix} C_{ij} \right\|_{I(S_{\phi})} \leq K \| [C_{ij}] \|_{I(S_{\phi})}.
\]

Then the dual norm \( \Phi' \) (see §1) satisfies

\[
\left\| \begin{bmatrix} \lambda_i - \mu_j \\ \lambda_i + \mu_j \end{bmatrix} C_{ij} \right\|_{I(S_{\phi})} = \sup_D \left\| \text{Tr} \left( \begin{bmatrix} \lambda_i - \mu_j \\ \lambda_i + \mu_j \end{bmatrix} [D_{ij}] \right) \right\|
\]

where the sup is taken over all \( n \times n \)-matrices \( D \) with \( \|D\|_{I(S_{\phi})} \leq 1 \). It is elementary to see

\[
\text{Tr} \left( \begin{bmatrix} \lambda_i - \mu_j \\ \lambda_i + \mu_j \end{bmatrix} [D_{ij}] \right) = -\text{Tr} \left( [C_{ij}] \begin{bmatrix} \mu_i - \lambda_j \\ \mu_i + \lambda_j \end{bmatrix} [D_{ij}] \right)
\]

Hence we get

\[
\left| \text{Tr} \left( \begin{bmatrix} \lambda_i - \mu_j \\ \lambda_i + \mu_j \end{bmatrix} C_{ij} \right) \right| \leq \| C \|_{I(S_{\phi})} \times \left\| \begin{bmatrix} \mu_i - \lambda_j \\ \mu_i + \lambda_j \end{bmatrix} D_{ij} \right\|_{I(S_{\phi})}
\leq \| C \|_{I(S_{\phi})} \times K \| D \|_{I(S_{\phi})} \quad \text{(by the assumption)},
\]
and
\[
\left\| \frac{\lambda_i - \mu_j}{\lambda_i + \mu_j} C_{ij} \right\|_{I(S^0)} \leq K \|C\|_{I(S^0)}.
\]

Proposition 8, (i), thus remains valid for the dual norm \(\|\cdot\|_{I(S^0)}\), and Theorem 9 shows that the separable \(I(S^0)\) is an interpolation space between \(C_{q_1}\) and \(C_{q_2}\), \(1 < q_2 < q_1 < +\infty\). But this means that \(I(S^0)^* = I(S_\phi)\) is an interpolation space between \(C_{p_1}\) and \(C_{p_2}\) with \(1 < p_1 < p_2 < +\infty\), \(p_1^{-1} + q_1^{-1} = 1\). Combining the above with other characterizations given in [4], we have proved the following main result in the article:

**Theorem 11.** Let \(I\) be either \(I(S^0)\) or \(I(S_\phi)\). The following conditions are equivalent:

(a) One of the seven estimates in Proposition 8 is valid (for example, \(\|A - B\| \leq \text{Const.} \|A - B\|\) for \(A, B \in I\)).

(b) \(I\) is an interpolation space between \(C_{p_1}\) and \(C_{p_2}\), \(1 < p_1 < p_2 < +\infty\).

(c) The triangle projection \(T\) is bounded relative to \(\|\cdot\|\).

(d) The Macaev theorem remains valid for \(I\), that is, whenever a Volterra operator \(A\) satisfies \(\text{Im} A \in I\), we must have \(\text{Re} A \in I\) and

\[
\|\text{Re} A\| \leq \text{Const.} \|\text{Im} A\|.
\]

(e) The Boyd indices (see [4] for details) of \(\Phi\) are non-trivial.

The last condition is very useful because one can check Lipschitz-continuity of the map \(A \to |A|\) by just looking at the norm \(\Phi\) on the sequence space \(f\). Define the discrete dilation operators \(D_m, D_{1/m}\) \((m = 1, 2, \ldots)\) on \(f\) by

\[
D_m(\xi) = \left(\xi_1, \ldots, \frac{\xi_1}{m}, \frac{\xi_2}{m}, \ldots, \frac{\xi_2}{m}, \ldots\right),
\]

\[
D_{1/m}(\xi) = \left(\frac{\xi_1}{m}, \ldots, \frac{\xi_1}{m}, \frac{\xi_2}{m}, \ldots, \frac{\xi_2}{m}, \ldots\right).
\]

Then compute the norms \(\|D_m\|\) and \(\|D_{1/m}\|\) (relative to \(\Phi(\cdot)\)). The Boyd indices \((p_\Phi, q_\Phi)\) are defined by

\[
p_\Phi = \sup_m \frac{\log m}{\log\|D_m\|} \quad \left(= \lim_{m \to \infty} \frac{\log m}{\log\|D_m\|}\right),
\]

\[
q_\Phi = \inf_m \frac{\log(1/m)}{\log\|D_{1/m}\|} \quad \left(= \lim_{m \to \infty} \frac{\log(1/m)}{\log\|D_{1/m}\|}\right).
\]

It is easy to see \(1 \leq p_\Phi \leq q_\Phi \leq +\infty\). (For \(\Phi_p\) corresponding to the Schatten ideal \(C_p\), we easily see \(p_{\Phi_p} = q_{\Phi_p} = p\).) Non-triviality in the last condition (e) means \(1 \nleq p_\Phi \nleq q_\Phi \nleq +\infty\).
Let $C_{pq}$ $(1 \leq p \leq +\infty, 1 \leq q \leq +\infty)$ be the non-commutative analogue of the Lorentz space (see [4] or [19]) consisting of all compact operators such that
\[
\|A\|_{pq} = \left( \sum_{i=1}^{\infty} i^{(q/p)-1} s_i(A)^q \right)^{1/q}
\]
\[
(= \sup_i (i^{1/p} s_i(A)) \text{ if } q = +\infty)
\]
is finite. Note that $\|\cdot\|_{pq}$ is a norm only if $q \leq p$, but when $p > 1$ there is a norm on $C_{pq}$ equivalent to $\|\cdot\|_{pq}$. It is well-known that $C_{pq}$ $(1 < p < \infty, 1 \leq q \leq +\infty)$ is an interpolation space (the K-method can be used, [5]) between $C_{p_1}$ and $C_{p_2}$ $(1 < p_1 < p < p_2 < +\infty)$. Therefore, the map $A \to |A|$ is Lipschitz-continuous relative to $\|\cdot\|_{pq}$ $(1 < p < +\infty, 1 \leq q \leq +\infty)$.

The above characterization roughly says that the map $A \to |A|$ is Lipschitz-continuous when the “geometry” of a symmetrically normed ideal in question is “good”. However, the example presented after Corollary 4.6, [4], shows: there exists a non-uniformly convex symmetrically normed ideal in which the map $A \to |A|$ is Lipschitz-continuous.

§ 4. Estimates in the Operator and Trace Class Norms

As was mentioned in §0, the map $A \to |A|$ is not Lipschitz-continuous for $\|\cdot\|$ and $\|\cdot\|_1$. Instead the following estimates are known ([13], [16]):
\[
\begin{align*}
\| |A| - |B| \| & \leq \frac{2}{\pi} \| A - B \| \left( 2 + \log \frac{\|A\| + \|B\|}{\|A - B\|} \right); \quad A, B \in B(H), \\
\| |A| - |B| \|_1 & \leq \sqrt{2} \| A + B \|^{1/2} \| A - B \|^{1/2}; \quad A, B \in C_1.
\end{align*}
\]
In this section we obtain different (and probably more natural) estimates for $|A| - |B|$ by making use of the ideals introduced by V. I. Macaev.

For a sequence $\xi = \{\xi_i\}_{i=1,2,...}$, let $\{\xi^*_i\}_{i=1,2,...}$ be the non-increasing rearrangement of $\{|\xi_1|, |\xi_2|, \ldots\}$. We introduce the (dual) symmetric norms $\Phi_\Omega$, $\Phi_\omega$ (on $f$) defined by
\[
\Phi_\Omega(\xi) = \sup_n \left( \sum_{i=1}^{n} \frac{\xi_i}{(2i - 1)^{-1}} \right),
\]
\[
\Phi_\omega(\xi) = \sum_{i=1}^{\infty} (2i - 1)^{-1} \xi^*_i.
\]
The corresponding symmetrically normed ideals $I(S_{\Phi_\omega}) (= I(S_{\Phi_\omega}^0))$ and $I(S_{\Phi_\omega}) \supset I(S_{\Phi_\omega})^0 = \{ A \in C_\infty : \lim_{n \to \infty} \left( \sum_{i=1}^{n} s_i(A) / \sum_{i=1}^{n} (2i - 1)^{-1} \right) = 0 \}$ satisfy
\[
\begin{align*}
I(S_{\Phi_\omega}^0)^* & = I(S_{\Phi_\omega}), \\
I(S_{\Phi_\omega})^* & = I(S_{\Phi_\omega}).
\end{align*}
\]
These ideals were introduced by V. I. Macaev and play important roles in analysis of compact operators (see [11], [12] for details and typical applications). Notice that $I(S_{\phi_0})$ (resp. $I(S_{\phi_a})$) is "slightly" larger (resp. smaller) than $C_1$ (resp. $C_\alpha$):

$$\begin{align*}
I(S_{\phi_0}) &\subseteq C_p, & p > 1, \\
C_p &\subseteq I(S_{\phi_\alpha}), & p < +\infty.
\end{align*}$$

In what follows, the norms of $I(S_{\phi_0})$ and $I(S_{\phi_a})$ will be denoted by $\| \cdot \|_\Omega$ and $\| \cdot \|_{\infty}$ respectively. Recall that the proof of (1) (in §2) was based on Theorem 6.3 in Chap. III, [12]. If one starts from Theorem 2.2 in Chap. III, [12], instead, one obtains

$$\|P_{n+4}(A)\|_\Omega \leq \|A\|_1.$$ 

Therefore, by repeating the arguments in p. 150, [9], we get

(4)

$$\left\| \frac{\lambda_i - \mu_j}{\lambda_i + \mu_j} C_{ij} \right\|_{\Omega} \leq \text{Const.} \|C\|_1.$$ 

(Here the obvious fact $\| \cdot \|_\Omega \leq \| \cdot \|_1$ is used.) Hence, the same arguments as in §2 show (among other commutator estimates) the next perturbation result.

**Theorem 12.** If $A, B \in B(H)$ satisfy $A - B \in C_1$ then $|A| - |B|$ belongs to the ideal $I(S_{\phi_0})$ and

$$\|A| - |B|\|_{\Omega} \leq \text{Const.} \|A - B\|_1.$$ 

By the obvious modification of the proof of (3), from (4) we get

$$\left\| \frac{\lambda_i - \mu_j}{\lambda_i + \mu_j} C_{ij} \right\|_{\infty} \leq \text{Const.} \|C\|_{\infty}.$$ 

We then would like to show a dual version of the previous theorem. However, notice that Lemma 3 is not valid for $C_\infty$ the compact operators. Starting from the assumption $A = A^* \in I(S_{\phi_0}), X \in B(H)$, we get

$$\|p_n[|A|, X]p_n\| \leq \text{Const.} \|[A, X]\|_{\infty}$$

as in the proof of Theorem 6. Since $Y \in B(H) \rightarrow \|Y\| = \sup \{\|Y\|_H : \xi \in H, \|\xi\|_H \leq 1\}$ is lower semi-continuous relative to the strong operator topology, (without using Lemma 3) we conclude

$$\|[|A|, X]\| \leq \liminf_{n \rightarrow \infty} \|p_n[|A|, X]p_n\|$$

$$\leq \text{Const.} \|[A, X]\|_{\infty}.$$ 

Hence, (2) in the proof of Theorem 6 is still valid and we get

$$\| |A| - |B| \| \leq \text{Const.} \|A - B\|_{\infty}; \quad A, B \in I(S_{\phi_0}).$$
**Theorem 13.** If $A, B \in B(H)$ satisfy $A - B \in I(S_{\omega})$, then $|A| - |B|$ is a compact operator and

$$\| |A| - |B| \| \leq \text{Const.} \| A - B \|_{\omega}.$$ 

**Proof.** The arguments in the second half of the proof of Corollary 7 (but Lemma 3 is replaced by the above-mentioned lower semi-continuity of $Y \to \| Y \|$) show the desired inequality. The compactness of $|A| - |B|$ follows from the following standard argument: Let $B(H)/C_\omega$ be the Calkin algebra and $\pi: B(H) \to B(H)/C_\omega$ be the natural projection. We have $\pi(A) = \pi(B)$ because $A - B$ is compact. Since $\pi$ is a $C^*$-algebra homomorphism, we conclude $\pi(|A|) = |\pi(A)| = |\pi(B)| = \pi(|B|)$, i.e., $|A| - |B| \in C_\omega$. (Q.E.D.)

When $A$ is an $n \times n$-matrix, $s_i(A) = 0$ for $i \geq n + 1$. Consequently we get

$$\left\{ \sum_{i=1}^{n} s_i(A) \right\} / \sum_{i=1}^{n} (2i - 1)^{-1} \left( = \| A \|_1 / \sum_{i=1}^{n} (2i - 1)^{-1} \right) \leq \| A \|_2,$$

$$\| A \|_\omega = \sum_{i=1}^{n} (2i - 1)^{-1} s_i(A) \leq \| A \| \sum_{i=1}^{n} (2i - 1)^{-1}.$$ 

For the second estimate the obvious fact $s_1(A) = \| A \| \geq s_2(A) \geq s_3(A) \geq \cdots$ was used. We thus get the next result for (finite) matrices.

**Corollary 14.** (Theorem 14, [9]) There exists a constant $K$ such that for any $n \times n$-matrices $A, B$ ($n \geq 2$) we have

$$\left\{ \| |A| - |B| \| \leq (\log n) K \| A - B \| \right.$$ 

$$\left\{ \| |A| - |B| \|_1 \leq (\log n) K \| A - B \|_1 \right.$$ .

**References**


The Map $A \to |A|$


