On the Whitney-Schwartz Theorem

By

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Let \( F \) be a closed set in \( \mathbb{R}^n \). Then, according to L. Schwartz [6], \( F \) is called regular if for each \( x \in F \) there are numbers \( d(>0) \), \( \alpha(\geq 0) \) and \( q(\geq 1) \) such that any two points \( y, z \) of \( F \) with \( r_{yz} \leq d \) and \( r_{xz} \leq d \), are joined by a rectifiable curve in \( F \), of length not greater than \( \alpha r_{yz}^{1/q} \) where \( r_{xy} \) is the distance between \( x \) and \( y \). This definition is a generalization of “Property (P) local” by H. Whitney ([9]). Schwartz stated in [6] the following theorem without proof.

**Theorem (Whitney-Schwartz).** Let \( T \) be a distribution in \( \mathbb{R}^n \) of order \( m \) whose support is contained in a compact regular set \( F \). Then

(A) \( \langle T, \varphi \rangle \to 0 \) provided \( \varphi \in C^\infty(\mathbb{R}^n) \) and their derivatives of order not greater than \( m' \) converge to zero uniformly on \( F \), where \( m' \) is any integer \( \geq q(F)m \) and \( q(F) \) is a number \( \geq 1 \), depending on \( F \).

(B) \( T \) is represented by a finite sum of derivatives of measures whose supports are contained in \( F \).

A similar result to the part (A) of Theorem was given for a general compact set \( F \) by G. Glaeser, in such a sense that it has an advantage not making the behavior of \( \varphi \) interfered in a ‘neighborhood’ of \( F \) (see Proposition II, Chap III in [2]). We shall give an elementary proof of Theorem for a distribution in an open set \( \Omega \) of \( \mathbb{R}^n \). For the proof we make use the reproduction of Whitney’s extension theorem by L. Hörmander ([4]). The key lemma is the following:

**Lemma.** Let \( u \) be a distribution of order \( m \) in \( \Omega \) with support in a compact regular set \( F \subset \Omega \). Then there is a constant \( C \) depending on \( m' \) and \( F \) such that for any \( \varphi \in C^\infty(\Omega) \)

\[
|\langle u, \varphi \rangle| \leq C \|\varphi\|_{m', F} \tag{1}
\]

where \( q = q(F) \) is a positive number depending on \( F \), \( m' \) is any integer \( \geq qm \), and

\[
\|\varphi\|_{k, F} = \sum_{|\alpha| \leq k} \sup_{x \in F} |(\partial^\alpha / \partial x^\alpha)\varphi(x)|.
\]
In case $F$ is a closed ball, a simple proof of (1) was given by S. Mizohata in proving that evolution equations with finite propagation speed should be of kowalevskian type (see [5]). Now we shall restate the Whitney-Schwartz theorem in our form.

**Theorem.** Let $u$ be a distribution in $\mathcal{D}$ of order $m$ with support in a compact regular set $F \subset \mathcal{D}$. Then

(A) $\langle u, \varphi \rangle \to 0$ provided $\varphi \in C^m(\mathcal{D})$ and their derivatives of order not greater than $m'$ converge to zero uniformly on $F$, where $m'$ is any integer $\geq q(F)m$.

(B) $u$ is represented by a finite sum of derivatives of measures in $\mathcal{D}$ whose supports are in $F$.

Proof of (A) is immediate from Lemma. For proving (B) we can apply the Hahn-Banach theorem to the inequality (1) through the well-known method. We omit the details (see [6]).

Before proceeding to prove our lemma, we shall give a sketch of the partition of unity by Whitney in [8], following the reproduction by Hörmander.

Let $A$ be a closed set in $\mathbb{R}^n$. The partition of unity is constructed as follows. First, divide $\mathbb{R}^n$ into $n$-cubes of side 1, and let $K_0$ be the set of all those cubes whose distance from $A$ are at least $\sqrt{n}$. Next, divide the remaining cubes into $2^n$ cubes of side $1/2$, and let $K_1$ be the set of those distant from $A$ at least $\sqrt{n}/2$. Repeating such a division process, we have a series of the sets $\{K_0, K_1, \ldots\}$ where the union of all cubes of them is $\mathbb{R}^n \setminus A$. Arrange all cubes in order of a series $\mathcal{Q}_1, \mathcal{Q}_2, \ldots$; the center and side of each $\mathcal{Q}_j$ are denoted by $y_j$ and $s_j$, respectively. Now take $\chi_0 \in C^\infty_0$ being equal to 1 on the cube

$$|x_i| \leq 1/2, \quad i = 1, \ldots, n$$

and vanishing outside the cube

$$|x_i| \leq 1/2 + 1/8, \quad i = 1, \ldots, n.$$ Then define $\chi_j \in C^\infty_0(\mathbb{R}^n)$ by

$$\chi_j(x) = \frac{\chi_0\left(\frac{x - y_j}{s_j}\right)}{\sum_{k=1}^m \chi_0\left(\frac{x - y_k}{s_k}\right)}, \quad j = 1, 2, \ldots.$$ As for the denominator, it is verified

$$1 \leq \sum_{i=1}^m \chi_0\left(\frac{x - y_i}{s_i}\right) \leq 4^n.$$ The sequence $\chi_j$ in $C^\infty_0(\mathbb{R}^n)$ is locally finite in $\mathbb{R}^n \setminus A$ and has the properties:

(i) $\chi_j \geq 0$; $\sum_{j=1}^m \chi_j(x) = 1$ for $x \in \mathbb{R}^n \setminus A$

(ii) for each $\alpha$, there is a constant $C_\alpha$ such that

$$\sum_{j=1}^m |D^\alpha \chi_j(x)| \leq C_\alpha(d(x, A)^{-|\alpha|} + 1)$$
for \( x \in R^n \setminus A \) where \( D = \partial / \partial x \)

(iii) (the diameter of \( \text{supp} \chi_j \)) \( \leq C d(\text{supp} \chi_j, A), \quad j = 1, 2, \ldots \)

for some constant \( C \).

In the following we quote each of (i), (ii), (iii) as the property of \( \chi_j \). Let \( x \in \text{supp} \chi_j \). Then it can be easily verified \( d(x, A) > 1 \), provided \( s_j = 1 \) (see \([4]\))

So we note \( d(x, A) \leq 1 \) implies \( s_j < 1 \).

**Proof of Lemma.** Since \( F \) is regular, to each \( x \in F \) there corresponds an open ball \( B_d(x) \) of center \( x \), with radius \( d \) in \( \Omega \) such that any two points \( y, z \) of \( F \cap \overline{B_d(x)} \) can be joined by a rectifiable curve in \( F \). Here we note the radius \( d \) depends on \( x \). Since \( F \) is compact, we can choose a finite family \( \{B_{d_1}(x_1), \ldots, B_{d_m}(x_m)\} \) from the open cover \( \{B_d(x) \mid x \in F\} \) of \( F \) so that

\[
F \subseteq B_{d_1}(x_1) \cup \cdots \cup B_{d_m}(x_m).
\]

Take a partition of unity \( \phi_j \) subordinate to the finite open cover \( \{B_{d_j}(x_j)\} \). Then \( u \) is represented in the form

\[
u = \phi_1 u + \cdots + \phi_m u = u_1 + \cdots + u_m
\]

where \( u_j = \text{supp} \phi_j u \subseteq F_j = F \cap \overline{B_{d_j}(x_j)} \), \( F_j \) being also compact and regular.

Suppose the estimate (1) is valid for each \( u_j, F_j, q \), instead of \( u, F, q \). Then for \( \varphi \in C^\infty(\Omega) \), there is a constant \( C_j \) such that

\[
|\langle u_j, \varphi \rangle| \leq C_j \|\varphi\|_{m_j, F_j}.
\]

where \( m_j \) is any integer \( \geq m q_j, q_j \) being a number \( \geq 1 \) related to the regularity of \( F_j \). Clearly, Lemma is a consequence of (2) and (3), with \( q = q(F) = \sup_{1 \leq j \leq m} q_j \) and \( C = \sup_{1 \leq j \leq m} C_j \). So it suffices to derive (3) for each \( \varphi \in C^\infty(\Omega) \), in which we write \( F = F_j \) and \( q = q_j \), dropping the subscript \( j \) for simplicity of notation.

Take \( \varphi \) in \( C^\infty(\Omega) \) and extend it to a function in \( C^\infty(\Omega^n) \) by setting zero outside \( \Omega \), which we denote \( \varphi \) again. We shall give a function \( \phi \in C^\infty(\Omega) \) so that \( \phi(\alpha)(x) = \varphi(\alpha)(x) \) in \( F \) when \( |\alpha| \leq m \) where \( D^a f = f^{(a)} \). This can be carried out by the method of Whitney's extension theorem, as follows. Making use of the partition of unity \( \chi_j \) in \( R^n \setminus F \) just constructed above, we define a function \( \phi \) by

\[
\phi(x) = \begin{cases} \varphi (x) & \text{for } x \in F \\ \sum_j \chi_j(x) \varphi_m (x; y^i) & \text{for } x \in R^n \setminus F \end{cases}
\]

where \( y^i \in F \) is taken so that

\[
d(\text{supp} \chi_j, F) = d(\text{supp} \chi_j, y^i)
\]

\[
\varphi_m (x; y) = \sum_{|\alpha| \leq m} \frac{1}{\alpha !} \varphi^{(a)}(y)(x-y) \alpha
\]

and \( \sum_j \) stands for the sum with \( s_j < 1 \). Then \( \phi \in C^\infty(\Omega^n) \) and satisfies
\[ D^n \phi = D^n \varphi \quad \text{in} \quad F \]

when \(|\alpha| \leq m\) ([6]). What we are going to obtain is the estimate

\[ ||\varphi||_{m, F} \leq C ||\varphi||_{m', F} \quad (5) \]

where \(C\) is a constant depending only on \(m, q\) and \(F\). Let \(x \in Q \setminus F\) be fixed. Then \(d(x, F) > 0\). To derive (5) we divide the case into (1) \(d(x, F) > 1\) and (2) \(d(x, F) \leq 1\). It is enough to show for the case where \(m \geq 1\).

1) \(d(x, F) > 1\).

Differentiation of \(\varphi\) in (4) gives, by Leibniz’s formula,

\[ \phi^{(\alpha)}(x) = \sum_{\beta \in \mathbb{N}^m} \sum_{\gamma \in \mathbb{N}} \chi^{(\beta)}(x) \varphi_{m}(x; y^\gamma). \]

Since \(\sum_{\beta} |\chi^{(\beta)}(x)| \leq 2C\beta\) by the property (ii) of \(\{\chi_j\}\), we have

\[ ||\phi^{(\alpha)}(x)|| \leq C' ||\varphi||_{m, F} \]

for a constant \(C'\) when \(|\alpha| \leq m\), which implies (5).

2) \(0 < d(x, F) \leq 1\).

In this case, as we noted before, \(s_j < 1\) provided \(x \in \text{supp} \chi_j\). Hence we have \(\sum_{\beta} \chi^{(\beta)}(x) = \sum_{\gamma \in \mathbb{N}} \chi_j(x) = 1\), which gives

\[ \phi(x) = \varphi_m(x; y) + \sum_{\gamma} \chi_j(x) [\varphi_m(x; y^\gamma) - \varphi_m(x; y)] \quad (6) \]

where \(y \in F\) is so chosen as \(d(x, F) = |x - y|\). Further we take \(x' \in \text{supp} \chi_j\) so as to satisfy \(d(\text{supp} \chi_j, F) = |x' - y|\). Then

\[ |x - x'| \leq \text{diam}(\text{supp} \chi_j) \leq C d(\text{supp} \chi_j, F) \leq C d(x, F) \]

for some constant \(C\) where we used the property (iii) of \(\{\chi_j\}\). Thus in view of the definitions of \(y, x'\) and \(y'\), we get the inequalities

\[ |x - y'| \leq |x - x'| + |x' - y'| \leq (C + 1) d(x, F) \]
\[ |y - y'| \leq |y - x'| + |x - y'| \leq (C + 2) d(x, F) \quad (7) \]

which will be needed later. Denoting by \(R_m(x; y)\) the remainder term of Taylor’s formula at \(y\), we have

\[ \varphi(x) = \varphi_m(x; y) + R_m(x; y). \quad (8) \]

Our basic concern is to estimate the derivatives of the difference \(\varphi_m(x; y') - \varphi_m(x; y)\) in (6). The Taylor polynomial of \(\varphi^{(\gamma)}(z^\beta)\)

\[ \varphi^{(\gamma)}_m(z^\beta; z') = \sum_{|\beta| \leq m, |\gamma| \leq |\beta|} \frac{1}{\beta!} \varphi^{(\beta + \gamma)}_m(z^\beta - z')^\gamma \]

combined with the formula for any \(z, z', z''\)

\[ \varphi^{(\beta + \gamma)}_m(z') = \varphi^{(\beta + \gamma)}_m(z'; z) + R_m^{(\beta + \gamma)}(z'; z) \]

\[ \varphi^{(\beta + \gamma)}_m(z') = \varphi^{(\beta + \gamma)}_m(z'; z) + R_m^{(\beta + \gamma)}(z'; z) \]
obtained by differentiating (8), gives
\[ \varphi^{(\gamma + \delta)}_m(z^\prime; z) = \sum_{|\delta| \leq m-1, |\eta| \leq \eta-1} \frac{1}{\delta!} [\varphi^{(\gamma \delta)}_m(z^\prime; z) + R^{(\gamma \delta)}_m(z^\prime; z)](z^\prime - z)^\delta. \]  
(9)

On the other hand
\[ \varphi^{(\gamma + \delta)}_m(z^\prime; z) = \sum_{|\delta| \leq m-1} \frac{1}{\delta!} \varphi^{(\gamma + \delta)}_m(z^\prime; z)(z^\prime - z)^\delta. \]  
(10)

Thus the subtraction of (10) from (9) yields
\[ \varphi^{(\gamma)}_m(z^\prime; z^\prime) - \varphi^{(\gamma)}_m(z^\prime; z) = \sum_{|\delta| \leq m-1} \frac{1}{\delta!} R^{(\gamma + \delta)}_m(z^\prime; z)(z^\prime - z)^\delta, \]  
(11)

so that changing \(z, z', z''\) to \(y, y', x\) gives
\[ |\varphi^{(\gamma)}_m(x; y') - \varphi^{(\gamma)}_m(x; y)| \leq \sum_{|\delta| \leq m-1} |R^{(\gamma + \delta)}_m(y'; y)(x - y')^\delta| \]  
(12)

since \(R_m(x; y) = \varphi_m(x; y) - \varphi_m(x; y) + R_m'\) where \(m'\) is any integer \(\geq m\). So we are left with estimation of \(R^{(\gamma + \delta)}_m(y'; y)\). This will be worked out by a technical modification of [7]. As \(y, y' \in F\), there is a rectifiable curve \(C\) in \(F\) of length, say \(L\), joining \(y\) and \(y'\). Let \(\Delta: y = z^0, z^1, \ldots, z^p = y'\) be a subdivision of \(C\) in \(F\) and let \(|\Delta| = \sup \text{lengths of } z^i - z^{i-1}|.\) Note that
\[ \varphi^{(\gamma)}_m(z^\prime; z') - \varphi^{(\gamma)}_m(z^\prime; z) = R^{(\gamma)}_m(z^\prime; z) - R^{(\gamma)}_m(z^\prime; z') \]  

since
\[ \varphi^{(\gamma)}_m(z^\prime; z') = \varphi^{(\gamma)}_m(z^\prime; z) + R^{(\gamma)}_m(z^\prime; z) \]  

Thus we get by (11)
\[ R^{(\gamma + \delta)}_m(z^\prime; z') - R^{(\gamma + \delta)}_m(z^\prime; z) = \sum_{|\delta| \leq m-1, |\eta| \leq \eta-1} \frac{1}{\delta!} R^{(\gamma + \delta + \eta)}_m(z^\prime; z)(z^\prime - z)^\delta. \]  
Changing \(z, z', z''\) to \(z^{-1}, z^1, y\) in this equation, summing over \(i\) and noting \(R^{(\gamma)}_m(y'\); \(y^0 = 0\) when \(|\gamma| \leq m'\), we consequently have
\[ R^{(\gamma + \delta + \eta)}_m(y'; y) = \sum_{i=1}^{p} \sum_{|\delta| \leq m-1, |\eta| \leq \eta-1} \frac{1}{\delta!} R^{(\gamma + \delta + \eta)}_m(z^i; z^{i-1})(y^i - y')^\delta. \]  
(13)

Note that by the classical formula for the remainder term
\[ |R^{(\gamma + \delta + \eta)}_m(z^i; z^{i-1})| \leq |z^i - z^{i-1}|^{m' - |\gamma + \delta + \eta|} \varepsilon(|z^i - z^{i-1}|) \]  
(14)

where \(\varepsilon(h) \to 0 \) when \(h \to 0\).

Now split the sum for \(\delta\) in (13) into the sums for \(|\delta| < m' - |\gamma + \eta|\) and for \(|\delta| = m' - |\gamma + \eta|\), and then denote the former by \(J_d\) and the latter by \(J_d', \) re-
pectively. Since \(|z^t - z^{t-1}| \leq L\) and \(|y^j - z^t| \leq L\), in view of (14) we have

\[
|I_d| \leq \sum_{i=1}^{p} \frac{1}{\delta!} L^{m' - |\gamma + \eta|} \sum_{i=1}^{p} |z^t - z^{t-1}| \varepsilon(||d||)
\]

which tends to 0 when \(|d| \to 0\).

On the other hand

\[
J_d = \sum_{i=1}^{p} \sum_{\delta = m' - |\gamma + \eta|} \frac{1}{\delta!} \left[ \varphi_m^{(r + \delta + \eta)}(z^t) - \varphi_m^{(r + \delta + \gamma)}(z^{t-1}) \right](y^j - z^t)\delta
\]

since for \(|\delta| = m' - |\gamma + \eta|\),

\[
\varphi_m^{(r + \delta + \eta)}(z^t; z^{t-1}) = \varphi_m^{(r + \delta + \gamma)}(z^{t-1})
\]

and so

\[
\varphi_m^{(r + \delta + \eta)}(z^t) = \varphi_m^{(r + \delta + \gamma)}(z^{t-1}) + R_m^{(r + \delta + \eta)}(z^t; z^{t-1}).
\]

Now for each fixed \(\delta\) we have

\[
\sum_{i=1}^{p} \left[ \varphi_m^{(r + \delta + \eta)}(z^t) - \varphi_m^{(r + \delta + \gamma)}(z^{t-1}) \right](y^j - z^t)\delta
\]

which tends to a Stieltjes integral

\[
- \int_{s}^{\delta} \left[ \varphi_m^{(r + \delta + \eta)}(z(s)) - \varphi_m^{(r + \delta + \gamma)}(z(s)) \right] d(y^j - z(s))\delta
\]

(15)

when \(|d| \to 0\), where \(z(s)\) denotes the point on the curve \(C\) of length \(s\) along \(C\) from \(y\).

After the differentiation in the integral, (15) becomes

\[
\sum_{i=1}^{p} \frac{\delta!}{(\delta - \kappa)!} \int_{y^j}^{y^j} \left[ \varphi_m^{(r + \delta + \eta)}(z) - \varphi_m^{(r + \delta + \gamma)}(y) \right] (y^j - z^t)\delta_{\kappa}(dz)^t.
\]

(16)

Denote the sum of integrals (16) by \(I_{\gamma, \delta, \eta}\). Then we have

\[
R_m^{(r + \eta)}(y^j; y) = \sum_{\delta = m' - |\gamma + \eta|} \frac{1}{\delta!} I_{\gamma, \delta, \eta} = \lim_{|d| \to 0} J_d.
\]

Hence, taking the regularity of \(F\) into consideration, we have the estimates

\[
|R_m^{(r + \eta)}(y^j; y)| \leq C_1 L^{m' - |\eta + \gamma|} \|\varphi\|_{m', F}
\]

\[
\leq C_2 d(x, F)^{m' - |\eta + \gamma|} / \|\varphi\|_{m', F}
\]

for some constant \(C_1\) and for any \(\gamma\) and \(\eta\) with \(|\gamma + \eta| \leq m'\). The last estimate combined with (7) and (12) implies
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\[ |\varphi_m(t; y') - \varphi_m(t; y)| \leq C_2 \| \varphi \|_{m', r} \sum_{\gamma \leq m' - |\gamma|} d(x, F)^{m' - |\gamma|/q} d(x, F)^{|\gamma|} \]

where \( C_2 \) is a constant depending only on \( m' \) and \( F \). Now, the differentiation of \( \psi \) in (6) with respect to \( x \) gives

\[ \psi^{(\alpha)}(x) = \varphi_m^{(\alpha)}(x; y) + \sum_{\beta + \gamma = \alpha} \sum_{f=1}^\infty X(\beta)(x)[\varphi_m^{(1)}(x; y') - \varphi_m^{(1)}(x; y)]. \]

Thus in view of the property (ii) of \( \{ \mathcal{A}_j \} \),

\[ |\varphi^{(\alpha)}(x)| \leq \| \varphi \|_{m', r} \sum_{\beta + \gamma = \alpha} (d(x, F)^{-|\beta|} + 1)^{\sum_{\eta \leq m' - |\eta|} d(x, F)^{(m' - |\gamma|)/q} |\eta|}. \]

As for the exponent of \( d(x, F) \), if \( |\alpha| \leq m \)

\[ ((m' - |\gamma + \eta|)/q + |\eta| - |\beta|) \geq \frac{1}{q} (mq - |\alpha - \beta| + (q - 1)|\eta| - q|\beta|) \]

\[ = \frac{1}{q} (\eta(m - |\beta| + |\gamma|) - (|\alpha| - |\beta| + |\gamma|)) \]

\[ \geq \frac{q - 1}{q} (m - |\beta| + |\gamma|) \geq 0. \]

Since \( d(x, F) \leq 1 \), we finally have

\[ |\psi^{(\alpha)}(x)| \leq C \| \varphi \|_{m', r} \]

when \( |\alpha| \leq m \), where \( C \) is a constant depending only on \( m' \) and \( F \). Collecting the results obtained so far, we consequently proved the estimate (5). Recall a property of distributions with compact support that if \( \mathcal{A} \in C^0_m(\Omega) \) and its derivatives of order up to \( m \) vanish on \( F \), then \( \langle u, \mathcal{A} \rangle = 0 \) (cf. [6]). Suppose \( \eta \in C^0_m(\Omega) \) is equal to 1 on a neighborhood of \( F \). Then \( \eta \psi \) can be regarded as a function in \( C^0_m(\Omega) \) and \( (\eta \psi)^{\alpha} \) on \( F \) when \( |\alpha| \leq m \). Thus we get

\[ \langle u, \eta \psi \rangle = \langle u, \eta \phi \rangle, \]

and so

\[ |\langle u, \varphi \rangle| = |\langle u, \eta \varphi \rangle| = |\langle u, \eta \phi \rangle| \]

\[ \leq C_1 \| \phi \|_m \Omega \]

\[ \leq C_2 \| \varphi \|_{m', r} \quad \text{(by (5))} \]

for any integer \( m' \geq mq \) where \( C_1, C_2 \) are constants depending only on \( m' \) and \( F \) which completes the proof of Lemma.

Remark. A typical example of regular set is a convex set, where \( q=1 \). In this particular case, the proof of Lemma is carried out much more readily than the above, since it is enough to use the classical formula for \( R_m^{(\gamma)}(y'; y) \) in (12). Today, we know a large family of regular sets, that is, compact sub-
analytic sets in $\mathbb{R}^n$ (or in real analytic manifolds) (see [1], [3]).

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References