On Multi-valued Analytic Solutions of First Order Non-linear Cauchy Problems

Dedicated to Professor Shigetake Matsuura on the sixtieth anniversary of his birthday

By

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Introduction

In this article we study the first order non-linear Cauchy problems of the form

\[
\begin{align*}
F(x; d u(x), u(x)) = 0 \\
|_s = \phi
\end{align*}
\]

(1)

where (1) are defined in a complex domain \( M \) in \( C^n, n \geq 2 \). Our aims are to find analytic solutions of (1) multi-valued in general, which ramify around a fixed point \( x^0 \) in \( M \), and to calculate their ramification degrees there.


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We always consider (1) in the following situation (2):

\begin{align*}
\text{(a) } F(x; \xi, z) & \text{ is holomorphic in an open neighborhood of a fixed base point } \\
\text{where we denote by } J^1M = \bigcup_{x \in M} J^1_xM, \text{ the first order complex jet bundle over } M, \text{ which can be identified to the product space } T^*M \times C.
\end{align*}

\begin{align*}
\text{(b) } S & \text{ is a non-singular complex hypersurface of } M \text{ passing through the point } x^o \in M. S \text{ is defined by a holomorphic germ } s \in \mathcal{O}_{M, x^o} \text{ locally at } x^o, \text{ that is, } S = \{x \in M; s(x) = 0\} \text{ near } x^o. \\
\text{(c) } \phi \in \mathcal{O}_{S, x^o} & \text{ is a holomorphic Cauchy data on } S \text{ at } x^o \text{ satisfying } (d\phi(x^o), \phi(x^o)) = (\iota^*\xi^o, \phi(x^o)).
\end{align*}

In (2), \( \iota^*: T^*M|_S \to T^*S \) denotes the dual bundle map of the injective tangent map \( \iota_*: TS \to TM|_S \) induced by the inclusion map \( \iota: S \subseteq M \).

We assume the following three conditions [A.1], [A.2] and [A.3]:

The first condition is

\begin{equation}
[A.1] \quad \sum_{j=1}^n |\partial_{\xi_j} F(e^o)| \neq 0
\end{equation}

where \( (\xi_1, \ldots, \xi_n) \) is the dual coordinate system of a local coordinate system \( (x_1, \ldots, x_n) \) of \( M \) around the point \( x^o \in M \). We note that \( (x_1, \ldots, x_n; \xi_1, \ldots, \xi_n, z) \) forms a local coordinate system of \( J^1M \) around \( e^o = (x^o; \xi^o, z^o) \). We remark that the condition [A.1] is independent of a choice of local coordinate systems.

The second condition is

\begin{equation}
[A.2] \quad \text{The function } \tau \to F(x^o; \tau ds(x^o) + \xi^o, z^o) \text{ of the one variable } \tau \text{ vanishes with a finite vanishing order } p \geq 1 \text{ at } \tau = 0.
\end{equation}

We note that the special case \( p = 1 \) is nothing but the case the following condition holds:

\begin{equation}
[A.3] \quad 0 \neq \partial_{\xi} \{F(x^o; \tau ds(x^o) + \xi^o, z^o)\}|_{\tau = 0} = \sum_{j=1}^n \partial_{\xi_j} F(e^o) \partial_{x^o} s(x^o).
\end{equation}

We call \( S \) is non-characteristic for \( F \) micro-locally at \( e^o \), if the condition (3) holds. Thus our condition [A.2] involves the non-characteristic case.

The third condition is, roughly speaking, stated as the following form:

\begin{equation}
[A.3] \quad \text{There exists a holomorphic approximate solution } \Phi \in \mathcal{O}_{M, x^o} \text{ of the Cauchy problem (1) such that } \Phi \text{ has several "good" properties.}
\end{equation}

These "good" properties of \( \Phi \) in [A.3] are stated by means of the Newton polygon of the function

\[ f^\Phi(y, \tau) := F(y; \tau ds(y) + d\Phi(y), \Phi(y)) \in \mathcal{O}_{\mathcal{S} \times \mathcal{C}, (x^o, \Phi)}. \]
In this article such an approximate solution $\Phi$ with "good" properties is called by the name of a *good extension* of the Cauchy data $\phi$.

This naming comes from the following definition: We call a germ $\Phi \in \mathcal{O}_{M, x \circ}$ an *approximate solution of* (1) *at* $e \circ$ *of the approximation order* $k \in \mathbb{N} \cup \{\infty\}$ if

$$
\begin{align*}
\operatorname{ord}_{x \circ}[F(x \circ d \Phi(x), \Phi(x))] &= k \\
\Phi|_{S} &= \phi \quad (\Phi \text{ is an extension of } \phi) \\
(x \circ d \Phi(x \circ), \Phi(x \circ)) &= e \circ
\end{align*}
$$

where the notation $\operatorname{ord}_{x \circ}[f(x \circ)]$ denotes the vanishing order of $f$ at $x = x \circ$.

Note that the condition [A.3] can be said for short the following form:

[A.3]' There exists a good extension $\Phi$ of the Cauchy data $\phi$.

For a precise definition of the good extensions, see §2 (Definition 2.16).

Now we assume the conditions [A.1]-[A.3]. Let $\Phi$ be a good extension of the Cauchy data $\phi$ of (1). We consider a map germ

$$
\begin{align*}
\gamma_{\phi}: (S \times C, (x \circ, 0)) &\longrightarrow (\mathcal{I}^{M}, e \circ) \\
\gamma_{\phi}(y, \tau) &= (y; \tau ds(y) + d\Phi(y), \Phi(y))
\end{align*}
$$

and the pull-back $f^{\phi}$ of $F$ by $\gamma_{\phi}$:

$$
f^{\phi}(y, \tau) := (\gamma_{\phi}^{*}F)(y, \tau) = F(y; \tau ds(y) + d\Phi(y), \Phi(y)).
$$

Taking the Taylor expansion

$$
f^{\phi}(y, \tau) = \sum_{\nu=0}^{\infty} c_{\nu}(y) \tau^{\nu} \quad (c_{\nu} \in \mathcal{O}_{S, x \circ} \text{ for } \nu = 0, 1, 2, \ldots)
$$

of $f^{\phi}$ along $\{\tau = 0\}$, we define the *Newton polygon* $\mathcal{N}(f^{\phi})$ of $f^{\phi}$ at $(x \circ, 0)$ by

$$
\mathcal{N}(f^{\phi}) := \text{ch} \left[ \bigcup_{c_{\nu} \neq 0} \{\operatorname{ord}_{x \circ}[c_{\nu}], \nu \} + \mathcal{R}^{+} \right]
$$

where the notation $\text{ch}[A]$ for a subset $A$ of $R^{2}$ denotes the convex hull of $A$, and where $\mathcal{R}^{+}$ denotes the set of non-negative real numbers.

In order to construct solutions of the Cauchy problem (1), we utilize the classical theory of characteristic curves. Let

$$
f^{\phi}(y, \tau) = \prod_{j=1}^{r} f_{j}^{\phi}(y, \tau)^{x_{j}}
$$

be the irreducible decomposition of $f^{\phi}$ in the local ring $\mathcal{O}_{S \times C(x \circ, 0)}$. We set germs $V_{j}$ ($1 \leq j \leq r$) of analytic sets of $(C, 0)_{\circ} \times (S \times C, (x \circ, 0))_{(y, \tau)}$ by

$$
V_{j} := \{(t, y, \tau); f^{\phi}_{j}(y, \tau) = 0\}.
$$

Let $t \mapsto \Psi(t, y, \tau)$ be the *characteristic curve* of $F$ (the integral curve of the
characteristic vector field $Y_F$ associated to $F$ satisfying the initial condition $\Psi(0, y, \tau) = \gamma_0(y, \tau) \in (J^1M, \varepsilon^0)$, where $Y_F$ is given by

$$Y_F = \sum_{j=1}^{n} (\partial_x F_j) \partial x_j - \sum_{j=1}^{n} \xi_j (\partial_x F) + \partial_x F \partial_{\xi_j} + \left( \sum_{j=1}^{n} \xi_j \partial_{\xi_j} F \right) \partial_x .$$

Then we have the following induced map germs $\Psi$ and $\pi_j$ for $1 \leq j \leq r$:

$$\begin{array}{cccc}
V_j & \Psi_j & (F^{-1}(0), \varepsilon^0) \\
\pi_j & (C, 0) \times (S \times C, (x^0, 0)) & \rightarrow & (M, x^0) \\
projection & projection & \downarrow & u_j \\
 & (C, x^0) & \rightarrow & (C, x^0)
\end{array}$$

Indeed, the following property of the characteristic curves

$$\partial_t \{ F(\Psi(t, y, \tau) \} = 0$$

yields $\Psi(V_j) \subset F^{-1}(0)$. Thus we have the induced map germs $\Psi_j$ $(1 \leq j \leq r)$. Our main result is the following

**Theorem 4.2.** Assume the conditions [A.1], [A.2] and [A.3]. Then, for any $1 \leq j \leq r$, the following statements 1) and 2) hold:

1) The map germ $\pi_j$ is a germ of an analytic covering of $(M, x^0)$ such that its ramification degree at $x^0$ is the positive integer $v_j$ which can be obtained from the Newton polygon $N(f^0)$ by means of the formula (4.7) stated in § 4.

2) Let $\Sigma_j$ be the critical locus of the germ $\pi_j$ of an analytic covering of $(M, x^0)$ (see § 3). Then there exists a multi-valued germ $u_j$ on $(M - \Sigma_j, x^0)$ which makes the diagram (11) commute, such that

(a) $F(x; du_j(x), u_j(x)) = 0$ and

(b) $u_j$ is exactly $v_j$-valued on $(M - \Sigma_j, x^0)$

(the ramification degree of $u_j$ at the point $x^0$ is equal to $v_j$).

We remark that the assertion of the main theorem involves the classical result in the case $p=1$ (Theorem 1.6), which says that if $p=1$ then the ramification degree is equal to one (unramified), see Remark 4.4.

Our program proceeds as follows:

In § 0, we give an example in $C^4$, which is a prototype of our theory.

In Chapter I, we give preliminaries to state the main result. In § 1, we
summarize the classical theory of characteristic curves from our viewpoint. In §2, we give a precise definition of the good extensions. In §3, we prepare several geometric notions such as finite holomorphic maps and germs of analytic (ramified) coverings.

In chapter II, we state the main theorem and its direct corollaries. The first corollary is related to the analytic continuation of holomorphic local solutions of the Cauchy problem (1) at generic points \( y \) in \( S-\{x^0\} \). The second corollary asserts the necessity of the non-charactericity (3) for the existence of holomorphic local solution of (1) at \( x^0 \), under \([A.1],[A.2]\) and \([A.3]\).

In chapter III, we give a proof of the main theorem. In §5, we carry out a reduction of the main theorem to a simpler Theorem 5.1. In §6, we introduce map germs \( \pi^\gamma_j (1 \leq j \leq r) \) and their decompositions. By virtue of these decompositions, our proof of Theorem 5.1 can be reduced to those of Theorems 6.10 and 6.11. In §§7-10, we prove these theorems.

In chapter IV, we give proofs of several basic facts which are assumed in chapter III.

The logical relations among the sections in Chapters III and IV except for §11 are as follows (the content of §11 is used almost everywhere):

\[
\begin{align*}
\text{§12} & \quad \rightarrow \quad \text{§7} & \rightarrow & \quad \text{§9} \\
\text{§14} & \quad \rightarrow & \quad \text{§6} & \rightarrow & \quad \text{§5} & \rightarrow & \quad \text{§4 (Main Theorem 4.2)} \\
\text{§13} & \quad \rightarrow & \quad \text{§8} & \rightarrow & \quad \text{§10} \\
\end{align*}
\]

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§0. A Typical Example

In this section we give a simple example which is a prototype of our general theory.

Example 0.1. In \( \mathbb{C}^2 \), we consider a Cauchy problem

\[
\begin{cases}
\prod_{\mu=1}^{m} (\partial_{z_1} u)^{p(\mu)} - x_2 q(\mu) - \partial_{z_2} u = 0 \\
u(0, x_2) = \phi(x_2) = 0
\end{cases}
\]

under the following assumptions (a), (b) and (c) for the positive integers \( p(\mu) \) and \( q(\mu) \):

(a) \( p(1)/q(1) > \cdots > p(m)/q(m) \).

(b) \( p(\mu) \) and \( q(\mu) \) are coprime for \( 1 \leq \mu \leq m \).
If we put \( a(0):=0, \ a(\mu):=q(1)+\cdots+q(\mu) \) for \( 1 \leq \mu \leq m \),
then \( q(\mu) \) and \( a(\mu-1)+1 \) are coprime for \( 1 \leq \mu \leq m \).

We fix a base point \( e^0:=(0, 0; 0, 0, 0) \in f^*C^\circ \cap F^{-1}(0) \), where \( F \) is given by
\[
F(x; \xi, z)=\prod_{\mu=1}^{m}(\xi^{p(\mu)}-x^{q(\mu)})-\xi^s.
\]

We note that the assumptions [A.1] and [A.2] of the main theorem are satisfied in this example, since we have
\[
\begin{align*}
\left\{ \frac{\partial_{\xi s}F}{F} & =-1 \quad \text{and} \\
\text{ord}_s[F(0, 0; \tau dx_1, 0)] & =p(1)+\cdots+p(m)<\infty.
\end{align*}
\]

We take an extension \( \Phi(x):=0 \) of the Cauchy data \( \phi(x):=0 \), and consider the function \( f(y, \tau) \) defined by
\[
(0.3) \quad f(y, \tau):=f^0(y, \tau)=F(0, y; \tau dx_1+d\Phi(0, y), \Phi(0, y))
= F(0, y; \tau, 0, 0)=\prod_{\mu=1}^{m}(\tau^{p(\mu)}-y^{q(\mu)}).
\]

Since Newton polygons have the additivity property
\[
N(gh)=N(g)+N(h) \quad \text{(see §11, Proposition 11.3)}
\]
we have
\[
(0.4) \quad N(f)=\sum_{\mu=1}^{m}N(\tau^{p(\mu)}-y^{q(\mu)}).
\]

**Lemma 0.2.** For positive integers \( p(\mu), q(\mu) \), we put
\[
N_{q(\mu), p(\mu)}:=(c, d)/(c/q(\mu))+(d/p(\mu)) \in \mathbb{N}. \]

Let \( N:=\sum_{\mu=1}^{m}N_{q(\mu), p(\mu)} \) be the vector sum of these \( \{N_{q(\mu), p(\mu)}\} \) in \( \mathbb{R}^2 \). If the finite sequences \( \{p(\mu)\}_{\mu=1, \ldots, m} \) and \( \{q(\mu)\}_{\mu=1, \ldots, m} \) satisfy the condition (a), then the vertices of \( N \) are given by
\[
(0.5) \quad \{(a(\mu), b(\mu)); 0 \leq \mu \leq m\} \subseteq \mathbb{R}^2
\]
where we define \( a(\mu) \) as in the assumption (c), and we put
\[
\begin{cases}
\begin{align*}
b(0) & :=0, \quad \text{and} \\
b(\mu) & :=p(1)+\cdots+p(\mu) \quad \text{for } \mu \geq 1.
\end{align*}
\end{cases}
\]

Proof. Let \( (c, d)=\sum_{\mu=1}^{m}(c_{\mu}, d_{\mu}) \in \mathbb{N} \) with \( (c_{\mu}, d_{\mu}) \in N_{q(\mu), p(\mu)} \) for \( 1 \leq \mu \leq m \). We can write \( (c_{\mu}, d_{\mu}) \) as follows:
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\[
(c_\mu, d_\mu) \subseteq (c_\mu', d_\mu') + R^+_c
\]
\[
(c_\mu', d_\mu') \subseteq \text{the segment joining } (0, p(\mu)) \text{ and } (q(\mu), 0) .
\]

For our aim, it suffices to show the following facts:

\[
p(\mu)c + q(\mu)d \geq p(\mu)a(\mu) + q(\mu)\{p - b(\mu)\} \quad \text{for } 1 \leq \mu \leq m
\]
and the equality holds in (0.7) if and only if

\[
(c_\lambda, d_\lambda) \begin{cases} 
= (q(\lambda), 0) & \text{if } \lambda < \mu \\
= (0, p(\lambda)) & \text{if } \lambda > \mu \\
\subseteq \text{the segment joining } (0, p(\mu)) \text{ and } (q(\mu), 0) & \text{if } \lambda = \mu.
\end{cases}
\]

By the expression (0.6), we have

\[
p(\mu)c + q(\mu)d = \sum_{\lambda=1}^{m} \{p(\mu)c_\lambda + q(\mu)d_\lambda\} = \sum_{\lambda=1}^{m} \{p(\mu)c(\lambda) + q(\mu)\{p - b(\lambda)\}\}
\]
\[
= \sum_{\lambda=1}^{m} [p(\mu)c(\lambda) + q(\mu)\{(p(\lambda)/q(\lambda))c(\lambda) + p(\lambda)\}]
\]
\[
= q(\mu) \sum_{\lambda=1}^{m} \{(p(\mu)/q(\mu)) - (p(\lambda)/q(\lambda))\}c(\lambda) + p(\lambda)\}.
\]

Since the assumption (a) yields

\[
(p(\mu)/q(\mu)) - (p(\lambda)/q(\lambda)) \begin{cases}
< 0 & \text{if } \lambda < \mu \\
= 0 & \text{if } \lambda = \mu \\
> 0 & \text{if } \lambda > \mu
\end{cases}
\]
the rightest hand of (0.9) (hence also \(p(\mu)c + q(\mu)d\)) is minimized only if the condition (0.8) holds. Conversely, if (0.8) holds then we have

\[
p(\mu)c + q(\mu)d = p(\mu)\{a(\mu - 1) + c_\mu\} + q(\mu)\{p - b(\mu) + d_\mu\}
\]
\[
= p(\mu)a(\mu) + q(\mu)(p - b(\mu)).
\]

Hence we get Lemma 0.2. Q. E. D.

Since it is clear that \(N(c^{p(\mu)} - y^{q(\mu)}) = N_{\mu=1}^m\) for \(1 \leq \mu \leq m\), the equality (0.4) and Lemma 0.2 yield the following figure of \(N(f)\):
The aim of this section is to show the following

Proposition 0.3. For $1 \leq \mu \leq m$, we define a positive integer $v(\mu)$ by

$$v(\mu) := p(\mu)\{a(\mu-1)+1\} + q(\mu)\{p-b(\mu-1)+1\} .$$

Then the Cauchy problem (0.1) has a $v(\mu)$-valued analytic solution $u(\mu)(x)$ around the origin of $C^s$ for $1 \leq \mu \leq m$.

To show Proposition 0.3, we utilize the classical theory of characteristic curves. Let $t \rightarrow (X; \Xi, Z)(t, y, \tau)$ be the characteristic curve of $F(x; \xi, z)$ given by (0.2), passing through a point $(0, y, \tau, 0, 0) \in F^{-1}(0)$ at the initial time $t=0$. Note that the definition (0.3) of $f(y, \tau)$ yields

$$(0, y; \tau, 0, 0) \in F^{-1}(0) \iff (y, \tau) \in f^{-1}(0).$$

We set complex curves $D(\mu)$ ($1 \leq \mu \leq m$) by

$$D(\mu) := \{(y, \tau); \tau^{p(\mu)} - \tau^{q(\mu)} = 0\}.$$

Then we have the following irreducible decomposition:

$$f^{-1}(0) = \bigcup_{\mu=1}^{m} D(\mu).$$

We construct solution $u(\mu)(x)$ of (0.1) by the following diagram:

$$(C, 0)_{t} \times (D(\mu), (0, 0))_{y, \tau} \xrightarrow{X} (X; \Xi, Z) \xrightarrow{F^{-1}(0)} (C^s, \xi, z)$$

We must show that the diagram (0.12) determines a multi-valued germ $u(\mu)(x)$ around the origin. It suffices to show the map

$$X: (C, 0)_{t} \times (D(\mu), (0, 0))_{y, \tau} \longrightarrow (C^s, (0, 0))_{x}$$

is a germ of an analytic covering of $(C^s, (0, 0))_{x}$ (for the terminology, see § 3).

To verify this fact, we observe the components $(X_{t}, X_{z}, \Xi)$ which satisfy

$$\partial_{t}X_{t} = \partial_{t}F = (\partial_{t}f)(X_{t}, \Xi) = \partial_{t}X_{z} = \partial_{t}F = -1$$

and

$$\partial_{z}X_{z} = -\partial_{z}F - \partial_{z}F = 0.$$

We solve (0.14)-(0.15) explicitly as follows. First we obviously have
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\begin{align}
X_{\xi}(t, y, \tau) &= y - t \\
E_{\xi}(t, y, \tau) &= \tau.
\end{align}

Then the first equation of (0.14) can be written as the form

\begin{align}
\partial_t X_{\xi} &= \partial_{\xi} \left[ \prod_{\lambda=1}^{\infty} \{ y^{p_{\lambda}} - (y - t)^{q_{\lambda}} \} \right], \\
X_{\xi}(0, y, \tau) &= 0.
\end{align}

We put

\begin{align}
g(y, \tau; t) &= \prod_{\lambda=1}^{\infty} \{ y^{p_{\lambda}} - (y - t)^{q_{\lambda}} \} \\
&= f(y - t, \tau) = \sum_{i,j=0} \gamma t^{i-1}.
\end{align}

where the coefficients \( c_{i,j} \in C \) satisfy

\begin{align}
c_{i,j} \neq 0 \quad \text{only if } (i, j) \in \mathbb{N}.\end{align}

Using the function \( g \) and its expansion, we solve (0.17) as

\begin{align}
X_{\xi}(t, y, \tau) &= \sum_{i,j=0} \{ j c_{i,j} / (i + 1) \} \{ y^{i+1} - (y - t)^{i+1} \} t^{i-1}.
\end{align}

Note that, by virtue of the assumption b), the curve \( D(\mu) \) defined by (0.11) has a resolution of singularity of the form

\begin{align}
\rho : (C, 0) \ni \theta \longrightarrow (y, \tau) = (\theta^{p_{\mu}}, \theta^{q_{\mu}}) \in D(\mu).
\end{align}

We define \( (X^{-}, E^{-}, Z^{-}) \) as the pull-back of \( (X, E, Z) \) by the mapping

\begin{align}
(C, 0) \times (C, 0) \ni \frac{1 \times \rho}{(C, 0) \times (D(\mu), (0, 0)) y, \tau}.
\end{align}

Then we have the following expressions (0.16) and (0.20):

\begin{align}
(0.16)^{-}
\begin{cases}
X^{-}_{\xi}(t, \theta) = \theta^{p_{\xi}} - t \\
E^{-}_{\xi}(t, \theta) = \theta^{q_{\xi}}
\end{cases}
\end{align}

\begin{align}
(0.20)^{-}
X^{-}_{\xi}(t, \theta) &= \sum_{i,j=0} \{ j c_{i,j} / (i + 1) \} \{ \theta^{p_{\xi}}(i+1) - X^{-}_{\xi}(t, \theta)^{i+1} \} \theta^{q_{\xi}(i-1)}.
\end{align}

In order to count the ramification degrees of \( (t, \theta) \) as a multi-valued function of \( (x_1, x_2) \), we consider the following equation in \( (t, \theta) \):

\begin{align}
(0.22)
\begin{cases}
X^{-}_{\xi}(t, \theta) = x_1 \\
X^{-}_{\xi}(t, \theta) = x_2.
\end{cases}
\end{align}

Note that \( X^{-}_{\xi}(t, \theta) \) involves the variable \( t \) only of the form \( X^{-}_{\xi}(t, \theta) \). Thus we can write \( X^{-}_{\xi}(t, \theta) \) as the form
where $H_\mu(\theta, x_2)$ is given by

$$H_\mu(\theta, x_2) := \sum_{i,j=0}^{n} \{jc_{i,j}/(i+1)\} \{\theta^{p(\mu)}(i+1) - x_i^{i+1}\} \theta^{q(\mu)}(j-1).$$

In this situation we have the following

**Lemma 0.4.** Assume (a), (b) and (c). We set

$$h_\mu(x_1, x_3, \theta) := H_\mu(\theta, x_2) - x_1$$

Then it follows that

$$\text{ord}[h_\mu(0, 0, \theta)] = v(\mu) = p(\mu)\{a(\mu - 1) + 1\} + q(\mu)\{p - b(\mu - 1)\}.$$

**Proof.** Note that

$$h_\mu(0, 0, \theta) = H_\mu(\theta, 0) = \sum_{i,j=0}^{n} \{jc_{i,j}/(i+1)\} \theta^{p(\mu)}(i+1) + q(\mu)\theta^{q(\mu)}(j-1)$$

which yields the inequality

$$\text{ord}[h_\mu(0, 0, \theta)] \geq p(\mu) - q(\mu) + \min\{p(\mu)i + q(\mu)j; c_{i,j} \neq 0, j \geq 1\}.$$  

For Lemma 0.4, it suffices to show the following (0.27) and (0.28):

$$\min\{p(\mu)i + q(\mu)j; c_{i,j} \neq 0, j \geq 1\} = p(\mu)a(\mu - 1) + q(\mu)\{p - b(\mu - 1)\}.$$  

$$\sum_{c_{i,j} \neq 0} jc_{i,j}/(i+1) \neq 0$$

where $I \subseteq Z^+$ denotes the set of $(i, j)$ attaining the minimum value (0.27).

By virtue of (0.19), we consider the linear functional

$$k_\mu: \mathbb{R}^2 \ni (i, j) \mapsto p(\mu)i + q(\mu)j \in \mathbb{R}$$

and observe that the minimum value of $k_\mu$ on $N(f) \setminus \{(i, j); j \geq 1\}$ is given by the right hand side of (0.27). Note that, for any $c \in \mathbb{R}$, the level set $k_\mu^{-1}(c)$ forms a line with the slope $-p(\mu)/q(\mu)$. Thus the minimum value is attained if and only if the level set $k_\mu^{-1}(c)$ coincides with the line joining $(a(\mu - 1), p - b(\mu - 1))$ and $(a(\mu), p - b(\mu))$. Hence the assumption (b) and the condition $j \geq 1$ yield that the minimum value can be attained by $(i, j)$ if and only if the following (0.30) holds:

$$(i, j) = \begin{cases} 
(a(\mu - 1), p - b(\mu - 1)) \text{ or } (a(\mu), p - b(\mu)) & \text{if } \mu < m, \\
(a(m - 1), p - b(m - 1)) & \text{if } \mu = m. 
\end{cases}$$

Thus we have

$$\min k_\mu = k_\mu(a(\mu - 1), p - b(\mu - 1)) = p(\mu)a(\mu - 1) + q(\mu)\{p - b(\mu - 1)\}.$$
which shows (0.27).

Now we prove (0.28). First we claim

\[(0.31) \sum_{(i,j)\in I} j c_{i,j}/(i+1) \]

\[= \begin{cases} (-1)^{m-1} \left( (p-b(m-1))/(a(m-1)+1) - (p-b(m))/a(m+1) \right) & \text{if } \mu < m. \\ (-1)^{m-1} \left( (p-b(m-1))/(a(m-1)+1) \right) & \text{if } \mu = m. \end{cases} \]

**Proof of (0.31).** Since \(c_{i,j}\) is the Taylor coefficient of \(y^i z^j\) in

\[f(y, \tau) = \sum_{k=1}^{\infty} \{ \tau^p q^{\ell} - y^q q^{\ell} \} \]

we especially have

\[(0.32) c_{a, p-b(\mu)} = (-1)^\mu \text{ for } 0 \leq \mu \leq m.\]

Indeed, Lemma 0.2 yields that \((a(\mu), p-b(\mu))\) is a vertex of \(N(f)\). Thus the condition (0.8) in the proof of Lemma 0.2 holds, which derives the following implications:

\[\exists (i, j) \in N(\tau^p q^{\ell} - y^q q^{\ell}), \quad 1 \leq \lambda \leq m, \text{ such that} \]

\[\langle a(\mu), p-b(\mu) \rangle = \sum_{\lambda=1}^{\infty} (i, j) \]

\[\implies (i, j) = \begin{cases} (q(\lambda), 0) & \text{if } \lambda < \mu \\ (0, p(\lambda)) & \text{if } \lambda > \mu \end{cases} \]

Hence we get

\[c_{a(\mu), p-b(\mu)} = \begin{cases} 1 \# \{1; \lambda > \mu\} (-1)^{\# \{1; \lambda \leq \mu\}} = (-1)^\mu \end{cases}.\]

Thus we have (0.32). Then it is obvious that (0.30) and (0.32) yield the desired (0.31). Q.E.D.

We continue the proof of (0.28). But this is easily verified from (0.31) since \(a(\mu)\) and \(b(\mu)\) satisfy the inequalities

\[a(\mu-1) < a(\mu), \quad \text{and} \quad p-b(\mu-1) > p-b(\mu).\]

The proof of (0.28), hence of Lemma 0.4, is complete. Q.E.D.

Since \(h(\mu)(x_1, x_2, \theta) = H(\mu)(\theta, x_2) - x_1\), vanishes at \((0, 0, 0)\) with order one, \(h(\mu)(x_1, x_2, \theta)\) is irreducible at the origin. Hence Lemma 0.4 yields that the function \(\theta(x_1, x_2)\) determined by the equation

\[(0.33) h(\mu)(x_1, x_2, \theta) = 0\]

is exactly \(v(\mu)\)-valued. We therefore have the following at most \(v(\mu)\)-valued inverse \((t(x), \theta(x))\) of the mapping \(\phi: (C, 0) \times (C, 0) \to (C^2, 0)\), which gives the solutions of the equation (0.22): Indeed, if we put
then we have
\[
\begin{align*}
X_1(t(x), \theta(x)) &= H_\mu(\theta(x), X_2(t(x), \theta(x))) = h_\mu(x_1, x_3, \theta(x)) + x_1 = x_1 \\
X_2(t(x), \theta(x)) &= \theta^{p(\mu)} - t(x) = x_2.
\end{align*}
\]

Thus we have the following diagram:

\[
\begin{array}{c}
(C, 0)_x \times (D(\mu), (0, 0)) \xrightarrow{1 \times \rho} (X; \mathcal{E}, Z) \\
\downarrow (X^\sim; \mathcal{E}^\sim, Z^\sim) \xrightarrow{F^{-1}(0), \rho^\psi} (F^{-1}(0), \rho^\psi) \\
\end{array}
\]

\[X \xrightarrow{\text{projection}} (t(x), \theta(x)) \xrightarrow{\text{projection}} (F^1C^\psi, \rho^\psi)_{x: \xi, z} \xrightarrow{u_{\rho}(x)} (C, 0)_x
\]

Hence the multi-valued germ \( u_{\rho}(x) \) can be defined by the diagram (0.34), or by the diagram (0.12), such that \( u_{\rho}(x) \) is at most \( v(\mu) \)-valued and satisfies the equation
\[
F(x; d u_{\rho}(x), u_{\rho}(x)) = \prod_{i=1}^{m} \left( \partial_x u_{\rho}(x) - x_2 \right) = 0.
\]

It remains to show that the germ \( u_{\rho}(x) \) is exactly \( v(\mu) \)-valued.

To verify this we use the well-known relation
\[
\partial_x u_{\rho}(x) = \xi_1(x)
\]
where the function \( \xi_1(x) \) is given by the following diagram (0.36):

\[
\begin{array}{c}
(C, 0)_x \times (C, 0)_x \xrightarrow{(X_1, X_2, \mathcal{E}_1)} (C^2, (0, 0))_x \times (C, 0)_{t_1} \xrightarrow{(C, 0)_{t_1}} (C, 0)_{t_1}
\end{array}
\]

\[
\xi_1(x) \xrightarrow{(t(x), \theta(x))} (C^2, (0, 0))_x \xrightarrow{(C, 0)_{t_1}} \xi_1(x) \xrightarrow{(C, 0)_{t_1}}
\]

Lemma 0.5. The function \( \xi_1(x) \) defined by (0.36) is also \( v(\mu) \)-valued.

Proof. By the definition of \( \mathcal{E}_1(t, \theta) \), we have
\[
\mathcal{E}_1(t, \theta) = \mathcal{E}_1(t, y, \tau)|_{y, \tau} = (\theta^{p(\mu)}, \theta^{q(\mu)}) = \theta^{q(\mu)}
\]
which implies
\[
\xi_1(x) = \theta(x)^{q(\mu)}.
\]
Thus when $\theta(x)$ rounds its singular points one-time, $\xi_i(x)$ rounds its singular points $q(\mu)$-times.

Note that, for $1 \leq \mu \leq m$, the assumptions (b), (c) yield that

\begin{equation}
q(\mu) \text{ and } v(\mu) \text{ are coprime.}
\end{equation}

Indeed, if we denote the greatest common divisor of $a, b \in \mathbb{Z}$ by $(a, b)$, then we have

\begin{align*}
(q(\mu), v(\mu)) &= (q(\mu), p(\mu)\{a(\mu-1)+1\}+q(\mu)\{p-b(\mu-1)-1\}) \\
&= (q(\mu), a(\mu-1)+1) \\
&= 1. \\
&\quad [\therefore (q(\mu), p(\mu)) = 1]
\end{align*}

Hence we have the following implication:

\begin{equation}
q(\mu)k \in v(\mu)\mathbb{Z} \implies k \in v(\mu)\mathbb{Z}.
\end{equation}

Hence we conclude that the function $\xi_i(x)$ is exactly $v(\mu)$-valued. Q.E.D.

As a consequence of Lemma 0.5 and the relation (0.35) we get that the multi-valued solution $u^\mu$ is exactly $v(\mu)$-valued as desired.

The proof of Proposition 0.3 is complete. Q.E.D.

We conclude this section to give the simple

\textbf{Corollary 0.6.} We consider a very special case that

\begin{equation}
m=1, \quad p=p(1)=2 \quad \text{and} \quad q(1)=1
\end{equation}

hold in Example 0.1:

\begin{equation}
\begin{cases}
(\partial_{z_1}u)^3 - x_z - \partial_{z_2}u = 0. \\
u(0, x_z) = 0.
\end{cases}
\end{equation}

Then the Cauchy problem (0.41) has a 3-valued analytic solution.

\textbf{Proof.} We only have to verify $v(1)=3$. The assumption (0.40) yields

\begin{align*}
v(1) &= p(1)\{a(0)+1\} + q(1)\{p-b(0)-1\} \\
&= p(1)+q(1) \\
&= 3.
\end{align*}

Q.E.D.

\textbf{Remark 0.7.} By a direct computation, we have the following explicit expressions of the functions $X^\gamma(t, \theta)$ and $\theta(x)$ of the Cauchy problem (0.41):

\begin{align*}
X_1^\gamma(t, \theta) &= 2\theta t, \\
X_2^\gamma(t, \theta) &= \theta^2 - t
\end{align*}
Thus the 3-valuedness of the function
\[ \partial_{x_1} u(x) = \xi_1(x) = \theta(x^{(1)}) = \theta(x) \]
is a consequence of the 3-sheetedness of the following mapping \( \pi \):
\begin{equation}
\pi : h^{-1}_1(0) \hookrightarrow (C^3, 0)_x \xrightarrow{\text{projection}} (C^2, 0)_x.
\end{equation}

We give an illustration of the surface \( h^{-1}_1(0) \cap R^3 \) as follows:

Note that this kind of singularity of the map germ \( \pi \) is called by the name of "Whitney's tuck" (see e.g. [Ar: Appendix 12, Lagrangian singularities]).

Chapter I. Preliminaries
§ 1. Classical Theory of Characteristic Curves

In this section we give a summary of the classical theory of characteristic curves from our viewpoint, by introducing an affine bundle \( E = E(\phi) \) over the initial hypersurface \( S \). This bundle \( E \) is, roughly speaking, a space of jets where \( @^0 M \), \( x^0 \) runs through all holomorphic extensions of the data \( \phi \in \mathcal{O}_S, x^0 \) of the Cauchy problem (1).

Let us recall the Cauchy problem (1) with the condition c) in (2):
\[ (d\phi(x^0), \phi(x^0)) = (\xi_{x^0}, x^0) \in J^1_s S \]
where \( e^0 = (x^0; \xi^0, z^0) \) is the base point lying in a neighborhood the equation \( F(x, \xi, z) \) is defined, and where \( e^*: T^*M |_S \to T^*S \) denotes the dual bundle map
Definition 1.1. We define a subset \( E = E(\phi) \) by setting
\[
E = E(\phi) := \{(y; \xi, \phi(y)) \in J^1 M; y \in S, \xi \cdot \xi = d \phi(y)\}
\]
and we also set the fiber of \( E \) by
\[
E_y := E \cap J^1_y M \quad \text{for} \quad y \in S.
\]

The meaning of \( E \) is clarified by the

Lemma 1.2. 1) For any local extension \( \Phi \in \mathcal{O}_{M, y^0} \) of the data \( \phi \) at \( y^0 \in S \) we have
\[
(d \Phi(y), \phi(y)) \in E_y \quad \text{for} \quad \forall y \in S, y^0
\]
where \((S, y^0)\) denotes the germ of \( S \) at \( y^0 \), that is, the set which consists of all \( y \in S \) sufficiently near \( y^0 \).

2) For \( y \in S \), the set \( E_y \) forms an one-dimensional affine subspace of the \((n+1)\)-dimensional complex vector space \( \mathcal{J}^1_y M = T^* y M \times C \). More precisely, for any local extension \( \Phi \in \mathcal{O}_{M, y} \) of \( \phi \) at \( y \), the following equality holds:
\[
E_y = \{ (\tau d s(y) + d \Phi(y), \phi(y)) \in \mathcal{J}^1_y M; \tau \in C \}.
\]

Proof. The first assertion (1.3) is a direct consequence of the commutativity of the pull-back \( \tau^* \) and the exterior derivativation \( d \):
\[
\iota_y^* d \Phi(y) = d(\iota^* \Phi)(y) = d \phi(y) \quad \text{if} \quad \Phi|_S = \iota^* \Phi = \phi.
\]
Note that (1.3) and \( s = 0 \) on \( S \) imply the inclusion
\[
E_y \supseteq \{ (\tau d s(y) + d \Phi(y), \phi(y)); \tau \in C \}
\]
for any holomorphic extension \( \Phi \) of \( \phi \). Hence it suffices for (1.4) to show the converse inclusion of (1.4). Let \( (\xi, \phi(y)) \in E_y \). Then (1.3) yields
\[
\iota_y^*(\xi - d \Phi(y)) = d \phi(y) - d \phi(y) = 0, \quad \text{that is}, \quad \langle \xi - d \Phi(y), \iota_y^*(T_y S) \rangle = 0.
\]
Hence we get \( \xi - d \Phi(y) \in C d s(y) \) which shows the equality (1.4). Q.E.D.

Corollary 1.3. Let \( u \in \mathcal{O}_{M, y^0} \) be a holomorphic local solution of the Cauchy problem (1). Then it follows
\[
(y; d u(y), u(y)) \in E \cap F^{-1}(0) \quad \text{for} \quad \forall y \in (S, y^0).
\]

We shall give a summary of the theory of characteristic curves, by concerning geometric nature of \( E \cap F^{-1}(0) \) as a hypersurface of \( E \).

Let us recall the characteristic vector field \( Y_F \) on the germ \((J^1 M, e^0)\) associate with \( F \), which can be written as the form
(1.6) \[ Y_F = \sum_{j=1}^{n} (\partial_j F) \partial x_j - \sum_{j=1}^{n} (\xi_j \partial_j F + \partial x_j F) \partial x_j + \left( \sum_{j=1}^{n} \xi_j \partial_j F \right) \partial_x \]

by means of any local coordinate system of the form \((x_1, \ldots, x_n; \xi_1, \ldots, \xi_n, z)\).

**Notation 1.4.** 1) We denote by
\[ (C, 0) \ni t \mapsto \Psi(t, e) = (X(t, e), Z(t, e)) \in (J^1 M, e^0) \]
the characteristic curve of \(F\), passing through a point \(e \in (E, e^0)\) at the initial time \(t=0\), that is, a uniquely determined integral curve of \(Y_F\) passing through \(e\). This family of characteristic curves determines a holomorphic map germ
\[ \Psi^\sim: (C, 0) \times (E, e^0) \longrightarrow (J^1 M, e^0) \]

2) We define an analytic set \(V\) by
\[ V := (C, 0) \times (E \cap F^{-1}(0), e^0) \]  
Restricting the map germ \(\Psi^\sim\) on \(V\), we have the induced map germ
\[ \Psi: (V, (0, e^0)) \longrightarrow (F^{-1}(0), e^0) \]
since the characteristic curve \(\Psi^\sim\) satisfies
\[ \partial_t \{F(\Psi^\sim(t, e))\} = 0. \]
Note that the induced map \(\Psi\) is holomorphic as a map between analytic sets (see Definition 3.1).

3) We define a tangent vector \(L_F(e^0)\) \(\in T_{x^0} M\) by
\[ L_F(e^0) := \sum_{j=1}^{n} \partial_j F(e^0) \partial x_j. \]
Note that this vector is nothing but the image vector of the characteristic vector \(Y_F(e^0)\) \(\in T_{x^0}(J^1 M)\) at \(e^0\) under the tangent bundle map \(\pi_{*,x^0}: T_{e^0}(J^1 M) \longrightarrow T_{x^0} M\) of the natural projection \(\pi: J^1 M \longrightarrow M\).

Using these notations, our conditions [A.1] [A.2] can be written as
[A.1]' \[ L_F(e^0) \neq 0. \]
[A.2]' \[ \text{ord}_{e^0}[F|_{E^x}] := \rho \in [1, \infty). \]

Note that [A.2]' implies \(F|_{E^x} \neq 0\), hence the intersection \(E \cap F^{-1}(0)\) is a complex hypersurface of \(E\). Thus the germ \((V, (0, e^0))\) defined by (1.8) is a \(n\)-dimensional hypersurface of \((C, 0) \times (E, e^0)\), which has singular points \(V_{\text{sing}}\) containing \(x^0\) if
\[ 1 < \text{ord}_{e^0}[F|_{E^x}] \leq \rho \]  
Note that [A.2]' implies \(F|_{E^x} \neq 0\), hence the intersection \(E \cap F^{-1}(0)\) is a complex hypersurface of \(E\). Thus the germ \((V, (0, e^0))\) defined by (1.8) is a \(n\)-dimensional hypersurface of \((C, 0) \times (E, e^0)\), which has singular points \(V_{\text{sing}}\) containing \(x^0\) if
\[ 1 < \text{ord}_{e^0}[F|_{E^x}] \leq \rho \]  
The classical theory of characteristic curves is based on the fact (1.10) and
the following well-known

**Lemma 1.5.** We denote by $V_{\text{reg}} := V - V_{\text{sing}}$ the regular part of $V$ which forms a $n$-dimensional complex manifold. Then it follows that the pull-back of the fundamental 1-form $dz - \sum_{j=1}^{n} \xi_j dx_j$ on $J^1M$ by $\Psi$ vanishes on $V_{\text{reg}}$:

$$W^* (dz - \sum_{j=1}^{n} \xi_j dx_j) = 0 \quad \text{on} \quad V_{\text{reg}}.$$

Let $\pi_V: (V, (0, e^0)) \rightarrow (M, x^0)$ be a holomorphic map germ determined by the following diagram:

$$
\begin{array}{ccc}
(V, (0, e^0)) & \xrightarrow{\Psi} & (F^{-1}(0), e^0) \\
\pi_V \downarrow & & \downarrow \text{projection} \\
(M, x^0) & \xleftarrow{\text{projection}} & (J^1M, e^0)
\end{array}
$$

The above properties (1.10) and (1.12) of characteristic curves derive the following classical existence theorem:

**Theorem 1.6** (see, for example, [Ar: Appendix 4 M]). There are the implications $1) \Rightarrow 2) \Rightarrow 3)$ for the following conditions:

1) The induced map germ

$$\pi_V|_{F^{-1}(0), e^0} : (E \cap F^{-1}(0), e^0) \hookrightarrow (J^1M|_S, e^0) \rightarrow (S, x^0)$$

is locally biholomorphic at $e^0$, that is, the vanishing order $p$ in [A.2] is equal to one ($S$ is non-characteristic for $F$ micro-locally at $e^0$).

2) The map germ $\pi_V$ is locally biholomorphic at $(0, e^0)$.

3) There exists a unique holomorphic local solution $u \in \mathcal{O}_{M, x^0}$ of the Cauchy problem (1) at $x^0$ satisfying

$$\sum (x^0; d\xi(x^0), u(x^0)) = e^0.
$$

This unique holomorphic solution $u$ is determined by the following diagram:

$$
\begin{array}{ccc}
(V, (0, e^0)) & \xrightarrow{\Psi} & (F^{-1}(0), e^0) \\
\pi_V \downarrow & & \downarrow \text{projection} \\
(M, x^0) & \xleftarrow{\text{projection}} & (C, z^0)
\end{array}
$$

It is our starting point of this article to consider the
Problem 1.7. If we weaken the condition 1) in Theorem 1.6 to our condition [A.2] (or [A.2]) then what kind of solutions of the Cauchy problem (1) do appear around the point $x^0 \in M$?

§ 2. Definition of Good Extensions

In this section we give a precise definition of “good extension” in our condition [A.3]. First we recall the

Definition 2.1. A holomorphic germ $\Phi \in \mathcal{O}_{\mathbb{C}, x^0}$ is called a holomorphic approximate solution of the Cauchy problem (1) of the approximation order $k$ at $x^0=(x^0, \xi^0, z^0)$ if $\Phi$ satisfies the condition (4) in the introduction.

From now on we call such a $\Phi$ an approximate solution for short.

Notation 2.2. Let $\Phi$ be an approximate solution of (1). 1) We set a map germ $\gamma_\Phi: (S \times C, (x^0, 0)) \to (E, e^0)$ by

$$\gamma_\Phi(y, \tau) := \gamma(y; \tau ds(y) + d\Phi(y), \Phi(y)).$$

By virtue of Lemma 1.2, the map germ $\gamma_\Phi$ is locally biholomorphic.

2) We define a germ $f^\Phi$ of a function as the pull-back of the restriction $F|_E$ by the biholomorphic map germ $\gamma_\Phi$:

$$f^\Phi := \gamma_\Phi(F|_E) \in \mathcal{O}_{S \times C, (x^0, 0)}$$

Definition 2.3. Let $f^\Phi(y, \tau) = \sum_{y \geq 0} c_\nu(y) \tau^\nu$ be the Taylor expansion along $\tau=0$.

1) We define a Newton polygon $N(f^\Phi)$ of $f^\Phi$ at $(x^0, 0)$ by

$$N(f^\Phi) := \text{convex hull } \bigcup_{\nu \geq 0} [(\text{ord}(c_\nu), \nu) + \overline{R}_+]$$

where ord($c_\nu$) denotes the vanishing order of $c_\nu(y)$ at $y=x^0$ and we put

$$\overline{R}_+ := \{ t \in \mathbb{R}; t \geq 0 \}.$$

2) For a Newton polygon $N$ we define its strict boundary $\partial^0 N$ by

$$\partial^0 N := \{ A \in N; [A + (-\overline{R}_+)] \cap N = \{ A \} \}.$$ 

Note that $\partial^0 N$ consists of either only one point or a union of finitely many segments, where we call a subset $\sigma \subset \partial^0 N$ by the name of a segment of $N$ if there exists a line $\sigma^\perp$ in $\mathbb{R}^2$ such that $\sigma = \sigma^\perp \cap \partial^0 N$ with $\# \sigma \geq 3$.

3) A point $A \in \partial^0 N$ is called a vertex of $N$ if the following implication (2.5) holds for any $B, C \in N$ with $B \neq C$ and for $t \in [0, 1]$:

$$A = tB + (1-t)C \implies t = 0 \text{ or } t = 1.$$ 

Note that the assumption [A.2] yields the point $(0, p) \in \mathbb{R}^2$ is always a vertex of $N(f^\Phi)$, since we have ord$_{y=x^0} [f^\Phi(x^0, \tau)] = \text{ord}_{x^0} [F|_{x^0}] = p$. 

Notation 2.4. 1) We denote by \( \text{Seg} N \) [or \( \text{Ver} N \) respectively] the set of all segments [vertices] of a Newton polygon \( N \).

2) Let \( \Phi \in \mathcal{O}_{S, x^0} \) be an approximate solution of \( (1) \). We set
\[
\begin{align*}
& m := \# \text{Seg} N(f^0) \\
& \{ A(\mu) = (a(\mu), \ p - b(\mu)); \ 0 \leq \mu \leq m \} := \text{Ver} N(f^0)
\end{align*}
\]
where sequences \( \{ a(\mu) \} \) \( \{ b(\mu) \} \) are arranged as strictly monotone increasing:
\[
\begin{align*}
0 = a(0) < a(1) < \cdots < a(m) \\
0 = b(0) < b(1) < \cdots < b(m) \leq p.
\end{align*}
\]

3) For \( 1 \leq \mu \leq m \), we define positive integers \( q(\mu) \), \( \rho(\mu) \) and a positive rational number \( \kappa(\mu) \) as follows:
\[
\begin{align*}
q(\mu) & := a(\mu) - a(\mu - 1), \ \rho(\mu) := b(\mu) - b(\mu - 1) \\
\kappa(\mu) & := \rho(\mu) / q(\mu).
\end{align*}
\]
Note that \( \kappa(\mu) \) represents the slope of the \( \mu \)-th segment of \( N(f^0) \), thus we get
\[
\kappa(1) > \kappa(2) > \cdots > \kappa(m) > 0.
\]

Definition 2.5. We say a Newton polygon \( N(f^0) \) satisfies the coprimeness condition if
1) \( f^0(y, 0) = 0 \), that is, the Newton polygon \( N(f^0) \) intersects the horizontal axis \( R \times 0 \).
2) For \( 1 \leq \mu \leq m \), the integers \( \rho(\mu) \) and \( q(\mu) \) are coprime.

Definition 2.6. Let \( c \in \mathcal{O}_{S, x^0} \) be a germ. For a local coordinate system \( (y_1, \cdots, y_{n-1}) \) of \( S \) at \( x^0 \), let \( c(y) = \sum a_y y^a \) be the Taylor expansion of \( c(y) \) with respect to the coordinate system. We define the localization \( \text{Loc}[c] : T_{x^0} S \to \mathbb{C} \) of the germ \( c \) at \( x^0 \in S \) by
\[
\text{Loc}[c](\sum_{j=1}^{n-1} Y_j \partial_{y_j}) := \sum_{|a| = \text{ord}(c)} a_y Y^a.
\]
Note that the localization \( \text{Loc}[c] \) is determined, as a homogeneous polynomial function on \( T_{x^0} S \), independently of a choice of local coordinate systems.

Remark 2.7. If \( p := \text{ord}_{x^0}[F|_{x^0}] \geq 2 \), then the tangent vector \( L_F(e^0) \) defined by \( (1.11) \) can be regarded as a non-zero vector in \( T_{x^0} S \).

Indeed, we have
\[
\langle ds(x^a), \ L_F(e^0) \rangle = \partial_x \{ F(x^a; \tau d s(x^a) + e^0, z^a) \} |_{z = 0} = 0
\]
if \( p \geq 2 \). Hence it follows \( L_F(e^0) \in \mathcal{O}_{x^0}(T_{x^0} S) \underset{t_{x^0}}{\sim} T_{x^0} S \).
Definition 2.8. Let $L$ be a non-zero vector in $T_{x^0}S$. We say a Newton polygon $N(f^\theta)$ is stable in a direction of $L$, if
\[(2.10) \quad \text{Loc}[c_v](L)\neq 0 \quad \text{for all } \nu \text{ satisfying } (\text{ord}(c_v), \nu) \in \text{Ver } N(f^\theta)\]
where $c_v(y)$ is the $\nu$-th Taylor coefficient of $f^\theta(y, \tau)$.

Remark 2.9. Let $(C, 0) \ni \theta \mapsto y(\theta) \in (S, x^0)$ be a complex curve satisfying
\[y(0) = x^0, \quad \text{and} \quad y'(0) \in (C - \{0\})L\]
where $L$ is a non-zero vector in Definition 2.8. For such a curve $y(\theta)$ we set
\[g(\theta, \tau) := f^\theta(y(\theta), \tau).\]

Then the condition (2.10) is equivalent to
\[(2.10)' \quad N(f^\theta) = N(g).\]

Proof. Since $y(\theta)$ can be expanded as
\[y(\theta) = y'(0)\theta + O(\theta^2) = kL\theta + O(\theta^2) \quad \exists k \in C - \{0\}\]
we have
\[c_v(y(\theta)) = (k\theta)^{\text{ord}(c_v)}\text{Loc}[c_v](L) + O(\theta^{\text{ord}(c_v)+1}),\]
which yields the equivalence between (2.10) and (2.10)'. Q.E.D.

Notation 2.10. 1) Now we denote an irreducible decomposition of $F|_E$ in the local ring $\mathcal{O}_{E, \xi_0}$ by
\[(2.11) \quad F|_E = \bigoplus_{j=1}^r F_j^{(j)}\]
where $r, \nu(j)$ are positive integers and where $F_j \subset \mathcal{O}_{E, \xi_0}$ are irreducible such that $F_j \neq gF_k$ for any germ $g \subset \mathcal{O}_{E, \xi_0}$ if $j \neq k$.

2) For an approximate solution $\Phi$ of the Cauchy problem (1), we set
\[(2.12) \quad f^\theta := \gamma^\theta F_j \subset \mathcal{O}_{S \times C, (x^0, 0)} \quad \text{for } 1 \leq j \leq r.\]

3) For positive integers $p(\mu), q(\mu)$ in Notation 2.4, we put
\[N_{q(\mu), p(\mu)} := \{(s, t) \in \mathbb{R}^3 : s, t \geq 0, (s/q(\mu)) + t/p(\mu) \geq 1\}.\]

Proposition 2.11. Under Notations 2.4 and 2.10, it follows that
\[(2.13) \quad N(f^\theta) = \sum_{j=1}^r \nu(j) N(f^\theta) = \sum_{\mu=1}^\infty N_{q(\mu), p(\mu)}.\]

Proposition 2.12. Assume that $N(f^\theta)$ satisfies the coprimeness condition. Then we have
1) $\nu(j) = 1$ for all $j$. 

2) \( N(f^q_j) \) satisfies the coprimeness condition for all \( j \).

3) There exist subsets \( M_j \) of \( \{1, 2, \ldots, m\} \), \( 1 \leq j \leq r \), such that the following (2.14)–(2.16) hold:

\[
2.14 \quad M_j \cap M_k = \emptyset \quad \text{if} \ j \neq k.
\]

\[
2.15 \quad \{1, 2, \ldots, m\} = \bigcup_{j=1}^{r} M_j \quad \text{(disjoint union).}
\]

\[
2.16 \quad N(f^q_j) = \sum_{\mu \in M_j} N(f^q_j, \mu) \quad \text{for} \ 1 \leq j \leq r.
\]

The proofs of Propositions 2.11 and 2.12 are given in §11.

Remark 2.13. In Proposition 2.12, the assertion 2) follows from the assertion 3). Indeed, the equality (2.16) and the coprimeness of \( N(f^q_j) \) imply the coprimeness of \( N(f^q_j) \) for \( 1 \leq j \leq r \).

Definition 2.14. We say a subset \( M_j \) defined by 3) in Proposition 2.12 is a nice subset if the following condition (2.17) holds:

\[
2.17 \quad \text{GCD} \left[ \bigcup_{\mu \in M_j} \{a(\mu-1)+1, q(\mu)\} \right] = 1
\]

where \( \text{GCD}[B] \) denotes the greatest common divisor of a finite subset \( B \subset \mathbb{Z} \).

Remark 2.15. Let \( j^* \) be the integer satisfying \( 1 \in M_{j^*} \). Then \( M_{j^*} \) is a nice subset, since

\[
\bigcup_{\mu \in M_{j^*}} \{a(\mu-1)+1, q(\mu)\} \supseteq a(0)+1 = 1.
\]

In particular, if the germ \( F|^x \) is irreducible, we have that \( M_i = \{1, \ldots, m\} \) is a nice subset.

Now we can give a precise definition of “good extension” as follows:

Definition 2.16. Let \( \Phi \in Q_{U, \epsilon^0} \) be an approximate solution of the Cauchy problem (1) at \( \epsilon^0 \) of a finite approximation order \( k \in \mathbb{N} \).

1° In the case \( p=1 \), we say \( \Phi \) is a good extension of the Cauchy data \( \phi \) if the Newton polygon \( N(f^q) \) satisfies the coprimeness condition (Definition 2.5).

2° In the case \( p \geq 2 \), we say \( \Phi \) is a good extension of the Cauchy data \( \phi \) if the following conditions 1)–4) holds:

1) The Newton polygon \( N(f^q) \) satisfies the coprimeness condition.
2) \( N(f^q) \) is stable in the direction of the tangent vector \( L_{f^q}(\epsilon^0) \in T_{\epsilon^0}S \) (Definition 2.8 and Remark 2.7).
3) The subsets \( M_j \) are all nice for \( 1 \leq j \leq r \) (Definition 2.14).
4) The approximation order \( k \) of \( \Phi \) is greater than \( \kappa(m)^{-1} \):

\[
2.18 \quad k := \text{ord}_{x^0}[F(x; d\Phi(x), \Phi(x))] > \kappa(m)^{-1}
\]
where $-\kappa(m)$ is the slope of the rightest segment of $N(f^\phi)$.

**Remark 2.17.** In the case that $p=1$, there exists an good extention of the Cauchy data $\phi$. Thus our assumption [A.3] is trivial if $p=1$ in [A.2].

**Proof.** Since $p=1$ it suffices to find $\Phi$ satisfying the first condition of Definition 2.5, that is, $\text{ord}_{x_0}[f^\phi(y, 0)]<\infty$. Recall that, by Theorem 1.6, we can find a unique holomorphic solution $u \in \mathcal{O}_{M, x_0}$ of (1) such that

\[(x^0; du(x^0), u(x^0)) = e^0.\]

We construct a desired approximate solution $\Phi$ of (1) of the following form:

\[\Phi(x) = u(x) + s(x)w(x) \quad (w \in \mathcal{O}_{M, x_0}).\]

Since $d\Phi(y) = du(y) + w(y)ds(y)$ on $S$, if we choose $w$ as $w(x^0) = 0$ then $\Phi$ is an approximate solution of (1) at $e^0$. By the Taylor expansion and the equation $F(x; du(x), u(x)) = 0$, we have

\[F(y; d\Phi(y), \phi(y)) = \sum_{n=0}^{\infty} \langle \partial_x^n F \rangle(y; du(y), \phi(y))w(y)^n \{ds(y)\}^n = \langle L_P(y; du(y), \phi(y)), ds(y)w(y) + O(w(y)^2) \quad (y \in S).\]

Recall that $\langle L_P(e^0), ds(x^0) \rangle \neq 0$ if $p=1$, which yields that

\[\text{ord}_{x_0}[f^\phi(y, 0)] = \text{ord}_{x_0}[F(y; d\Phi(y), \phi(y))] = \text{ord}_{x_0}[w(y)].\]

Hence we get a desired $\Phi$ if we choose $w$ as $\text{ord}[w|_S] < \infty$.

The proof of Remark 2.17 is complete. Q. E. D.

**Example 2.18.** Recall the typical Example 0.1 under the assumptions (a), (b) and (c). Then $\Phi(x_1, x_2) = 0$ is a good extension of the data $\phi(x_1) := 0$.

**Proof.** By Remark 2.17 we may assume $p \geq 2$. Since

\[F(x_1, x_2; \partial_{x_1} \Phi, \partial_{x_2} \Phi) = \prod_{\mu=1}^{m} \langle \xi_1^{p(\mu)} - x_1^{q(\mu)} \rangle - \xi_2^{q(\mu)} \mid (\xi_1, \xi_2) = (0, 0)\]

the approximation order of $\Phi \equiv 0$ is $q := q(1) + \cdots + q(m)$. Note that the inequality $q \leq \kappa(m)^{-1}$ implies $p := p(1) + \cdots + p(m) = 1$. Hence we have the inequality (2.18) in the case $p \geq 2$. By the definition, we also have

\[f^\phi(y, \tau) = F(0, y; \tau, 0) = \prod_{\mu=1}^{m} \langle \tau^{p(\mu)} - y^{q(\mu)} \rangle.\]

Then the assumption (b) [or (c) resp.] means that $N(f^\phi)$ satisfies the coprimeness condition [or, that $M_\mu = \{\mu\}$ is all nice for $1 \leq \mu \leq m$]. On the other hand, the stability of $N(f^\phi)$ in the direction of $L_P(e^0) = -\partial_{x_2}$ is trivial because $S = \{x_1 = 0\}$.
is one-dimensional.

Q. E. D.

Example 2.19. In $C^3$, we consider the following Cauchy problem

\[ F(y, z; u_x, u_y) := u_x^5 - y(1 + y^5)u_x^3 + (y^4 + z^4)u_x^2 + y^4u_x - u_y = 0 \]

with a base point $e^0 := (0; 0, 0) \in J^1C^3 \cap F^{-1}(0)$. Then, a local extension

\[ \Phi(x, y, z) = xA(x, y, z) + \phi(y, z) \quad (A \in O_{cs, 0}) \]

is a good extension of the data $\phi$ if and only if the germ

\[ \beta_i \in (y, z)^{3-i} \quad \text{for } i = 0, 1, 2 \]

satisfies

\[ \alpha_0 \in (z)^3. \]

Proof. We put $a(y, z) := A(0, y, z)$. Then we have

\[ f^\theta(y, z, \xi) = \{ (\xi + a)^3 - y^3 \} \{ (\xi + a)^3 - y(\xi + a) + y^4 + z^4 \} \]

which gives an irreducible decomposition of $f^\theta$, since

\[ \begin{cases} f_1^\theta := (\xi + a)^3 - y^3 \\ f_2^\theta := (\xi + a)^3 - y(\xi + a) + y^4 + z^4 \end{cases} \]

are both irreducible in $O_{s \times c, (0, 0)}$.

Claim (1). $N(f^\theta)$ satisfies the coprimeness condition if and only if

\[ a \in (y, z)^3. \]

Indeed, the necessity of (2.22) is obtained since, if we assume that (2.22) is not true, then it follows

\[ N(f^\theta) = \{ (s, t); s + t \geq 2, s \geq 0, t \geq 0 \} \]

which does not satisfy the coprimeness condition. This contrasts the assertion 2) in Proposition 2.12. Conversely if we assume (2.22), then the Newton polygon $N(f^\theta) = N(f_1^\theta) + N(f_2^\theta)$ is given by one of the following (2.23). Hence we have the coprimeness condition of $N(f^\theta)$:
The case \( a \in (y, z)^3 - (y, z)^3 \)

The case \( a \in (y, z)^3 \)

Note that, under the condition (2.22), the subsets \( M_i = \{2\} \), \( M_2 = \{1, 3\} \) of \( \{1, 2, 3\} \) are both nice subsets.

\textbf{Claim (2).} Under the condition (2.22), \( N(f^g) \) is stable in the direction of \( L_P(e^g) = -\partial_y \) if and only if one of the following (2.24)-(2.26) holds:

\begin{align*}
(2.24) & \quad a \in (y, z)^3 - (y, z)^3 \quad \text{and} \quad \text{Loc}_{[a]} (3y)^0. \\
(2.25) & \quad a \in (y, z)^3 - (y, z)^3 \quad \text{and} \quad \text{Loc}_{[a]} (3y)^1. \\
(2.26) & \quad a \in (y, z)^3.
\end{align*}

Indeed, since we easily observe (2.27) \( \text{Loc}_{[f^g]} (y, z, 0)](\partial_y) \neq 0 \) for \( i = 1, 3, 5 \) we only have to consider (2.27) for \( i = 0 \). Note that

\[ \text{Loc}_{[f^g]} (y, z, 0)](Y\partial_y + Z\partial_z) = \{ \text{Loc}_{[a]} (3y)^0 \} (Y\partial_y + Z\partial_z) \]

\[ = \left\{ \begin{array}{ll} Y^4 \text{Loc}_{[a]} (3y)^0 & \text{if } a \in (y, z)^3 - (y, z)^3. \\
Y^4 \{ Y \text{Loc}_{[a]} (Y\partial_y + Z\partial_z) - Y^4 - Z^4 \} & \text{if } a \in (y, z)^3 - (y, z)^3. \\
-Y^4 (Y^4 + Z^4) & \text{if } a \in (y, z)^3.
\end{array} \right. \]

Thus we get Claim (2) as desired.

\textbf{Claim (3).} An extension \( \Phi = xA + \phi \) has an approximation order greater than \( \alpha(3)^{-1} \) if and only if either the following (2.28) or (2.29) holds:

\begin{align*}
(2.28) & \quad a \in (y, z)^3 - (y, z)^3 \quad \text{and} \quad \partial_y A \in (x, y, z)^3. \\
(2.29) & \quad a \in (y, z)^3 \quad \text{and} \quad \partial_y A \in (x, y, z)^3.
\end{align*}

Indeed, we easily observe

\[ F(y, z; \partial_z \Phi, \partial_y \Phi) = -x\partial_y A \quad \text{mod}(x, y, z)^4. \]

On the other hand, by the figure (2.23), we have
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1/κ(3) = \begin{cases} 
2 & \text{if } a \in (y, z)^3 - (y, z)^3 \\
3 & \text{if } a \in (y, z)^3. 
\end{cases}

Hence we get Claim (3).

Note that (2.24) and (2.28) are not compatible since, under (2.24), we have \(|\partial_y A| < |\partial_y a| = 1\). Thus it suffices for the conclusion (2.21) to consider the compatibility conditions of (2.25) and (2.29), or of (2.26) and (2.29). Note that, for a germ \(A(x, y, z)\) of the form (2.20), we have

\[(2.29) \iff a \in (y, z)^3 \quad \text{and} \quad y \partial_y \beta_i + \beta_i \in (y, z)^{3-i} \quad (i = 0, 1, 2)\]

since the operator \(y \partial_y + 1\) preserves the vanishing order of \(\beta_i\). In particular we have \(\beta_i \in (y, z)^3\) which yields the equivalence

\[(2.31) \quad a = a_0(y, z) + y \beta_0(y, z) \in (y, z)^3 - (y, z)^4 \iff a_0 \in (z)^3 - (z)^4.\]

Hence we get

\[(2.32) \quad \text{Loc}[a](5, -1) = 0 \quad (\neq 1) \quad \text{if } a \in (y, z)^3 - (y, z)^4.\]

From these (2.30)–(2.32) we conclude that \(\Phi\) is a good extension if and only if the condition (2.21) holds, as desired.

Thus the assertion of Example 2.19 is proved. Q. E. D.

§ 3. Germs of Analytic Coverings

In this section we prepare several geometric notions such as finite holomorphic maps, germs of analytic coverings, which are needed to state our main result in § 4. We refer [Gr-Re] for this section.

Definition 3.1. Let \(X \text{[or, } Y \text{ resp.]} \) be an analytic set of a domain \(D \text{[or, } D' \text{]} \) in \(C^N \text{[or, } C^{N'} \text{]}, \) that is, locally at any \(x \in X \text{[or, } y \in Y \text{]}, \) \(X \text{[or, } Y \text{]} \) is defined as a common zero set of finitely many holomorphic germs

\[g_1, \ldots, g_i \in \mathcal{O}_{D, z} \text{[or, } h_1, \ldots, h_j \in \mathcal{O}_{D', y} \text{].}\]

A continuous map \(f : X \to Y\) is called a holomorphic map if there exists a holomorphic map \(g : D \to D'\) in the sense of theory of complex manifolds such that the map \(f\) is induced by \(g\), that is, the following diagram commutes:

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
D & \xrightarrow{g} & D'
\end{array}
\]
Example 3.2. The map germ \( \Psi : (V, (0, e^0)) \rightarrow (F^{-1}(0), e^0) \) defined by (1.9) is a holomorphic map since it is induced by the map germ \( \Psi \) defined by (1.7).

Definition 3.3. Let \( X, Y \) be analytic sets, and let \( f : X \rightarrow Y \) be a holomorphic map. We call \( f \) is a finite map if \( f \) is a closed map and each fiber \( f^{-1}(y) \) of \( y \in Y \) is a finite subset in \( X \).

Lemma 3.4 [Gr-Re; Proposition 3.1.2, p. 63]. Let \( f : X \rightarrow Y \) be a holomorphic map. Suppose \( x \in X \) is an isolated point of the fiber \( f^{-1}(f(x)) \). Then there exist open neighborhoods \( U \) of \( x \) in \( X \) and \( V \) of \( y \) in \( Y \) with \( f(U) \subseteq V \) such that the induced map

\[
    f_{U,V} : U \rightarrow V
\]

is a finite map.

This Lemma 3.4 asserts that the notion of finite maps is localizable, that is, a notion of finite holomorphic map germs makes sense:

Definition 3.5. A holomorphic map germ \( f : (X, x) \rightarrow (Y, y) \) is called a finite holomorphic map germ if there exist open neighborhoods \( U \) of \( x \) in \( X \) and \( V \) of \( y \) in \( Y \) with \( f(U) \subseteq V \) such that the germ \( f \) has a finite holomorphic representative \( f_{U,V} : U \rightarrow V \).

Criterion 3.6. A holomorphic map germ \( f : (X, x) \rightarrow (Y, y) \) is a finite map germ if and only if \( x \) is isolated in the fiber \( f^{-1}(y) \).

Proof. The “only if” part is trivial since any finite holomorphic representative \( f_{U,V} \) of \( f \) has a finite fiber \( (f_{U,V})^{-1}(y) \). The “if” part is easily obtained by applying Lemma 3.4 to a holomorphic representative of \( f \). Q.E.D.

Definition 3.7. A holomorphic map germ \( f : (X, x) \rightarrow (Y, y) \) is called an open holomorphic map germ if there exist open neighborhoods \( U \) of \( x \) in \( X \) and \( V \) of \( y \) in \( Y \) with \( f(U) \subseteq V \) such that the germ \( f \) has a holomorphic representative \( f_{U,V} : U \rightarrow V \) which is open at \( x \), that is, for any open neighborhood \( U' \) of \( x \) in \( X \) with \( U' \subseteq U \), the image \( f_{U,V}(U') \) is an open neighborhood of \( y \) in \( Y \).

Example 3.8. Let

\[
    w(x, z) = z^k + \sum_{j=1}^{k} w_j(x) z^{k-j} \in \mathbb{C}[[x, z]]
\]

be a Weierstrass polynomial in \( z \) of degree \( k \), that is, \( w_j(0) = 0 \) \( (1 \leq j \leq k) \). We set \( X := w^{-1}(0) \) and \( Y := \mathbb{C}^n \). Then the holomorphic map germ \( f \) defined by
Definition 3.9. Let \( f : (X, x) \rightarrow (Y, y) \) be a finite open holomorphic map germ. The germ \( f \) is called a germ of an analytic covering of \( (Y, y) \), if there exists a germ \( (\Sigma, y) \) of a nowhere dense analytic subset of \( Y \) at \( y \) such that
1) \( f^{-1}(\Sigma) \) is a germ of a nowhere dense analytic subset of \( X \) at \( x \).
2) The induced map germ
\[
\tilde{f} : (X - f^{-1}(\Sigma), x) \rightarrow (Y - \Sigma, y)
\]
is a locally biholomorphic map germ.

Remark 3.10. Let \( f : (X, x) \rightarrow (Y, y) \) be a germ of an analytic covering of \( (Y, y) \), and let \( (\Sigma_i, y) \) be germs of nowhere dense analytic subsets of \( Y \) at \( y \) satisfying the conditions 1) and 2) in Definition 3.9 for \( i = 1, 2 \). Then the intersection germ \( (\Sigma_1 \cap \Sigma_2, y) \) also satisfies the conditions 1) and 2).

Definition 3.11. By virtue of Remark 3.10 and of the Noether property of the ring \( \mathcal{O}_{Y, y} \) [Gr-Re; Corollary 2.2.1, p. 44], there exists a unique germ \( (\Sigma_0, y) \) of a nowhere dense analytic subset of \( Y \) at \( y \) such that \( (\Sigma_0, y) \) is minimal in such germs \( (\Sigma, y) \) satisfying the conditions 1) and 2) in Definition 3.9. This germ \( (\Sigma_0, y) \) is called the critical locus of the germ \( f \) of an analytic covering.

Definition 3.12. Let \( f : (X, x) \rightarrow (Y, y) \) be a germ of an analytic covering of \( (Y, y) \) with a critical locus \( (\Sigma, y) \). Then the following germ \( v(z) \) of a function determined by
\[
v(z) := \# f^{-1}(z) \quad \text{for} \quad z \in (Y - \Sigma, y)
\]
is locally constant. In particular, if \( (Y - \Sigma, y) \) is connected, then \( v(z) \) is constant there:
\[
v(z) = \exists \nu \in \mathbb{N} \quad \text{on} \quad (Y - \Sigma, y).
\]
When this (3.1) occurs, we call that \( f \) is a \( v \)-sheeted germ of an analytic covering of \( (Y, y) \).

Remark 3.13. It is known that if \( (Y, y) \) is a germ of a complex manifold then, for any germ \( (\Sigma, y) \) of a nowhere dense analytic subset of \( Y \) at \( y \), it follows that \( (Y - \Sigma, y) \) is connected. This fact is a direct consequence of the following
Riemann's Extension Theorem [Gr-Re; Theorem 7.1.3, p.132]. Let $X$ be a complex manifold, let $A$ be a nowhere dense analytic subset of $X$ and let $f \in \mathcal{O}(X-A)$ be a holomorphic function on $X-A$. Assume that $f$ is bounded near $A$. Then $f$ has a unique holomorphic extension $f^*$ to $X$.

Example 3.14. Let $X := \mathbb{C}$, and let $Y$ be a complex curve in $\mathbb{C}^2$ defined by $Y := \{(y, z); y^2 - z^2 = 0\}$.

If the integers $p$ and $q$ are coprime, then the following map germ

$$f : X \to Y \quad f(x) := (x^p, x^q)$$

is a one-sheeted germ of an analytic covering of $(Y, (0, 0))$ with a critical locus $\Sigma = \{(0, 0)\}$.

Proof. It suffices to show the existence of an inverse map germ

$$g : Y - \{(0, 0)\} \to X - \{0\}$$

of $f|_{X-\{0\}}$. We first remark that

$$(3.2) \quad Y - \{(0, 0)\} \subset Y \cap \{(y, z); yz \neq 0\}.$$ 

Since $p$ and $q$ are coprime, we can find integers $a, b$ such that

$$(3.3) \quad ap + bq = 1.$$ 

We define $g(y, z) := y^b z^a$. Then (3.2) yields $g \in \mathcal{O}(Y - \{(0, 0)\})$. Moreover (3.3) implies that

$$(g \circ f)(x) = (x^p)^a (x^q)^b = x^{ap+bq} = x \quad \text{for any } x \in X - \{0\}.$$ 

On the other hand, since $y^p = z^q$ on $Y$, we also have

$$(f \circ g)(y, z) = ((y^b z^a)^a, (y^b z^a)^b) = (y^{bq} (z^a)^a, (y^p)^a z^a) = (y^{a p + bq}, z^a) = (y, z) \quad \text{for any } (y, z) \in Y - \{(0, 0)\}.$$ 

Thus the assertion is proved as desired. Q.E.D.

Chapter II. Results

§ 4. Statement of the Main Result

In this section we state our main result (Main Theorem 4.2) and show its corollaries.

We return to the situation at where the classical Theorem 1.6 is stated. We assume the conditions [A.1, 2, 3] and recall the diagram in Theorem 1.6:
We also recall the irreducible decomposition
\[ F = \prod_{j=1}^{n} F_{j}^{\nu(j)} = \prod_{j=1}^{n} F_{j} \] (\nu(j)=1, see Proposition 2.12)
later at \( e^{0} \in E \).

We define germs \((V_{j}, (0, e^{0}))\) for \( 1 \leq j \leq r \), of analytic hypersurfaces of \( C \times E \) at \((0, e^{0})\) by setting
\[ (V_{j}, (0, e^{0})) := (C, 0) \times (F_{j}^{-1}(0), e^{0}). \]
Then the germ \((V, (0, e^{0}))\) can be decomposed into the following union of irreducible components at \((0, e^{0})\):
\[ (V, (0, e^{0})) = \bigcup_{j=1}^{r} (V_{j}, (0, e^{0})). \]

Now we consider the following \( r \)-diagrams instead of (4.1):
\[ \left( V_{j}, (0, e^{0}) \right) \xrightarrow{\psi_j} \left( F^{-1}(0), e^{0} \right) \]
\[ \left( M, x^{0} \right) \xrightarrow{\pi_j} \left( J^{1}M, e^{0} \right) \text{ for } 1 \leq j \leq r. \]

Our result asserts that the map germs \( \pi_j : (V_{j}, (0, e^{0})): \rightarrow (M, x^{0}) \) are germs of analytic coverings of \((M, x^{0})\) and that their numbers of sheets are calculable by means of the Newton polygon \( N(f^{0}) \), where \( \Phi \) is a good extension at \( e^{0} \) of the Cauchy data \( \Phi \).

Definition 4.1. 1) For \( 1 \leq \mu \leq m := \# \text{Seg} N(f^{0}) \) we set
\[ v(\mu) := p(\mu)(a(\mu-1)+1) + q(\mu)(p-b(\mu-1)-1) \]
where \( a(\mu), b(\mu), p(\mu) \) and \( q(\mu) \) are the integers determined by Notation 2.4.

2) We define integers \( v_j \) for \( 1 \leq j \leq r \) by
\[ v_j := \sum_{\mu \in M_j} v(\mu) \]
where \( M_j \subset \{1, 2, \ldots, m\} \) is the subset which is defined by the assertion 3) in Proposition 2.12.

Now the time has come to state our main result:
Main Theorem 4.2. Assume the conditions \([A.1],[A.2]\) and \([A.3]\). Then, for \(1 \leq j \leq r\), the following statements hold:

1) The holomorphic map germ \(\pi_j: (V_j, \langle 0, e^0 \rangle) \to (M, x^0)\) determined by the diagram (4.5) is a \(v_j\)-sheeted germ of an analytic covering of \((M, x^0)\).

2) Let \((\Sigma_j, x^0)\) be the critical locus of \(\pi_j\). We define a germ \(u_j(x)\) on \((M-\Sigma_j, x^0)\) of a multi-valued analytic function by the following diagram (4.8):

\[
\begin{array}{c}
(V_j-\pi_j'(\Sigma_j), \langle 0, e^0 \rangle) \xrightarrow{\mathcal{F}_j} (V_j, \langle 0, e^0 \rangle) \xrightarrow{\pi_j} (F^{-1}(0), e^0) \\
\end{array}
\]

Then the germ \(u_j\) is exactly \(v_j\)-valued, that is, for \(\forall x \in (M-\Sigma_j, x^0)\) the multi-valued germ \(u_j\) has \(v_j\)-branches \(u_j^{i}, (1 \leq i \leq v_j)\) such that any two branches of \(u_j\) can be continued each other along a path in \((M-\Sigma_j, x^0)\).

Remark 4.3. If the assertion 1) of Main Theorem 4.2 is established then the multi-valued germ \(u_j\) is well-defined by the diagram (4.8), since the induced map germ \(\pi_j: (V_j-\pi_j'(\Sigma_j), \langle 0, e^0 \rangle) \to (M-\Sigma_j, x^0)\) is a locally biholomorphic map germ.

Remark 4.4. Main Theorem 4.2 includes Theorem 1.6 as a special case. Indeed, if \(p=1\) then there exists an approximate solution \(\Phi\) of (1) such that \(\gamma:=\text{ord}_{x^0}[f \Phi] < \infty\) (Remark 2.17). Thus we have \(N(f^0)=N_{q,1}\) (Notation 2.10). Hence Theorem 4.2 yields that the ramification degree is given by

\[v_1=v(1)=p(1)a(0)+q(1)(p-b(0)-1)\]

\[=p+q(p-1)\]

\[=1.\]

In the remaining part of this section, we state and show the following Corollaries 4.6 and 4.7.

The first one is related to the analytic continuations of holomorphic local solutions of the Cauchy problem (1). To state this, we prepare the

Lemma 4.5. Assume \([A.1,2,3]\). We define a germ \(\Omega\) by

\[
\Omega := \{ e = (y; \xi, \Phi(y)) \in (E \cap F^{-1}(0), e^0) ; \text{ord}_x[F_{E_y}] = 1 \}.
\]

Then the following 1) and 2) hold:

1) \(\Omega\) is a non-empty germ at \(e^0\).
2) For any $e \in \Omega$ there exists a unique $j \ (1 \leq j \leq r)$ such that

$$e \in F_j^{-1}(0) - \bigcup_{i \neq j} F_i^{-1}(0).$$

Lemma 4.5, Theorem 1.6 and Main Theorem 4.2 immediately yield the

**Corollary 4.6.** Let $e = (y; \xi, \phi(y)) \in \Omega$ and let $j$ be the unique number satisfying (4.10). Then the following 1) and 2) hold:

1) There exists a unique holomorphic local solution $u_j \in \mathcal{O}_{M, y}$ of the Cauchy problem (1) satisfying $(y; du_j, u_j(y)) = e$.

2) The holomorphic local solution $u_j \in \mathcal{O}_{M, y}$ mentioned in 1) can be continued analytically to the multi-valued germ $u_j$ determined by (4.8) on $(M - r, x^s)$. Hence the analytic continuation of $u_j$ around the point $x^s \in M$ is exactly $v_j$-valued.

**Proof of Lemma 4.5.** We only have to verify the assertion 1), since the assertion 2) is a consequence of the fact

$$\Omega \cap \bigcup_{i \neq j} (F_i^{-1}(0) \cap F_j^{-1}(0)) = \emptyset.$$ 

It suffices for the assertion 1) to show that, for any open neighborhood $U$ of $(x^0, 0)$ in $S \times C$, and for $1 \leq j \leq r$, it follows that

$$f_j^{-1}(0) \cup ([\partial_x f_j]^{-1}(0) \cup \bigcup_{i \neq j} f_i^{-1}(0)) \neq \emptyset$$

where we set

$$f_j(y, \tau) := (y \in F_j(y, \tau) = F_j(y; \tau ds(y) + d\Phi(y), \phi(y)).$$

Recall that the coprimeness condition yields that the germ $F_j|_E$ has no multiple factor $F_j$ (see Proposition 2.12). Therefore the factorization

$$f^\phi(y, \tau) = \prod_{j=1}^{r} f_j(y, \tau)$$

is an irreducible decomposition of the germ $f^\phi := \pi^\phi(F|_E)$.

We show (4.11) by contradiction. If we assume that (4.11) is not true then we can find an open neighborhood $U$ and a number $j$ such that

$$f_j^{-1}(0) \cap U \subseteq ([\partial_x f_j]^{-1}(0) \cup \bigcup_{i \neq j} f_i^{-1}(0)).$$

We set $h(y, \tau) := \partial_x f_j(t, \tau) \times \prod_{i \neq j} f_i(y, \tau)$. Then (4.13) yields

$$h|_{f_j^{-1}(0)} \equiv 0.$$ 

Hence, by virtue of the Rückert's Nullstellensatz (see § 13), we have

$$h \in \text{Rad} [(f_j)] := \{g \in \mathcal{O}_{S \times C, (x^0, 0)}; \exists k \in \mathbb{N}, g^k \in (f_j)\}.$$ 

Since the ideal $(f_j)$ is a prime ideal, it follows that $h \in (f_j)$ thus we have
This contradicts the facts that \( f_j \) is irreducible and that (4.12) is an irreducible decomposition. Hence (4.11) follows. Q. E. D.

Our second corollary (Corollary 4.7 below) asserts the converse of the classical Theorem 1.6 under the assumptions \([A.1, 2, 3]\).

**Corollary 4.7.** Assume the conditions \([A.1, 2, 3]\). If there exists a holomorphic local solution \( u \in \mathcal{O}_{\mathbb{R}, x^0} \) of the Cauchy problem (1) satisfying \( (x^0; du(x^0), u(x^0)) = e^a \), then it follows that

\[ \text{ord}_{e_0}[F|_{E_{x^0}}] = 1. \]

**Proof.** By virtue of Main Theorem 4.2, the assumption of Corollary 4.7 yields that there exists a number \( j (1 \leq j \leq r) \) such that

\[ (4.14) \quad v_j = \sum_{\mu \in M_j} v(\mu) = 1. \]

By the definition (4.6) of \( v(\mu) \) we have

\[ v(\mu) = p(\mu)(a(\mu - 1) + 1) + q(\mu)(p - b(\mu - 1) - 1) \geq p(\mu)(a(\mu - 1) + 1) \geq p(\mu) \geq 1 \]

because \( p(\mu), q(\mu) > 0 \) and \( a(\mu - 1), p - b(\mu - 1) - 1 \geq 0 \). Hence (4.14) implies

\[ \{ \# M_j = 1, \text{ that is, } M_j = \{ \exists \mu^a \} \text{ and } \]

\[ p - b(\mu^a - 1) - 1 = a(\mu^a - 1) = 0 \text{ and } p(\mu^a) = 1 \]

which yield

\[ \mu^a = m = 1 \text{ and } p = p(1) = 1. \]

Hence we get Corollary 4.7 as desired. Q. E. D.

**Example 4.8.** Let us recall Example 2.19. We calculate the ramification degrees \( v_j (j = 1, 2) \) of multi-valued analytic solutions of the Cauchy problem (2.19) as follows: By the condition (2.21), any good extension \( \Phi = xA(x, y, z) + \phi(y, z) \) of the data \( \phi \) satisfies

\[ a(y, z) := A(0, y, z) \in (y, z)^a. \]

Hence, by the right figure of (2.23), it follows that

\[ v(1) = 6, \quad v(2) = 10 \quad \text{and} \quad v(3) = 5. \]

Since we have \( M_1 = \{2\}, M_4 = \{1, 3\} \), we get
which shows that

\[ v_1 = v(2) = 10, \quad v_2 = v(1) + v(3) = 11 \]

which shows that

\[
\{ \text{the Cauchy problem (2.19) has two sorts of multi-valued analytic solutions } u_1 \text{ and } u_2 \text{ of 10-valued and 11-valued respectively.} \}
\]

Chapter III. Proof

§ 5. Reduction of the Main Theorem to Theorem 5.1

In this section we reduce the proof of Main Theorem 4.2 to that of the following Theorem 5.1.

We consider the following Cauchy problem (5.1) which is defined in an open neighborhood \( M \) of the origin of \( \mathbb{C}^n \) with the zero Cauchy data:

\[
\begin{aligned}
F(x; u(x), u(x)) &= G(x; \partial_{x_1} u, \partial_{x_2} u, u, u) - \partial_{x_n} u = 0 \\
u |_{x_1 = 0} &= 0
\end{aligned}
\]

where \((x_1, x_2, x_n) \in \mathbb{C} \times \mathbb{C}^{n-2} \times \mathbb{C} \). We treat (5.1) with a base point \( e^0 = (0; 0, 0) \in \mathcal{F}^{-1}(0) \)
and with the following assumptions \([B.1]-[B.4]\):

\[ [B.1] \quad 0(0; ^*, r, 0) \in (f, f^*) \]

\[ [B.2] \quad \text{ord}_0 [G(0; ^*, 0, 0)] = : p \in [2, \infty) \]

\[ [B.3] \quad \phi(x) := 0 \text{ is a good extension of the data } \phi(x') := 0. \]

\[ [B.4] \quad \text{The approximation order } \text{ord}_0[F(x; 0, 0)] \text{ of } \Phi = 0 \text{ is equal to the order } q := \text{ord}_0[f(x') = 0]. \]

Note that the condition \([B.4]\) is stronger than the inequality

\[ \text{ord}_0[F(x; 0, 0)] > \kappa(m)^{-1} \quad \text{(the fourth condition of } [B.3]) \]

since the condition \( p \geq 2 \) implies \( q > \kappa(m)^{-1} \).

**Theorem 5.1.** Under the assumptions \([B.1]-[B.4]\), the conclusions 1) and 2) of Main Theorem 4.2 hold for the reduced Cauchy problem (5.1).

Let us return the situation of Main Theorem 4.2.

We must show that Theorem 5.1 implies Main Theorem 4.2. Note that, since in the case \( p = 1 \) the assertion of Main Theorem 4.2 is contained in that of Theorem 1.6, we may assume \( p \geq 2 \), where \( p \) is the vanishing order in the condition \([A.2]\), that is, \( p = \text{ord}_r[F(x; \tau d s(x^0) + \xi^0, z^0)] \).

Our reduction starts from a simple
Lemma 5.2. There exists a local coordinate system \((x_1, \ldots, x_n)\) of \(M\) around \(x^0\) such that

\[
S = \{x_1 = 0\} \quad \text{and} \quad L_F(x^0) = -\partial_{x_n}.
\]

Proof. Taking a coordinate system as \(x_1 = s(x)\), we may assume that a system satisfying \((5.2)\) is already chosen. The assumption \(\rho \geq 2\) yields \(\partial_{x_i}F(x^0) = 0\), hence, by the condition \([A.1]\), we may assume \(\partial_{x_n}F(x^0) \neq 0\). We take a linear coordinate transformation of the form

\[
x \mapsto x^\ast := \begin{bmatrix} I_{n-1} & 0 \\ \lambda^\ast & \lambda_n \end{bmatrix} x
\]

and choose \((\lambda^\ast, \lambda_n)\) as

\[
\lambda_n := -[\partial_{x_n}F(x^0)]^{-1}, \quad \lambda_j := \lambda_n \partial_{x_j}F(x^0) \quad \text{for} \quad 2 \leq j \leq n-1.
\]

Then it is easily verified that the coordinate system \((x^\ast_1, \ldots, x^\ast_n)\) satisfies the conditions \((5.2)\), \((5.3)\) as desired. Q.E.D.

The next step of our reduction is to show the existence of another good extension \(\Phi^\ast\) with a “better” approximation order than that of the original \(\Phi\):

Proposition 5.3. Let \(\Phi\) be a good extension of \(\phi\). For the irreducible decomposition \((4.2)\) of \(F\) locally at \(x^0\), we set

\[
(5.4) \quad \begin{cases} f^\Phi(x', \xi_1) := F(0, x'; \xi_1 d x_1 + d \Phi(0, x'), \phi(x')) = \sum_{\xi=0} c_{\nu}(x') \xi_1. \\
f_{\nu}^\Phi(x', \xi_1) := F(0, x'; \xi_1 d x_1 + d \Phi(0, x'), \phi(x')) = \sum_{\xi=0} c_{\nu}(x') \xi_1.
\end{cases}
\]

Then there exists a good extension \(\Phi^\ast\) of \(\phi\) such that

1) \(N(f_{\nu}^\Phi) = N(f_{\nu}^{\Phi^\ast})\) for all \(j = 1, 2, \ldots, r\).
2) \(\text{Loc}[c_{\nu}] = \text{Loc}[c_{\nu}^{\ast}]\)
   \(\text{for all } \nu\) satisfying \((\text{ord}[c_{\nu}], \nu) \in \text{Ver}(f^\Phi) (= \text{Ver}(f^{\Phi^\ast}))\).
3) \(\text{ord}_{\nu}(F(x'; d \Phi^\ast(x), \Phi^\ast(x))) = q\) \((:= \text{ord}_{\nu}(f^\Phi(x', 0)))\)

where we use the analogous expression of \(f^\Phi^\ast\) or \(f_{\nu}^{\Phi^\ast}\) resp. which is gained by replacement of \((\Phi, c_{\nu})\) \([((\Phi, c_{\nu}),)]\) in \((5.4)\) with \((\Phi^\ast, c_{\nu}^\ast)\) \([((\Phi^\ast, c_{\nu}^\ast),)]\).

Proof. We seek the desired “better” extension \(\Phi^\ast\) as the form
where \((x_{1}, \cdots, x_{n})\) is a local coordinate system of \(M\) at \(x^{0}\), which is obtained by Lemma 5.2. Note that the assumption \(p \geq 2\) implies

\[ q > \kappa(m)^{-1}. \tag{5.5} \]

We denote the approximation order of \(\Phi\) by \(k\):

\[ k := \text{ord}_{x^{0}}[F(x; d\Phi(x), \Phi(x))] > \kappa(m)^{-1}. \tag{5.6} \]

Since Proposition 5.3 clearly holds for \(\Phi^{-} := \Phi\) if \(q \leq k\), we may assume

\[ q > k. \tag{5.7} \]

We first give a sufficient condition of \(w\) for the assertions of Proposition 5.3 except for the assertion 3):

**Lemma 5.4.** If \(w \in \mathcal{O}_{M, x^{0}}\) satisfies the condition

\[ \text{ord}_{x^{0}}[w] \geq k \tag{5.8} \]

then the assertions of Proposition 5.3 except for the assertion 3) follow.

**Proof.** Note that \(d \Phi^{-}(0, x') = w(0, x') dx_{1} + d\Phi(0, x')\) yields that the germ \(\Phi^{-}\) is a holomorphic approximate solution of the Cauchy problem (1), since the inequalities (5.6) and (5.8) imply \(w(0, 0) = 0\). Moreover we have

\[ f^{\Phi^{-}}(x', \xi_{i}) = f^{\Phi}(x', \xi_{i} + w(0, x')) = \sum_{\nu=0}^{\infty} c_{\nu}(x')(\xi_{i} + w(0, x'))^{\nu}, \tag{5.9} \]

where \(c_{\nu}(x', x'') = \sum_{\lambda=0}^{\nu} C(\nu, \lambda) c_{\nu}(x') w(0, x')^{\nu-\lambda} \xi_{i}^{\lambda}. \tag{5.9}'\]

We fix \(f\) and first show the following inclusion:

\[ N(f^{\Phi^{-}}) \subseteq N(f^{\Phi}). \tag{5.10} \]

We define positive integers \(p_{j}, a_{j}(\mu)\) and \(b_{j}(\mu)\) for \(\mu \in M_{j}\) by

\[
\left\{
\begin{array}{l}
p_{j} := \text{ord}[f^{\Phi}(0, \xi_{i})] \\
a_{j}(\mu) := \sum_{\lambda_{j} \leq \mu} q(\lambda), \quad b_{j}(\mu) := \sum_{\lambda_{j} < \mu} p(\lambda).
\end{array}\right.
\]

Then Lemma 0.2 and the assertion 3) of Proposition 2.12 yield that
Hence (ord[c_v, j], v) ∈ N(f_θ^j) implies the following inequality:

\[ \nu \geq -\kappa(\mu) \{ \text{ord}[c_v, j] - a(\mu) \} + p_j - b(\mu) \quad \text{for } \mu \in M_j ; \nu = 0, 1, \ldots \]

where \( \kappa(\mu) = p(\mu)/q(\mu) \). Thus (5.9)' yields the following inequality:

\[ -\kappa(\mu) \{ \text{ord}[c_v, j] - a(\mu) \} + p_j - b(\mu) \]

\[ \leq \max_{\nu \geq \lambda} \left[ -\kappa(\mu) \{ \text{ord}[c_v, j] - a(\mu) \} + p_j - b(\mu) \right] \]

\[ \leq \max_{\nu \geq \lambda} \left[ \nu - \kappa(\mu)(\nu - \lambda) \right] \text{ord}[w(0, x')] \]

\[ = \{ 1 - \kappa(\mu) \text{ord}[w(0, x')] \} \min \{ \nu \} + \kappa(\mu) \text{ord}[w(0, x')] \lambda \]

\[ \leq \lambda \]

since (5.8), (5.6) imply \( \kappa(\mu) \text{ord}[w(0, x')] \geq \kappa(m)k > 1 \). Hence (5.10) follows.

Interchanging the roles of \( \Phi \) and \( \Phi^* \), we also have the converse inclusion of (5.10). Hence we conclude the assertion 1) of Proposition 5.3.

Note that this assertion 1) implies that

\[ N(f^\Phi) = \sum_{j=1}^r N(f^\Theta_j) = \sum_{j=1}^r N(f^{\Theta_j^*}) = N(f^\Phi^*) \quad \text{and} \quad M_j = M_j^* \quad (1 \leq j \leq r) \]

which show that the Newton polygon \( N(f^{\Theta_j^*}) \) satisfies the coprimeness condition and that all the subsets \( M_j^* \) (1 ≤ j ≤ r) are nice subsets of \{1, 2, ..., m\}.

For the proof of the assertion 2), it suffices to show

\[ \text{ord}[c_{p-b(\mu)}] = a(\mu) < \min_{\nu \geq \nu_p-b(\mu)} \{ \text{ord}[c_v] + (\nu - p + b(\mu)) \text{ord}[w(0, x')] \} \]

for 1 ≤ \( \mu \leq m \)

since we have

\[ c_{p-b(\mu)}(x') = c_{p-b(\mu)}(x') + \sum_{\nu \geq \nu_p-b(\mu)} C(\nu, p-b(\mu))c_v(x')w(0, x')^{p - b(\mu)} \]

by (5.9). By virtue of \( \text{ord}[w(0, x')] \geq k \), it suffices for (5.12) to derive

\[ a(\mu) < \min_{\nu \geq \nu_p-b(\mu)} \{ \text{ord}[c_v] + (\nu - p + b(\mu))k \} \quad \text{for } 1 \leq \mu \leq m. \]

From (ord[c_v], v) ∈ N(f_θ), we have the inequality

\[ \nu \geq -\kappa(\mu) \{ \text{ord}[c_v] - a(\mu) \} + p - b(\mu) \quad \text{for } 1 \leq \mu \leq m ; \nu = 0, 1, \ldots \]

which is equivalent to \( \text{ord}[c_v] \geq a(\mu) - \kappa(\mu)^{-1}(\nu - p + b(\mu)) \). Hence it follows

\[ \text{the right hand side of (5.13)} \geq a(\mu) + \min_{\nu \geq \nu_p-b(\mu)} \{ \nu - p + b(\mu) \} \{ k - \kappa(\mu)^{-1} \}. \]

Thus, by virtue of \( k > \kappa(m)^{-1} \geq \kappa(\mu)^{-1} \), we get (5.13).

Note that the assertion 2) of Proposition 5.3 implies the stability of the
Newton polygon $N(f^\infty)$ in the direction of the tangent vector $L_F(e^\infty)$. Thus if we establish the assertion 3) of Proposition 5.3, then the holomorphic approximate solution $\Phi^\infty$ is a good extension of the Cauchy data $\phi(x^\prime)$.

The proof of Lemma 5.4 is complete. Q. E. D.

We continue the proof of Proposition 5.3. If suffices to find a germ $w(x)$ satisfying (5.8) and the assertion 3) in Proposition 5.3 when we set $\Phi^\infty$ as

$$\Phi^\infty := x_1w(x) + \Phi.$$ 

Taking the Taylor expansion of $F$ at $(x; \xi, z) = (x; d\Phi(x), \Phi(x))$, we have

$$F(x; d\Phi^\infty(x), \Phi^\infty(x)) = \sum_{\alpha_1 \geq 0} (\alpha_1!)^{-1} \partial_{\xi_1}^\alpha F(x; d\Phi(x), \Phi(x))(x_1\partial_{x_1} w + w)^{\alpha_1}$$

$$+ \sum_{\alpha_1 + \beta_1 + \gamma_1 = 0} (\alpha_1 \beta_1 \gamma_1)^{-1} \partial_{\xi_1}^\alpha \partial_{\xi_1}^\beta \partial_{\xi_1}^\gamma F(x; d\Phi(x), \Phi(x))$$

$$\times (x_1\partial_{x_1} w + w)^{\alpha_1}(x_1\partial_z w)^{\beta_1}(x_1 w)^{\gamma_1}. $$

Note that the coefficients of the first term in (5.14) have the following expressions (5.15): Since there exists a holomorphic germ $g(x, \xi_1)$ such that

$$F(x; \xi_1 d x_1 + d\Phi(x), \Phi(x)) = F(0, x^\prime; \xi_1 d x_1 + d\Phi(0, x^\prime), \Phi(x^\prime) + x_1 g(x, \xi_1))$$

$$= f^0(x^\prime, \xi_1) + x_1 g(x, \xi_1),$$

we have

$$\partial_{\xi_1}^\alpha F(x; d\Phi(x), \Phi(x)) = \partial_{\xi_1}^\alpha f^0(x^\prime, 0) + x_1 \partial_{\xi_1}^\alpha g(x, 0)$$

$$= (\alpha_1! \{c_{a_1}(x^\prime) + x_1 \mathbb{E} g_{a_1}(x)\}) \text{ for } \alpha_1 \geq 0.$$ 

In particular, taking $\alpha_1 = 0$, we note

$$\text{ord}[g_{a_1}] = \text{ord}[F(x; d\Phi(x), \Phi(x)) - c_{a_1}(x^\prime)] - 1 \geq k - 1.$$ 

On the other hand, the second term in the right hand side of (5.14) can be divided by $x_1$, thus it can be written as the form

$$x_1 \left[ \sum_{j \geq 2} \partial_{\xi_1}^j F(x; d\Phi(x), \Phi(x)) \partial_{x_1} w + \partial_{\xi_1} F(x; d\Phi(x), \Phi(x)) w $$

$$+ \mathbb{E} K(x; x_1 \partial_{x_1} w + w, \partial_{\xi_1} w, w) \right]$$

where $K$ satisfies

$$K(x; x_1 \xi_1 + z, x_1 \xi_1 + z, \xi^\prime, z) = (x_1 \xi_1 + z) (x_1 \xi_1 + z)^3.$$ 

Hence, from (5.15) and (5.17), we can write (5.14) as the following form:
(5.19) \[ F(x ; d\Phi^{-}(x), \Phi^{-}(x)) = \sum_{a_1=0}^{\infty} \{ c_{a_1}(x') + x_1 g_{a_1}(x) \} (x, \partial z_1 w + w)^{a_1} \]
\[ + x_1 \left[ \sum_{j=1}^{n} \partial_j F(x ; d\Phi(x), \Phi(x)) \partial z_1 w + \partial_j F(x ; d\Phi(x), \Phi(x)) w \right. \]
\[ \left. + K(x ; x, \partial z_1 w + w, \partial x \cdot w, w) \right] . \]

We claim that the following inequality holds under (5.8):

\[ \text{ord}[c_{a_1}(x')(x, \partial z_1 w + w)] \geq q \quad \text{for all } a_1 \geq 0 . \]

Indeed, from the inclusion

\[ (\text{ord}[c_{a_1}], a_1) \in N(f^c) \subseteq \{(s, t) ; t \geq -\kappa(m)(s-q)\} \]

and (5.8), we easily have

*the left hand side of (5.20) \geq q - \kappa(m)^{-1} + \alpha, \text{ord}[w] \]
\[ \geq q + \alpha \{ k - \kappa(m)^{-1} \} \]
\[ \geq q . \]

By virtue of (5.19) and (5.20), it suffices for Proposition 5.3 to show the existence of a holomorphic germ \( w(x) \) satisfying (5.8) and the following equation (5.21) for some \( h_i(x) \in (x)^{q-1} \):

\[ \sum_{a_1=0}^{\infty} g_{a_1}(x)(x, \partial z_1 w + w)^{a_1} + \sum_{j=1}^{n} \partial_j F(x ; d\Phi(x), \Phi(x)) \partial z_1 w \]
\[ + \partial_j F(x ; d\Phi(x), \Phi(x)) w + K(x ; x, \partial z_1 w + w, \partial x \cdot w, w) = \exists h_i(x) . \]

We set

\[ h(x) := g_{a_1}(x) - h_i(x) \quad \text{and} \]

\[ H(x ; \xi, z) := \sum_{a_1=0}^{\infty} g_{a_1}(x)(x, \xi_1 + z)^{a_1} + \sum_{j=1}^{n} \partial_j F(x ; d\Phi(x), \Phi(x)) \xi_j \]
\[ + \partial_j F(x ; d\Phi(x), \Phi(x)) z + K(x ; x, \xi_1 + z, \xi', z) + h(x) . \]

Note that (5.16) yields

\[ \text{ord}[h] \geq \min \{ q, k \} - 1 = k - 1 . \]

We seek a solution \( w \) of the non-linear equation

\[ H(x ; \partial z w, w) = 0 \]

with the condition (5.8), that is, \( \text{ord}[w] \geq k \). Since the condition (5.18) yields \( K(\subseteq x_1, z) \), we note

\[ H(0 ; \xi, 0) = \sum_{j=1}^{n} \partial_j F(e^0 \xi_j + h(0)) = -\xi + h(0) . \]
Then the Weierstrass's preparation theorem leads us to

\[(5.25) \quad H(x; \xi, z) = \{H^*(x; \xi, \xi', z) - \xi_n\} e(x; \xi, z)\]

locally at \((x; \xi, z) = (0; 0, \ldots, 0, h(0), 0)\) for some holomorphic germs

\[H^* \in O_{\mathbb{C}^n \times \mathbb{C}^{n-1} \times \mathbb{C}(0; 0, 0)}, \quad e \in O_{\mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}(0; 0, \ldots, 0, h(0), 0)}\]

with \(H^*(0; 0, 0) = h(0), \quad e(0; 0, \ldots, 0, h(0), 0) \neq 0\). From (5.25) we have the

**Claim.** For any germ \(w(x)\) satisfying \((d w(0), w(0)) = (0, \ldots, 0, h(0), 0)\), the equation (5.21)' is equivalent to the following normal formed one:

\[(5.26) \quad \partial_{x_n} w = H^*(x; \partial_{x_n}, w, \partial_{x_n}, w)\]

where \(H^*(x; \xi, \xi', z)\) satisfies

\[(5.27) \quad \text{ord}[H^*(0; 0, 0, h(0))] \geq \max\{k - 1, 1\}.\]

**Proof.** It only remains to verify (5.27). Setting \((\xi, z) = (0, \ldots, 0, h(0), 0)\) in (5.25), we have

\[(5.28) \quad H(x; 0, \ldots, 0, h(0), 0) = \{H^*(x; 0, 0, 0) - h(0)\} e(x; 0, \ldots, 0, h(0), 0).\]

On the other hand, the definition (5.22) of \(H\) yields that

\[(5.29) \quad H(x; 0, \ldots, 0, h(0), 0) = \partial_{x_n} F(x; d \Phi(x), \Phi(x)) h(0) + h(x) + K(x; x_n, 0+0, 0, \ldots, 0, h(0), 0).\]

Since \(H(0; 0, \ldots, 0, h(0), 0) = 0\) by (5.24), it follows that

\[\text{ord}[H(x; 0, \ldots, 0, h(0), 0)] \geq 1 = \max\{k - 1, 1\} \quad \text{if} \quad k = 1.\]

If \(k \geq 2\) then \(h(0) = 0\) since \(h \in (x)^{k-1}\). Therefore, (5.18) and (5.29) yield

\[H(x; 0, \ldots, 0, h(0), 0) = h(x) + K|_{x_n = 0, h(0)} = h(x).\]

Thus we have

\[\text{ord}[H(x; 0, \ldots, 0, h(0), 0)] = \text{ord}[h] \geq k - 1 = \max\{k - 1, 1\}.\]

Hence, from (5.28), we get (5.27). The proof of Claim is complete. Q.E.D.

Since the equation (5.26) can be solved with any holomorphic data on \(\{x_n = 0\}\), we can find a unique holomorphic solution \(w(x; w_0)\) of (5.26) satisfying the following data:

\[(5.30) \quad w(x; w_0)|_{x_n = \theta} = w_0(x_1, x^*) \in (x_1, x^*)^\theta.\]

It suffices for the proof of Proposition 5.3 to show

\[(5.31) \quad \text{ord}[w(x; w_0)] \geq k.\]
We write $H^{-}(x; \xi, \xi^*, z)$ as the form

$$H^{-}(x; \xi, \xi^*, z) = \sum_{j=1}^{n-1} a_j(x; \xi, \xi^*, z)z_j + b(x; \xi, \xi^*, z)z + H^{-}(x; 0, 0, 0) - h(0) + h(0).$$

Then the solution $w(x; w_0)$ satisfies the following linear equation

(5.32) $$\partial_{x_n} w = \sum_{j=1}^{n-1} A_j(x) \partial_{x_j} w + B(x) w + C(x) + h(0)$$

where we put

$$A_j(x) := a_j(x; \xi, \xi^*, z) \partial_{x_j} w(x; w_0), \quad w(x; w_0) \quad 1 \leq j \leq n-1$$
$$B(x) := b(x; \xi, \xi^*, z) \partial_{x_j} w(x; w_0), \quad w(x; w_0)$$
$$C(x) := H^{-}(x; 0, 0, 0) - h(0).$$

By virtue of (5.27), it suffices for (5.31) to verify the

Lemma 5.5. Let $w(x)$ be the unique holomorphic solution of the linear Cauchy problem (5.32) with the data (5.30). Let $k \geq 1$ and assume

(5.33) $\text{ord}_{x_0}[C] \geq \max\{k-1, 1\}$.

Then it follows that

(5.8) $\text{ord}_{x_0}[w] \geq k$.

Proof. We expand $w(x)$ in $x_n$ as $w(x) = \sum_{\mu=0}^{\infty} w_\mu(x_1, x^*) x_n^\mu$. Then the equation (5.32) yields the following equalities:

(5.34) $$\begin{cases} w_\mu(x_1, x^*) = C(x_1, x^*, 0) + h(0) \mod \sum_{j=1}^{n-1} (\partial_{x_j} w_0) + (w_0). \\ \mu w_\mu(x_1, x^*) = (\mu-1)!^{-1} \partial_{x_n}^{\mu-1} C(x_1, x^*, 0) \mod \sum_{j=1}^{n-1} \left( \sum_{j=1}^{n-1} (\partial_{x_j} w_0) + (w_0) \right) \end{cases}$$

for $\mu \geq 2$.

We prove

(5.35) $\text{ord}[w_\mu(x_1, x^*)] \geq k - \mu$ for $0 \leq \mu \leq k$

by induction on $\mu$. Note that if $k = 1$ then there is nothing to prove. Thus we may assume that $k \geq 2$, hence $h(0) = 0$. In the case $\mu = 1$, the equality (5.34) with the assumptions (5.33) and (5.30) yields that $\text{ord}[w_1] \geq k-1$.

Now let $k \geq \mu \geq 2$, and assume

(5.36) $\text{ord}[w_\lambda] \geq k - \lambda$ for $0 \leq \lambda \leq \mu - 1$.

Then (5.34) with the assumptions (5.33) and (5.36) yields that
Hence the proof of Lemma 5.5 is complete. Q. E. D.

By virtue of Lemmata 5.4 and 5.5, we get Proposition 5.3. Thus the proof of Proposition 5.3 is complete. Q. E. D.

The third step of our reduction is the changing of unknown function from $u(x)$ to the following $\hat{u}(x)$. We denote by $\Phi$ the good extension $\Phi^-$ which is gained by Proposition 5.3. We set $\hat{u}(x)$ and $F^-\left(x; \xi^-, z^-\right)$ by

\[
\begin{align*}
\hat{u} & := u - \Phi, \\
F^-\left(x; \xi^-, z^-\right) & := F\left(x; \xi^- + d\Phi(x), z^- + \Phi(x)\right).
\end{align*}
\]

Note that if $\hat{u}$ is a solution of the Cauchy problem

\[
\begin{align*}
F^-\left(x; d\hat{u}(x), \hat{u}(x)\right) & = 0 \\
\hat{u}(0, x') & = 0
\end{align*}
\]

then $u = \hat{u} + \Phi$ is a solution of the original Cauchy problem (1).

**Lemma 5.6.** Let $t \mapsto \Psi^-\left(t, \hat{e}\right) = \left(X^-(t, \hat{e}); \Xi^-(t, \hat{e}), Z^-(t, \hat{e})\right)$ be the characteristic curve of $F^-$ which passes through

\[
\hat{e} := (0, y'; \eta^- d x_1, 0) \in \hat{E} := T^* \mathbb{R}^n \times \{0\}
\]

at the initial time $t = 0$. We set a biholomorphic map $\lambda: \hat{E} \rightarrow E$ by

\[
\lambda(\hat{e}) := (0, y'; \eta^- d x_1 + d\Phi(0, y'), \Phi(0, y'))
\]

and we put

\[
\Psi_\Phi(t, \lambda(\hat{e})) := \left(X^+; \Xi^- + d\Phi(X^-), Z^+ + \Phi(X^-)\right)(t, \hat{e}).
\]

Then $t \mapsto \Psi_\Phi(t, \lambda(\hat{e}))$ is a characteristic curve of $F$ passing through $\lambda(\hat{e})$ at the initial time $t = 0$.

**Proof.** Since we have

\[
\begin{align*}
\partial_x F^- & = \partial_{x_j} F + \sum_{i=1}^{n} (\partial_{\xi_i} F) \partial_{x_i} \partial_{x_j} \Phi + (\partial_{\xi_j} F) \partial_{x_i} \Phi \\
\partial_{\xi_j} F^- & = \partial_{\xi_j} F, \quad \text{and} \quad \partial_{x_i} F^- = \partial_{x_i} F
\end{align*}
\]

it follows that...
\[\partial_t X^j = \partial_{x_j} F(\Psi^e(t, \lambda(\hat{e})))\]
\[\partial_t \{E^j + \partial_{x_j} \Phi(X^\ast)\} = -\partial_x F^\ast - \partial_{x_j} F^\ast + \sum_{i=1}^n (\partial_{x_i} \partial_{x_j} \Phi) \partial_{x_i} X^j\]
\[= \{-\partial_x F^\ast - \partial_{x_j} F^\ast\}(\Psi^e(t, \lambda(\hat{e})))\]
\[\partial_t \{Z^\ast + \Phi(X^\ast)\} = \sum_{j=1}^n \partial_{x_j} F^\ast + \sum_{j=1}^n (\partial_{x_j} \Phi) \partial_{x_j} X^j\]
\[= \sum_{j=1}^n \{\partial_{x_j} \Phi \partial_{x_j} F(\Psi^e(t, \lambda(\hat{e})))\}.
\]
Hence we get Lemma 5.6. Q. E. D.

Let us recall the irreducible decomposition
\[(2.11)\]
\[F_{|E} = \prod_{j=1}^r F_j\]
and we set
\[F_j^\circ := \lambda_* F_j \quad (1 \leq j \leq r)\]
where \(\lambda : \hat{E} \ni (0, y') : \eta_1 dx_1 + d\Phi(0, y') \ni E\) is the map germ which is introduced in Lemma 5.6. Since the map germ \(\lambda : \hat{E} \to E\) is a biholomorphic map germ, we have the following irreducible decompositions:
\[(5.39)\]
\[F^\ast_{|E} = \prod_{j=1}^r F_j^\circ\]
\[(5.40)\]
\[V^\circ := (C, 0) \times (\hat{E} \wedge F^{-1}(0), \hat{e}^\circ) = \bigcup_{j=1}^r V_j^\circ\]
where we set \(V_j^\circ := (C, 0) \times (F_j^{-1}(0), \hat{e}^\circ)\) and \(\hat{e}^\circ := (x^0; 0, 0)\).

**Definition 5.7.** Let \(f_i : (X_i, x_i) \to (Y, y)\) be germs of analytic coverings of \((Y, y)\) for \(i = 1, 2\). We call \(f_1\) and \(f_2\) are equivalent if there exists a biholomorphic map germ \(g : (X_1, x_1) \simeq (X_2, x_2)\) such that the following diagram commutes:

\[
\begin{array}{ccc}
(X_1, x_1) & \xrightarrow{g} & (X_2, x_2) \\
\downarrow f_1 & \sim & \downarrow f_2 \\
(Y, y) & \xrightarrow{f} & (Y, y)
\end{array}
\]

Note that if \(f_1\) and \(f_2\) are equivalent then they have the same critical locus \(\Sigma \subset Y\). On each connected component of \((Y - \Sigma, y)\), we have
\[\#f_1^{-1}(x) = \#f_2^{-1}(x)\].

Using this terminology, we have:

**Corollary 5.8.** We define the map germs \(\pi_j^\circ : (V_j^\circ, (0, \hat{e}^\circ)) \to (M, x^\circ)\) from \(\Psi_j^\circ := \Psi^\circ_{|V_j^\circ}\) like as the construction of \(\pi_j\) from \(\Psi_j\). Then the following diagrams
(5.41), commute for $1 \leq j \leq r$:

$$
(C, 0) \times (E, \theta^0) \overset{\pi_j}{\longrightarrow} (V_j, (0, \theta^0)) \overset{(M, x^0)}{\longrightarrow} \pi_j
$$

Hence the germs $\pi_j$ and $\pi_j^*$ of analytic coverings of $(M, x^0)$ are equivalent. In particular, if the conclusion of Main theorem 4.2 holds for $\pi_j^*$, then it also does for $\pi_j$.

**Remark 5.9.** The function $F^*(x; \xi^*, z^*) = F(x; \xi + d\Phi(x), z^* + \Phi(x))$ satisfies the following conditions [A.1]—[A.3] and [A.4]:

[A.1] $L_{\Phi^*}(\theta^0) = -\partial_{x_n}$

[A.2] $\text{ord}_0[F^*|_{E_{x_0}}] = \rho := \text{ord}_0[F|_{E_{x_0}}]$

[A.3] $\Phi^*(x) = 0$ is a good extension of $\phi^*(x') = 0$.

[A.4] $\text{ord}[F^*(x; 0, 0)] = q := \text{ord}[F^*(0, x'; 0, 0)]$

where $\theta^0 = \lambda^{-1}(\theta^0) = (x^0; 0, 0) \in E \cap F^{-1}(0)$.

By virtue of Corollary 5.8 and Remark 5.9, our main theorem for the Cauchy problem (1) is reduced to that for the Cauchy problem (5.38) under the assumptions [A.1]—[A.3] and [A.4].

Now we proceed the fourth step of our reduction, that is, we reduce $F^*$ to the form of (5.1). We denote by $F$ the function $F^*$ satisfying [A.1]—[A.3] and [A.4]. By virtue of [A.1] and of the Weierstrass's preparation theorem, we can find a holomorphic germ $G(x; \xi, \xi^*, z)$ and a unit $e(x; \xi, z)$ such that

$$
F(x; \xi, z) = [G(x; \xi, \xi^*, z) - e(x; \xi, z)] e(x; \xi, z)
$$

holds locally at $e^0 := (x^0; 0, 0)$. We set

$$
F^*(x; \xi, z) := G(x; \xi, \xi^*, z) - e(x; \xi, z).
$$

Note that $-F^*$ is a Weierstrass polynomial in $\xi_n$ of degree one.

**Notation 5.10.** We denote by $t \mapsto \Psi^*(t, e^0) = (X^*, \xi', Z^*); (t, e^0)$ the characteristic curve of $F^*$ passing through a point $e^0 := (0, y'; \xi, dx, 0)$ in the analytic set $E \cap F^{-1}(0)$, where $E := T_0^* M \times \{0\}$.

We consider the relation between the characteristic curves $\Psi^*$ of $F^*$ and $\Psi$ of $F$, and have the

**Lemma 5.11.** Let $g(t, e) \equiv \mathcal{O}_{C \to (0, e^0)}$ be the unique solution of
\[
\begin{align*}
\partial_t g &= \varepsilon(\Psi^{-}(g, e)) \\
g(0, e) &= 0.
\end{align*}
\]

Then the following 1), 2) hold:

1) The map germ \( g \times \text{id}_E : (C, 0) \times (E, e^o) \to (C, 0) \times (E, e^o) \) is a biholomorphic map germ.

2) We set

\[
\Psi_*(t, e) := \left[ \Psi^{-}(g \times \text{id}_E) \right](t, e) = \Psi^{-}(g(t, e), e).
\]

If \( e \in E \cap F^{-1}(0) \), then \( t \to \Psi_*(t, e) \) is the characteristic curve of \( F \) passing through \( e \) at \( t = 0 \), that is, we have \( \Psi_* = \Psi \) on \((C, 0) \times (E \cap F^{-1}(0), e^o)\).

**Proof.** Since \( g \) is a solution of (5.43) it follows that

\[
\varepsilon(x \mid , z) = 1.
\]

Hence the assertion 1) follows.

Since \( t \to \Psi^{-}(t, e) \) is a characteristic curve of \( F^{-} \), we have

\[
\partial_1 \{ \Psi^{-}(g(t, e), e) \} = (\partial_1 \Psi^{-})(g(t, e), e) \partial_1 g = \left[ (\partial_1 F^{-}; -E^{-} \partial_1 F - \partial_2 F^{-}, E^{-} \partial_2 F^{-}) \varepsilon(\Psi^{-}(g(t, e), e)) \right].
\]

The assumption \( e \in F^{-1}(0) = F^{-1}(0) \) yields \( \Psi^{-}(s, e) \in F^{-1}(0) \) for \( \forall s \), hence we have

\[
d_{x, 1} F(\Psi^{-}(g(t, e), e)) = (d_{x, 1} F^{-}) \varepsilon(\Psi^{-}(g(t, e), e))
\]

which implies

\[
\partial_1 \{ \Psi^{-}(g(t, e), e) \} = (\partial_1 F^{-}; -E^{-} \partial_1 F - \partial_2 F^{-}, E^{-} \partial_2 F^{-}) \Psi^{-}(g(t, e), e)).
\]

Since \( g(0, e) = 0 \), we also have \( \Psi_*(0, e) = \Psi^{-}(g(0, e), e) = \Psi^{-}(0, e) = e \). Thus the proof of Lemma 5.11 is complete. Q.E.D.

**Corollary 5.12.** We define the map germs \( \pi_j^\gamma : (V_j, (0, e^o)) \to (M, x^o) \) from \( \Psi_j^\gamma := \Psi^{-}|_{V_j} \) like as the construction of \( \pi_j \) from \( \Psi_j \). Then the following diagrams

\[
(5.45)_j
\]

commute for \( 1 \leq j \leq r \):

\[
\begin{align*}
(C, 0) \times (E, e^o) & \xrightarrow{g \times \text{id}_E} (V_j, (0, e^o)) \xrightarrow{\pi_j^\gamma} (M, x^o) \\
(C, 0) \times (E, e^o) & \xrightarrow{g \times \text{id}_E} (V_j, (0, e^o)) \xrightarrow{\pi_j} (M, x^o)
\end{align*}
\]

Hence the map germs \( \pi_j \) and \( \pi_j^\gamma \) are equivalent as germs of analytic coverings of \((M, x^o)\). Thus if the conclusion of Main Theorem 4.2 holds for \( \pi_j^\gamma \) then it also does for \( \pi_j \).
To complete the reduction of Main Theorem 4.2 to Theorem 5.1, it remains to verify the

**Lemma 5.13.** Let $F = F^\sim e$ be a germ satisfies the conditions [A.1]—[A.3] and [A.4] such that $-F^\sim$ is the Weierstrass polynomial of $F$ in $\xi_n$ of degree one:

$$F^\sim(x; \xi, z) = G(x; \xi_1, \xi', z) - \xi_n.$$  

Then $F^\sim$ satisfies the assumptions [B.1]—[B.4] of Theorem 5.1.

**Proof.** Since [B.2] and [B.4] are trivial by [A.2] and [A.4], we only have to show [B.1] and [B.3]. By virtue of [A.1], we have

$$-d\xi_n = d_x F(e^\varphi) = \varepsilon(e^\varphi) dx^\sim F^\sim(e^\varphi) = \varepsilon(e^\varphi) \left[ \sum_{j=1}^{n-1} \partial_{\xi_j} G(x^0; 0, 0, 0)d\xi_j - d\xi_n \right],$$

which shows

$$\partial_{\xi_j} G(x^0; 0, 0, 0) = 0 \quad \text{for} \quad 1 \leq j \leq n-1.$$  

Thus we have the condition [B.1].

Now we check the condition [B.3]. We set

$$f(x', \xi_1) := f^\sim(x', \xi_1) = F(0, x'; \xi_1 dx_1, 0)$$

and

$$f^\sim(x', \xi_1) := f^\sim(x', \xi_1) = F^\sim(0, x'; \xi_1 dx_1, 0) = G(0, x'; \xi_1 dx_1, 0).$$

Then we have $f = f^\sim e^\varphi$ ($e^\varphi$ is a unit) which yields

$$N(f) = N(f^\sim) + N(e^\varphi) = N(f^\sim)$$

since Newton polygons have the additivity $N(gh) = N(g) + N(h)$ (see § 11).

By (5.47), we know that $N(f^\sim)$ satisfies the coprimeness condition, and that each subset $M_\gamma = M_j$ of $\{1, 2, \ldots, m\}$ is a nice subset for $1 \leq j \leq r$.

Note that the condition [B.4] implies the inequality

$$\text{ord}[F^\sim(x; 0, 0)] = \text{ord}[f^\sim(x', 0)] = g > \kappa(m)^{-1}$$

since we assume $p \geq 2$.

It remains to show

$$N(f^\sim)$$

is stable in the direction of $L_{\varphi^\sim}(e^\varphi) = -\partial_{x_n}$.

To show (5.48) we utilize the following

**Claim 5.14.** Let

$$f(x', \xi_1) = \sum_{x'=0}^\infty c(x')^\xi_1, \quad f^\sim(x', \xi_1) = \sum_{x'=0}^\infty c^\sim(x')^\xi_1$$

be the Taylor expansions of the germs $f$ and $f^\sim$ given by (5.46). Then we have
\begin{equation}
\text{Loc}[c_v] = e^v \text{Loc}[c_v] \text{ for all } v \text{ satisfying } (\text{ord}[c_v], v) \in \text{Ver} N(f).
\end{equation}

**Proof.** Let $e^v(x', \xi_i) = \sum_{v=0}^{\infty} e_v(x') \xi_i^v$ be the Taylor expansion. Then we have $c_v = \sum_{v=0}^{\infty} c_v^v x_{v-1}$. Assume $(\text{ord}[c_v], v) \in \text{Ver} N(f) = \text{Ver} N(f^\nu)$ then it follows that 
\[\text{ord}[c_v^v] > \text{ord}[c_v] \text{ for all } v < \nu.\]
Hence we have
\[\text{ord}[c_v^v] = \text{ord}[c_v] + \text{ord}\left[\sum_{v=0}^{\infty} c_v^v x_{v-1}\right]\]
which shows (5.49) as desired. Q. E. D.

The assertion (5.48) immediately follows from Claim 5.14. Thus we complete the proof of Lemma 5.13.

Our reduction of the proof of Main Theorem 4.2 to that of Theorem 5.1 is also complete.

\section{Decomposition of Map Germs $\pi_j$}

We begin to prove Theorem 5.1. In this section we consider map germs $\pi_j: \text{graph}(\Psi_j) \to (M, 0)$ which are equivalent to $\pi_j: (V_j, e^0) \to (M, 0)$ for $1 \leq j \leq r$. Recall that we have reduced $x^0$ [or $e^0$ resp.] to the origin of $C^n[(x; \xi, z) = (0; 0, 0)]$ in \S 5.

An advantage of this consideration comes from the fact that $\pi_j$ is decomposable to a composition of three map germs as the form $\pi_j = \pi_j^3 \circ \pi_j^2 \circ \pi_j^1$. The aim of this section is to show that the first map germ $\pi_j^1$ is a biholomorphic map germ for $1 \leq j \leq r$. This fact yields that the analysis of $\pi_j$ can be reduced to those of $\pi_j^2$ and $\pi_j^3$ (see Theorems 6.10 and 6.11 at the end of this section).

Let us recall the diagrams
\begin{equation}
\begin{array}{ccc}
\langle V_j, (0, e^0) \rangle & \xrightarrow{\Psi_j} & \langle F^{-1}(0), e^0 \rangle \\
\downarrow \pi_j & & \downarrow \iota \\
(M, 0) & \xleftarrow{\pi_j^1 M} & (J^1 M, e^0)
\end{array}
\end{equation}
where $\langle V_j, (0, e^0) \rangle = \langle C, 0 \rangle \times (F^{-1}(0), e^0)$ is an irreducible component of $\langle V, (0, e^0) \rangle = \langle C, 0 \rangle \times (E \cap F^{-1}(0), e^0)$, for $1 \leq j \leq r$.

**Definition 6.1.** We define map germs $\pi_j$ for $1 \leq j \leq r$ by the following diagram:
\begin{equation}
\begin{array}{ccc}
\langle V_j, (0, e^0) \rangle & \xrightarrow{\sim} & \text{graph}(\Psi_j) \\
\downarrow \pi_j & & \downarrow \iota \\
(M, 0) & \xleftarrow{\text{projection}} & (J^1 M, e^0)
\end{array}
\end{equation}
Since \( \pi_j \) and \( \pi_{\tilde{j}} \) are equivalent as germs of analytic coverings of \((M, 0)\) in the sense of Definition 5.7, we may consider \( \pi_{\tilde{j}} \) instead of \( \pi_j \).

**Definition 6.2.** 1) We decompose the composite projection

\[
P: (C, 0) \times (E, e^0) \times (J^1M, e^0) \longrightarrow (J^1M, e^0) \longrightarrow (M, 0)
\]

as follows:

\[
(C, 0) \times (E, e^0) \times (J^1M, e^0) \xrightarrow{\pi_1} (C, 0) \times (M, 0) \times (C, 0) \xrightarrow{\pi_2} (M, 0)
\]

(6.3)

2) We define subsets \( V_{j1} \) and \( V_{j2} \) respectively by

\[
V_{j1} := (\pi_1 \circ \psi)(\text{graph}(\mathcal{W}_j)) \subset (C, 0) \times (M, 0) \times (C, 0)
\]

\[
V_{j2} := (\pi_2 \circ \tau)^{-1}(V_{j1}) \subset (M, 0) \times (C, 0)
\]

where \( \psi, \tau \) denote the following inclusion maps:

\[
\psi: \text{graph}(\mathcal{W}_j) \hookrightarrow (C, 0) \times (E, e^0) \times (J^1M, e^0)
\]

\[
\tau: \psi^{-1}(V_{j1}) \hookrightarrow (M, 0) \times (C, 0)
\]

3) According to the decomposition (6.3) of \( P \), we have the following decomposition (6.4) of the map germ \( \pi_{\tilde{j}} \):

\[
(C, 0) \times (E, e^0) \times (J^1M, e^0) \xrightarrow{\pi_1} (C, 0) \times (M, 0) \times (C, 0) \xrightarrow{\pi_2} (M, 0)
\]

(6.4)

**Notation 6.3.** We denote a characteristic curve of \( F \), which passes through a point \( \varepsilon = (0, y'; \eta_1 dx_1, 0) \in E = T^\mathcal{W} M \times \{0\} \), by

\[
\mathcal{W}(t, y', \eta_1) := (X(t, y', \eta_1); E(t, y', \eta_1), Z(t, y', \eta_1)).
\]

Of course, by the definition \( \mathcal{W}_j := \mathcal{W}_{[j]} \), we use the same expression \( (X; E, Z) \) to denote the value \( \mathcal{W}_j(t, y', \eta_1) \) for \( 1 \leq j \leq r \).

From now on, we show that the map germ \( \pi_{j1} : \text{graph}(\mathcal{W}_j) \to V_{j1} \) is a biholomorphic map germ for \( 1 \leq j \leq r \). First we prove
Claim 6.4. There exists a holomorphic map germ

\[(Y', H_1): (C, 0) \times (C^{n-1}, 0)_{x'} \times (C, 0)_{\xi_1} \longrightarrow (T^*M, (0, 0))_{y', \eta_1}\]

such that

\[(6.5) \quad \left[ \frac{X'}{\xi_1} \right](t, Y'(t, x', \xi_1), H_1(t, x', \xi_1)) = \left[ \frac{X'}{\xi_1} \right] \quad \text{and} \]

\[(6.6) \quad \left[ \frac{Y'}{H_1} \right](t, X'(t, y', \eta_1), \mathcal{E}_1(t, y', \eta_1)) = \left[ \frac{Y'}{\eta_1} \right].\]

Proof. We consider the following map germ

\[(6.7) \quad \left\{ \begin{array}{l}
Z: (C, 0) \times (T^*M, (0, 0))_{y', \eta_1} \longrightarrow (C, 0) \times (C^{n-1}, 0)_{x'} \times (C, 0)_{\xi_1} \\
Z(t, y', \eta_1) := (t, X'(t, y', \eta_1), \mathcal{E}_1(t, y', \eta_1)).
\end{array} \right.\]

Since \((X', \mathcal{E}_1)|_{t=0} = (y', \eta_1)\) holds, we have

\[\frac{\partial Z}{\partial (t, y', \eta_1)}(0, y', \eta_1) = \det \begin{bmatrix} 1 & 0 \\ I_n & 0 \end{bmatrix} = 0.\]

Hence, by the inverse mapping theorem, there exists the inverse map germ

\[X^{-1}(t, x', \xi_1) = (t, Y'(t, x', \xi_1), H_1(t, x', \xi_1))\]

which has the desired properties (6.5), (6.6). Q.E.D.

Notation 6.5. 1) Since \(X_1|_{t=0} = 0\), we can write \(X_1\) as the form

\[X_1(t, y', \eta_1) = tX_{\gamma}(t, y', \eta_1).\]

2) We set \(f_j(y', \eta_1) := (\gamma_{F_j}(y', \eta_1) \) where \(\gamma_{F_j}(y', \eta_1) = (0, y'; \eta_1 d x_1, 0)\).

3) Using these \(X_\gamma, f_j\) and the germ \((Y', H_1)\) mentioned in Claim 6.4, we define germ \(A\) and \(B_j\) as follows:

\[(6.8) \quad \left\{ \begin{array}{l}
A(t, x', \xi_1) := X_{\gamma}(t, Y'(t, x', \xi_1), H_1(t, x', \xi_1)) \\
B_j(t, x', \xi_1) := f_j(Y'(t, x', \xi_1), H_1(t, x', \xi_1)) \quad \text{for} \ 1 \leq j \leq r.
\end{array} \right.\]

Lemma 6.6. Set a germ \(H\) of a hypersurface of \(C_{t \times M_x \times C_{\xi_1}}\) by

\[H := \{(t, x, \xi_1) \in \mathcal{P} \mid x_1 = tA(t, x', \xi_1)\}.\]

Then the projection \(P_1\) induces a biholomorphic map germ \(h^-: \text{graph} (\mathcal{P}^-) \longrightarrow H\) as follows:

\[(6.9) \quad \begin{array}{ccc}
\text{graph} (\mathcal{P}^-) & \xrightarrow{h^-} & (C, 0)_{t \times (E, \mathcal{P}^y_{\eta_1})} \times (J^1M, \mathcal{P}^y_{x, \xi_1}) \\
\downarrow P_1 & & \downarrow \text{graph} (\mathcal{P}^-) \\
H & \xrightarrow{h^+} & (C, 0)_{t \times (M, 0)_x \times (C, 0)_{\xi_1}}
\end{array}\]
Proof. We first observe that \( P_1 \) in fact induces the map \( h^- \). Assume that 
\[
(t; y', \eta_1; x, \xi, z) = (t; y', \eta_1; (X, \Xi, Z)(t, y', \eta_1)) \in \text{graph}(\Psi^-) .
\]
It suffices to verify 
\[
X_i(t, y', \eta_1) = tA(t, X'(t, y', \eta_1), \Xi(t, y', \eta_1)).
\]
(6.10)

By the definition (6.8) of \( A \), the right hand side of (6.10) is equal to 
\[
X_i(t, Y'(t, X'(t, y', \eta_1), \Xi(t, y', \eta_1)), H_i(t, X'(t, y', \eta_1), \Xi(t, y', \eta_1))).
\]
Thus the identity (6.6) yields (6.10).

Note that (6.10) shows that the following diagram commutes:
\[
\begin{array}{ccc}
(C, 0) \times (T^s M, (0, 0)) & \cong & (C, 0) \times (E, e^s) \\
\chi & \sim & \text{graph}(\Psi^-) \\
\downarrow & & \downarrow h^- \\
(C, 0) \times (C^{n-1}, 0) \times (C, 0) & \cong & H
\end{array}
\]

where \( \chi \) is the map germ defined by (6.7). Since we know \( \chi \) is a biholomorphic germ, \( h^- \) also is. The proof of Lemma 6.6 is complete.

Q. E. D.

Proposition 6.7. 1) The map germ \( \pi_{j_1} : \text{graph}(\Psi_j) \to V_j \) is a biholomorphic map germ for \( 1 \leq j \leq r \).

2) The germ \( (V_j, (0, 0, 0)) \) is defined by the following two equations as a germ of analytic subset in \( (C, 0)^2 \times (M, 0) \times (C, 0)^2 \):

(6.11)

\[
\begin{cases}
  tA(t, x', \xi_1) - x_1 = 0 \\
  B_j(t, x', \xi_1) = 0 .
\end{cases}
\]

Proof. Since \( \Psi_j = \Psi^- \mid V_j \) implies that the map germ \( \pi_{j_1} \) is the restriction of the biholomorphic germ \( h^- : \text{graph}(\Psi^-) \to H \) on \( \text{graph}(\Psi_j) \), the assertion 1) follows.

By virtue of the identity (6.6), we have 
(6.12) 
\[
B_j(\chi(t, y', \eta_1)) = f_j(y', \eta_1) .
\]

Indeed, by the definition of \( B_j \), the left hand side of (6.12) is equal to 
\[
f_j(Y'(\chi(t, y', \eta_1)), H_j(\chi(t, y', \eta_1))).
\]
Then the identity (6.6) which is equivalent to 
\[
(Y'(\chi(t, y', \eta_1)), H_j(\chi(t, y', \eta_1))) = (y', \eta_1)
\]
yields (6.12) as desired.

From (6.12), we have the following commutative diagram:
The desired equations (6.11) immediately follows from this diagram. The proof of Proposition 6.7 is complete. Q.E.D.

**Remark 6.8.** Let us define ideals \( \mathcal{J}_j \) of the ring \( \mathcal{O}_{C \times M} \times C, (0, 0, 0) \) by

\[
\mathcal{J}_j := (\mathcal{A}(t, x', \xi_1) - x_1, B_j(t, x', \xi_1)).
\]

Then these ideals are prime for \( 1 \leq j \leq r \).

**Proof.** Assume \( g_1 g_2 \in \mathcal{J}_j \), \( g_1 \notin \mathcal{J}_j \). We set \( g_i(t, x', \xi_1) := g_i|_H \) \( (i = 1, 2) \). Then we have \( g_1 g_2 \in (B_j) \), \( g_1 \notin (B_j) \). We claim

\[
(6.14) \quad (B_j) \text{ is a prime ideal of the ring } \mathcal{O}_{C \times C^n \times C, (0, 0, 0)}.
\]

If (6.14) is established then we have \( g_1 \in \mathcal{J}_j \) which yields the desired fact \( g_1 \in \mathcal{J}_j \). Note that, by virtue of (6.12), it suffices for (6.14) to show that

\[
(6.15) \quad (f_j(y, \eta_1)) \text{ is a prime ideal of the ring } \mathcal{O}_{T \times T \times T, (0, 0, 0)}.
\]

Recall that \( f_j(y, \eta_1) \) is irreducible in \( \mathcal{O}_{T \times T, (0, 0)} \). Hence it suffices for (6.15) to show the following

**Claim 6.9.** Let \( f \in \mathcal{O}_{T \times T, (0, 0)} \) be an irreducible germ, and let \( \rho: C \times T \times T \rightarrow T \times T \) be the projection. Then the pull-back germ \( \rho^* f \) is irreducible in the ring \( \mathcal{O}_{C \times T \times T, (0, 0, 0)} \).

To show Claim 6.9, we note the following simple fact for \( f \in \mathcal{O}_{T \times T, (0, 0)} \):

\[
(6.16) \quad a(t, e) = \sum_{\nu} a_\nu(e) t^\nu \in (\rho^* f) \iff a_\nu \in (f) \quad \text{for all } \nu.
\]

Indeed, by the definition, \( a \in (\rho^* f) \) means that there exists a germ \( b(t, e) = \sum_{\nu} b_\nu(e) t^\nu \) such that \( a(t, e) = (\rho^* f)(t, e) b(t, e) = f(e) b(t, e) \). This is clearly equivalent to \( a_\nu(t, e) = f(e) b_\nu(e) \) for all \( \nu \). Hence we get (6.16).

We continue the proof of Claim 6.9. For \( a_i(t, e) = \sum_{\nu} a_i,\nu(t, e) t^\nu \) \((i = 1, 2)\) we
assume $a_1z \in (\rho^*f)$, $a_i \notin (\rho^*f)$. By (6.16), it follows that

\[ \sum_{\lambda=0}^{\gamma-1} a_{1,\gamma-\lambda} z_{\lambda} \in (f) \quad \text{for all } \gamma = 0, 1, 2, \ldots \]  

and that there exists $\mu$ such that

\[ a_{1,\mu} \notin (f) . \]

We choose $\mu$ as the minimum value of such $\mu$. We want to show

\[ a_{\nu} \in (f) \quad \text{for all } \nu. \]  

We prove (6.19) by induction on $\nu$ as follows:

1) The case $\nu=0$. Taking $\gamma$ in (6.17) as $\mu$, we have

\[ a_{1,\mu} a_{\nu,0} + \sum_{\lambda=1}^{\mu} a_{1,\mu-\lambda} z_{\lambda} \in (f). \]

Since $a_{1,\mu-\lambda} \in (f)$ for $\lambda \geq 1$, we have $a_{1,\mu} a_{\nu,0} \in (f)$. Then, by (6.18), it follows that $a_{\nu,0} \in (f)$ since the ideal $(f)$ is prime.

2) The case $\nu \geq 1$. Taking $\gamma$ in (6.17) as $\nu+\mu$, we have

\[ \sum_{\lambda=0}^{\nu-1} a_{1,\nu+\mu-\lambda} z_{\lambda} + a_{1,\mu} a_{\nu,\nu} + \sum_{\lambda=\nu+1}^{\nu+\mu} a_{1,\nu+\mu-\lambda} z_{\lambda} \in (f). \]

Since $a_{\nu,\nu} \in (f)$ for $\lambda \leq \nu-1$ by the inductive assumption, and since $a_{1,\nu+\mu-\lambda} \in (f)$ for $\lambda \geq \nu+1$, we have $a_{1,\mu} a_{\nu,\nu} \in (f)$. Thus (6.18) implies $a_{\nu,\nu} \in (f)$ as desired. Hence we get (6.19).

Since (6.16) and (6.19) yield $a_z \in (\rho^*f)$, we get Claim 6.9. Hence the proof of Remark 6.8 is complete. Q. E. D.

We conclude this section to show that, if we establish the following Theorems 6.10 and 6.11, then Theorem 5.1 follows:

We consider the following diagram under the assumptions [B.1]-[B.4] of Theorem 5.1:

\[ V_{j_1} = \{ tA - x_i = B_i = 0 \} \rightarrow (C, 0) \times (M, 0) \times (C, 0) \rightarrow P_2 \]

\[ (6.20) \]

For the diagram (6.20), we state the following two theorems which imply Theorem 5.1:

**Theorem 6.10.** Under the assumptions [B.1]-[B.4], there exists an irreducible
Weierstrass polynomial \( w_2(x, \xi_1) \subseteq O_{M, 0}[\xi_1] \) of the degree
\[ v_j = \sum_{\mu \in M_j} v(\mu) \]
such that the following equality holds locally at \((x, \xi_1) = (0, 0)\):
\[ (V_{j2}, (0, 0)) = (w_j^{-1}(0), (0, 0)). \]

**Theorem 6.11.** The map germ \( \pi_{j2}: V_{j1} \to V_{j2} \) is a germ of one-sheeted analytic covering of \( V_{j2} = (V_{j3}, (0, 0)) \).

We must show that Theorems 6.10 and 6.11 imply Theorem 5.1. Note that Theorem 6.10 yields that the map germ \( \pi_{j2}: V_{j2} \to (M, 0) \) is a germ of a \( v_j \)-sheeted analytic covering of \((M, 0)\). Hence, by Theorem 6.11 and the first assertion of Proposition 6.7, we conclude that

\[ (6.21) \quad \pi_{j2} = \pi_{j1} \circ \pi_{j2}^* \text{ is a germ of a } v_j \text{-sheeted analytic covering of } (M, 0) \]

which is the assertion 1) of Theorem 5.1.

To show the assertion 2) of Theorem 5.1, let \( u_j \) be the multi-valued germ defined by the diagram (4.8). Then, by our construction of the maps \( \pi_j \), it follows

\[ (6.22) \quad \partial_{x_1} u_j(X(t, y', \eta_1)) - \mathcal{E}_1(t, y', \eta_1) = 0 \]

for \( \forall (t; 0, y'; \eta_1; \partial_x x_1, 0) \in (V_{j1}, (0, \epsilon^0)). \)

Then, the irreducibility of the defining germ \( w_j(x, \xi_1) \) of \( V_{j2} \) yields that the germ \( \xi_1 = \mathcal{E}_1(t, y', \eta_1) \) is exactly \( v_j \)-valued as a germ of a function in \( x \). On the other hand, (6.21) shows that the germ \( u_j \) is at most \( v_j \)-valued around \( x = 0 \). Hence the relation (6.22) shows that the germ \( u_j \) itself is exactly \( v_j \)-valued.

Thus Theorem 5.1 follows if we establish Theorems 6.10 and 6.11.

We shall prove Theorem 6.10 in §9, and Theorem 6.11 in §10. Before to prove these theorems we need some preparation which is done in §§7 and 8.

**§ 7. Newton Polygons of \( A(t, 0, \xi_1) \) and \( B_j(t, 0, \xi_1) \)**

In this section we decide the “principal part” of the Newton polygons of the restrictions \( A|_{x' = 0} \) and \( B_j|_{x' = 0} \) of \( A(t, x', \xi_1) \) and \( B_j(t, x', \xi_1) \) for \( 1 \leq j \leq r \), where \( A \) and \( B_j \) are defined by (6.8) in Notation 6.5.

**Definition 7.1.** 1) Let \( N_i \) \((i = 1, 2)\) be Newton polygons. We say \( N_i \) is properly contained in \( N_i \) (we denote this by \( N_i \subseteq N_i \)) if

\[ (7.1) \quad N_i \subseteq N_i \quad \text{and} \quad N_i \cap \partial^o N_i = \emptyset \]

where \( \partial^o N \) denotes the strict boundary of a Newton polygon \( N \), which is defined
in 2) in Definition 2.3.

2) Let \((\Sigma, \sigma)\) be a germ of a complex manifold, and let \(f, g \in \mathcal{O}_{\Sigma \times C, (\sigma, 0)}\) be germs of functions. We say \(N(f)\) and \(N(g)\) have a same principal part if

\[
N(f - g) \equiv N(f) = N(g).
\]

3) Let \(f \in \mathcal{O}_{\Sigma \times C, (\sigma, 0)}\) be a germ and let \(f(x, y) = \sum_{\nu=0}^{m} c_\nu(x)y^\nu\) be the Taylor expansion of \(f\) along \(y = 0\) \((c_\nu \in \mathcal{O}_{\Sigma, \sigma} \text{ for } \nu = 0, 1, \ldots)\). We define the characteristic polynomial function \(\text{ch}(f)\) by

\[
\text{ch}(f)(X, y) := \sum_{(\text{ord}(c_\nu), \nu) \in \text{ann}(f_\nu)} \text{Loc}[c_\nu](X)y^\nu \quad ((X, y) \in T_\sigma \Sigma \times C)
\]

where \(\text{Loc}[c_\nu] : T_\sigma \Sigma \times C\) is the localization of \(c_\nu\) at \(\sigma\) (see Definition 2.6).

Note that it clearly follows that \(N(f)\) and \(N(\text{ch}(f))\) have a same principal part. Moreover \(N(f)\) and \(N(g)\) have a same principal part if and only if

\[
\text{ch}(f) = \text{ch}(g).
\]

For this reason, we call the characteristic polynomial function \(\text{ch}(f)\) by the name of the principal part of \(N(f)\).

Let us recall the irreducible decomposition of \(f\) locally at \((0, 0) \in T_\Sigma^* M:\)

\[
f(y', \eta_1) := F(0, y'; \eta_1 d x_1, 0) = \prod_{j=1}^{r} f_j(y', \eta_1) \quad (f_j \in \mathcal{O}_{T_\Sigma^* M, (0, 0)}).
\]

The aim of this section is to show the following

**Proposition 7.2.** The principal parts of the Newton polygons \(N(A(t, 0, \xi_1))\) and \(N(B_j(t, 0, \xi_1))\) \((1 \leq j \leq r)\), satisfy the following (7.4) and (7.5):

\[
\text{ch}(B_j(t, 0, \xi_1)) = \text{ch}(f_j)(-t L_F(e^\theta), \xi_1) \quad \text{for } 1 \leq j \leq r.
\]

\[
N[\text{ch}(A(t, 0, \xi_1)) - t^{-1} \frac{1}{\theta} \partial_\xi_1 \text{ch}(f)(-\theta L_F(e^\theta), \xi_1)d\theta] + N(\xi_1) \equiv N(f).
\]

**Remark 7.3.** By the assumptions [B.1] and [B.3] of Theorem 5.1, we have

\[
N(f) = N(f(0, \ldots, 0, y_n, \eta_1)).
\]

\[
N(f_j) = N(f_j(0, \ldots, 0, y_n, \eta_1)).
\]

**Proof.** The first equality (7.6) is nothing but the stability of \(N(f)\) in the direction of \(L_F(e^\theta) = -\partial_{x_n}\). Since \(\Phi(x') = 0\) is a good extension of the Cauchy data \(\phi(x') = 0\), (7.6) follows.

Note that there is a trivial inclusion

\[
N(f_j(y', \eta_1)) \equiv N(f_j(0, \ldots, 0, y_n, \eta_1)).
\]

On the other hand, the additivity property of Newton polygons yields
Hence we have, by (7.6) and (7.9), the following equality:

\[(7.10) \quad \sum_{j=1}^{r} N(f_j) = N(f) = N(f(0, \cdots, 0, x_n, \eta_1)) = \sum_{j=1}^{r} N(f_j(0, \cdots, 0, x_n, \eta_1)).\]

By (7.8) and (7.10), we conclude (7.7) as desired. \(\text{Q.E.D.}\)

Let us recall the map germ

\[\Psi = (X; E, Z): (C, 0) \times (E, e^0) \longrightarrow (J^1 M, e^0)\]

which is induced by the following family of characteristic curves of \(F:\)

\[\{ t \rightarrow \Psi(t, e); \Psi(0, e) = e = (0, y'; \eta, d \xi, 0) \in (E, e^0) \}.\]

We expand \(\Psi\) with respect to \(t\) as the following form:

\[(7.11) \quad X(t, y', \eta_1) = (0, y') + \sum_{i=1}^{m} (i!)^{-1} t^i \partial X(0, y', \eta_1).
\]

\[\quad E(t, y', \eta_1) = (\eta_1, 0) + \sum_{i=1}^{m} (i!)^{-1} t^i \partial E(0, y', \eta_1).
\]

\[\quad Z(t, y', \eta_1) = \sum_{i=1}^{m} (i!)^{-1} t^i \partial i Z(0, y', \eta_1).
\]

Let \(q := \text{ord}[F(0, x'; 0, 0)]\), which is equal to the approximation order \(\text{ord}[F(x, 0, 0)]\) of the good extension \(\Phi \equiv 0\), by virtue of the assumption [B.4].

**Notation 7.4.** For germs \(g, h \in \mathcal{O}_{S \times C(0, e)}\) and for an ideal \(\mathcal{I}\) of the ring \(\mathcal{O}_{S \times C(0, e)}\), we denote by \(N(\mathcal{I}g) \subseteq N(h)\) [or \(N(\mathcal{I}g) \equiv N(h)\) resp.] if

\[N(\mathcal{I}g) \subseteq N(h)\] [\(N(\mathcal{I}g) \equiv N(h)\)] for any \(a \in \mathcal{I}\).

**Lemma 7.5.** Let \(f(y', \eta_1) = F(0, y'; \eta_1 d \xi, 0)\), and let \((y') \in \mathcal{O}_{S \times C(0, e)}\) be the defining ideal of \(\{ 0 \} \times C \subseteq S \times C\). Assume the conditions [B.1]–[B.4] of Theorem 5.1. Then the following 1–6) hold for all \(i \geq 1:\)

1) \[N[(y')^{i-1}(\eta_1)\{ \partial X(0, y', \eta_1) = (y')^{i-1}(\eta_1) \} \subseteq N(f).\]
2) \[\mathcal{N}[(y')^{i-1}(\partial E(0, y', \eta_1))] \subseteq N(f) \quad \text{for} \ 2 \leq j \leq n.\]
3) \[\mathcal{N}[(y')^{i-1}(\partial Z(0, y', \eta_1))] \subseteq N(f).\]
4) \[(y')^{i-1}(\partial X(0, y', \eta_1)) \subseteq (y', \eta_1) \quad \text{for} \ 2 \leq j \leq n-1.\]
5) \[(y')^{i-1}(\partial E(0, y', \eta_1)) \subseteq (y')^{i-1}+(\eta_1).\]
6) \[X_n(t, y', \eta_1) = y_n - t.\]

The proof of Lemma 7.5 will be given in § 12.
In this section we shall prove Proposition 7.2, under the assumption that Lemma 7.5 is true.

**Lemma 7.6.** Let \((Y', H_i):(C, 0) \times (C^{n-1}, 0) \rightarrow (T^*M, (0, 0))\) be the map germ determined by Claim 6.4. Then \((Y', H_i)\) has the following expression:

\[
\begin{align*}
    Y'(t, x', \xi_1) &= x' + tY''(t, x', \xi_1) = \left[\begin{array}{c} x'' \\ x_n \end{array}\right] + t\left[\begin{array}{c} Y'' \\ 1 \end{array}\right]
    \\
    H_i(t, x', \xi_1) &= \xi_1 + tH_1(t, x', \xi_1)
\end{align*}
\]

such that

\[
\begin{align*}
    Y''(t, 0, \xi_1) &= (t, \xi_1) \\
    H_1(t, 0, 0) &= (t)^{n-1}.
\end{align*}
\]

**Proof.** By the identity (6.5) and the expansion (7.11), it follows that

\[
\begin{align*}
    \left[\begin{array}{c} x' \\ \xi_1 \end{array}\right] &= \left[\begin{array}{c} X' \\ \xi_1 \end{array}\right] + t\sum_{i=1}^{\infty} (i!)^{-1} \partial_{Y_i} \left[\begin{array}{c} X' \\ \xi_1 \end{array}\right] \\
    &= \left[\begin{array}{c} Y' \\ H_1 \end{array}\right] + t\sum_{i=1}^{\infty} (i!)^{-1} \partial_{Y_i} \left[\begin{array}{c} Y' \\ H_1 \end{array}\right].
\end{align*}
\]

Setting \(t = 0\) in (7.14), we have \((Y', H_i)|_{t=0} = (x', \xi_1)\). Hence we get

\[
\begin{align*}
    \left[\begin{array}{c} x' \\ \xi_1 \end{array}\right] &= \left[\begin{array}{c} x' \\ \xi_1 \end{array}\right] + t\sum_{i=1}^{\infty} (i!)^{-1} \partial_{Y_i} \left[\begin{array}{c} x' \\ \xi_1 \end{array}\right].
\end{align*}
\]

Note that the assertion 6) in Lemma 7.5 implies

\[
\begin{align*}
    x_n &= y_n(t, x', \xi_1) - t = x_n + tY''(t, x', \xi_1) - t
\end{align*}
\]

which shows

\[
Y_n(t, x', \xi_1) \equiv 1.
\]

Substituting (7.15) into (7.14), with using (7.16) and

\[
\partial_i(X', \xi_1)(0, y', \eta_1) = (\partial_x F, -\eta, \partial_{y} F - \partial_{x} F)(0, y'; \eta_1 d x_1, 0)
\]

we have the following identities:

\[
\begin{align*}
    \left[\begin{array}{c} x'' \\ x_n \end{array}\right] &= \left[\begin{array}{c} x'' \\ x_n \end{array}\right] + t\left[\begin{array}{c} Y'' \\ 1 \end{array}\right] + t\left[\begin{array}{c} \partial_{x} F \\ -1 \end{array}\right] \\
    &+ \sum_{i=1}^{\infty} (i!)^{-1} \partial_{Y_i} \left[\begin{array}{c} x'' \\ x_n \end{array}\right] \\
    \xi_1 &= \xi_1 + tH_1 + t\left[-\eta \partial_{x} F - \partial_{x} F\right](0, Y'; H_i d x_1, 0)
\end{align*}
\]

Substituting (7.17) into (7.18), we get

\[
\begin{align*}
    x_n &= x_n + tY''(t, x', \xi_1) - t
\end{align*}
\]

Substituting (7.15) into (7.14), with using (7.16) and

\[
\partial_i(X', \xi_1)(0, y', \eta_1) = (\partial_x F, -\eta, \partial_{y} F - \partial_{x} F)(0, y'; \eta_1 d x_1, 0)
\]

we have the following identities:

\[
\begin{align*}
    \left[\begin{array}{c} x'' \\ x_n \end{array}\right] &= \left[\begin{array}{c} x'' \\ x_n \end{array}\right] + t\left[\begin{array}{c} Y'' \\ 1 \end{array}\right] + t\left[\begin{array}{c} \partial_{x} F \\ -1 \end{array}\right] \\
    &+ \sum_{i=1}^{\infty} (i!)^{-1} \partial_{Y_i} \left[\begin{array}{c} x'' \\ x_n \end{array}\right] \\
    \xi_1 &= \xi_1 + tH_1 + t\left[-\eta \partial_{x} F - \partial_{x} F\right](0, Y'; H_i d x_1, 0)
\end{align*}
\]
Setting $x'=0$ in (7.17) and dividing it by $t$, we get

$$-Y^\sim(t, 0, \xi_1) = \partial_\xi F(0, tY^\sim(t, 0, \xi_1), t; \{\xi_1 + tH_\gamma(t, 0, \xi_1)\} dx_1, 0)$$

$$+ \sum_{i=2}^{\infty} (i!)^{-1} t^{i-1} \partial_\xi X^\sim(0, tY^\sim(t, 0, \xi_1), t, \xi_1 + tH_\gamma(t, 0, \xi_1)).$$

Thus, by $\partial_\xi F \in (x, \xi, z)$ which is a consequence of [B.1], the first assertion of (7.13) follows.

Setting $(x', \xi_1)=(0, 0)$ in (7.18) and dividing it by $t$, it follows

$$H_\gamma(t, 0, 0) = -tH_\gamma(t, 0, 0) \partial_\xi F(0, tY^\sim(t, 0, 0); tH_\gamma(t, 0, 0) dx_1, 0)$$

$$- \partial_\xi F(0, tY^\sim(t, 0, 0); tH_\gamma(t, 0, 0) dx_1, 0)$$

$$+ \sum_{i=2}^{\infty} (i!)^{-1} t^{i-1} \partial_\xi X^\sim(0, tY^\sim(t, 0, 0), tH_\gamma(t, 0, 0)).$$

Then the assumption [B.4], that is, $F(x; 0, 0) \in (x)^p$ yields

$$\partial_\xi F(0, tY^\sim(t, 0, 0); 0, 0) \in (t)^{p-1}. \tag{7.20}$$

By the assertion 5) in Lemma 7.5, we also have

$$t^{i-1} \partial_\xi X^\sim(0, tY^\sim(t, 0, 0), 0) \in (t)^{p-1}. \tag{7.21}$$

From (7.19)-(7.21), we get

$$-H_\gamma(t, 0, 0) \in (t)^{p-1} + (tH_\gamma(t, 0, 0))$$

that is, there exist germs $a(t), b(t)$ such that

$$-H_\gamma(t, 0, 0) = t^{p-1} a(t) + tH_\gamma(t, 0, 0) b(t).$$

Since $1+tb(t)$ is an invertible germ, we get the second assertion of (7.13). The proof of Lemma 7.6 is complete. Q. E. D.

**Corollary 7.7.** There exist map germs $y^\sim(t, \xi_1), \eta_\gamma(t)$ and $\sigma(t, \xi_1)$ such that

$$\begin{cases}
Y'(t, 0, \xi_1) = t(y^\sim(t, \xi_1), 1) \\
H_\gamma(t, 0, \xi_1) = \xi_1 \sigma(t, \xi_1) + \eta_\gamma(t)
\end{cases}$$

with the following properties:

$$\begin{cases}
y^\sim(t, \xi_1) \in (t, \xi_1). \\
\eta_\gamma(t) \in (t)^q. \\
\sigma(0, \xi_1) \equiv 1.
\end{cases} \tag{7.22}
$$

**Proof.** We set

$$y^\sim(t, \xi_1) := Y^\sim(t, 0, \xi_1), \eta_\gamma(t) := tH_\gamma(t, 0, 0).$$

Then, Lemma 7.6 yields that $y^\sim \in (t, \xi_1)$, $\text{ord}[\eta_\gamma] \geq q$ and $Y'|_{x'=0} = t(y^\sim, 1)$. On...
the other hand, it follows
\[
H(t, 0, \xi) = \xi + tH(t, 0, \xi) \\
= \xi + t\{H(t, 0, 0) + \xi^2 \sigma(t, \xi)\} \\
= \xi(1 + t\sigma(t, \xi)) + \eta(t).
\]
Thus it suffices to set \(\sigma(t, \xi) := 1 + t\sigma(t, \xi)\).

Q.E.D.

Now we prove the assertion \((7.4)\) of Proposition 7.2.

For \(1 \leq j \leq r\), Corollary 7.7 yields
\[
B(t, 0, \xi) := f_j(Y(t, 0, \xi), H(t, 0, \xi)) \\
= f_j(t(y^\sim(t, \xi), 1), \xi, \sigma(t, \xi) + \eta(t)).
\]

Let \(f_j(y', \eta_1) = \sum c_{j,\eta_1} y^{\eta_1}\) be the Taylor expansion of \(f_j\). Then we have
\[
(7.23) \quad B(t, 0, \xi) = \sum c_{j,\eta_1} (t(y^\sim(t, \xi), 1))\eta_1 \sigma(t, \xi) + \eta(t).^r
\]

Claim 7.8. Let us put \(q_j := \text{ord}[f_j(y', 0)] = \text{ord}[c_{j,\eta_1}]\). Then we have
\[
B(t, 0, 0) = \text{Loc}[c_{j,\eta_1}](\partial_\eta^r y^{s_j}(1 + O(t)).
\]

Proof. Set \(\xi = 0\) in \((7.23)\), then we have
\[
(7.24) \quad B(t, 0, 0) = \sum c_{j,\eta_1} (t(y^\sim(t, 0), 1)\eta_1 \sigma(t, \xi) + \eta(t)).
\]

since \(\text{ord}[\eta_1] \geq q\) by \((7.22)\). Since \(\text{ord}[c_{j,\eta_1}] = q_j\) and \(\text{ord}[y^\sim(t, 0)] \geq 1\), it follows that the second term in the right hand side of \((7.24)\) has a vanishing order at least
\[
\min\{q_j - |\alpha| + 2|\alpha|; |\alpha| \geq 1\} > q_j.
\]

On the third term in the right hand side of \((7.24)\), we note:
\[
(7.25) \quad q_j < q \quad \text{if} \quad c_{j,\eta_1}(y') \quad \text{is a unit in} \quad \mathcal{O}_{S,0}.
\]

Indeed, if \(c_{j,\eta_1}\) is a unit then \(p_j := \text{ord}[f_j(0, \eta_1)] = 1\). Since we assume \(p = \sum_{j=1}^r p_j \geq 2\), we get \(r \geq 2\). Hence \((7.25)\) follows. By \((7.25)\) we have
\[
\text{ord}[c_{j,\eta_1}(t(y^\sim(t, 0), 1)\eta_1(t))] \geq \text{ord}[c_{j,\eta_1}] + q > q_j.
\]

Hence we conclude
which shows Claim 7.8. Q. E. D.

From (7.23), we can find a germ $g_j(t, \xi_i)$ such that

\[ B_j(t, 0, 0) = c_0(0, \cdots, 0, t) \in (t)^{q+1} \]

Claim 7.9. Let us put

\[ B_j(t, \xi_i) := \sum_{\nu=0}^{m_j} c_{\nu}(t(y^{\nu}(t, \xi_i), 1))\{\xi_i, \sigma(t, \xi_i)\}^\nu + \eta_i(t) g_j(t, \xi_i). \]

Then it follows that

\[ N(B_j) \subseteq N(f_j) \quad \text{for } 1 \leq j \leq r. \]

Proof. We write $B_j(t, \xi_i)$ as $B_j(t, \xi_i) = \sum_{\mu \geq 0} \sum_{\nu \in S} c_{\mu, \nu} (t)^{\mu} (\xi_i)^{\nu}$. Then it suffices for Claim 7.9 to derive

\[ (\text{ord} [c_{\mu, \nu}], \mu) \in N(f_j) \quad \text{for all } \mu. \]

From the definition (7.27) of $B_j$, the coefficients $c_{\mu, \nu}(t)$ are given by

\[ c_{\mu, \nu}(t) = \sum_{\nu=0}^{\mu} ((\mu - \nu)!)^{-1} \partial_{\xi_1}^{-\nu} \{c_{\nu}(t(y^{\nu}(t, \xi_i), 1))\sigma(t, \xi_i)^\nu \mid \xi_1 = 0 \}
\]

which yields

\[ \text{ord} [c_{\mu, \nu}] \geq \min_{0 \leq \nu \leq \mu} \text{ord} [c_{\nu}]. \]

Hence the fact

\[ (\text{ord} [c_{\mu, \nu}], \mu) = (\text{ord} [c_{\mu, \nu}], \nu) + (0, \mu - \nu) \in N(f_j) \quad \text{for } 0 \leq \nu \leq \mu \]

implies the desired (7.28). Thus Claim 7.9 follows. Q. E. D.

Claim 7.10. For $1 \leq j \leq r$, it follows that

\[ \text{ch}(B_j(t, \xi_i)) = \text{ch}(f_j)(-t L_F(e^0), \xi_i). \]

Proof. Let us denote the set $\text{Ver} N(f_j)$ of vertices of $N(f_j)$ as

\[ \text{Ver} N(f_j) = \{(a_j(\lambda), p_j - b_j(\lambda)); 0 \leq \lambda \leq m_j\} \]

where the sequences $\{a_j(\lambda)\}$, $\{b_j(\lambda)\}$ are arranged as monotonely increasing in $\lambda$.

Since the Newton polygon $N(f_j)$ satisfies the coprimeness condition, it follows that

\[ \text{ch}(f_j)(-t L_F(e^0), \xi_i) = \text{ch}(f_j)(0, \cdots, 0, t, \xi_i) \]

\[ = \sum_{\ell = 0}^{\frac{m_j}{2}} \text{Loc}[c_{\ell, \nu}^{(2)}(t)^{\nu}(\xi_i)^{\nu}] \partial_{\nu}^{(2)} \hat{a}_{\nu}^{(2)}(\xi_i)^{b_j(2)}. \]

Let $e_{ij} \in C$ be the coefficient of $t^{e_j(2)}(\xi_i)^{b_j(2)}$ in $B_j(t, \xi_i)$. Since
by Claim 7.9 and Remark 7.3, the expression (7.31) yields that it suffices for
Claim 7.10 to verify
\[(7.32)\quad e_{\lambda k} = \text{Loc}[c_{p, j - b_j, \lambda} \partial_{\nu_\lambda}^j] \quad \text{for} \quad 0 \leq \lambda \leq m_j.\]

We fix \(\lambda\). It is clear from (7.27) that \(e_{\lambda k}\) depends only on the terms
\[c_{p, j - b_j, \lambda} t(t y^{\alpha} - (t, \xi_\lambda), 1) \{\xi_\lambda \sigma(t, \xi_\lambda)\}^\nu \quad \text{for} \quad 0 \leq \nu \leq p_j - b_j(\lambda).\]

Since \((a_\lambda(\lambda), p_j - b_j(\lambda)) \in \text{Ver} \{N(f)\}\) it follows that
\[\text{ord}[c_{p, j}] > \text{ord}[c_{p, j - b_j, \lambda}] = a_\lambda(\lambda) \quad \text{if} \quad \nu < p_j - b_j(\lambda).\]

Thus \(e_{\lambda k}\) depends only on the following term:
\[c_{p, j - b_j, \lambda} t(t y^{\alpha} - (t, \xi_\lambda), 1) \{\xi_\lambda \sigma(t, \xi_\lambda)\}^\nu \quad \text{for} \quad 0 \leq \nu \leq p_j - b_j(\lambda).\]

Then, using the inequality
\[\text{ord}[\partial_{\nu_\lambda}^j c_{p, j - b_j, \lambda}] \{t y^{\alpha} - (t, 0)\}^\alpha \geq a_\lambda(\lambda) - |\alpha| + 2|\alpha| > a_\lambda(\lambda)\]
for \(|\alpha| \geq 1\), we get
\[e_{\lambda k} t^{a_\lambda(\lambda)} \xi_\lambda \sigma(t, \xi_\lambda)^\nu = \text{Loc}[c_{p, j - b_j, \lambda} \partial_{\nu_\lambda}^j \xi_\lambda \sigma(t, \xi_\lambda)^\nu].\]

Thus, by \(\sigma(t, \xi_\lambda) \equiv 1\) (Corollary 7.7), we conclude (7.32).

The proof of Claim 7.10 is complete. Q. E. D.

**Proof of (7.4).** By virtue of Claim 7.10, it suffices to show
\[(7.33)\quad \text{ch}[B_j(t, 0, \xi_1)] = \text{ch}(B_j(t, \xi_1)).\]

By the definition (7.27), the equality (7.26) can be written as
\[(7.34)\quad B_j(t, 0, \xi_1) = B_j(t, \xi_1) + \eta_1(t) g_j(t, \xi_1).\]

Since \(N(B_j) = N(f_j)\) which is a consequence of Claims 7.9 and 7.10, it suffices for (7.33) to verify
\[(7.35)\quad N(\eta_1(t) g_j(t, \xi_1)) \equiv N(f_j).\]

Putting \(\xi_1 = 0\) in (7.34), we have \(B_j(t, 0, 0) = B_j(t, 0) + \eta_1(t) g_j(t, 0)\). Since Claims 7.8 and 7.10 lead us to
\[\text{Loc}[B_j(t, 0, 0)] = \text{Loc}[B_j(t, 0)] = \text{Loc}[c_{0, j}] \partial_{\nu_0}^j = 0\]
we have \(\eta_1(t) g_j(t, 0) \equiv 0\). Then, by \(q \leq q_j\), we have
which shows the desired (7.35).

The proof of (7.4) in Proposition 7.2 is complete. Q. E. D.

Now we prove the second assertion (7.5) in Proposition 7.2.

It suffices for (7.5) to show

\[(7.36)\]

\[N_{\xi} \left\{ \left. A(t, 0, \xi) - t^{-1} \sum_{i=1}^{n} \partial_{\eta_i} \exp \left( -\int_{0}^{t} L P(e^s) \, ds \right) \right|_{\xi} \right\} \equiv N(f).\]

We recall the definition

\[A(t, x', \xi) := X_{\gamma}(t, Y(t, x', \xi), H_{\xi}(t, x', \xi))\]

where \(X_{\gamma}(t, y', \eta, \xi) = tX_{\gamma}(t, y', \xi)\) is the first component of the characteristic curve \(t \to Y(t, y', \xi)\) of \(F\). Since \(X_{\gamma}\) can be expanded as the form

\[
X_{\gamma}(t, y', \eta, \xi) = \partial_{\eta} f(0, y'; \eta, \xi) + \sum_{i=2}^{\infty} (i!)^{-1} t^{i-1} \partial_{\eta} X_{\gamma}(0, y', \eta, \xi) = \partial_{\eta} f(y', \eta, \xi) + \sum_{i=2}^{\infty} (i!)^{-1} t^{i-1} \partial_{\eta} X_{\gamma}(0, y', \eta, \xi),
\]

Corollary 7.7 yields that \(A(t, 0, \xi)\) can be written as the form:

\[(7.37)\]

\[A(t, 0, \xi) = X_{\gamma}(t, t(y_{\gamma}(t, \xi), 1), \xi_1 \sigma(t, \xi) + \eta_{\gamma}(t)) = \partial_{\eta} f(t(y_{\gamma}(t, \xi), 1), \xi_1 \sigma(t, \xi) + \eta_{\gamma}(t)) + \sum_{i=2}^{\infty} (i!)^{-1} t^{i-1} \partial_{\eta} X_{\gamma}(0, t(y_{\gamma}(t, \xi), 1), \xi_1 \sigma(t, \xi) + \eta_{\gamma}(t)).\]

Note that the expression (7.37) and the inclusion

\[(7.38)\]

\[N(\eta_{\gamma}(t) \xi) \subset N(t \xi)\]

imply the following equality:

\[(7.39)\]

\[\xi A(t, 0, \xi) = \xi \sum_{i=1}^{\infty} (i!)^{-1} t^{i-1} (-1)^{i-1} \partial_{\eta_i} f(t(y_{\gamma}(t, \xi), 1), \xi_1 \sigma(t, \xi)) = \xi \sum_{i=1}^{\infty} (i!)^{-1} t^{i-1} (\partial_{\eta_i} X_{\gamma}(0, t(y_{\gamma}(t, \xi), 1), \xi_1 \sigma(t, \xi)) - (-1)^{i-1} \partial_{\eta_i} f(t(y_{\gamma}(t, \xi), 1), \xi_1 \sigma(t, \xi))).\]

By virtue of the assertion 1) in Lemma 7.5, we have

\[(7.40) N[\text{the second term in the right hand side of (7.39)}] \equiv N(f).\]

Hence, if we define \(\Gamma(t, \xi)\) by

\[(7.41)\]

\[\Gamma(t, \xi) := \xi \sum_{i=1}^{\infty} (i!)^{-1} t^{i-1} (-1)^{i-1} \partial_{\eta_i} f(t(y_{\gamma}(t, \xi), 1), \xi_1 \sigma(t, \xi))\]
then (7.39), (7.40) and the inclusion $N(t^k \xi) \subseteq N(f(t', \eta))$ imply

\[(7.42) \quad N(\xi_1 A(t, 0, \xi) - \Gamma(t, \xi)) \subseteq N(f). \]

Thus it suffices for (7.36) to derive

\[(7.43) \quad N(\Gamma(t, \xi) - \xi_1 t^{-1} \sum_0^1 \partial_{\xi_1} ch(f)(-\theta L_{\nu}(t^p), \xi_1) d\theta) \subseteq N(f). \]

Let $f(t', \eta) = \sum v \xi_1(t') \eta_1^v$ be the Taylor expansion of $f$. Then (7.41) can be written as

\[(7.44) \quad \Gamma(t, \xi) = \sum \frac{(i!)}{v^v(-1)^{i-1}} \sum \{\partial_{\xi_1}^v c_{v}(t(\eta(t', \xi), 1))\nu \xi_1 \eta_1^v(\sigma(t, \xi))\nu^{-1}. \]

Put $\Gamma(t, \xi) := \sum k_i(t) \xi_1^i$. Then $k_2(t)$ can be written as the form

\[(7.45) \quad k_2(t) = \sum k_{2i}(t) \quad \text{where} \]

\[(7.46) \quad k_{2i}(t) := \sum \frac{(i!)}{v^v(-1)^{i-1}} \sum \{\partial_{\xi_1}^v c_{v}(t(\eta(t', \xi), 1))\nu \xi_1 \eta_1^v(\sigma(t, \xi))\nu^{-1} \eta_1^{v-1}. \]

Since ord $[k_{2i}] \geq i-1 + \text{ord}[\xi_1] - (i-1) = \text{ord}[c_{\nu}]$, if follows

\[(7.47) \quad \text{ord}[k_{2i}] \subseteq \text{ord}[c_{\nu}] \subseteq \text{ord}[c_{\nu}]. \]

Note that (7.42) and (7.45) yield

\[(7.48) \quad N(\xi_1 A(t, 0, \xi)) \subseteq \text{convex hull} \{N(\xi_1 A(t, 0, \xi) - \Gamma(t, \xi)) \cup N(\Gamma(t, \xi))\} \subseteq N(f). \]

Moreover (7.46) yields that, if we put

\[(7.49) \quad \Gamma(t, \xi) := \sum_{\lambda \geq 1, \text{ord}[c_{\lambda}] \lambda} k_{2i}(t) \xi_1^i \]

then we have

\[(7.50) \quad N(\Gamma(t, \xi) - \Gamma(t, \xi)) \subseteq N(f). \]

For $\lambda$ satisfying (ord $[c_{\lambda}]$, $\lambda) \subseteq \text{Ver} N(f)$, we consider $k_{2i}(t)$: Setting $\nu = \lambda$ in (7.44), we have

\[(7.51) \quad k_{2i}(t) = \sum \frac{(i!)^{-1} i^{-1}}{v^v(-1)^{i-1} \lambda} \{\partial_{\xi_1}^v c_{\lambda}(t(\eta(t', t, 0), 1))\sigma(t, 0)^{i-1}. \]

We take the expansion
\[
\partial_{y_2}^{-1} c_1(t(y^* \sim (t, 0), 1)) = \partial_{y_2}^{-1} c_1(0, \cdots, 0, t) + \sum_{i \geq 1} (\alpha_i !)^{-1} \partial_{y_2}^{-1} \partial_{y_i}^\alpha c_1(0, \cdots, 0, t) (t(y^* \sim (t, 0))^\alpha,
\]
and we put
\[
(7.51) \quad k_{y_2}(t) := \sum_{i=1}^{q} (i!)^{-1} i^{i-1} (t)^{t-1} \lambda \{ \partial_{y_2}^{-1} c_1(0, \cdots, 0, t) \} \sigma(t, 0)^{t-1}.
\]
Then the inequality \(\text{ord}[t^y \sim (t, 0)] \geq 2\) implies
\[
\text{ord}[k_{y_2}(t) - k_{y_2}^{\lambda}(t)] \geq i - 1 + \text{ord}[c_1] - (i-1) + \min_{i \geq 1} \{-|\alpha| + 2|\alpha|\}
\]
\[> \text{ord}[c_1].\]
Thus, if we put
\[
(7.52) \quad \Gamma_s(t, \xi_i) := \sum_{\lambda \geq 1, \langle \text{ord}[c_1], \lambda \rangle \in \text{VerN}(f)} k_{y_2}^{\lambda}(t) \xi_i^l
\]
then we have
\[
(7.53) \quad N(\Gamma_s(t, \xi_i) - \Gamma_s(t, \xi_i)) \in \text{N}(f).
\]
By virtue of (7.49) and (7.53), it suffices for (7.43) to show
\[
(7.54) \quad N(\Gamma_s(t, \xi_i) - \Gamma_s(t, \xi_i)) \in \text{N}(f).
\]
We denote the set \(\text{VerN}(f)\) by \(\{(a(\mu), p-b(\mu); 0 \leq \mu \leq m)\} \). Then, substituting \(\lambda = p-b(\mu)\) in (7.51), (7.52) can be written as
\[
\Gamma_s(t, \xi_i) = \sum_{\mu=0}^{m-1} k_{p-b(\mu), p-b(\mu)}(t) \xi_i^{p-b(\mu)}
\]
\[
= \sum_{\mu=0}^{m-1} \sum_{i=1}^{q} (i!)^{-1} i^{i-1} (t)^{i-1}(p-b(\mu))
\]
\[\times \{(\partial_{y_2}^{-1} c_{p-b(\mu)}(0, \cdots, 0, t)) \sigma(t, 0)^{p-b(\mu)-1} \xi_i^{p-b(\mu)}\}.
\]
Note that
\[
c_{p-b(\mu)}(y^*) = y_2^{\alpha(\mu)} \text{Loc}[c_{p-b(\mu)}](\partial_{y_2})(1 + O(y_2)) \mod(y_2^*)
\]
which implies
\[
\partial_{y_2}^{-1} c_{p-b(\mu)}(0, \cdots, 0, t) = \partial_{y_2}^{-1} [e_\mu t^{a(\mu)}(1 + O(t))]
\]
where \(e_\mu \in C\) denotes the non-zero constant \(\text{Loc}[c_{p-b(\mu)}](\partial_{y_2})\).
Since \(N(t^{a(\mu)+1} \xi_i^{p-b(\mu)}) \in \text{N}(f)\), if we put
\[
\Gamma_s(t, \xi_i) := \sum_{\mu=0}^{m-1} \sum_{i=1}^{q} (i!)^{-1} i^{i-1} (t)^{i-1}(p-b(\mu)) \partial_{y_2}^{-1} e_\mu t^{a(\mu)} \xi_i^{p-b(\mu)}
\]
then we have
\[
(7.55) \quad N(\Gamma_s(t, \xi_i) - \Gamma_s(t, \xi_i)) \in \text{N}(f).
\]
Hence it suffices for (7.54) to show
(7.56) \[ \Gamma_s(t, \xi) = t^{-1} \xi \int_0^t \partial_{\xi} \text{ch}(f)(-\theta L_{\theta}(e^\theta), \xi) d\theta. \]

We verify (7.56): Since \( \partial_i t^{-i} a_{\mu}(p) = 0 \) if \( i > a(\mu) + 1 \), we have the following expression of \( \Gamma_s(t, \xi) \):

\[
\Gamma_s(t, \xi) = \sum_{\mu=0}^{m-1} \sum_{i=1} a(\mu) + 1 \cdot (i!)^{-i-1} \left(-1\right)^{i-1}(p-b(\mu))
\times \frac{a(\mu) + 1}{\left(a(\mu) - i + 1\right)!} t^a(\mu) - i + 1 \right) \xi_1^{a(\mu) - b(\mu) - 1} \frac{1}{a(\mu) + 1} \left(a(\mu) + 1\right)
\times \sum_{i=1}^{a(\mu) + 1} C(a(\mu) + 1, i) (-1)^{i-1}
\]

where we set \( C(N, i) := \frac{N!}{i!(N-i)!} \) for \( 0 \leq i \leq N \). Note that

\[
\sum_{i=1}^{N} C(N, i) (-1)^{i-1} = -(1-1)^N + C(N, 0) = 1
\]

which implies

\[
\Gamma_s(t, \xi) = t^{-1} \xi \int_0^t \partial_{\xi} \left[ \sum_{\mu=0}^{m-1} a(\mu) \xi_1^{a(\mu) - b(\mu) - 1} \right] d\theta
\]

\[
= t^{-1} \xi \int_0^t \partial_{\xi} \text{ch}(f)(0, \cdots, 0, \theta, \xi) d\theta
\]

\[
= t^{-1} \xi \int_0^t \partial_{\xi} \text{ch}(f)(-\theta L_{\theta}(e^\theta), \xi) d\theta.
\]

Hence (7.56) follows.

The proof of (7.5) in Proposition 7.2 is complete. Q. E. D.

§ 8. Proof of \( V_{j_1} = R_j(0) \) as Germs of Hypersurfaces

In this section we prove that the image set \( V_{j_1} = \pi_{j_1}(V_{j_1}) \) has only one irreducible component locally at \( (0, 0) \in (M, 0_x) \times (C, 0_{\xi}), \) by means of the theory of resultant. We also show that this irreducible component is given by a zero set of a resultant \( R_j(x, \xi) \) (see Proposition 8.9 and Theorem 8.10).

We first replace the defining germs \( tA-x_1 \) and \( B_j \) of \( V_{j_1} \), which are obtained by Proposition 6.7, with suitable polynomials \( tP-x_1 \) and \( Q_j \) with respect to the variable \( t \):

**Lemma 8.1.** 1) For \( 1 \leq j \leq r \), there exist a Weierstrass polynomial \( Q_j \), with
respect to $t$ of degree $q_j := \text{ord}[f_j(y', 0)]$, and a unit $\varepsilon_j$ such that

$$B_j(t, x', \xi_i) = Q_j(t, x', \xi_i)\varepsilon_j(t, x', \xi_i) \quad \text{in} \quad \mathcal{O}_{C \times M \times C, (t, 0, \xi)}$$

where $Q_j$ and $\varepsilon_j$ are uniquely determined by $B_j$.

2) There exist a germ $\tilde{A}$ and a polynomial $P$ with respect to $t$ of degree at most $q - 1 = \sum_{j=1}^{r} q_j - 1$ such that

$$A(t, x', \xi_i) = \tilde{A}(t, x', \xi_i) + \sum_{j=1}^{r} Q_j(t, x', \xi_i) + P(t, x', \xi_i)$$

where $\tilde{A}$ and $P$ are uniquely determined by $A$.

Proof. By Proposition 7.2, we have $N(B_j(t, 0, \xi)) = N(f_j)$ which yields

$$\text{ord}[B_j(t, 0, 0)] = \text{ord}[f_j(y', 0)] = q_j < \infty.$$ 

Then the assertion 1) follows from the Weierstrass's preparation theorem. The assertion 2) is a consequence of the Weierstrass's division theorem. Q.E.D.

Corollary 8.2. It follows that

$$tA - x_1, B_j) = (tP - x_1, Q_j)$$

as ideals in the ring $\mathcal{O}_{C \times M \times C, (t, 0, \xi)}$ for $1 \leq j \leq r$.

Proof. Let $f, g$ be germs in $\mathcal{O}_{C \times M \times C, (t, 0, \xi)}$. Then we have

$$f(tA - x_1) + gB_j = f\left(t\left(\tilde{A}\prod_{i=1}^{r} Q_i + P\right) - x_1\right) + gQ_j\varepsilon_j$$

$$= f(tP - x_1) + (g\varepsilon_j + t\tilde{A}\prod_{i=1}^{r} Q_i)Q_j.$$ 

Note that a transformation

$$(f, g) \longrightarrow (f, g\varepsilon_j + t\tilde{A}\prod_{i=1}^{r} Q_i)$$

is invertible, since $\varepsilon_j$ is a unit. Hence Corollary 8.2 follows. Q.E.D.

By this corollary we can take $tP - x_1$ and $Q_j$ as defining functions of $V_{j, l}$ locally at $(0, 0, 0) \in C \times M \times C$:

$$(V_{j, l}, (0, 0, 0)) = \{(t, x, \xi_i); tP(t, x', \xi_i) - x_1 = Q_j(t, x', \xi_i) = 0\}.$$

Lemma 8.3. The principal part of $N(P(t, 0, \xi))$ has the same property (7.5) of the principal part of $N(A(t, 0, \xi))$, that is, it follows that

$$N[N(P(t, 0, \xi)) - t^{-1}\partial_{\xi_i} N(f) \cdot (-\theta L_{\xi_i}(e^\theta), \xi_i) d\theta] + N(\xi_i) \subseteq N(f).$$

Proof. By virtue of (7.5) in Proposition 7.2, it suffices to derive

$$N(\xi_i | A(t, 0, \xi_i) - P(t, 0, \xi_i)) \subseteq N(f).$$
By the definition of $P$ we have
\begin{equation}
A(t, 0, \xi_i) - P(t, 0, \xi_i) = \tilde{A}(t, 0, \xi_i) \prod_{j=1}^k Q_j(t, 0, \xi_i).
\end{equation}

Note that the additivity of Newton polygons (Proposition 11.3) yields
\[ N(B_j) = N(Q_j \varepsilon_j) = N(Q_j) + N(\varepsilon_j) = N(Q_j) \]
since $\varepsilon_j$ is a unit. Hence it follows that
\[ N(\prod_{j=1}^k Q_j(t, 0, \xi_i)) = \sum_{j=1}^k N(Q_j(t, 0, \xi_i)) = \sum_{j=1}^k N(B_j(t, 0, \xi_i)). \]

Then, by (7.4) in Proposition 7.2, we have
\[ N(\prod_{j=1}^k Q_j(t, 0, \xi_i)) = \sum_{j=1}^k N(f_j) = N(\prod_{j=1}^k f_j) = N(f). \]

Thus the equality (8.2)' yields
\[ N(\xi_1 \{A(t, 0, \xi_i) - P(t, 0, \xi_i)\}) \subset N(\xi_1) + N(f) \equiv N(f). \]

The proof of Lemma 8.3 is complete. Q.E.D.

**Remark 8.4.** It follows that
\[ q - 1 \geq \deg_t(P) \geq a(m-1) = q(1) + \cdots + q(m-1) \]
where $(a(m-1), p - b(m-1))$ is the rightest vertex of $N(f)$ except for $(a(m), 0)$.

**Proof.** Note that
\begin{equation}
(a(m-1), p - b(m-1) - 1) \in \text{Ver} N(t^{-1} \int_0^t \partial_{\xi_i} \text{ch}(f)(-\theta L_P(e^\theta), \xi_i) d\theta).
\end{equation}

Indeed, for a germ $g \in \mathcal{O}_{\xi_0, (a, 0)}$ and for a vertex $(a, b) \in \text{Ver} N(g)$ with $b \geq 1$, it follows that $(a, b-1) \in \text{Ver} N(\partial_{\xi_i} g)$, and that the operator
\[ g \mapsto t^{-1} \int_0^t g(\theta, \xi_i) d\theta \]
preserves the Newton polygon $N(g)$. Hence (8.7) follows. Then, by Lemma 8.3, we conclude $\deg_t(P) \geq a(m-1)$. Since the other inequality $q - 1 \geq \deg_t(P)$ is trivial by the definition of $P$, the proof of Remark 8.4 is complete. Q.E.D.

Now we recall the

**Definition 8.5.** Let $\mathcal{O}$ be an integral domain, and let $f(t), g(t) \in \mathcal{O}[t]$ be polynomials with $\mathcal{O}$-coefficients of degree $m, n$ respectively as follows:
\[
f(t) = \sum_{i=0}^m a_i t^i, \quad g(t) = \sum_{j=0}^n b_j t^j.
\]

We define a $(n+m)$-square matrix $D(f, g)$ by
and we also define a resultant \( r(f, g) \in \mathcal{O} \) of \( f(t) \) and \( g(t) \) by

\[
r(f, g) := \det D(f, g).
\]

The following Proposition 8.6 is well-known (see, for example: \cite[Theorem 4.11.1, p. 164]{Na}):

**Proposition 8.6.** Let \( \mathcal{O}, f, g \) be as same as in Definition 8.5, and \( \delta_j \) be the \((j, n+m)\)-cofactor of the matrix \( D(f, g) \). Then:

1) If either \( f \) or \( g \) is a monic polynomial then the following (a) and (b) are equivalent as statements for an element \( c \in \mathcal{O} \):
   (a) \( c \in \langle f, g \rangle \mathcal{O}[t] \).
   (b) There exist \( e_j \in \mathcal{O} \) \((1 \leq j \leq n+m)\), \( e \in \mathcal{O} \), and \( d \in \mathcal{O} - \{0\} \) such that

\[
e_j = \frac{e}{d} \delta_j \text{ for } 1 \leq j \leq n+m.
\]

2) Let \( \mathcal{K} \) be an algebraic closed field containing the ring \( \mathcal{O} \). Let

\[
f(t) = a_n \prod_{i=1}^{n} (t - \alpha_i), \quad g(t) = b_n \prod_{j=1}^{m} (t - \beta_j)
\]

be the factorizations of \( f \) and \( g \) in the ring \( \mathcal{K}[t] \). Then it follows that

\[
r(f, g) = a_n b_n \prod_{i,j} (\beta_j - \alpha_i).
\]

**Corollary 8.7.** It follows that

\[
r(f, g) \in \langle f, g \rangle \mathcal{O}[t].
\]

**Proof.** Let \( D := D(f, g) \) be the \((n+m)\)-square matrix defined by (8.8), and let \( D^c \) be the cofactor matrix of \( D \). Since \( D^c D = (\det D) I_{n+m} \), we have

\[
\delta_n a_n + \delta_{n+m} b_n = \det D = r(f, g).
\]

Hence the condition (b) of the assertion 1) in Proposition 8.6 holds if we take

\[
e_j := \delta_j \quad (1 \leq j \leq n+m) \quad \text{and} \quad d = e := 1.
\]

Thus Corollary 8.7 follows. Q. E. D.
Now we return to our problem. The first aim in this section is to show Proposition 8.9 stated below. Before stating this, we introduce the

**Notation 8.8.** We set $D_j$, $R_j$ respectively as

\begin{equation}
D_j := D(Q_j, tP - x_j) \tag{8.14}
\end{equation}

\begin{equation}
R_j := r(Q_j, tP - x_j) = \det D_j \quad \text{for} \quad 1 \leq j \leq r. \tag{8.15}
\end{equation}

By Corollary 8.7, it follows that

$$R_j \in \mathcal{O} \cap (Q_j, tP - x_j) \mathcal{O}[t]$$

where we put $\mathcal{O} := \mathcal{O}_{M \times C, (i, o)}$. Hence we conclude

\begin{equation}
(V_{ij}, (0, 0)) \subset (R_j^{-1}(0), (0, 0)). \tag{8.16}
\end{equation}

**Proposition 8.9.** The germ $(R_j^{-1}(0), (0, 0))$ of a hypersurface of $M \times C$ at $(0, 0)$ has only one irreducible component, that is, if

\begin{equation}
R_j(x, \xi_i) = \prod_{i=1}^{k(j)} S_{j,i}(x, \xi_i)^{\mu_{i,j}} \tag{8.17}
\end{equation}

is an irreducible decomposition of $R_j$ at $(0, 0)$, then $k(j) = 1$ follows.

**Proof.** Since (8.17) is an irreducible decomposition, we note

\begin{equation}
\begin{cases}
   a) \hspace{1em} \mu(\lambda, j) \geq 1, \\
   b) \hspace{1em} S_{j,i} (1 \leq \lambda \leq k(j)) \text{ are irreducible in } \mathcal{O}_{M \times C, (i, o)}, \text{ and} \\
   c) \hspace{1em} \text{there is no germ } g(x, \xi_i) \text{ satisfying } S_{j,i} = gS_{j,i}, (\lambda \neq \lambda').
\end{cases} \tag{8.18}
\end{equation}

**Claim 1.** Set $X_j := (S_{j,i}^{-1}(0), (0, 0))$. Then, for any $1 \leq \lambda \leq k(j)$ and for any sufficiently small open neighborhood $U$ of $(0, 0)$ in $M \times C$, it follows that

$$X_j \cap U \supset \bigcup_{\lambda \neq \lambda'} X_{j'}. \tag{8.19}$$

**Proof.** If we assume that Claim 1) is not true then there exist $\lambda$ and an open neighborhood $U$ such that

\begin{equation}
X_j \cap U \subset \bigcup_{\lambda \neq \lambda'} X_{j'}. \tag{8.19}
\end{equation}

We set $T := \prod_{\lambda \neq \lambda'} S_{j,i}$. Then (8.19) yields $T \mid_{x_j} \equiv 0$. Hence the Rückert's Nullstellensatz ($\S$ 13) implies that

$$T \in \text{Rad}[(S_{j,i})] := \{g \in \mathcal{O}_{M \times C, (i, o)}; \exists i, g^{\dagger} \in (S_{j,i})\}.$$ 

Since $(S_{j,i})$ is a prime ideal, we have $T \in (S_{j,i})$ which implies that there exists $\lambda' (\neq \lambda)$ such that $S_{j,i} \in (S_{j,i})$. This contradicts the condition c) in (8.18). Hence Claim 1) follows.
Claim 2. Let \( d := \deg(P) \), and let us write
\[
P(t, x', \xi_i) = \sum_{i=0}^d P_i(x', \xi_i) t^i.
\]
Then \( P_d | x_2 \neq 0 \) for any \( 1 \leq \lambda \leq k(j) \) where we set \( X_2 := (S_{2j}(0), (0, 0)) \).

Proof. If Claim 2 is not true then the Rückert's Nullstellensatz yields for some \( \lambda \)
\[
P_d \in \text{Rad}[(S_{2j})] = (S_{2j}) \text{ for some } \lambda.
\]
We note that
\[
\text{ord}[S_{2j}(x_1, 0, \cdots, 0)] = s(\lambda, j) < \infty.
\]
Indeed, since \( Q_j \) is a Weierstrass polynomial, we have
\[
R_j(x_1, 0, \cdots, 0) = \det \begin{vmatrix} J_{d+1} & 0 & \cdots & 0 \\ 0 & -x_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & -x_1 \end{vmatrix} = (-x_1)^{q_j}
\]
which yields
\[
\sum_{j=1}^{k(j)} s(\lambda, j) \nu(\lambda, j) = q_j < \infty.
\]
Hence (8.20) follows.

Since \( P_d(x', \xi_i) \) is independent of the variable \( x_1 \), the following division
\[
P_d(x', \xi_i) = 0 \times S_{2j} + P_d
\]
of \( P_d \) by \( S_{2j}(x, \xi_i) \) with respect to the variable \( x_1 \) is a Weierstrass division.
On the other hand, the condition \( P_d \ni (S_{2j}) \) yields
\[
P_d(x', \xi_i) = a(x, \xi_i) S_{2j} + 0
\]
which is also a Weierstrass division of \( P_d \) by \( S_{2j} \) with respect to \( x_1 \). Then the uniqueness of Weierstrass divisions implies \( a=0 \) thus we have \( P_d \equiv 0 \). This contradicts the definition of \( P_d \). Hence Claim 2 follows.

According to Claims 1) and 2), we have:
\[
(8.21) \quad [X_2 - (\cup_{\lambda} X_2 \cup P_{\lambda}^{-1}(0))] \cap U \neq \emptyset \quad \text{for any } \lambda, U.
\]
On the other hand, the assertion 2) in Proposition 8.6 yields
\[
(8.22) \quad [X_2 - P_{\lambda}^{-1}(0)] \cap U \subset \pi_{j}(V_{j}) = V_{j}\quad \text{for any } \lambda, U.
\]
Indeed, for any fixed \((x_1, \xi_i) \in [X_2 - P_{\lambda}^{-1}(0)] \cap U\), it follows that
\[
(8.23) \quad 0 = R_j(x_1, \xi_i) = P_d(x_1, \xi_i) \prod_{i, k}(\beta_k - \alpha_k).
\]
where \( \{\alpha_i\} \) [or \( \{\beta_i\} \) resp.] denotes the roots of \( Q_j(t, x', \xi^i) \) \( [tP(t, x', \xi^i) - x^i] \) in the algebraic closed field \( \mathbb{C} \). Since \( P_d(x', \xi^i) \neq 0 \), (8.23) yields that there exists a common root \( t^o := \alpha_1 = \beta_1 \). Hence we have \((x^o, \xi^i) \in V_{j^o} \).

Note that (8.21) and (8.22) yield

\[
\pi_{j^o}^i(x_i) \neq \emptyset \quad \text{and} \quad \pi_{j^o}^i(x_i) \not\subset \pi_{j^o}^i(x_i) \quad \text{for any} \; \lambda, \lambda' \; (\lambda \neq \lambda').
\]

On the other hand, the inclusion (8.16) implies

\[
V_{j^o} = \bigcup_{\lambda=1}^{k(j)} \pi_{j^o}^i(x_i).
\]

Thus we conclude that if \( k(j) \geq 2 \) then the analytic set \( V_{j^o} \) is reducible at \((0, 0, 0) \in \mathbb{C} \times M \times \mathbb{C} \). But we have already shown that \( V_{j^o} \) is irreducible by Corollary 8.2 and Remark 6.8. Hence we get \( k(j) = 1 \) as desired.

The proof of Proposition 8.9 is complete. Q.E.D.

By virtue of Proposition 8.9 and of (8.16), for any \( j \) \((1 \leq j \leq r)\) and for any open sufficiently small neighborhood \( U \) of \((0, 0) \) in \( M \times \mathbb{C} \), we have:

1) There exist an irreducible germ \( S_j(x, \xi_i) \) and an integer \( \nu(j) \geq 1 \) such that

\[
R_j(x, \xi_i) = S_j(x, \xi_i)^{x^{\nu(j)}} \quad \text{on} \; U.
\]

2) Let \( P_d(x', \xi_i) \) be the leading coefficient in the polynomial \( P(t, x', \xi_i) \). Then the following inclusions hold:

\[
\emptyset \neq [R_j^{-1}(0) - P_d^{-1}(0)] \cap U \subset V_{j^o} \cap U \subset R_j^{-1}(0) \cap U.
\]

The second aim of this section is to show

**Theorem 8.10.** It follows that

\[
(V_{j^o}, (0, 0, 0)) = (R_j^{-1}(0), (0, 0, 0))
\]

as germs of analytic subsets of \( M \times \mathbb{C} \) at \((0, 0)\).

Our proof of Theorem 8.10 is based on the local dimension theory of analytic sets which is summarized in §13. We first remark a simple

**Claim 8.11.** The map germ \( \pi_{j^o} : (V_{j^o}, (0, 0, 0)) \rightarrow (V_{j^o}, (0, 0, 0)) \) is a finite holomorphic map germ.

**Proof.** Since \((V_{j^o}, (0, 0, 0)) = \{(t, x, \xi_i) ; tP - x_i = Q_j = 0\}\), the map germ \( \pi_{j^o} \) is a restriction of a map germ \( \pi_{j^o} \) defined by the diagram:

\[
\begin{array}{ccc}
(Q_j^{-1}(0), (0, 0, 0)) & \longrightarrow & (C \times M \times C, (0, 0, 0)) \\
\pi_{j^o} & \downarrow & \text{projection} \\
(M \times C, (0, 0))
\end{array}
\]
Since \( Q_j(t, x', \xi, \eta) \) is a Weierstrass polynomial in \( t \), the map germ \( \pi_{j8} \) is finite. Hence Criterion 3.6 implies Claim 8.11. Q.E.D.

Now we recall a way of regarding an analytic subset \( X \) of a domain \( D \) in \( \mathbb{C}^N \) as a reduced complex space \( (X, \mathcal{O}_X) \) in the sense of [Gr-Re]. According to the summary of this way in § 13, we set

\[
\mathcal{O}_X := (\mathcal{O}_D / i(X)) | x
\]

where \( i(X) \) is the ideal sheaf of the analytic set \( X \). Note that the Rückert's Nullstellensatz asserts that, if \( X \) is defined as a common zero set of \( f_i \in \mathcal{O}_D \) \((1 \leq i \leq m)\) (we denote this by \( X = \text{Null}(f_1, \ldots, f_m) \)) then

\[
i(X) = i(\text{Null}(f_1, \ldots, f_m)) = \text{Rad}[(f_1, \ldots, f_m)] \quad \text{(see § 13)}.
\]

Hence any stalk \( \mathcal{O}_{x, x} \) is a reduced ring, that is, \( \mathcal{O}_{x, x} \) has no nilpotent element.

**Lemma 8.12.** Let \( (V_{j1}, \mathcal{O}_{V_{j1}}) \) [or \( (\text{Null}(R_j), \mathcal{O}_{\text{Null}(R_j)}) \) resp.] be the reduced complex space which is obtained from \( V_{j1}[\text{Null}(R_j)] \) by the above way. Then these complex spaces are irreducible locally at \((0, 0, 0)\) or at \((0, 0)\) respectively, that is, the following (8.30) holds:

\[
\mathcal{O}_{V_{j1}, (0, 0, 0)} \quad \text{and} \quad \mathcal{O}_{\text{Null}(R_j), (0, 0)} \quad \text{are integral domains}.
\]

**Proof.** By virtue of \( V_{j1} = \text{Null}(tP - x_1, Q_j) \) and of the irreducibility of the ideal \((tP - x_1, Q_j)\) at \((0, 0, 0)\), we have

\[
i(V_{j1}, (0, 0, 0)) = i(\text{Null}(tP - x_1, Q_j))_{(0, 0, 0)}
\]

\[
= \text{Rad}[(tP - x_1, Q_j)]_{(0, 0, 0)}
\]

\[
= (tP - x_1, Q_j)_{(0, 0, 0)}.
\]

Hence we have

\[
\mathcal{O}_{V_{j1}, (0, 0, 0)} = \mathcal{O}_{\mathbb{C}^N \times \mathbb{C}, (0, 0, 0)} / (tP - x_1, Q_j)_{(0, 0, 0)}
\]

which shows the first assertion of (8.30).

By virtue of Proposition 8.9, we similarly have

\[
i(\text{Null}(R_j))_{(0, 0)} = \text{Rad}[(R_j)]_{(0, 0)} = (S_j)_{(0, 0)}
\]

where \( R_j \) and \( S_j \) are related as (8.26). Since \( S_j(x, \xi, \eta) \) is irreducible at \((0, 0)\) we conclude that

\[
\mathcal{O}_{\text{Null}(R_j), (0, 0)} = \mathcal{O}_{\mathbb{C}^N \times \mathbb{C}, (0, 0)} / (S_j)_{(0, 0)}
\]

is also an integral domain. The proof of Lemma 8.12 is complete. Q.E.D.

Let \( X, Y \) be analytic sets and \( f : X \rightarrow Y \) be a holomorphic map in the sense of Definition 3.1. If we regard \( X \) [or \( Y \) resp.] as a reduced complex space \((X, \mathcal{O}_X) \) [(\( Y, \mathcal{O}_Y \))] then the map \( f \) can be regarded as a morphism of complex
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spaces \((f, f^-) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)\) where \(f^- : \mathcal{O}_Y \to \mathcal{O}_X\) is a sheaf map on \(Y\), which is defined in the canonical way mentioned in §13 (Lemma 13.5).

Now we recall the definition of analytic subset \(Z\) of a complex space \((X, \mathcal{O}_X)\) and its local dimension (Definitions 13.9 and 13.10). We regard the map germ \(\pi_{jz} : V_{jz} \to \text{Null}(R_j) = R_j^{-1}(0)\) as a germ of a finite morphism

\[
\pi_{jz} : (X, \mathcal{O}_X) \to (\text{Null}(R_j), \mathcal{O}_{\text{Null}(R_j)})
\]

and apply the local dimension theory to the finite morphism \(\pi_{jz}\).

We first note that the finite mapping theorem (Theorem 13.14) yields that the image \(V_{jz} = \pi_{jz}(V_{j1})\) is an analytic set of the complex space \((\text{Null}(R_j), \mathcal{O}_{\text{Null}(R_j)})\). Then we have

\[
(8.31) \quad \dim_{\text{co}, 0}(V_{jz}, \mathcal{O}_{\pi_{jz}}) \leq \dim_{\text{co}, 0}(V_{jz}) \leq \dim_{\text{co}, 0}(\text{Null}(R_j), \mathcal{O}_{\text{Null}(R_j)})
\]

by virtue of Proposition 13.11. On the other hand, Proposition 13.12 yields

\[
(8.32) \quad \dim_{\text{co}, 0}(\text{Null}(R_j), \mathcal{O}_{\text{Null}(R_j)}) = \dim_{\text{co}, 0}(M \times C) - 1 = n.
\]

By the isomorphic map germs

\[
\pi_{jz} : (V_{j1}, 0, 0, 0) \mapsto (\text{Null}(R_j), 0, 0, 0)
\]

we have

\[
(8.33) \quad \dim_{\text{co}, 0}(V_{jz}, \mathcal{O}_{\pi_{jz}}) = \dim C + \dim E - 1 = n.
\]

Combining these \((8.31)-(8.33)\), we get

\[
(8.34) \quad \dim_{\text{co}, 0} V_{jz} = \dim_{\text{co}, 0}(\text{Null}(R_j), \mathcal{O}_{\text{Null}(R_j)}).
\]

Since we have observed that the complex space \((\text{Null}(R_j), \mathcal{O}_{\text{Null}(R_j)})\) is irreducible at \((0, 0)\) in Lemma 8.12, we conclude that there exists an open neighborhood \(U\) of \((0, 0)\) in \(M \times C\) such that

\[
V_{jz} \cap U = \text{Null}(R_j) \cap U
\]

as a consequence of Proposition 13.13.

The proof of Theorem 8.10 is complete.

Q. E. D.

§ 9. Irreducibility of \(R_j(x, \xi_i)\)

In this section we complete the proof of Theorem 6.10. By virtue of Theorem 8.10, it suffices to show the following

**Theorem 9.1.** Under the assumptions \([B.1]-[B.4]\), for the resultants \(R_j(x, \xi_i) = r(Q_j(t, x', \xi_i), tP(t, x', \xi_i) - x_i)\), \(1 \leq j \leq r\), it follows:

1) \(R_j(x, \xi_i)\) is locally irreducible at \((0, 0)\).

2) \(R_j(x, \xi_i)\) has the finite order \(v_j := \sum_{\mu \in \mathcal{M}_j} v(\mu)\) with respect to \(\xi_i\):
We derive Theorem 9.1 from the following Proposition 9.2. The Newton polygon $N(R_j(x_1, 0, \ldots, 0, \xi_i))$ of the restriction $R_j|_{x' = 0}$ is given by

$$N(R_j(x_1, 0, \ldots, 0, \xi_i)) = \sum_{\rho \in \mathbb{K}_j} N_{q(\rho), v(\rho)}$$

where we denote by $N_{a, b}$ the following Newton polygon (see Notation 2.10):

$$N_{a, b} = \{(s, 0; s \geq 0, t \geq 0, (s/a) + (t/b) \geq 1)\}.$$ 


Proof. Since the assertion 2) in Theorem 9.1 is a direct consequence of (9.2), we only have to show the assertion 1) in Theorem 9.1.

By virtue of Proposition 8.9, we have known that $R_j(x, \xi_i)$ has only one irreducible component at $(0, 0)$, that is, there exists an irreducible germ $S_j$ at $(0, 0)$ such that the following (8.26) holds:

$$R_j(x, \xi_i) = S_j(x, \xi_i)^{v(f)}.$$ 

Thus it suffices for Lemma 9.3 to show

$$\nu(j) = 1 \quad \text{for } 1 \leq j \leq r.$$ 

We first observe the Claim 9.4. The finite sequence $\{v(\mu)/q(\mu) ; 1 \leq \mu \leq m := \# \text{Seg} N(f)\}$ is monotonely decreasing in $\mu$:

$$v(1)/q(1) > v(2)/q(2) > \cdots > v(m)/q(m).$$

Proof (of Claim 9.4). Recall the definition

$$v(\mu) = p(\mu) (a(\mu - 1) + 1) + q(\mu) (p - b(\mu - 1) - 1)$$

in Definition 4.1, which yields

$$v(\mu)/q(\mu) = \kappa(\mu) (a(\mu - 1) + 1) + p - b(\mu - 1) - 1$$

where $\kappa(\mu) := p(\mu)/q(\mu)$. Then we have

$$v(\mu)/q(\mu) - v(\mu + 1)/q(\mu + 1)$$

$$= \kappa(\mu) (a(\mu - 1) + 1) - \kappa(\mu + 1) (a(\mu) + 1) + b(\mu) - b(\mu - 1)$$

$$= \kappa(\mu) (a(\mu) + 1 - q(\mu)) - \kappa(\mu + 1) (a(\mu) + 1) + p(\mu)$$

$$= (\kappa(\mu) - \kappa(\mu + 1)) (a(\mu) + 1) - \kappa(\mu) q(\mu) + p(\mu)$$

Thus it suffices for Lemma 9.3 to show

$$\nu(j) = 1 \quad \text{for } 1 \leq j \leq r.$$
Since $p(\mu) = \kappa(\mu)q(\mu)$ and $\kappa(\mu) - \kappa(\mu+1) > 0$, we get

$$v(\mu)/q(\mu) - v(\mu+1)/q(\mu+1) > 0$$

which shows Claim 9.4. Q.E.D.

For $1 \leq j \leq r$, we denote the subset $M_j \subseteq \{1, 2, \ldots, m\}$ by

$$M_j = \{\mu_j(k); 1 \leq k \leq m_j := \# M_j\} \text{ with } 1 \leq \mu_j(1) < \cdots < \mu_j(m_j) \leq m.$$ Then it follows that

$$(9.6) \quad \text{Ver} N(R_j(x_1, 0, \ldots, 0, \xi_i)) = \{(s, t); s = q(\mu_j(1)) + \cdots + q(\mu_j(k)),
\quad t = v(\mu_j(k+1)) + \cdots + v(\mu_j(m_j)), 0 \leq k \leq m_j\}$$

Indeed, (9.6) easily follows from (9.5) and Lemma 0.2, under the assumption that Proposition 9.2 is true.

Proof of Lemma 9.3 (continued). Setting $x' = 0$ in (8.26) we have

$$R_j(x_1, 0, \ldots, 0, \xi_i) = S_j(x_1, 0, \ldots, 0, \xi_i)^{y_{(j)}}.$$ Thus, by the additivity of Newton polygons, it follows

$$(9.8) \quad N(R_j(x_1, 0, \ldots, 0, \xi_i)) = \nu(j) N(S_j(x_1, 0, \ldots, 0, \xi_i)).$$

Note that the Newton polygon $N(S_j(x_1, 0, \ldots, 0, \xi_i))$ can be written as the form

$$N(S_j(x_1, 0, \ldots, 0, \xi_i)) = \sum_{i} N_{a(i), \beta(i)} \quad (N_{a, \beta} \text{ is defined by (9.3)})$$

for some positive integers $a(i), \beta(i)$ satisfying

$$\beta(1)/a(1) > \cdots > \beta(i(j))/a(i(j)) > 0.$$ Thus, from (9.8), we have

$$(9.9) \quad N(R_j(x_1, 0, \ldots, 0, \xi_i)) = \sum_{i} N_{\nu(j), a(i), \beta(i)}.$$ Comparing (9.9) with (9.2) we get

$$\sum_{\mu \in M_j} N_{q(\mu), \nu(\mu)} = \sum_{i=1}^{(i)} N_{\nu(j), a(i), \nu(j), \beta(i)}$$

which implies

$$\begin{cases} i(j) = m_j, \quad \nu(j) = q(\mu_j(i)) \quad \text{and} \quad \nu(j) \leq \mu_j(i) \quad \text{for } 1 \leq i \leq m_j \end{cases}$$

since $\{\nu(\mu)/q(\mu)\}$ and $\{\beta(i)/\alpha(i)\}$ are monotonely decreasing. Hence we have
(9.10) \( \nu(j) \) is a common divisor of \( \bigcup_{\mu \in \mathbb{M}_j} \{ q(\mu), v(\mu) \} \).

Note that the greatest common divisor \( (q(\mu), v(\mu)) \) is given by
\[
(q(\mu), v(\mu)) = (q(\mu), p(\mu)\{a(\mu-1)+1\} + q(\mu)\{p-b(\mu-1)-1\}) = (q(\mu), p(\mu)\{a(\mu-1)+1\}).
\]

Hence the coprimeness condition \( (q(\mu), p(\mu)) = 1 \) yields
\[
(q(\mu), v(\mu)) = (q(\mu), a(\mu-1)+1).
\]

Thus the niceness of the subset \( \mathbb{M}_j \) implies
\[
\text{GCD} \bigcup_{\mu \in \mathbb{M}_j} \{ q(\mu), v(\mu) \} = \text{GCD} \bigcup_{\mu \in \mathbb{M}_j} \{ q(\mu), a(\mu-1)+1 \} = 1.
\]

Then (9.10) yields (9.4) as desired. The proof of Lemma 9.3 is complete. Q.E.D.

It remains to show Proposition 9.2. We note that it follows
\[
N(Q_j(t, 0, \xi)) = N(f_j(-tLp(\theta), \xi)) = N(f_j(y, \eta)) = \sum_{\mu \in \mathbb{M}_j} N_{q(\mu), p(\mu)}
\]
by virtue of Proposition 7.2, Remark 7.3 and Proposition 2.12.

Lemma 9.5. Under the following condition
\[
N(Q_j(t, 0, \xi)) = \sum_{\mu \in \mathbb{M}_j} N_{q(\mu), p(\mu)} \quad \text{for } 1 \leq j \leq r
\]
there exist irreducible Weierstrass polynomials \( Q_p(t, \xi) \in \mathbb{C}_c[t] \) (1 \( \leq \mu \leq m = \# \text{Seg } N(f) \)) such that
\[
Q_j(t, 0, \xi) = \prod_{\mu \in \mathbb{M}_j} Q_p(t, \xi) \quad \text{for } 1 \leq j \leq r, \quad \text{and}
\]
\[
N(Q_p) = N_{q(\mu), p(\mu)}.
\]

The proof of Lemma 9.5 will be given in § 14.

Let \( g_1(t), g_2(t) \) and \( h(t) \) be polynomials with coefficients in an integral domain \( \mathcal{O} \). Let \( r(g, h) \) be the resultant of \( g \) and \( h \) defined by (8.9). Then the assertion 2) in Proposition 8.6 yields \( r(g_1g_2, h) = r(g_1, h)r(g_2, h) \). Thus the assertion (9.12) in Lemma 9.5 implies
\[
R_j(x, 0, \xi) = r(Q_j(t, 0, \xi), tP(t, 0, \xi)-x) = r(\prod_{\mu \in \mathbb{M}_j} Q_p(t, \xi), tP(t, 0, \xi)-x) = \prod_{\mu \in \mathbb{M}_j} r(Q_p(t, \xi), tP(t, 0, \xi)-x) .
\]
By virtue of (9.14) and of the additivity of Newton polygons, it suffices for Proposition 9.2 to show the following

**Proposition 9.6.** For $1 \leq \mu \leq m$, we put

$$r_\mu(x_1, \xi_1) := r(Q_\mu^*(t, \xi_1), tP(t, 0, \xi_1) - x_1).$$

Then it follows that

$$N(r_\mu) = N_{q_\mu, s_\mu}.$$  (9.15)

The first step of the proof of Proposition 9.6 is to show the

**Lemma 9.7.** There exists the following inclusion:

$$N(r_\mu) \subseteq N_{q_\mu, s_\mu} \quad \text{for} \quad 1 \leq \mu \leq m.$$

**Proof.** We write the Weierstrass polynomial $Q_\mu$ as the form

$$Q_\mu^*(t, \xi_1) := \sum_{\nu=0}^{q_\mu} w_\nu(\xi_1)t^\nu \quad (w_{q_\mu}(\xi_1) \equiv 1 \text{ and } w_\nu(0) = 0 \text{ for } 0 \leq \nu < q(\mu)).$$

We also write

$$P(t, x', \xi_1) := \sum_{\nu=0}^{q_\mu} P_\nu(x', \xi_1)t^\nu \quad \text{and} \quad tP(t, 0, \xi_1) - x_1 := \sum_{\nu=0}^{q_\mu} s_\nu(\xi_1)t^\nu - x_1,$$

where we set

$$s_\nu(\xi_1) := P_\nu(0, \xi_1) \quad \text{for} \quad 0 \leq \nu \leq q - 1.$$

Remark that it is not necessarily that $s_{q-1} \equiv 0$. But if we define a $(q + q(\mu))$-square matrix $D_\mu(x_1, \xi_1)$ by

$$D_\mu(x_1, \xi_1) := \begin{bmatrix} \begin{array}{cccccccc} 1 & w_{q_\mu-1} & \cdots & w_0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & \cdots & 0 & 0 \\ s_{q-1} & s_{q-2} & \cdots & s_0 & x_1 \\ 0 & s_{q-1} & \cdots & s_0 & 0 \end{array} \end{bmatrix},$$

then we always have

$$r_\mu(x_1, \xi_1) = \det[D_\mu(x_1, \xi_1)]$$

since $Q_\mu^*(t, \xi_1)$ is a Weierstrass, hence a monic, polynomial.

We fix $\mu$ and denote the $(i, k)$-component of $D_\mu(x_1, \xi_1)$ by $d_{ik}(x_1, \xi_1)$. Then $d_{i,k}$ is given by the following (9.18) [or (9.19) resp.] for $1 \leq i \leq q$ [for $q + 1 \leq i \leq q + q(\mu)$.]
\[ q + q(\mu) \] : 
\[
(9.18) \quad d_{i,k} = \begin{cases} 
  w_{q(\mu)+1-k}(\xi_i) & \text{for } i \leq k \leq i+q(\mu) \\
  0 & \text{for } k < i \text{ or } k > i+q(\mu) 
\end{cases}
\]
\[
(9.19) \quad d_{i,k} = \begin{cases} 
  -x_i & \text{for } i = k \\
  s_{k-1-i}(\xi_i) & \text{for } i-q \leq k \leq i-1 \\
  0 & \text{for } k < i-q \text{ or } k > i 
\end{cases}
\]

By the expressions \( Q_\mu(t, \xi_i) = \sum_{\nu=0}^{q(\mu)} w_\nu(\xi_i)t^\nu \) and \( P(t, 0, \xi_i) = \sum_{\nu=0}^{q(\mu)} s_\nu(\xi_i)t^\nu \), we have
\[
(\nu, \text{ord}[w_\nu]) \in N(Q_\mu) \quad \text{and} \quad (\nu, \text{ord}[s_\nu]) \in N(P(t, 0, \xi_i)).
\]
Thus (9.13) in Lemma 9.5 and Lemma 8.3 yield the following inequalities:
\[
(9.20) \quad \begin{cases} 
  \text{ord}[w_\nu] \geq -\kappa(\mu)\nu + p(\mu) \\
  \text{ord}[s_\nu] \geq -\kappa(\mu)(\nu - a(\mu)) + p - b(\mu) - 1
\end{cases}
\]

Now we estimate the Newton polygon \( N(r_\mu) \). Since \( r_\mu(x_1, \xi_i) \) can be written as
\[
r_\mu(x_1, \xi_i) = \sum_{\pi \in \mathfrak{S}[q+q(\mu)]} \text{sgn}(\pi) \prod_{i=1}^{q+q(\mu)} d_{i,\pi(i)}(x_1, \xi_i)
\]
where \( \mathfrak{S}[n] \) denotes the symmetric group of order \( n \), it suffices to estimate
\[
N\left( \prod_{i=1}^{q+q(\mu)} d_{i,\pi(i)}(x_1, \xi_i) \right) \quad \text{for } \pi \in \mathfrak{S}[q+q(\mu)].
\]
Note that we may assume that \( \pi \in \mathfrak{S}[q+q(\mu)] \) satisfies
\[
(9.21) \quad \prod_{i=1}^{q+q(\mu)} d_{i,\pi(i)} \neq 0.
\]

Under the conditions (9.18)-(9.21), we have
\[
\prod_{i=1}^{q+q(\mu)} d_{i,\pi(i)} = \left( \prod_{i=1}^{q} w_{q(\mu)+i-\pi(i)} \right)(-x_i)^{\alpha(\pi)} \prod_{\pi(1) \neq 1}^{q+q(\mu)} s_{i-\pi(i)-1}(x_1)^{\beta(\pi)}
\]
such that
\[
(9.22) \quad \alpha(\pi) = \#\{i ; i \geq q+1, \pi(i) = i\}
\]
and that \( \beta(\pi) \) is estimated from below as follows:
\[
(9.23) \quad \beta(\pi) \geq \sum_{i=1}^{q+q(\mu)} [-\kappa(\mu)q(\mu)+i-\pi(i)] + p(\mu) + \sum_{\pi(1) \neq 1}^{q+q(\mu)} [-\kappa(\mu)i-\pi(i)-1-a(\mu)] + p - b(\mu) - 1.
\]

Note that (9.23) is equivalent to
\[
(9.24) \quad \beta(\pi) \geq \{\nu(\mu)/q(\mu)\} \{q(\mu) - \alpha(\pi)\}.
\]
Indeed, the right hand side of (9.23) is equal to
\[-\kappa(\mu) \sum_{i=1}^{q(\mu)} \{i - \pi(i)\} + \sum_{i=1}^{q} \{-\kappa(\mu)q(\mu) + p(\mu)\} + \sum_{\pi(i) \neq i} \kappa(\mu)\{a(\mu)+1\} + p - b(\mu) - 1].\]

Since the first and second terms vanish, (9.23) is equivalent to
\[(9.23)' \quad \beta(\pi) \equiv \kappa(\mu)\{a(\mu)+1\} + p - b(\mu) - 1] \times \#\{i \geq q+1, \pi(i) \neq i\}
= [\kappa(\mu)\{a(\mu)+1\} + p - b(\mu) - 1]\{q(\mu) - \alpha(\pi)\}.
\]

Note that the identity \(\kappa(\mu)q(\mu) = p(\mu)\) implies
\[\kappa(\mu)a(\mu) - b(\mu) = \kappa(\mu)a(\mu - 1) - b(\mu - 1).\]

Hence (9.23)' is equivalent to
\[\beta(\pi) \equiv [\kappa(\mu)\{a(\mu - 1)+1\} + p - b(\mu - 1) - 1]\{q(\mu) - \alpha(\pi)\}
= q(\mu)^{-1}[\{p(\mu)\{a(\mu - 1)+1\} + q(\mu)\{p - b(\mu - 1) - 1\}\}{q(\mu) - \alpha(\pi)}
= \{v(\mu)/q(\mu)\}{q(\mu) - \alpha(\pi)}\]
which shows that (9.23) is equivalent to (9.24).

Note that (9.24) is also equivalent to
\[(9.25) \quad \{\alpha(\pi)/q(\mu)\} + \{\beta(\pi)/v(\mu)\} \geq 1\]
which shows
\[N\left(\prod_{i=1}^{q(\mu)} d_{i, \pi(i)}\right) \subseteq N(x_{1}^{\alpha(\pi)}\xi_{1}^{\beta(\pi)}) \subseteq N_{q(\mu), \pi(\mu)} \quad \text{for } \pi \in \mathbb{S}[q+q(\mu)].\]

Hence the proof of Lemma 9.7 is complete. Q. E. D.

In order to complete the proof of Proposition 9.6, we must show the converse inclusion of Lemma 9.7. For this purpose it suffices to show
\[(9.26) \quad (q(\mu), 0) \subseteq N(\mu) \quad \text{and} \quad (9.27) \quad (0, v(\mu)) \subseteq N(\mu).\]

Since \(Q_{\mu}(t, \xi, 1)\) is a Weierstrass polynomial in \(t\), it follows
\[r_{\mu}(x_{1}, 0) = \det \begin{bmatrix} I_{q} & 0 \\ \ast & \ddots \\ \ast & \ast & 0 \end{bmatrix} = (-x_{1})^{q(\mu)}\]
which shows (9.26).

To show (9.27) we must consider the case that the equality holds in the inequality (9.23) with \( \alpha(\pi) = \# \{ i ; i \geq q+1, \pi(i) = i \} = 0 \). Thus we consider

\[
(9.28) \quad \text{ord}[\Sigma' \text{sgn}(\pi) \prod_{i=1}^{q+\kappa(\mu)} d_{i, \pi(i)}(x_i, \xi_i)]
\]

where \( \Sigma' \) denotes the sum of \( \pi \in \mathbb{S}[q+q(\mu)] \) satisfying

\[
(9.29) \quad \begin{cases} 
\text{ord}[w_{q(\mu)+i-\pi(i)}] = -\kappa(\mu) \cdot q(\mu) + i - \pi(i) + q(\mu) \\
i \leq \pi(i) \leq i + q(\mu) \quad \text{for all } 1 \leq i \leq q.
\end{cases}
\]

\[
(9.30) \quad \begin{cases} 
\text{ord}[s_{i-\pi(i)-1}] = -\kappa(\mu) \cdot (i - \pi(i) - 1 - a(\mu)) + p - b(\mu) - 1 \\
i - q \leq \pi(i) \leq i \quad \text{for all } q+1 \leq i \leq q+q(\mu).
\end{cases}
\]

Note that the coprimeness condition of \( N(f) \) yields that (9.29), (9.30) are equivalent respectively to

\[
(9.29)' \quad q(\mu) + i - \pi(i) = 0 \quad \text{or} \quad q(\mu) \quad (1 \leq i \leq q).
\]

\[
(9.30)' \quad i - \pi(i) - 1 = \begin{cases} 
a(\mu-1) \text{ or } a(\mu) \quad \text{if } 1 \leq \mu < m \\
a(m-1) \quad \text{if } \mu = m \quad (q+1 \leq i \leq q+q(\mu)).
\end{cases}
\]

**Case 1.** When \( \mu < m \). In this case the sum (9.28) contributes to \( r_\mu = \det D_\mu \) as the following form:

\[
(9.31) \quad (9.28) = \text{ord}[\varepsilon(\det A)(\det B)(\det C)] \quad \text{where } \varepsilon = \pm 1
\]

such that \( A, B \) and \( C \) are determined by the following (9.32)-(9.35):

\[
(9.32) \quad D_\mu = \begin{cases} 
1 \quad \text{A appears here.} \\
1 \quad \text{B appears here.} \\
1 \quad \text{C appears here.}
\end{cases}
\]

\[
(9.33) \quad q - a(\mu) \quad 2q(\mu) \quad a(\mu-1) + 1
\]

\[
(9.34) \quad w_0 \quad w_0 \quad w_0
\]

\[
(9.35) \quad s_{a(\mu)} \quad s_{a(\mu)-1}
\]
From (9.31)-(9.35) we have
\[ \text{ord}[(\det A)(\det B)(\det C)] = \{a(\mu - 1) + 1\}\text{ord}[w_0] + \text{ord}[(s_{a(\mu - 1)} - s_{a(\mu)}w_0)I_{q(\mu)}]. \]

Since \( \text{ord}[w_0] = p(\mu) \), the right hand side can be written as
\[ \{a(\mu - 1) + 1\}p(\mu) + q(\mu)\text{ord}[(s_{a(\mu - 1)} - s_{a(\mu)}w_0)]. \]

Hence, by \( \nu(\mu) := p(\mu)(a(\mu - 1) + 1) + q(\mu)(p - b(\mu - 1) - 1) \), it suffices for (9.27) to show
\[ \text{ord}[(s_{a(\mu - 1)} - s_{a(\mu)}w_0)] = p - b(\mu - 1) - 1. \]

Recall that Lemma 8.3 asserts
\[ (8.5) \quad N\left[ P(t, 0, \xi_1) - t^1\partial_{\xi_1} \text{ch}(f(-\theta L_F(e^\theta), \xi_1))d\theta \right] + N(\xi_1) \equiv N(f) \]
which implies
\[ (9.37) \quad s_{a(\mu)}(\xi_1) = \frac{p - b(\mu)}{a(\mu) + 1} \text{Loc}[c_{p-b(\mu)}](\partial_{y_n})\xi_1^{p-b(\mu)-1}(1 + O(\xi_1)) \quad \text{for } \mu < m \]
where \( c_{\nu}(y') \) denotes the \( \nu \)-th Taylor coefficient of \( f(y', \eta_1) \).

To derive (9.36) we need to write the value \( A_\mu \in C \setminus \{0\} \) defined by
\[ (9.38) \quad w_0(\xi_1) = Q_p(0, \xi_1) = A_\mu \xi_1^{p(\mu)}(1 + O(\xi_1)), \]
by means of the terms \( \{\text{Loc}[c_{p-b(\lambda)}](\partial_{y_n}) : 0 \leq \lambda \leq m\} \).

We shall prove the following

**Lemma 9.8.** It follows that
\[ (9.39) \quad A_\mu = \text{Loc}[c_{p-b(\mu - 1)}](\partial_{y_n})/\text{Loc}[c_{p-b(\mu)}](\partial_{y_n}) \quad \text{for } \mu \geq 1. \]

In order to show Lemma 9.8, we use the following properties of characteristic polynomial functions:

**Lemma 9.9.** Let \( (\Sigma, \sigma) \) be a germ of \( (n-1) \)-dimensional complex manifold at a point \( \sigma \) with \( n \geq 2 \). Let \( f, g \) be holomorphic germs on \( \Sigma \times C \) at \( (\sigma, 0) \), and let \( \text{ch}(f), \text{ch}(g) \) be the characteristic polynomial functions of \( f, g \) which are defined by Definition 7.1. Then we have
1) \( \text{ch}(\text{ch}(f)) = \text{ch}(f) \).
2) \( \text{ch}(fg) = \text{ch}(f \text{ch}(g)) = \text{ch}(g \text{ch}(f)) \).

**Proof.** Recall the following equivalence
\[
\text{ch}(g) = \text{ch}(h) \iff N(g - h) \equiv N(g) = N(h).
\]
Then the assertion 1) immediately follows from
\[
N(g - \text{ch}(f)) \equiv N(f) = N(\text{ch}(f)).
\]
Note that (9.40) and the additivity property of Newton polygons yield
\[
N(fg - f \text{ch}(g)) = N(f) + N(g - \text{ch}(g))
\]
\[
\equiv N(f) + N(g) = N(f) + N(\text{ch}(g)).
\]
Hence we have
\[
N(fg - f \text{ch}(g)) \equiv N(fg) = N(f \text{ch}(g)).
\]
Thus the assertion 2) also holds. **Q. E. D.**

**Corollary 9.10.** For holomorphic germs \( g_1, \ldots, g_k \) on \( \Sigma \times \mathbb{C} \) at \((\sigma, 0)\), it follows that
\[
(9.41) \quad \text{ch}\left( \prod_{j=1}^k g_j \right) = \text{ch}\left( \prod_{j=1}^k \text{ch}(g_j) \right).
\]

**Proof.** When \( k=1 \), (9.41) is nothing but the assertion 1) in Lemma 9.9. Thus we may assume \( k \geq 2 \). Then the assertion 2) in Lemma 9.9 implies
\[
\text{ch}\left( \prod_{j=1}^k g_j \right) = \text{ch}\left[ \left( \prod_{j=1}^{k-1} g_j \right) \text{ch}(g_k) \right].
\]
Regarding \( \left( \prod_{j=1}^{k-1} g_j \right) \text{ch}(g_k) = g_{k-1} \times \left( \prod_{j=1}^{k-1} g_j \right) \text{ch}(g_k) \), and applying 2) in Lemma 9.9, we have
\[
\text{ch}\left( \prod_{j=1}^k g_j \right) = \text{ch}\left[ \left( \prod_{j=1}^{k-2} g_j \right) \text{ch}(g_{k-1}) \text{ch}(g_k) \right].
\]
Repeating such processes, we get (9.41) as desired. **Q. E. D.**

**Proof of Lemma 9.8.** Applying Corollary 9.10 to
\[
f(0, \ldots, 0, t, \xi) = \prod_{j=1}^k f_j(0, \ldots, 0, t, \xi),
\]
we have
\[
(9.42) \quad \text{ch}[f(0, \ldots, 0, t, \xi)] = \text{ch}\left[ \prod_{j=1}^k \text{ch}\{f_j(0, \ldots, 0, t, \xi)\} \right].
\]
Note that Remark 7.3 implies

\[(9.43) \begin{cases} \text{ch}[f(0, \cdots, 0, t, \xi_i)] = \text{ch}(f(0, \cdots, 0, t, \xi_i)) \\ \text{ch}[f_j(0, \cdots, 0, t, \xi_i)] = \text{ch}(f_j(0, \cdots, 0, t, \xi_i)) \end{cases} \text{ for } 1 \leq j \leq r.\]

Recall that Proposition 7.2 asserts

\[(7.4) \quad \text{ch}(f_j(0, \cdots, 0, t, \xi_i)) = \text{ch}(B_j(t, 0, \xi_i)).\]

Since \(B_j = Q_j \epsilon_j\) with \(\epsilon_j(0) \neq 0\), Corollary 9.10 yields

\[(9.44) \quad \text{ch}(B_j(t, 0, \xi_i)) = \text{ch}[\text{ch}(\epsilon_j(t, 0, \xi_i))\text{ch}(Q_j(t, 0, \xi_i))] = \text{ch}[\epsilon_j(0)\text{ch}(Q_j(t, 0, \xi_i))].\]

Since, for a non-zero constant \(a \in C\), it is easily verified that

\[\text{ch}(ag) = a \text{ch}(g)\]

by the definition (7.3) of characteristic polynomial functions, (9.44) derives

\[(9.45) \quad \text{ch}(B_j(t, 0, \xi_i)) = \epsilon_j(0)\text{ch}(Q_j(t, 0, \xi_i)).\]

By virtue of (9.43), (9.45) and (7.4), we can write (9.42) as

\[(9.46) \quad \text{ch}(f)(0, \cdots, 0, t, \xi_i) = \text{ch} \left[ \prod_{j=1}^{r} \text{ch}(B_j(t, 0, \xi_i)) \right] = \text{ch} \left[ \prod_{j=1}^{r} \{ \epsilon_j(0)\text{ch}(Q_j(t, 0, \xi_i)) \} \right] = \epsilon \text{ch} \left[ \prod_{j=1}^{r} \text{ch}(Q_j(t, 0, \xi_i)) \right]\]

where we put \(\epsilon := \prod_{j=1}^{r} \epsilon_j(0) \in C - \{0\}\).

Now, using the irreducible decomposition

\[(9.12) \quad Q_j(t, 0, \xi_i) = \prod_{\mu \in M_j} Q_{\mu}(t, \xi_i)\]

in Lemma 9.5, we claim

\[(9.47) \quad \text{ch}(f)(0, \cdots, 0, t, \xi_i) = \epsilon \text{ch} \left[ \prod_{\mu=1}^{m} \text{ch}(Q_{\mu}(t, \xi_i)) \right].\]

Indeed, by (9.46), (9.12) and Corollary 9.10, we have

\[\text{ch}(f)(0, \cdots, 0, t, \xi_i) = \epsilon \text{ch} \left[ \prod_{j=1}^{r} Q_j(t, 0, \xi_i) \right] = \epsilon \text{ch} \left[ \prod_{j=1}^{r} \prod_{\mu \in M_j} Q_{\mu}(t, \xi_i) \right] = \epsilon \text{ch} \left[ \prod_{\mu=1}^{m} \text{ch}(Q_{\mu}(t, \xi_i)) \right].\]

Hence (9.47) follows.
Since \( N(Q_p) = N_q(p, p) \), the coprimeness condition yields

\[(9.48) \quad \text{ch}(Q_p) = t_p + A_p \xi_1(p) \]

where \( A_p \) is the non-zero constant defined by (9.38).

We note that Lemma 0.2 implies

\[(9.49) \quad \text{ch} \left[ \prod_{\mu=1}^m \text{ch}(Q_p(t, \xi_1)) \right] = \text{ch} \left[ \prod_{\mu=1}^m (t_p + A_p \xi_1(p)) \right] = \sum_{\mu=1}^m \left( \prod_{\lambda=1}^\mu g_\lambda \right) \left( \prod_{\lambda=\mu+1}^m A_\lambda \xi_1(p) \right). \]

On the other hand, we have the following expression:

\[(9.50) \quad \text{ch}(f)(0, \ldots, 0, t, t, t) = \sum_{\mu=0}^m \text{Loc}[c_{p-b(\mu)}(\partial_{g_\mu})^a(p) \xi_1^{p-b(\mu)}] \]

where \( c_v(y) \) is the \( v \)-th Taylor coefficient of \( f(y', \eta) \).

By virtue of (9.49) and (9.50), we conclude that (9.47) is equivalent to

\[\text{Loc}[c_{p-b(\mu)}(\partial_{g_\mu})^a(p) \xi_1^{p-b(\mu)}] = \xi_1^{p-b(\mu)-1} \left( A_{\mu+1} A_{\mu+2} \cdots A_m \right) \quad \text{for} \quad 0 \leq \mu \leq m.\]

Hence we get the formula (9.39) as desired.

The proof of Lemma 9.8 is complete.

Proof of Proposition 9.6 (continued). By virtue of the expression (9.37) of \( s_{a_\mu}(\xi) \) and of Lemma 9.8, we calculate the left hand side of (9.36) as follows:

\[s_{a_\mu}(\xi) - s_{a_\mu} \omega_0 = \frac{b-b(\mu-1)}{a(\mu-1)+1} \text{Loc}[c_{p-b(\mu-1)}(\partial_{g_\mu}) \xi_1^{p-b(\mu)-1}] + O(\xi_1) \]

\[\frac{b-b(\mu)}{a(\mu)+1} \text{Loc}[c_{p-b(\mu)}(\partial_{g_\mu}) \xi_1^{p-b(\mu)-1}] + O(\xi_1) \]

\[= \xi_1^{p-b(\mu)-1} \left[ \frac{b-b(\mu-1)}{a(\mu-1)+1} \text{Loc}[c_{p-b(\mu-1)}(\partial_{g_\mu})] \right] \text{Loc}[c_{p-b(\mu)}(\partial_{g_\mu})] \]

\[= \xi_1^{p-b(\mu)-1} \left[ \frac{b-b(\mu-1)}{a(\mu-1)+1} \frac{b-b(\mu)}{a(\mu)+1} \text{Loc}[c_{p-b(\mu-1)}(\partial_{g_\mu})] \right] \]

Thus it suffices for (9.36) to verify

\[(p-b(\mu-1))(a(\mu)+1)-(p-b(\mu))(a(\mu-1)+1) \neq 0.\]
But this is trivial since
\[ p - b(\mu - 1) > p - b(\mu) \quad \text{and} \quad a(\mu) + 1 > a(\mu - 1) + 1. \]
Hence we get (9.36) which implies (9.27) as desired in the case 1.

Case 2. When \( \mu = m \). In this case the sum (9.28) contributes to \( r_\mu \) as the form

\[
\begin{align*}
(9.51) & \quad (9.28) = \text{ord}[\varepsilon (\det I_{q(m-1)})(\det B)(\det C^-)] \\
(9.52) & \quad B = \omega B_{a(m-1)+1} + (\text{nilpotent}) \\
(9.53) & \quad C^- = s_{a(m-1)}I_{q(m)}
\end{align*}
\]

such that \( B \) and \( C^- \) are determined by the following (9.52)-(9.54):

\[
\begin{align*}
(9.52) & \quad B = \omega B_{a(m-1)+1} + (\text{nilpotent}) \\
(9.53) & \quad C^- = s_{a(m-1)}I_{q(m)}
\end{align*}
\]

From (9.52) and (9.53), we have

\[
\text{ord}[(\det I_{q(m-1)})(\det B)(\det C^-)] = \{a(m-1)+1\} \text{ord} [\omega B_{a(m-1)+1}] + q(m) \text{ord} [s_{a(m-1)}],
\]

\[
= \{a(m-1)+1\} \rho(m) + q(m) \{p - b(m-1) - 1\} = v(m).
\]

Thus (9.51) implies (9.27) as desired also in the case 2.

The proof of Proposition 9.6, hence that of Proposition 9.2, is complete.

Q.E.D.
§ 10. One-sheetedness of Map Germs $\pi_{j,2}$

In this section we prove Theorem 6.11 which asserts that the finite map germs $\pi_{j,2} : V_{j,1} \rightarrow V_{j,2}$ are germs of one-sheeted analytic coverings of $V_{j,2}$ for $1 \leq j \leq r$. Our proof starts from the following Euclidean algorithm:

**Definition 10.1.** Put $\mathcal{O} := \mathcal{O}_{M \times C, (x, \xi)}$ and let $\mathcal{K}$ be the quotient field of $\mathcal{O}$, that is, the field which consists of germs of meromorphic functions at $(x, \xi) = (0, 0)$. We fix $j$ and we define a finite sequence $\{s^*_\nu(t); 1 \leq \nu \leq k\}$ of polynomials in $\mathcal{K}[t]$ as follows: For $\nu = 1, 2$, we put

\[
\begin{cases}
    s^*_1(t) := tP(t, x', \xi), & s^*_2(t) := Q_j(t, x', \xi), \\
    s^*_1(t) := Q_j(t, x', \xi), & s^*_2(t) := tP(t, x', \xi) - x_1, 
\end{cases}
\]

if $1 + \deg tP \geq q_j = \deg Q_j$.

For $\nu \geq 3$, we inductively define $s^*_\nu(t), \sigma^*_\nu(t) \in \mathcal{K}[t]$ by the following division in $\mathcal{K}[t]$ (the Euclidean algorithm):

\[
s^*_{\nu} - s^*_{\nu-1} = \sigma^*_\nu s^*_{\nu-1} + s^*_\nu 
\]

such that $\deg(s^*_{\nu}) < \deg(s^*_{\nu-1})$.

Since $\mathcal{K}[t]$ is a Euclidean ring, the division (10.2) determines $\{(s^*_\nu, \sigma^*_\nu)\}$ for $3 \leq \nu \leq k$ where $k$ is the integer satisfying

\[
\deg(s^*_k) \leq \deg(s^*_k) < \cdots < \deg(s^*_1) \leq \deg(s^*_1).
\]

Since $\deg(s^*_k) = \min\{q_j, 1 + \deg tP\} \geq 1$, it follows that $k \geq 3$.

**Lemma 10.2.** For $3 \leq \nu \leq k$, there exist $f^*_\nu, g^*_\nu \in \mathcal{K}[t]$ such that

\[
s^*_\nu = f^*_\nu s^*_\nu + g^*_\nu s^*_1
\]

\[
\begin{cases}
    \deg(f^*_\nu) = \deg(s^*_1) - \deg(s^*_\nu-1) \\
    \deg(g^*_\nu) = \deg(s^*_1) - \deg(s^*_\nu-1). 
\end{cases}
\]

**Proof.** By induction on $\nu$: For $\nu = 3$, since

\[
s^*_3 = -\sigma_3 s^*_2 + s^*_1
\]

we can take $f^*_3 = -\sigma_3, \ g^*_3 = 1$. Indeed, $\deg(g^*_3) = \deg(1) = 0 = \deg(s^*_2) - \deg(s^*_3)$ is trivial, and the inequality

\[
\deg(s^*_3) < \deg(s^*_2) \leq \deg(s^*_1)
\]

implies

\[
\deg(f^*_3) = \deg(\sigma_3) = \deg(s^*_1) - \deg(s^*_3).
\]

Next, for $\nu = 4$, since

\[
s^*_4 = s^*_2 - \sigma_4 s^*_3 = s^*_2 - \sigma_4 (\sigma_3 s^*_2 + s^*_1) = (\sigma_3 \sigma_4 + 1) s^*_2 - \sigma_4 s^*_1
\]
We can take \( f^\sim = \sigma_4 \sigma_4 + 1, \ g^\sim = -\sigma_4 \). Indeed, we have
\[
\deg(f^\sim) = \deg(\sigma_4 \sigma_4 + 1) = \deg(s_\sim^2) - \deg(s_\sim^1) \quad \text{and} \\
\deg(g^\sim) = \deg(-\sigma_4) = \deg(s_\sim^2) - \deg(s_\sim^1).
\]

Now let \( \nu \geq 5 \). By the inductive assumption it follows that
\[
s^\sim_\nu = s^\sim_{\nu - 2} - \sigma_4 s^\sim_{\nu - 1}
\]
\[
= f^\sim_{\nu - 2} s^\sim_2 + g^\sim_{\nu - 2} s^\sim_3 - \sigma_4 (f^\sim_{\nu - 2} s^\sim_2 + g^\sim_{\nu - 2} s^\sim_3)
\]
\[
= (f^\sim_{\nu - 2} - \sigma_4 f^\sim_{\nu - 1}) s^\sim_2 + (g^\sim_{\nu - 2} - \sigma_4 g^\sim_{\nu - 1})s^\sim_3.
\]
Thus it suffices for (10.4) to put
\[
f^\sim_\nu := f^\sim_{\nu - 2} - \sigma_4 f^\sim_{\nu - 1}, \quad g^\sim_\nu := g^\sim_{\nu - 2} - \sigma_4 g^\sim_{\nu - 1}.
\]
Then the inductive assumption yields
\[
\deg(f^\sim_{\nu - 2}) = \deg(s^\sim_1) - \deg(s^\sim_{\nu - 2}) \quad \text{and} \\
\deg(\sigma_4 f^\sim_{\nu - 1}) = \deg(\sigma_4) + \deg(f^\sim_{\nu - 1})
\]
\[
= \deg(s^\sim_{\nu - 2}) - \deg(s^\sim_{\nu - 2}) + \deg(s^\sim_1) - \deg(s^\sim_{\nu - 2})
\]
\[
= \deg(s^\sim_1) - \deg(s^\sim_{\nu - 2}).
\]
Since the inequalities (10.3) and \( \nu - 3 \geq 2 \) imply
\[
\deg(s^\sim_{\nu - 1}) < \deg(s^\sim_{\nu - 2}) < \deg(s^\sim_{\nu - 3}),
\]
we get
\[
\deg(f^\sim_\nu) = \deg(\sigma_4 f^\sim_{\nu - 1}) = \deg(s^\sim_1) - \deg(s^\sim_{\nu - 1}).
\]
The similar argument also yields
\[
\deg(g^\sim_\nu) = \deg(\sigma_4 g^\sim_{\nu - 1}) = \deg(s^\sim_1) - \deg(s^\sim_{\nu - 1}).
\]
Hence we get Lemma 10.2. Q. E. D.

**Lemma 10.3.** For \( 3 \leq \nu \leq k \), there exist \( f_\nu, g_\nu \in \mathcal{O}[t] \) and \( c_\nu, d_\nu \in \mathcal{O} - \{0\} \) such that if we set \( s_\nu := (d_\nu/c_\nu) s^\sim_\nu \) then the following (10.6)–(10.8) hold:
\[
(10.6) \quad s_\nu = f_\nu s^\sim_2 + g_\nu s^\sim_3.
\]
\[
(10.7) \quad \deg(f_\nu) = \deg(s^\sim_1) - \deg(s^\sim_{\nu - 1}), \quad \deg(g_\nu) = \deg(s^\sim_1) - \deg(s^\sim_{\nu - 1}).
\]
\[
(10.8) \quad \text{The polynomial } h_\nu := t^{q_\nu} f_\nu + g_\nu \text{ is a primitive polynomial, that is, there exists no non-unit common divisor of all coefficients of } h_\nu \in \mathcal{O}[t], \text{ where we put } \quad q_\nu := \deg(s^\sim_1) = \min\{q_0, 1 + \deg(P)\}.
\]
Proof. Let $f_\sim, g_\sim \in \mathcal{O}[t]$ be the polynomials in Lemma 10.2. We put
\[ h_\sim := t^N f_\sim + g_\sim \in \mathcal{O}[t]. \]
Note that, by (10.5) in Lemma 10.2 and $\deg(s_{\sim i}) \geq 1$, we have
\[ \deg(g_\sim) \leq q^\sim - 1. \]
Since $\mathcal{O}$ is a unique factorization domain, this $h_\sim$ can be written as the form
\[ h_\sim(t) = \sum_{i=0}^{N} (b_{i\nu}/a_{i\nu}) t^{i} \quad (N := q^\sim + \deg(f_\sim)) \]
where $a_{i\nu}, b_{i\nu} \in \mathcal{O}$ are taken such as $a_{i\nu}$ and $b_{i\nu}$ have no non-unit common divisor, that is, they are coprime for $0 \leq i \leq N$. We take
\[ c_{\nu} := \text{GCD}\{b_{0\nu}, b_{1\nu}, \ldots, b_{N\nu}\} \]
where the notation GCD denotes the greatest common divisor (Note that such $c_{\nu}$ is uniquely determined up to unit elements, since $\mathcal{O}$ is a unique factorization domain.). Then we have
\[ h_\sim/c_{\nu} = \sum_{i=0}^{N} (b'_{i\nu}/a_{i\nu}) t^{i} \]
where $b'_{i\nu}$ and $a_{i\nu}$ are coprime for $0 \leq i \leq N$, and where
\[ \text{GCD}\{b'_{0\nu}, b'_{1\nu}, \ldots, b'_{N\nu}\} = 1 \quad \text{up to unit elements}. \]
Now we take
\[ d_{\nu} := \text{LCM}\{a_{0\nu}, a_{1\nu}, \ldots, a_{N\nu}\} \]
where LCM denotes the least common multiplier. Then it follows that
\[ (d_{\nu}/c_{\nu}) h_\sim \in \mathcal{O}[t] \quad \text{and is a primitive polynomial}. \]
Thus, if we set
\[ f_{\nu} := (d_{\nu}/c_{\nu}) f_\sim, \quad g_{\nu} := (d_{\nu}/c_{\nu}) g_\sim \]
then (10.9) and (10.10) yield $f_{\nu}, g_{\nu} \in \mathcal{O}[t]$. Such constructions of $c_{\nu}, d_{\nu}$ and of $f_{\nu}, g_{\nu}$ easily imply the desired conditions (10.6)-(10.8).

The proof of Lemma 10.3 is complete. Q. E. D.

Definition 10.4. We define $s_\nu(t) \in \mathcal{O}[t]$ for $1 \leq \nu \leq k$ as follows:
1) For $\nu = 1, 2$, we set
\[ s_{1} := s^\sim, \quad s_{2} := s^\sim. \]
2) For $3 \leq \nu \leq k$, we define $s_{\nu} := (d_{\nu}/c_{\nu}) s^\sim$ by Lemma 10.3, where $k$ and $s^\sim$ are defined by Definition 10.1.

Proposition 10.5. For the finite sequence $\{s_{\nu}(t); 1 \leq \nu \leq k\} \subset \mathcal{O}[t]$ given by Definition 10.4, it follows that
\[ \deg(s_{\nu-1}) = 1. \]
\[ 10.11 \]
Proof. We prove (10.11) by contradiction. Note that if \( q_j = \text{deg}(s_0) = 1 \) then \( k = 3 \) follows, from \( \text{deg}(s_2) = \text{deg}(s_0^2) < \text{deg}(s_0^2) = q_j = 1 \). Thus (10.11) holds if \( q_j = 1 \). Hence we may assume that \( q_j = \text{deg}(s_0^2) \geq 2 \).

We assume that the conclusion (10.11) is not true. Then (10.7) in Lemma 10.3 yields

\[
\begin{align*}
\text{deg}(f_k) = & \text{deg}(s_1) - \text{deg}(s_{k-1}) \leq \text{deg}(s_1) - 2 \\
\text{deg}(g_k) = & \text{deg}(s_2) - \text{deg}(s_{k-1}) \leq \text{deg}(s_2) - 2.
\end{align*}
\]

Note that the following inequalities hold:

\[
\begin{align*}
\text{deg}(s_1) = & \text{deg}(s_0^2) = \begin{cases} \text{deg}(tP-x_1) \leq q & \text{if } s_1 = tP-x_1, \\
\text{deg}(Q_j) \leq q_j & \text{if } s_1 = Q_j. 
\end{cases} \\
\text{deg}(s_2) = & \text{deg}(s_0^2) = \begin{cases} \text{deg}(Q_j) \leq q_j & \text{if } s_2 = tP-x_1, \\
\text{deg}(tP-x_1) \leq q & \text{if } s_2 = Q_j. 
\end{cases}
\end{align*}
\]

By virtue of (10.12)-(10.14), we can write \( f_k(t)s(t) + g_k(t)s(t) \) as the following form:

\[
f_k s_2 + g_k s_1 = \begin{cases} f_k Q_j + g_k (tP-x_1) & \text{if } s_1 = tP-x_1, \ s_2 = Q_j, \\
f_k (tP-x_1) + g_k Q_j & \text{if } s_2 = Q_j, \ s_2 = tP-x_1 
\end{cases}
= \left( \sum_{i=0}^{q-2} e_i t^i \right) Q_j + \left( \sum_{i=0}^{q-2} e_i' t^i \right) (tP-x_1) \quad (\text{where } e_i, e_i' \in \mathcal{O}).
\]

Let us recall the \((q+q_j)\)-square matrix \( D_j = D(Q_j, tP-x_1) \) defined by Notation 8.8 and Definition 8.5. We note the

Remark 10.6. Let \( a_i \in \mathcal{O} \ (0 \leq i \leq q-1) \) and \( b_i \in \mathcal{O} \ (0 \leq i \leq q_j-1) \). Then the following 1) and 2) are equivalent:

1) \( \left( \sum_{i=0}^{q-1} a_i t^i \right) Q_j + \left( \sum_{i=0}^{q-1} b_i t^i \right) (tP-x_1) = \sum_{i=0}^{q+q_j-1} c_i t^i. \)

2) \( (c_{q+q_j-1}, c_{q+q_j-2}, \ldots, c_0) = (a_{q-1}, \ldots, a_0; b_{q_j-1}, \ldots, b_0) D_j. \)

Remark 10.6 and (10.15) yield that the relation

\[
s_k = f_k s_2 + g_k s_1 \in \mathcal{O} \cap (s_1, s_2) \mathcal{O} \left[ t \right],
\]

which is (10.6) for \( \nu = k \), can be written as the following form:

\[
(0, \ldots, 0, s_k) = (0, e_{q-2}, \ldots, e_0; 0, e_{q_j-2}, \ldots, e_0') D_j.
\]

Let \( \delta_i \) be the \((i, q+q_j)\)-cofactor of \( D_j \) and let \( D_j \) be the cofactor matrix of \( D_j \). Operating \( D_j \) to (10.16) we have

\[
(0, \ldots, 0) D_j = (0, e_{q-2}, \ldots, e_0; 0, e_{q_j-2}, \ldots, e_0') R_j I_{q+q_j}.
\]
since $D_jD_j^t=(\det D_j)I_{q+q_j}=R_jI_{q+q_j}$.

On the other hand, $D_jD_j=D_jI_{q+q_j}$ yields
\[ \delta \eta Q_j^t-\delta \eta Q_jx_1=R_j, \]
where $Q_j(x', \xi')$ denotes the coefficient of degree 0 in $t$ of $Q_j(t, x', \xi')$. Hence we have
\[ (10.17) \quad \delta \eta \not\in (R_j) \text{ or } \delta \eta Q_jx_1 \not\in (R_j). \]
Indeed, if we assume that (10.17) is false then we can find $\delta \eta$, $\delta \eta Q_jx_1$ such that $\delta \eta = R_j \delta \eta$, $\delta \eta Q_jx_1 = R_j \delta \eta Q_jx_1$. Hence we get
\[ (1-\delta \eta Q_jx_1)R_j=0. \]
Since $R_j \neq 0$, it follows $1-\delta \eta Q_jx_1 \neq 0$. This is a contradiction because $Q_j(0, 0)=0$. Thus (10.17) is true.

Note that (10.16)' is equivalent to
\[ (10.16)^* \quad s_b(\delta_1, \cdots, \delta_{q+q_j})=R_j(0, e_{q-2}, \cdots, e_0; 0, e_{q-2}', \cdots, e_0'). \]
Thus (10.17) yields
\[ (10.18) \quad s_b \subseteq (R_j) \]
since the ideal $(R_j)$ is a prime ideal of $\mathcal{O}$ (Theorem 9.1).

We return to the equality (10.16). Let $D_j'$ be the $(q+q_j-2)$-square matrix which is obtained by excluding from $D_j$ the first column and row and the $(q+1)$-th column and row. Then (10.16) can be written as the form
\[ (10.19) \quad (0, \cdots, 0, s_b)=(e_{q-2}, \cdots, e_0; e_{q-2}', \cdots, e_0')D_j'. \]
Restricting (10.19) on $\{R_j=0\}$, and using (10.18), we have
\[ (10.20) \quad 0=(e_{q-2}, \cdots, e_0; e_{q-2}', \cdots, e_0')D_j'|_{R_j=0}. \]
Since
\[ h_b=t\delta f_b+g_b=\sum_{i=0}^{q-2} e_it^{t+q}+\sum_{i=0}^{q-2} e_it^t \quad (\text{if } s_1=tP-x_1) \]
\[ \quad \sum_{i=0}^{q-2} e_it^{t+1+\deg P}+\sum_{i=0}^{\deg P-1} e_it^t \quad (\text{if } s_1=Q_j) \]
is a primitive polynomial (Lemma 10.3), it follows that
\[ (10.21) \quad (e_{q-2}, \cdots, e_0; e_{q-2}', \cdots, e_0')|_{(R_j=0)} \neq 0. \]
By virtue of (10.20) and (10.21), we get
\[ \det(D_j)|_{(R_j=0)} \equiv 0. \]
Hence the Rückert’s Nullstellensatz yields $\det(D_j) \subseteq \text{Rad}[(R_j)]$. Since $(R_j)$ is a prime ideal, we get
But (10.22) is a contradiction. Indeed, by the definition of the matrix $D'_j$, it follows that
\[
\text{ord}[(\det D'_j(x_1, x', \xi_i)|(x', \xi_i) = 0)] = q_j - 1
\]
since $Q_j(t, x', \xi_i)$ is a Weierstrass polynomial in $t$. On the other hand, we know
\[
\text{ord}[R_j(x_1, 0, 0)] = q_j.
\]
Thus we have
\[
(\det D'_j(x_1, 0, 0) \not\in (R_j(x_1, 0, 0))
\]
which contradicts (10.22). This contradiction comes from our assumption $\deg(s_{j-1}) \geq 2$. Hence it follows that $\deg(s_{j-1}) = 1$ as desired.

The proof of Proposition 10.5 is complete. Q. E. D.

By virtue of Proposition 10.5, the polynomial $s_{k-1}(t)$ can be written as
\[
s_{k-1}(t, x, \xi_i) = a(x, \xi_i)t + b(x, \xi_i).
\]
In this situation we have the

**Proposition 10.7.** It follows
\[
a(x, \xi_i)|_{x_i=0} \equiv 0.
\]

*Proof.* We first show the assertion (10.24) in the case $k=3$.
If $k=3$ then Proposition 10.5 yields
\[
\deg(s_2) = 1.
\]
By Definitions 10.4 and 10.1, we have
\[
s_2 = \begin{cases} 
Q_j & \text{if } q_j \leq 1 + \deg_j P, \\
-tP - x_1 & \text{if } q_j > 1 + \deg_j P.
\end{cases}
\]
Thus, in the case $q_j \leq 1 + \deg_j P$, we conclude that the Weierstrass polynomial $Q_j$ has degree one. Hence $a(x, \xi_i) \equiv 1$ holds. On the other hand, in the case $q_j > 1 + \deg_j P$, we get $\deg P = 0$ which derives $a(x, \xi_i) \equiv P$. Since the Claim 2) in the proof of Proposition 8.9 shows that the leading coefficient of $P$ does not vanish identically on $\{R_j = 0\}$, (10.24) also holds in the case $q_j > 1 + \deg_j P$. Hence Proposition 10.7 holds if $k=3$.

Now we prove Proposition 10.7 in the case $k \geq 4$. We assume that the conclusion (10.24) is not true. Then, since $(R_j)$ is a prime ideal of $\mathcal{O} := \mathcal{O}_{M, A, (0, 0)}$, the Rückert's Nullstellensatz yields $a(x, \xi_i) \in (R_j)$. Hence we can find a germ $\tilde{a} \in \mathcal{O}$ such that
\[
a = R_j\tilde{a}.
\]
By virtue of (10.25) with the assumption \( k-1 \geq 3 \), Lemma 10.3 yields
\[ R_j \partial_t + b = s_{k-1} = f_{k-1}s_2 + g_{k-1}s_1 \in (Q_j, tP-x_1)\mathcal{O}[t]. \]
Since \( R_j \in (Q_j, tP-x_1)\mathcal{O}[t] \) (Corollary 8.7), we get
\[ (10.26) \quad b \in \mathcal{O} \cap (Q_j, tP-x_1)\mathcal{O}[t]. \]
By Proposition 8.6, (10.26) yields that there exist \( e_i \in \mathcal{O} \) \((1 \leq i \leq q+q_j)\), \( e \in \mathcal{O} \) and \( d \in \mathcal{O} - \{0\} \) such that
\[ (10.27) \quad b = Q_j e_q - x_1 e_{q+q_j} \quad \text{and} \quad (10.28) \quad e_i = (e/d) \delta_i \quad \text{for } 1 \leq i \leq q+q_j \]
where \( \delta_i \) denotes the \((i, q+q_j)\)-cofactor of the matrix \( D_j \) in Notation 8.8.

Recall the relation
\[ (10.29) \quad R_j = \delta Q_j \delta - \delta_{q+q_j} x_1 \]
which is a consequence of \( D_j^2 D_j = R_j D_{q+q_j} \) \((D^2 \text{ denotes the cofactor matrix of } D_j)\).
By (10.27)-(10.29) we have
\[ (10.30) \quad db = Q_j e_q - x_1 d e_{q+q_j} = Q_j e_q - x_1 e Q_j \delta = e R_j. \]

Claim 10.8. It follows that \( b \in (R_j) \).

Proof. Since \((R_j)\) is a prime ideal of \( \mathcal{O} \), if we assume that Claim 10.8 is false then (10.30) yields \( d \in (R_j) \), that is, there exists a germ \( d^\sim \in \mathcal{O} \) such that \( d = R_j d^\sim \). Thus (10.28) can be written as
\[ R_j d^\sim e_i = d e_i = e \delta_i \quad \text{for } 1 \leq i \leq q+q_j. \]
Hence, by (10.17), we get \( e \in (R_j) \). But we can choose \( d, e \in \mathcal{O} \) in (10.28) such as \( d \) and \( e \) are coprime since \( \mathcal{O} \) is a unique factorization domain. Thus it is a contradiction that both \( d \) and \( e \) lie in \((R_j)\). Hence Claim 10.8 follows.

Q.E.D.

We continue the proof of Proposition 10.7. Recall the relation
\[ (10.31) \quad s_{k-1} = f_{k-1}s_2 + g_{k-1}s_1 \]
which is a consequence of Lemma 10.3. Since the assumption \( k \geq 4 \) yields \( \deg(s_{k-2}) > \deg(s_{k-1}) = 1 \), we have
\[
\begin{cases}
\quad \deg(f_{k-1}) = \deg(s_{k-1}) - \deg(s_{k-2}) \leq \deg(s_1) - 2 \\
\quad \deg(g_{k-1}) = \deg(s_{k-1}) - \deg(s_{k-2}) \leq \deg(s_2) - 2.
\end{cases}
\]
Thus the inequalities (10.13, 14) imply that there exist \( c_i, c_i^\prime \in \mathcal{O} \) such that
\[ f_{k-1}s_2 + g_{k-1}s_1 = \left( \sum_{t=0}^{q-2} c_i t^i \right) Q_j + \left( \sum_{t=0}^{q-2} c_i^\prime t^i \right) (tP-x_1). \]
Then Remark 10.6 yields that (10.31) can be written as

\[(0, \ldots, 0, a, b) = (0, c_{q-2}, \ldots, c_s; 0, c'_{q-2}, \ldots, c')D_j.\]

Recall the \((q+q_j-2)\)-square matrix \(D'_j\) obtained by excluding the first column and row, and the \((q+1)\)-th column and row of \(D_j\), which is used in the proof of Proposition 10.5. Then the relation (10.32) can be written as the form

\[(0, \ldots, 0, a, b) = (c_{q-2}, \ldots, c_s; c'_{q-2}, \ldots, c')D'_j.\]

Restricting (10.32)' on \(\{R_j=0\}\) and using \(a, b \in (R_j)\) we have

\[0 = (c_{q-2}, \ldots, c_s; c'_{q-2}, \ldots, c')D'_j \mid_{R_j=0}\]

which yields

\[(10.33) \quad (\det D'_j) \mid_{R_j=0} = 0\]

since the polynomial \(h_{k-1} = t^{q_j}f_{k-1} + g_{k-1}\) is a primitive polynomial. Then Rückert's Nullstellensatz and the primeness of \((R_j)\) imply that (10.33) is equivalent to

\[(10.34) \quad \det D'_j \in (R_j).\]

But (10.34) is a contradiction since

\[\det D'_j(x_1, 0, 0) \in (x_1)^{q_j-1}, \notin (R_j(x_1, 0, 0)) = (x_1)^{q_j}\]

as like as in the proof of Proposition 10.5. This contradiction comes from our assumption that Proposition 10.7 is not true. Hence Proposition 10.7 follows.

The proof of Proposition 10.7 is complete. Q. E. D.

As a corollary of Propositions 10.5 and 10.7, we get Theorem 6.11:

**Proof of Theorem 6.11.** For the polynomial \(s_{k-1}(t) = a(x, \xi) + b(x, \xi)\), we define a map germ \(\rho : (M \times C, (0, 0)) \to (C \times M \times C, (0, 0))\) by setting as

\[(10.35) \quad \rho(x, \xi) := \left(-\left\{b(x, \xi)/a(x, \xi)\right\}, x, \xi\right).\]

We show that \(\rho\) induces a meromorphic inverse \(V_{j_2} \to V_{j_1}\) of the map germ \(\pi_{j_2} : V_{j_1} = \{tP-x_1 = Q_j = 0\} \to V_{j_2} = \{R_j = 0\}\).

Note that \(V_{j_2} - \{a(x, \xi) = 0\} \neq \emptyset\) (Proposition 10.7) implies that the intersection germ \(\Sigma_j := V_{j_2} \cap \{a = 0\}\) is a germ of nowhere dense analytic subset of \(V_{j_2}\). Let \((x^0, \xi^0) \in V_{j_2} - \Sigma_j\). Since \(\pi_{j_2}\) is an open map germ, that is, \(\pi_{j_2}\) is surjective to \((V_{j_2}, (0, 0))\), there exists \(t^0 \in (C, 0)\) such that \((t^0, x^0, \xi^0) \in V_{j_1}\). Then we have

\[(10.36) \quad a(x^0, \xi^0)t^0 + b(x^0, \xi^0) = s_{k-1}(t^0, x^0, \xi^0) = 0\]

by virtue of \(s_{k-1} = f_{k-1}s_{k} + g_{k-1}s_1 \in (tP-x_1, Q_j)\). Since \(a(x^0, \xi^0) \neq 0\), the equation (10.36) yields

\[(t^0, x^0, \xi^0) = \rho(x^0, \xi^0)\]
which shows that the induced map germ
\[ \pi_{j_2} : V_{j_1} - \pi_{j_2}(\Sigma_j) \rightarrow V_{j_2} - \Sigma_j \]
is a biholomorphic map germ.

It only remains to show that
\[ (10.37) \quad \pi_{j_2}(\Sigma_j) \] is a germ of a nowhere dense analytic subset of \( V_{j_1} \).

But this is easy: Since \( \pi_{j_2} \) is an open map germ at \((0, 0, 0)\), we have
\[ \pi_{j_2}(\Sigma_j) \neq V_{j_1}. \]

Then the irreducibility of \( V_{j_1} \) at \((0, 0, 0)\) yields (10.37).

The proof of Theorem 6.11 is complete. Q.E.D.

Chapter IV. Appendices

§ 11. Generalities of Newton Polygons

In this section we summarize basic facts on Newton polygons. The aim of this section is to give proofs of Propositions 2.11 and 2.12.

Let \( S \) be a domain in \( C^{n-1}(n \geq 2) \) which contains the origin throughout this section. For a holomorphic germ
\[ f(y, \tau) = \sum_{\nu \in \mathbb{N}^n} c_\nu(y)\tau^\nu \in \mathcal{O}_{S \times \mathbb{C}, (0, \circ)}(c_\nu \in \mathcal{O}_{S, \circ}) \]
we define its Newton polygon \( N(f) \), the strict boundary \( \partial N(f) \) of \( N(f) \), and segments and vertices of \( N(f) \), by Definition 2.3. We also use Notation 2.4.

**Definition 11.1.** Let \( N \) be a Newton polygon. For a vertex \( A \in \text{Ver } N \):

1) We define the left \([\text{or right, resp.}]\) segment \( L(A) \) \([R(A)\text{, resp.}]\) of \( A \) as follows:

We arrange vertices of \( N \) as
\[ \{L(A) = (a(\mu), p-b(\mu)) : 0 \leq \mu \leq m\} \]
where finite sequences \( \{a(\mu)\}, \{b(\mu)\} \) are monotonely increasing in \( \mu \). We set
\[ L(A(\mu)) = \begin{cases} \{tA(\mu)+(1-t)A(\mu-1) : 0 \leq t \leq 1\} & \text{if } \mu \geq 1 \\ A(0)+0 \times \mathbb{R}_+ & \text{if } \mu = 0. \end{cases} \]
\[ R(A(\mu)) = \begin{cases} \{tA(\mu)+(1-t)A(\mu+1) : 0 \leq t \leq 1\} & \text{if } \mu < m \\ A(m)+\mathbb{R}_+ \times 0 & \text{if } \mu = m. \end{cases} \]

2) We set \( \kappa(L(A)) \) \([\text{or, } \kappa(R(A))\text{, resp.}] \in \mathbb{Q}_+ \cup \{\infty\} \) by
\[ \kappa(L(A(\mu))) = \begin{cases} \kappa(\mu) & \text{if } \mu \geq 1 \\ \infty & \text{if } \mu = 0 \end{cases} \quad \kappa(R(A(\mu))) = \begin{cases} \kappa(\mu) & \text{if } \mu < m \\ 0 & \text{if } \mu = m \end{cases} \]
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First we show the

**Proposition 11.2.** Let \( f, g \) be germs on \( (S \times C, (0, 0)) \) and let \( A \equiv \text{N}(f), B \equiv \text{N}(g) \). Then the following statements are equivalent:

1) \( A + B \in \text{Ver}(\text{N}(f) + \text{N}(g)) \).
2) \( A \in \text{Ver} \, \text{N}(f), B \in \text{Ver} \, \text{N}(g) \) such that

\[
\min \{ \kappa(L(A)), \kappa(L(B)) \} > \max \{ \kappa(R(A)), \kappa(R(B)) \}.
\]  

**Proof.** Note that the vector sum \( \text{N}(f) + \text{N}(g) \) is a closed, \( \mathbb{R}^n \)-invariant, convex set, thus we can define \( \text{Ver}(\text{N}(f) + \text{N}(g)) \).

We first show 1) \( \Rightarrow \) 2). Since 1) yields

\[
A + B \in \text{Ver}(\text{N}(f) + \text{N}(g)) \subseteq \partial(\text{N}(f) + \text{N}(g))
\]

we easily have \( A \in \partial \text{N}(f) \) and \( B \in \partial \text{N}(g) \).

To show \( A \in \text{Ver} \, \text{N}(f) \), we derive the following implication (11.2):

\[
A = tA' + (1 - t)A''(A', A'' \in \text{N}(f) \text{ with } A' \neq A'' \text{ and } t \in [0, 1])
\]

\[
\Rightarrow t = 0 \text{ or } t = 1.
\]

Indeed, \( A + B \in \text{Ver}(\text{N}(f) + \text{N}(g)) \) yields the implication

\[
A + B = t(A' + B) + (1 - t)(A'' + B) \Rightarrow t = 0 \text{ or } t = 1
\]

since \( A' + B, A'' + B \in \text{N}(f) + \text{N}(g) \) with \( A' + B \neq A'' + B \) and \( t \in [0, 1] \). Hence (11.2) follows, that is, \( A \in \text{Ver} \, \text{N}(f) \). We similarly have \( B \in \text{Ver} \, \text{N}(g) \).

Now we show the inequality (11.1) under the assumptions

\[
A \in \text{Ver} \, \text{N}(f), B \in \text{Ver} \, \text{N}(g) \text{ and } A + B \in \text{Ver}(\text{N}(f) + \text{N}(g))
\]

Since \( \kappa(L(A)) > \kappa(R(A)) \) and \( \kappa(L(B)) > \kappa(R(B)) \) are trivial, (11.1) is equivalent to

\[
\kappa(L(A)) > \kappa(R(B)) \text{ and } \kappa(L(B)) > \kappa(R(A)).
\]

By the symmetricity of \( A \) and \( B \), we only have to check the first inequality of (11.4). Note that in the case \( \kappa(L(A)) = \infty \) or \( \kappa(R(B)) = 0 \) the assertion is trivial, hence we may assume \( \kappa(L(A)) < \infty \) and \( \kappa(R(B)) > 0 \), that is, both \( L(A) \) and \( R(B) \) are segments in the sense of Definition 2.3.

We can therefore write \( L(A) \) and \( R(B) \) as

\[
L(A) = \{ tA + (1 - t)A' ; t \in [0, 1] \}
\]

\[
R(B) = \{ tB + (1 - t)B' ; t \in [0, 1] \}
\]

where \( A' \in \text{Ver} \, \text{N}(f), B' \in \text{Ver} \, \text{N}(g) \) with \( A \neq A', B \neq B' \). Note that the first inequality of (11.4) is equivalent to the following (11.5) (see figure):
(11.5) \( A+B \in \text{the closed half plane given by the closure of the upper side of the line joining } \ A' + B \text{ and } A+B' \).

We note that \( A+B \in \text{Ver}(N(f)+N(g)) \) implies (11.5), since \( A'+B, A+B' \) both belong to \( N(f)+N(g) \). Hence we have the first inequality of (11.4) as desired. Thus 1) implies 2).

Conversely we assume 2). It suffices for 1) to show that there exist two lines \( \sigma, \sigma' \) passing through \( A+B \) with non-positive distinct slopes such that

\[
(11.6) \quad N(f)+N(g) \subseteq \text{the closure of } \sigma \cap \text{(the upper side of } \sigma').
\]

We construct \( \sigma, \sigma' \) as the following forms:

\[
\begin{align*}
\sigma & := A+B + \{(x,y); y = -cx\} \\
\sigma' & := A+B + \{(x,y); y = -c'x\} \\
c & := \min\{\kappa(L(A)), \kappa(L(B))\} > c' := \max\{\kappa(R(A)), \kappa(R(B))\}
\end{align*}
\]

where, in the case \( c = \infty \), the line “\( y = -cx \)” denotes the vertical line \( x = 0 \), and “the upper side of \( \sigma \)” denotes the right side of \( \sigma \).

Let \( \nu \) or \( \nu' \) resp. be a linear functional on \( \mathbb{R}^2 \) which takes positive values on the upper side of the line \( y = -cx \) \( y = -c'x \). Then the inequality (11.1) yields the following inclusions:

\[
\begin{align*}
N(f) & \subseteq \{A'; \nu(A') \geq \nu(A) \text{ and } \nu'(A') \geq \nu'(A)\} \\
N(g) & \subseteq \{B'; \nu(B') \geq \nu(B) \text{ and } \nu'(B') \geq \nu'(B)\}
\end{align*}
\]

which imply

\[
N(f)+N(g) \subseteq \{A'+B'; \nu(A'+B') \geq \nu(A+B) \text{ and } \nu'(A'+B') \geq \nu'(A+B)\}.
\]

Hence we get (11.6) as desired.

The proof of Proposition 11.2 is complete. Q.E.D.

**Proposition 11.3** (the additivity property of Newton polygons). For any holomorphic germs \( f \) and \( g \) on \( (S \times C, (0,0)) \), it follows that

\[
(11.7) \quad N(fg) = N(f) + N(g).
\]

**Proof.** We take the Taylor expansions
\[ f(y, \tau) = \sum_{\nu=0}^{\infty} c_\nu(y)\tau^\nu, \quad g(y, \tau) = \sum_{\nu=0}^{\infty} d_\nu(y)\tau^\nu. \]

Of course we have \((fg)(y, \tau) = \sum_{\nu=0}^{\infty} \left( \sum_{\lambda=0}^{\nu} c_\lambda d_{\nu-\lambda} \right)\tau^\nu.\]

We first show

\[(11.8) \quad N(fg) \subseteq N(f) + N(g). \]

It suffices for \((11.8)\) to prove

\[(11.8)' \quad \left( \text{ord}\left[ \sum_{\lambda=0}^{\nu} c_\lambda d_{\nu-\lambda} \right], \nu \right) \in N(f) + N(g) \quad \text{if} \quad \sum_{\lambda=0}^{\nu} c_\lambda d_{\nu-\lambda} \neq 0 \]

since \(N(f) + N(g)\) is a convex, \(R^+_\nu\)-invariant set.

Assume \(\sum_{\lambda=0}^{\nu} c_\lambda d_{\nu-\lambda} \neq 0\). Then we have

\[(11.9) \quad \sum_{\lambda=0}^{\nu} c_\lambda d_{\nu-\lambda} \geq \min \{ \text{ord}[c_\lambda] + \text{ord}[d_{\nu-\lambda}] \}. \]

Choosing \(\lambda'(0 \leq \lambda' \leq \nu)\) to attain the right hand side of \((11.9)\), we have

\[
\left( \text{ord}\left[ \sum_{\lambda=0}^{\nu} c_\lambda d_{\nu-\lambda} \right], \nu \right) \in (\text{ord}[c_{\lambda'}], \lambda') + (\text{ord}[d_{\nu-\lambda'}], \nu - \lambda') + R^+_\nu \subseteq N(f) + N(g). \]

Hence the inclusion \((11.8)\) follows.

Now we prove the converse inclusion of \((11.8)\). It suffices to show

\[(11.10) \quad \text{Ver}(N(f) + N(g)) \subseteq N(fg). \]

Let \(A \in N(f), B \in N(g)\) satisfy \(A + B \in \text{Ver}(N(f) + N(g))\). By virtue of Proposition 11.2, we may assume there exist \(\lambda'(0 \leq \lambda' \leq \nu)\) such that

\[ A = (\text{ord}[c_{\lambda'}], \lambda') \in \text{Ver} N(f), \quad B = (\text{ord}[d_{\nu-\lambda'}], \nu - \lambda') \in \text{Ver} N(g). \]

We must show

\[(11.11) \quad \text{ord}\left[ \sum_{\lambda=0}^{\nu} c_\lambda d_{\nu-\lambda} \right] = \text{ord}[c_{\lambda'}] + \text{ord}[d_{\nu-\lambda'}]. \]

Note that it suffices for \((11.11)\) to verify

\[(11.12) \quad \text{ord}[c_\lambda] + \text{ord}[d_{\nu-\lambda}] > \text{ord}[c_{\lambda'}] + \text{ord}[d_{\nu-\lambda'}] \]

for all \(\lambda(\neq \lambda'), 0 \leq \lambda \leq \nu\).

Since \((\text{ord}[c_\lambda], \lambda) \subseteq N(f), (\text{ord}[d_{\nu-\lambda}], \nu - \lambda) \subseteq N(g)\), we have the following inequalities:

\[(11.13) \quad \begin{cases} \lambda \geq -\kappa(L(A))(\text{ord}[c_{\lambda'}] - \text{ord}[c_{\lambda'}]) + \lambda' \quad \text{and} \\ \nu - \lambda \geq -\kappa(R(B))(\text{ord}[d_{\nu-\lambda'}] - \text{ord}[d_{\nu-\lambda'}]) + \nu - \lambda' \end{cases} \quad \text{for} \quad \lambda' \leq \forall \lambda \leq \nu. \]
Consider the case $\lambda > \lambda'$. The inequalities (11.13) yield
\[
\text{ord}[c_{\lambda}] + \text{ord}[d_{\lambda'}] \
\geq \text{ord}[c_{\lambda'}] + \text{ord}[d_{\lambda}] + (\lambda - \lambda') \{1/\kappa(R(B)) - 1/\kappa(L(A))\}
\]
\[
> \text{ord}[c_{\lambda}] + \text{ord}[d_{\lambda}]
\]
since $\kappa(L(A)) > \kappa(R(B))$. In the case $\lambda < \lambda'$, the inequalities (11.14) imply (11.12) as similar as the case $\lambda > \lambda'$. Hence (11.11) follows. Thus we have (11.10) as desired. The proof of Proposition 11.3 is complete. Q. E. D.

**Corollary 11.4.** Let $A \in N(f)$, $B \in N(g)$ such that
\[
A + B \in \text{Ver}(N(f) + N(g)) = \text{Ver} N(fg).
\]
Then it follows that the left segment $L(A + B)$ of $A + B$ is given by
\[
L(A + B) = \begin{cases} 
A + L(B) & \text{if } \kappa(L(A)) > \kappa(L(B)), \\
B + L(A) & \text{if } \kappa(L(A)) = \kappa(L(B)), \\
L(A) + L(B) & \text{if } \kappa(L(A)) = \kappa(L(B)) = \infty.
\end{cases}
\]

**Proof.** We classify the proof in the following three cases:

- **Case 1** $\kappa(L(A)) \neq \kappa(L(B))$, **Case 2** $\kappa(L(A)) = \kappa(L(B)) < \infty$ and
- **Case 3** $\kappa(L(A)) = \kappa(L(B)) = \infty$.

First we consider the case 1. By the symmetricity of the roles $f$ and $g$, we may assume $\kappa(L(A)) > \kappa(L(B))$. Since $\kappa(L(B)) < \infty$, $L(B)$ is a segment of $N(g)$. Hence we can find $B' \in \text{Ver} N(g)$ such that $L(B)$ can be written as
\[
L(B) = \{tB + (1-t)B' \mid t \in [0,1]\}.
\]
Since the proof of 2)⇒1) in Proposition 11.2 shows that
\[
N(f) + N(g) \subseteq A + B + \{(x, y) \mid y \geq -\kappa(L(B))x\}
\]
it suffices for $L(A + B) = A + L(B)$ to show
\[
A + B' \in \text{Ver}(N(f) + N(g)).
\]
The relation $R(B') = L(B)$ and the inequality (11.1) yield
\[
\kappa(L(A)) > \kappa(L(B)) = \kappa(R(B'))
\]
and
\[
\kappa(L(B')) > \kappa(R(B')) = \kappa(L(B)) > \kappa(R(A)).
\]
Hence, by Proposition 11.2, we get (11.16) as desired.

Next we consider the case 2. Since both $L(A)$, $L(B)$ are segments, we can write them as

$$L(A) = \{tA + (1-t)A'; \ t \in [0, 1]\} \quad \text{and} \quad L(B) = \{tB + (1-t)B'; \ t \in [0, 1]\}$$

where $A' \neq A \in \text{Ver}(N(f))$, $B' \neq B \in \text{Ver}(N(g))$. Then it follows

$$\kappa(L(A')) > \kappa(L(A)) = \kappa(L(B)) = \kappa(R(B'))$$

and

$$\kappa(L(B')) > \kappa(L(B)) = \kappa(L(A)) = \kappa(R(A')).$$

Hence Proposition 11.2 yields that $A' + B' \in \text{Ver}(N(f) + N(g))$ as desired.

In the case 3, it is trivial that

$$(11.17) \quad N(f) + N(g) \subset A + B + \mathbb{R}_+ \times \mathbb{R}$$

which shows $L(A + B) = A + B + 0 \times \mathbb{R}_+ = L(A) + L(B)$.

The proof of Corollary 11.4 is complete. Q. E. D.

Remark 11.5. Proposition 2.11 follows from Proposition 11.3 and Lemma 0.2.

Proof. The additivity property of Newton polygons immediately implies

$$N\left(\prod_{j=1}^{r} f_j(y, \tau)^{\nu(j)}\right) = \sum_{j=1}^{r} \nu(j)N(f_j).$$

Thus the left equality of Proposition 2.11 is a direct consequence of Proposition 11.3. Hence it suffices for Proposition 2.11 to verify

$$(11.17) \quad N(f^g) = \sum_{\rho=1}^{m} N_{q(\rho), p(\rho)}.$$

But this immediately follows from Lemma 0.2. Indeed, since we assume $p(1)/q(1) > \cdots > p(m)/q(m)$ in Notation 2.4, we get

$$\text{Ver}\left(\sum_{\rho=1}^{m} N_{q(\rho), p(\rho)}\right) = \text{Ver}(f^g)$$

which shows (11.17). Q. E. D.

Now we recall the coprimeness condition (Definition 2.5).

Proposition 11.6. Let $f$, $g$ be holomorphic germs on $(S \times C, (0, 0))$ satisfying $f(0, \tau)g(0, \tau) \neq 0$. Then the following statements are equivalent:

1) $N(fg)$ satisfies the coprimeness condition.

2) Both $N(f)$, $N(g)$ satisfy the coprimeness condition, and the following condition holds:

$$\kappa(L(A)) \neq \kappa(L(B))$$

(11.18) for all $A \in \text{Ver}(N(f))$, $B \in \text{Ver}(N(g)$ satisfying

$$\kappa(L(A)) < \infty, \quad \kappa(L(B)) < \infty.$$
Proof. Since $(fg)(y, 0) = f(y, 0)g(y, 0)$, we only have to consider the second condition in Definition 2.5.

We assume that $N(fg)$ satisfies the coprimeness condition. We first show the condition (11.18) by contradiction. We assume that there exist $A \in N(f)$, $B \in N(g)$ such that

\[(11.19) \quad \kappa(L(A)) = \kappa(L(B)) < \infty.\]

Then it follows

\[\kappa(L(A)) = \kappa(L(B)) > \kappa(R(B))\]

and

\[\kappa(L(B)) = \kappa(L(A)) > \kappa(R(A)).\]

Hence $A + B \in \text{Ver } N(fg)$ by Propositions 11.2 and 11.3. Then Corollary 11.4 yields $L(A + B) = L(A) + L(B)$, which shows that $N(fg)$ does not satisfy the coprimeness condition. This contradicts the assumption. Hence we have the implication “$1 \Rightarrow (11.18)$” in Proposition 11.6.

Next we prove, under (11.18), that $N(f)$ and $N(g)$ both satisfy the coprimeness condition if and only if $N(fg)$ satisfies the same condition. Note that $N(f)$ and $N(g)$ can be written as the form (11.20), since $f(y, 0)g(y, 0) \not= 0$ and $f(0, \tau)g(0, \tau) \not= 0:

\[(11.20) \quad \begin{cases} N(f) = \sum_{\rho = 1}^{m_1} N_{q_1(\rho), p_1(\rho)}, & N(g) = \sum_{\nu = 1}^{m_2} N_{q_2(\nu), p_2(\nu)}, \\ p_1(1)/q_1(1) > \cdots > p_1(m_1)/q_1(m_1) > 0, \\ p_2(1)/q_2(1) > \cdots > p_2(m_2)/q_2(m_2) > 0. \end{cases}\]

Since the condition (11.18) is equivalent to

\[(11.21) \quad p_1(\mu)/q_1(\mu) \not= p_2(\nu)/q_2(\nu) \quad \text{for all} \quad 1 \leq \mu \leq m_1, \ 1 \leq \nu \leq m_2,

Proposition 11.3 yields that

\[(11.22) \quad N(fg) = N(f) + N(g) = \sum_{\rho = 1}^{m_1} N_{q_1(\rho), p_1(\rho)} + \sum_{\nu = 1}^{m_2} N_{q_2(\nu), p_2(\nu)}\]

with the distinct ratios $\{p_1(\mu)/q_1(\mu)\}_{1 \leq \mu \leq m_1} \cup \{p_2(\nu)/q_2(\nu)\}_{1 \leq \nu \leq m_2}$. Hence we get the following equivalence under the condition (11.18):

Both $N(f)$ and $N(g)$ satisfy the coprimeness condition.

\[(11.23) \quad \Leftrightarrow \begin{cases} \text{Both } p_1(\mu) \text{ and } q_1(\mu) \text{ are coprime for } 1 \leq \mu \leq m_1, \text{ and} \\ \text{Both } p_2(\nu) \text{ and } q_2(\nu) \text{ are coprime for } 1 \leq \nu \leq m_2. \end{cases}\]

\[\Leftrightarrow \quad N(fg) = N(f) + N(g) \text{ satisfies the coprimeness condition.}\]

By virtue of (11.23) and the implication “the condition 1)$\Rightarrow$(11.18)” we get the desired equivalence between 1) and 2) in Proposition 11.6.

The proof of Proposition 11.6 is complete. Q. E. D.
Proof of Proposition 2.12. We may assume our situation be as follows:

Let $S$ be a domain in $\mathbb{C}^n \setminus \{0\}$ containing the origin. Let $f(y, \tau) \in \mathcal{O}_{S \times \mathbb{C}, (0,0)}$ be a holomorphic germ which has a finite order $p \in [1, \infty)$ with respect to $\tau$.

Let us denote the irreducible decomposition of $f$ locally at $(0, 0)$ by

\begin{equation}
(11.24) \quad f = \prod_{j=1}^r f^{(j)}_y.
\end{equation}

We assume that the Newton polygon $N(f)$ satisfies the coprimeness condition, that is, the following two conditions hold:

\begin{equation}
(11.25) \quad N(f) \cap (\mathbb{R} \times 0) \neq \emptyset.
\end{equation}

\begin{equation}
(11.26) \quad p(\mu) \text{ and } q(\mu) \text{ are coprime for } 1 \leq \mu \leq m := \# \text{Seg } N(f).
\end{equation}

Recall that the integers $p(\mu)$ and $q(\mu)$ in (11.26) are defined by

\begin{equation}
(11.27) \quad \text{Write } \text{Ver } N(f) = \{(a(\mu), p-b(\mu)); 0 \leq \mu \leq m\} \text{ with }
\end{equation}

\begin{equation}
\begin{aligned}
0 = a(0) < a(1) < \cdots < a(m) = q = \text{ord}_0[f(y, 0)] \\
0 = b(0) < b(1) < \cdots < b(m) = p = \text{ord}_0[f(0, \tau)]
\end{aligned}
\end{equation}

and put $p(\mu) = b(\mu) - b(\mu - 1)$ and $q(\mu) = a(\mu) - a(\mu - 1)$.

It suffices for Proposition 2.12, to show the following

\begin{equation}
(11.28) \quad \begin{cases}
1) \quad \nu(j) = 1 \quad \text{for } 1 \leq j \leq \tau, \\
2) \quad N(f_j) \text{ satisfies the coprimeness condition for } 1 \leq j \leq \tau, \\
3) \quad \text{There exist subsets } M_j \text{ of } \{1, 2, \ldots, m\} (1 \leq j \leq \tau) \text{ such that}
\end{cases}
\end{equation}

\begin{equation}
\begin{cases}
(a) \quad M_j \cap M_k = \emptyset \quad \text{if } j \neq k, \\
(b) \quad \{1, 2, \ldots, m\} = \bigcup_{j=1}^\tau M_j, \\
(c) \quad N(f_j) = \sum_{\mu \in M_j} N_{q(\mu), p(\mu)} \quad \text{for all } 1 \leq j \leq \tau.
\end{cases}
\end{equation}

We first show the assertion 2) in (11.28). We fix $j$, and put

\begin{equation}
(11.29) \quad g_j := f^{(j)}_y - \prod_{\mu \in M_j} f^{(\mu)}_y.
\end{equation}

Then, applying Proposition 11.6 to $f = f_y g_j$, we get the coprimeness of $N(f_j)$.

Next we show the assertion 1). Assume that 1) is not true, then there is a number $j (1 \leq j \leq \tau)$ such that $\nu(j) \geq 2$. For such a number $j$, we put

\begin{equation}
(11.30) \quad g^*_j := f^{(j)}_y - \prod_{\mu \in M_j} f^{(\mu)}_y.
\end{equation}

Then, applying Proposition 11.6 to $f = f^*_y g^*_j$, we have

\begin{equation}
(11.31) \quad N(f^*_j) \text{ satisfies the coprimeness condition.}
\end{equation}

Then, applying Proposition 11.6 to $f^*_j$, we get
(11.30) \[ \{ \kappa(L(A)) \neq \kappa(L(B)) \] for all \( A, B \subseteq \text{Ver}(f_j) \) satisfying \( \kappa(L(A)), \kappa(L(B)) < \infty \).

But (11.30) is impossible for \( A = B \) satisfying \( \kappa(L(A)) < \infty \) (note that the existence of such a vertex \( A \) is a consequence of \( \#\text{Ver}(f_j) \geq 2 \)). This contradiction comes from the assumption \( \nu(j) \geq 2 \). Thus we get the assertion 1) in (11.28). Now we prove the assertion 3) in (11.28). By virtue of 1) in (11.28), we have the following irreducible decomposition of \( f \) locally at \( (0, 0) \):

(11.24) \[ f = \prod_{j=1}^r f_j. \]

Since each \( \text{N}(f_j) \) satisfies the coprimeness condition, we can write \( \text{N}(f_j) \) as the following form for \( 1 \leq j \leq r \):

(11.31) \[ \{ \text{N}(f_j) = \sum_{s=1}^{m_j} N_{s,(\nu_j)}, p_j(\nu_j) \} \]

\( p_j(1)/q_j(1) > p_j(2)/q_j(2) > \cdots > p_j(m_j)/q_j(m_j) > 0 \).

Claim 11.7. If \( j \neq k \) then it follows that

(11.32) \[ \{ p_j(\nu)/q_j(\nu) \}_{1 \leq \nu \leq m_j} \cap \{ p_k(\nu')/q_k(\nu') \}_{1 \leq \nu' \leq m_k} = \emptyset. \]

Proof. If contrary, there exist numbers \( j \) and \( k(j \neq k) \) and \( \nu, \nu' \) with \( 1 \leq \nu \leq m_j, 1 \leq \nu' \leq m_k \) such that

(11.33) \[ p_j(\nu)/q_j(\nu) = p_k(\nu')/q_k(\nu'). \]

Then Proposition 11.6 yields that \( \text{N}(f_j f_k) \) does not satisfy the coprimeness condition. Hence, regarding \( f = (f_j f_k \chi \prod_{l \neq j, k} f_l) \), Proposition 11.6 leads us to \( \text{N}(f) \) also does not satisfy the coprimeness condition, which contradicts the assumption of Proposition 2.12. Thus Claim 11.7 follows. Q.E.D.

By virtue of Proposition 11.3 and Claim 11.7, we get

(11.34) \[ \text{N}(f) = \sum_{j=1}^r \text{N}(f_j) = \sum_{j=1}^r \sum_{s=1}^{m_j} N_{s,(\nu_j), p_j(\nu)} \]

with the distinct ratios \( \bigcup_{j=1}^r \{ p_j(\nu)/q_j(\nu) \}_{1 \leq \nu \leq m_j} \).

Comparing (11.34) with

\[ \{ p(j)/q(j) \}_{1 \leq j \leq m} \]

we get the following disjoint decomposition of the set of ratios:

(11.35) \[ \{ p(\mu)/q(\mu) \}_{1 \leq \mu \leq m} = \bigcup_{j=1}^r \{ p_j(\nu)/q_j(\nu) \}_{1 \leq \nu \leq m_j}. \]
Note that the coprimeness conditions of $N(f)$ and $N(f_j)$ yield the following equivalence:

\[(p_1) = p_j(v) \quad \text{and} \quad (q_1) = q_j(v)\]

Thus, putting $M_j = \{1, 2, \ldots, m\}$ for $1 \leq j \leq r$ as

$$M_j = \{\mu; \exists v (1 \leq v \leq m_j) \text{ such that } p_1(q_1) = p_j(q_j)\},$$

we conclude the assertion 3) in (11.28) as follows: Indeed, the assertions (a) and (b) in 3) follow from the decomposition (11.35). Then the assertion (c) follows from (11.31) and (11.36).

The proof of Proposition 2.12 is complete. Q. E. D.

\section*{12. Proof of Lemma 7.5}

In this section we prove Lemma 7.5.

Recall the notation: let $t \mapsto \mathcal{H}^\infty(t, \gamma', \eta_1) = (X; \mathcal{E}, Z(t, \gamma', \eta_1)$ be the characteristic curve of $F(x; \xi, z) = G(x; \xi', \xi''; z) = -n$, such that

$$\mathcal{H}(0, \gamma', \eta_1) = (0, \gamma', \eta_1; \eta_1 d x_1, 0) \in E = T^{\#} M \times \{0\}.$$ We assume the assumptions \[B.1\]-\[B.4\] of Theorem 5.1. For simplicity of notations, we denote the variables $(\gamma', \eta_1) \in T^{\#} M \cong C^{n-1} \times C$ by $(y, \eta)$.

**Lemma 12.1.** We consider the following commutative diagram of holomorphic map germs:

\[
\begin{array}{ccc}
(C, t^0) & \xrightarrow{Y} & (C^N, y^0) \\
\downarrow h & & \downarrow H \\
(C, z^0) & & \\
\end{array}
\]

Then, for $\forall i \geq 1$ it follows

\[
(1/i! \partial)^i h(t) = \sum_{\alpha \in \mathbb{Z}^d} (1/\alpha!) (\partial^\alpha H)(Y(t))
\]

\[
\times \sum_{(i, j, \lambda)} \prod_{j=1}^N \prod_{\lambda=1}^{\alpha_j} (1/\alpha_j)(\partial_{(i, \lambda)} Y_j)(t)
\]

where $\{i(j, \lambda); 1 \leq j \leq N, 1 \leq \lambda \leq \alpha_j\}$ runs through the following set:

\[
\left\{ \begin{array}{l}
i(j, \lambda) \geq 1 \quad \text{for any } (j, \lambda), \text{ and} \\
\sum_{j=1}^N \sum_{\lambda=1}^{\alpha_j} i(j, \lambda) = i.
\end{array} \right.
\]

**Proof.** Let $t = (C, t^0)$, $y = Y(t) \in (C^N, y^0)$. We take the Taylor expansions of $H$ [or $Y$ resp.] at $y$ [at $t$]:

\[
\]
Substituting \( y_j = Y_j(t), y^ \equiv Y_j(C) \), we have the following expression of \( h(t) \):

\[
h(t) = H(Y(t)) + \sum_{i=1}^{N} (1/\alpha) \partial_y^i H(Y(t)) \{ Y(t) - Y(t^\prime) \}^\alpha
\]

Thus the coefficient of \((t-t^\prime)^i\) in the expansion of \( h(t) \) is given by

\[
\sum_{i=1}^{N} (1/\alpha) \partial_y^i H(Y(t^\prime)) \prod_{j=1}^{\sigma} \prod_{\lambda=1}^{\alpha_j} (1/\alpha(j, \lambda)) \partial_t^{(j, \lambda)} Y_j(t^\prime)(t-t^\prime)^{(j, \lambda)} \alpha_j.
\]

Hence \( \alpha \) runs through \( 1 \leq |\alpha| \leq i \).

The proof of Lemma 12.1 is complete. Q.E.D.

Now we prove Lemma 7.5. Since the assertion 6) in Lemma 7.5 is trivial by the fact \( \partial_{\xi} F = -1 \), we only have to prove the assertions 1)-5).

We use induction on \( i \geq 1 \). Note that

\[
i = \sum_{j=1}^{N} \sum_{\lambda=1}^{\alpha_j} i(j, \lambda) \geq \sum_{j=1}^{\alpha} \alpha_j = |\alpha|.
\]

Hence it suffices to prove Lemma 7.5 for \( 1 \leq i \leq q \).

\textbf{Step 0.} When \( i = 1 \).

1) Since \( \partial_x X(0, y, \eta) = \partial_x F(0, y, \eta, d x, 0) = \partial_y f(y, \eta) \), it follows

\[
N[(\eta) \{ \partial_x X(0, y, \eta) - \partial_y f(y, \eta) \}] = N(0) = \emptyset \subseteq N(f).
\]

2) For \( 2 \leq j \leq n \), we have \( \mathcal{E}_j(0, y, \eta) = 0 \) which yields

\[
\partial_y \mathcal{E}_j(0, y, \eta) = -\mathcal{E}_j(0, y, \eta) \partial_y f(0, y; \eta, d x, 0) - \partial_x f(0, y; \eta, d x, 0)
\]

\[
= -\partial_y f(y, \eta).
\]

Hence we get

\[
N[(\eta) \partial_y \mathcal{E}_j(0, y, \eta)] = N[(\eta) \partial_y f(y, \eta)] \subseteq N(f).
\]

3) Since \( \mathcal{E}_j(0, y, \eta) = \delta_{ij} \eta \) (\( \delta_{ij} \) is a Kronecker's delta) for \( 1 \leq j \leq n \), we get
\[ \partial_t Z(0, y, \eta) = \eta \partial_{x_1} F(0, y; \eta dx_1, 0) = \eta \partial_{y} f(y, \eta). \]

Thus we have

\[ N[\partial_t Z(0, y, \eta)] \subseteq N(f). \]

4) For \( 2 \leq j \leq n-1 \), the assumption [B.1] yields \( \partial_{x_j} F(0, y; \eta dx_1, 0) \in (y, \eta) \) hence we get

\[ \partial_t X_j(0, y, \eta) = \partial_{x_j} F(0, y; \eta dx_1, 0) \in (y, \eta). \]

5) Since \( \partial_t Z(0, y, \eta) = -\eta \partial_{x} F(0, y; \eta dx_1, 0) - \partial_{x_1} F(0, y; \eta dx_1, 0) \equiv -\partial_{x_1} F(0, y; 0, 0) \mod(\eta) \)

the assumption [B.4], that is, \( \text{ord}[F(x; 0, 0)] = q \) implies

\[ \partial_t Z(0, y, \eta) \in (y)^{q-1} + (\eta). \]

The assertions 1)-5) in Lemma 7.5 have been proved for \( i = 1 \).

Next let \( i \geq 2 \). We assume that Lemma 7.5 is true for \( i' \) (\( 1 \leq i' \leq i-1 \)).

Step 1. Proof of the assertion 1).

We use the following

Notation 12.2. Set \( Z_+ := N \cup \{0\} \). For a multi-index

\[ (\alpha; \beta, k) = (\alpha, \alpha_n; \beta_1, \beta'_1; k) \in Z^*_1 \times Z^*_n \times Z_+ \]

we set

\[ (12.4) F^{(\alpha, \beta')}_{(\alpha_n, \beta_1)}(y, \eta) := \partial_{x_{\alpha_n}} \partial_\xi \partial_{x_1} F(x; \xi, z)(x; \xi, z = (0, y; \eta dx_1, 0) \]

Note that the sub-index \( (\alpha_n, \beta_1) \) is associate with the tangential variables \( (x', x_n, \xi_1) \) of \( E = T^*_M \times \{0\} \equiv T^*_M \).

According to Lemma 12.1, it follows

\[ (12.5) (1/i!) \partial_t X_i(0, y, \eta) \]

\[ = (1/i) [(1/(i-1)!)] \partial_t^{i-1} \partial_{x_1} F(X; \xi, Z)] \big|_{t=0} \]

\[ = \frac{(1/i)}{\prod_{1 \leq |\alpha| + |\beta| + k \leq i-2}} (\alpha \beta k)^{-1} F^{(\alpha, \beta')}_{(\alpha_n, \beta_1)} \times \sum_{(u(f, \lambda), d(f, \lambda))} K(\alpha, \beta, k; \{u(j, \lambda), d(j, \lambda)\}) \]

where \( K(\alpha, \beta, k; \{u(j, \lambda), d(j, \lambda)\}) \) is given by
and where \( \{u(j, \lambda), d(j, \lambda)\} \) runs through the following set:

\[
\{ u(j, \lambda) \geq 1, \quad d(j, \lambda) \geq 1 \quad \text{and} \quad \sum_{j=1}^{n+1} u(j, \lambda) + \sum_{j=1}^{n+1} d(j, \lambda) = i-1 \}.
\]

We classify the proof of the step 1 into the following four cases:

- **Case 1.1.** \( \alpha + |\beta'| + k \geq 1 \).
- **Case 1.2.** \( \alpha + |\beta'| + k = 0, \quad \beta_i \geq 1 \).
- **Case 1.3.** \( \alpha + |\beta'| + k + \beta_i = 0, \quad |\alpha'| \geq 1 \).
- **Case 1.4.** \( \alpha + |\beta'| + k + \beta_i + |\alpha'| = 0 \).

**Case 1.1.** In this case we note that \( K(\alpha, \beta, k; \{u(j, \lambda), d(j, \lambda)\}) \) belongs to the ideal

\[
(\partial^{(1, \lambda)}_t X_{i-1} + \sum_{j=2}^{n} (\partial^{(j, \lambda)}_t \Xi_j) + (\partial^{(n+1, \lambda)}_t Z) |_{t=0}.
\]

Thus it follows

\[
(\partial^{(1, \lambda)}_t X_{i-1}) + \sum_{j=2}^{n} (\partial^{(j, \lambda)}_t \Xi_j) + (\partial^{(n+1, \lambda)}_t Z) |_{t=0}.
\]

Note that (12.7) yields

\[
i-1 \geq \sum_{j=1}^{n+1} u(j, \lambda) \geq u(j, \lambda) \quad \text{for} \quad 1 \leq j \leq n+1
\]

and that the three brackets \([ \ ]_{\omega_0} (\nu = 1, 2, 3)\) in the rightest hand side of (12.8) satisfy

\[
N([ \ ]_{\omega_0}) \subset N(f)
\]

which is a consequence of the inductive assumption. Hence we get
Case 1.2. In this case, note that

\[ K(\alpha, \beta, k; \{u(j, \lambda), d(j, \lambda)\}) \subseteq \bigcap_{j=1}^{\beta_1} (\partial_{1,1}^{\alpha(j, X)} \Xi_1 |_{t=0}). \]

Hence it follows

\[
(y)^{i-1}(\eta)(F(a^0, 0, 0, \alpha_n, \beta_{1+1})(K(\alpha, \beta, k; \{u(j, \lambda), d(j, \lambda)\})) \\
\subseteq (y)^{i-1} - \sum (1, 1, 1, 1) - (\eta)(F(a^0, 0, 0, \alpha_n, \beta_{1+1})) \bigcap_{j=1}^{\beta_1} \bigcap_{j=1}^{\beta_1} [(y)^{g-1} + (\eta)]
\]

by the inductive assumption. Note that

\[
\bigcap_{j=1}^{\beta_1} [(y)^{g-1} + (\eta)] \subseteq (y)^{g-1} + (\eta)^{\beta_1}
\]

which yields

\[
(12.9) (y)^{i-1}(\eta)(F(a^0, 0, 0, \alpha_n, \beta_{1+1})(K(\alpha, \beta, k; \{u(j, \lambda), d(j, \lambda)\})) \\
\Rightarrow (y)^{i-1} - \sum (1, 1, 1, 1) - (\eta)(F(a^0, 0, 0, \alpha_n, \beta_{1+1})) [(y)^{g-1} + (\eta)^{\beta_1}].
\]

We first observe

\[
(12.10) N[(y)^{i-1} - \sum (1, 1, 1, 1) - (\eta)(F(a^0, 0, 0, \alpha_n, \beta_{1+1})(\eta)^{\beta_1}] \subseteq N(f).
\]

Indeed, we can write

\[
(12.11) N[(y)^{i-1} - \sum (1, 1, 1, 1) - (\eta)(F(a^0, 0, 0, \alpha_n, \beta_{1+1})(\eta)^{\beta_1}] \\
= N[(y)^{i-1} - \sum (1, 1, 1, 1) + N[(\eta)^{\beta_1+1}F(a^0, 0, 0, \alpha_n, \beta_{1+1}] \\
= N[(y)^{i-1} - \sum (1, 1, 1, 1) + N[\beta_1+1] + N[y^\alpha y^\alpha + \eta \beta_1+1] \\
\]

by virtue of the additivity of Newton polygons. Note that

\[ i - 1 - \sum_{d} d(1, \lambda) - 1 |\alpha^n| - \alpha_n \geq i - 1 - \sum_{j,d} d(j, \lambda) + \beta_1 \geq 0 \]

which yields the first term in the rightest hand side of (12.11) is contained in N((y)\lambda). On the other hand, we have

\[ N[(y)^\alpha y^\alpha y^\alpha + \eta \beta_1+1] \subseteq N(f). \]

Hence (12.11) implies (12.10).
Indeed, we have

\[ (y)^{i-1-\Sigma_1(d(j, \lambda)-1)}(\eta)F^{(0,0,0)}_{(\alpha, \beta, \lambda)}(y)^{i-1} \leq N(f) \]

which yields (12.12).

By virtue of (12.10), (12.12), the inclusion (12.9) implies

\[ N[(y)^{i-1}(\eta)F^{(0,0,0)}_{(\alpha, \beta, \lambda)}(\alpha, \beta, \lambda ; \{u(j, \lambda), d(j, \lambda)\})] \leq N(f) \]

in the case 1.2 as desired.

**Case 1.3.** In this case we have

\[ K(\alpha, \beta, \lambda ; \{u(j, \lambda), d(j, \lambda)\}) \leq \prod_{j=1}^{n-1} \prod_{\lambda=1}^{d(j, \lambda)} (\partial_{x_j}^{(j, \lambda)} X_j) \mid_{t=0} \].

Thus it follows

\[ N[(y)^{i-1}(\eta)F^{(0,0,0)}_{(\alpha, \beta, \lambda)}(\alpha, \beta, \lambda ; \{u(j, \lambda), d(j, \lambda)\})] \leq N[(y)^{i-1-\Sigma_1(d(j, \lambda)-1)}(\eta)F^{(0,0,0)}_{(\alpha, \beta, \lambda)}(\alpha, \beta, \lambda ; \{u(j, \lambda), d(j, \lambda)\})] \]

\[ + N[(y)^{i-1-\Sigma_1(d(j, \lambda)-1)}(\eta)F^{(0,0,0)}_{(\alpha, \beta, \lambda)}(\alpha, \beta, \lambda ; \{u(j, \lambda), d(j, \lambda)\})] \]

(12.13)

which is a consequence of the inductive assumption. Since (12.7) yields

\[ i-1-\sum_{j=1}^{n-1} \sum_{\lambda=1}^{d(j, \lambda)} (d(j, \lambda)-1) = | \alpha^n | + i-1-\sum_{j=1}^{n-1} \sum_{\lambda=1}^{d(j, \lambda)} d(j, \lambda) \]

we have

\[ N[(y)^{i-1}(\eta)F^{(0,0,0)}_{(\alpha, \beta, \lambda)}(\alpha, \beta, \lambda ; \{u(j, \lambda), d(j, \lambda)\})] \leq N[(y)^{i-1-\Sigma_1(d(j, \lambda)-1)}(\eta)F^{(0,0,0)}_{(\alpha, \beta, \lambda)}(\alpha, \beta, \lambda ; \{u(j, \lambda), d(j, \lambda)\})] \]

\[ \leq N(f) \].

Hence (12.13) implies

\[ N[(y)^{i-1}(\eta)F^{(0,0,0)}_{(\alpha, \beta, \lambda)}(\alpha, \beta, \lambda ; \{u(j, \lambda), d(j, \lambda)\})] \leq N[(y)^{i-1}(\eta)F^{(0,0,0)}_{(\alpha, \beta, \lambda)}(\alpha, \beta, \lambda ; \{u(j, \lambda), d(j, \lambda)\})] + N(f) \]

\[ \leq N(f) \]

as desired, since \(| \alpha^n | \geq 1\) in the case 1.3.

**Case 1.4.** This case contributes to (12.5) as the following sum:

\[ (1/i) \sum_{\alpha_n=1}^{\infty} (1/\alpha_n!) F^{(0,0,0)}_{(\alpha_n, \alpha_n, \lambda)} \sum_{(d(n, \lambda))} \prod_{\lambda=1}^{d(n, \lambda)} (1/d(n, \lambda)) ! \partial_{x_\lambda}^{(n, \lambda)} X_n \mid_{t=0} \]

(12.14)
where \( \{d(n, \lambda); 1 \leq \lambda \leq \alpha_n \} \) runs through
\[
(12.15) \quad d(n, \lambda) \geq 1 \quad \text{and} \quad \sum_{\lambda=1}^{\alpha_n} d(n, \lambda) = i - 1.
\]
If \( \alpha_n < i - 1 \), then it follows
\[
N[(y)^{i-1}(\eta)F_{(0,0,0)}] \subseteq N[(y)^{i-1-\alpha_n}] + N[(y)^{\alpha_n}(\eta)F_{(0,0,0)}].
\]
\[
\subseteq N[(y)1] + N(f)
\]
\[
\subseteq N(f).
\]
It remains the terms in the case \( \alpha_n = i - 1 \). Note that (12.15) yields
\[
d(n, \lambda) = 1 \quad \text{for all} \quad 1 \leq \lambda \leq \alpha_n = i - 1
\]
in this case. Hence it suffices to observe the following term:
\[
(12.16) \quad \frac{1}{i!}(1/(i-1)!)F_{(0,0,0)} \{\partial_y X_n(0, y, \eta)\}^{i-1}.
\]
Since
\[
\partial_y X_n(0, y, \eta) = -1 \quad \text{and} \quad F_{(0,0,0)} = \partial_y^i \partial_y f(y, \eta)
\]
the term (12.16) is nothing but
\[
(1/i!)(-1)^{i-1} \partial_y^i \partial_y f(y, \eta).
\]
Thus the proof in the cases 1.1–1.4 leads us to
\[
N[(y)^{i-1}(\eta)] \{1/i! \partial_y X_n(0, y, \eta) - (1/i!)(-1)^{i-1} \partial_y^i \partial_y f(y, \eta)\} \subseteq N(f)
\]
which is the assertion 1) of Lemma 7.5.

**Step 2. Proof of the assertion 2).**

We use the following simpler notation than Notation 12.2:

**Notation 12.3.** For a multi-index \((\alpha; \beta, k) = (\alpha_1, \alpha'; \beta_1, \beta', k)\), we set
\[
(12.4)' \quad F_{(\alpha_1, \beta_1, 0)}(y, \eta) = \partial_x^\alpha \partial_y^\beta \partial_z^k F(x; \xi, z) \big|_{(x; \xi, z) = (0, \eta, z, 1, 0)}
\]
that is, we denote \((\alpha, \alpha')\) in Notation 12.2 by \(\alpha'\).

By virtue of Lemma 12.1, for \(2 \leq j \leq n\), we have the following expression:
\[
(1/i!) \partial_j Z_j(0, y, \eta)
\]
\[
= (1/i!)[(1/(i-1)!) \partial_j^{i-1} \{- \partial_j F(X; E, Z) - \partial_j F(X; E, Z)\}]_{i=0}
\]
\[
= (1/i) \sum_{1 \leq |\alpha| + |\beta| + k \leq i} \alpha! \beta! k! (\partial_j^\alpha \partial_y^\beta \partial_z^k \{- \partial_j F - \partial_j F\}]_{i=0}
\]
\[
\times \sum_{u(j, \lambda), d(j, \lambda)} K(\alpha, \beta, k; (j, \lambda), d(j, \lambda))
\]
where \( K(\alpha, \beta, k; \{u(j, \lambda), d(j, \lambda)\}) \) is defined by (12.6), and where the index 
\( \{u(j, \lambda), d(j, \lambda)\} \) runs through the set determined by (12.7).

We classify the proof in the step 2 into the following five cases:

\[
\begin{aligned}
\text{Case 2.1.} & \quad |\beta'| + k \geq 1. \\
\text{Case 2.2.} & \quad |\beta'| + k = 0, \quad \alpha_i \geq 1, \quad \text{and} \quad \beta_i \geq 1. \\
\text{Case 2.3.} & \quad |\beta'| + k + \beta_i = 0, \quad \alpha_i \geq 1. \\
\text{Case 2.4.} & \quad |\beta'| + k + \alpha_i = 0, \quad \beta_i \geq 1. \\
\text{Case 2.5.} & \quad |\beta'| + \alpha_i + \beta_i = 0.
\end{aligned}
\]

**Case 2.1.** Note that in this case we have

\[
K(\alpha, \beta, k; \{u(j, \lambda), d(j, \lambda)\}) = \sum_{j=0}^{n} (\partial^{u(j, \lambda)}_{\xi} X_j(t=0)) + (\partial^{u(n+1, \lambda)}_{\xi} Z)_{t=0}.
\]

Hence it follows

\[
(y)^j K(\alpha, \beta, k; \{u(j, \lambda), d(j, \lambda)\}) \subset \sum_{j=0}^{n} (y)^{u(j, \lambda)} (\partial^{u(j, \lambda)}_{\xi} X_j(t=0)) + (y)^{u(n+1, \lambda)} (\partial^{u(n+1, \lambda)}_{\xi} Z)_{t=0}.
\]

Since the inductive assumption yields the blackets \([ \cdot ]_{(\nu)}\) \((\nu = 1, 2)\) satisfy

\[
N([\cdots]_{(\nu)}) \subset N(f),
\]

we get

\[
N([y]^j K(\alpha, \beta, k; \{u(j, \lambda), d(j, \lambda)\}) \subset N(f)
\]

in the case 2.1.

**Case 2.2.** In this case we note that the inductive assumption yields

\[
(y)^j K(\alpha, \beta, k; \{u(j, \lambda), d(j, \lambda)\}) \subset (y)^j (\partial^{u(1, \lambda)}_{\xi} X_1(t=0, \eta)) (\partial^{u(1, \lambda)}_{\xi} X_1(t=0, \eta))
\]

\[
\subset (y)^j (\partial^{u(1, \lambda)}_{\xi} X_1(t=0, \eta)) (\partial^{u(1, \lambda)}_{\xi} X_1(t=0, \eta))
\]

\[
\times (y)^j (\partial^{u(1, \lambda)}_{\xi} X_1(t=0, \eta)) (y)^{u(1, \lambda)} (\eta)
\]

since \( i-1 \geq u(1, \lambda) + d(1, \lambda) \). Then the following facts

\[
N([y]^j) \subset N(f) \quad \text{and} \quad N([y]^j) \subset N(f)
\]

yield

\[
N([y]^j K(\alpha, \beta, k; \{u(j, \lambda), d(j, \lambda)\}) \subset N(f)
\]

in the case 2.2.
To consider the remaining Cases 2.3–2.5, we need the

Remark 12.4. If \(|\beta'| + k = 0\) and \(2 \leq j \leq n\), then it follows

\[(12.17)\]

\[\partial_{(\alpha', \beta')} (\xi \partial_x \eta \partial \partial_x^j \partial_x^j) \{0, y \eta \partial_x \partial \partial_x^j \partial_x^j \} = -F_{(\alpha' \beta')} (\xi \partial_x \eta \partial \partial_x^j \partial_x^j) \cdot \]

Proof. Since the derivation does not involve that of in the \(\xi\)-direction we have

\[\partial_{(\alpha', \beta')} (\xi \partial_x \eta \partial \partial_x^j \partial_x^j) = -\xi \partial_{(\alpha', \beta')} \partial_x^j \partial_x^j \eta \partial \partial_x^j \partial_x^j \cdot \]

Thus, restricting this on \(E = T^s \mathbb{M} \times \{0\}\) and using \(\Xi (0, y, \eta) = 0\), we get (12.17) as desired.

Q. E. D.

Case 2.3. In this case, Remark 12.4 yields

\[(12.18)\]

Thus it suffices to show

\[N[(y)^t F_{(\alpha' \beta')} (\xi \partial_x \eta \partial \partial_x^j \partial_x^j) X_1 (0, y, \eta)] \subset N(f) \cdot \]

We first note

\[(12.19)\]

Indeed, the assumption \([B.4]\) implies

\[\partial_{(\alpha' \beta')} F(x ; 0, 0) = (x)^{\alpha_1 - \alpha_1' + 1} \cdot \]

Hence we get

\[F_{(\alpha' \beta')} (y, \eta) \eta = 0 \subset (y)^{\alpha_1 - \alpha_1' - 1} \cdot \]

Thus (12.19) follows.

By virtue of (12.19) we have the inclusion

\[(12.20)\]

Note that

\[i + q - 1 - |\alpha' | - \alpha_1 \geq q + i - 1 - \sum_{j=1}^{s_1} \sum_{\lambda=1}^{a_1} d(j, \lambda) - \alpha_1 u(1, \lambda) = q \cdot \]

since \(|\beta'| + k + \beta_1 = 0\) in the case 2.3. Hence \(q := \text{ord}[f(y, \eta)]\) yields

\[N[(y)^{i + q - 1 - |\alpha' | - 1}] \subset N[(y)^q] \subset N(f) \cdot \]

On the other hand, it follows

\[N[(y)^{(i + q - 1 - |\alpha' | - 1)} X_1 \}_{t=0} \]

\[+ N[(y)^q (\xi \partial_x \partial \partial_x^j \partial_x^j) X_1 \}_{t=0} \]

by the inductive assumption. Thus (12.20) implies (12.18) as desired.

**Case 2.4.** In this case we have

\[ K(\alpha, \beta, \kappa; \{u(j, \lambda), d(j, \lambda)\}) \subseteq \prod_{k=1}^{\beta_1} (\partial_t^{(1, \lambda)} \Xi_1 |_{t=0}). \]

Thus, by Remark 12.4, it suffices for the proof in the case 2.4 to show

\[ N[(y)^{(0,0)} \prod_{(\alpha', \eta)} \partial_t^{(1, \lambda)} \Xi_1 |_{t=0}] \subseteq N(f). \]

The left hand side of (12.21) is contained in

\[ N[(y)^{(0,0)} \prod_{(\alpha', \eta)} \partial_t^{(1, \lambda)} \Xi_1 |_{t=0}] \subseteq N[(y)^{(0,0)} \prod_{(\alpha', \eta)} \partial_t^{(1, \lambda)} \Xi_1 |_{t=0}] \]

Thus, by Remark 12.4, it suffices for the proof in the case 2.4 to show

\[ N[(y)^{(0,0)} \prod_{(\alpha', \eta)} \partial_t^{(1, \lambda)} \Xi_1 |_{t=0}] \subseteq N(f). \]

Note that the inequality

\[ i - \sum_{k=1}^{\beta_1} \{d(1, \lambda) - 1\} \geq \beta_1 + 1 + i - 1 - \sum_{k=1}^{\beta_1} d(1, \lambda) \geq \beta_1 + 1 \]

yields

\[ N[(y)^{(0,0)} \prod_{(\alpha', \eta)} \partial_t^{(1, \lambda)} \Xi_1 |_{t=0}] \subseteq N[(y)^{(0,0)} \prod_{(\alpha', \eta)} \partial_t^{(1, \lambda)} \Xi_1 |_{t=0}] \]

On the other hand, we have

\[ N[(y)^{(0,0)} \prod_{(\alpha', \eta)} \partial_t^{(1, \lambda)} \Xi_1 |_{t=0}] \subseteq N[(y)^{(0,0)} \prod_{(\alpha', \eta)} \partial_t^{(1, \lambda)} \Xi_1 |_{t=0}] \]

where we note

\[ i - \sum_{k=1}^{\beta_1} \{d(1, \lambda) - 1\} - |\alpha'| - 1 \geq \beta_1 + i - 1 - \sum_{k=1}^{\beta_1} d(1, \lambda) - \sum_{j=1}^{\alpha_1} \sum_{k=1}^{\beta_1} d(j, \lambda) \]

\[ \geq \beta_1 \]

\[ > 0. \]

Thus we get

\[ N[(y)^{(0,0)} \prod_{(\alpha', \eta)} \partial_t^{(1, \lambda)} \Xi_1 |_{t=0}] \subseteq N[(y)^{(0,0)} \prod_{(\alpha', \eta)} \partial_t^{(1, \lambda)} \Xi_1 |_{t=0}] \]

\[ = N[(y)^{(0,0)} \prod_{(\alpha', \eta)} \partial_t^{(1, \lambda)} \Xi_1 |_{t=0}] \subseteq N(f). \]

Hence (12.22) implies (12.21) as desired.
Case 2.5. Since $|\beta'| + k + \alpha_1 + \beta_1 = 0$, Remark 12.4 yields that it suffices to show
\[(12.23) \quad N[(y)^{\ell} F_{(\alpha' + \epsilon_j, \eta)}] \subset N(f). \]
The left hand side is contained in
\[N[(y)^{\ell-1-i} \partial^i \partial^z y^{\alpha'} (y)^{\alpha' + 1} F_{(\alpha' + \epsilon_j, \eta)}] \subset N[(y)^{\alpha' + 1} F_{(\alpha' + \epsilon_j, \eta)}] \]
\[= N[(y)^{\alpha' + 1} \partial^i \partial^z y^{\alpha'} (y), \eta] \subset N(f). \]
Indeed, we have
\[i - 1 - |\alpha'| \geq i - 1 - \sum_{j=1}^{n} \sum_{k=1}^{s_j} d(j, \lambda) = 0 \]
which yields the first inclusion, and it is trivial that the second inclusion holds.

By the cases 2.1-2.5, we get
\[N[(y)^{\ell} \partial^i \partial^z y, \eta)] \subset N(f) \quad \text{for} \quad 2 \leq j \leq n \]
as desired.

Step 3. Proof of the assertion 3).
We use the Leibniz’s rule which makes our proof of 3) reduce to another assertions of Lemma 7.5.

We first note that the Leibniz’s rule yields
\[\frac{1}{i!} \partial^i Z(0, y, \eta) \]
\[= \frac{1}{i!} \partial^i \left\{ \sum_{j=1}^{n} \sum_{k=1}^{s_j} \partial^z y^{\alpha'} (y)^{\alpha' + 1} F_{(\alpha' + \epsilon_j, \eta)} \right\} \bigg|_{t=0} \]
\[= \frac{1}{i!} \sum_{j=1}^{n} \sum_{k=1}^{s_j} C(i-1, s) \partial^i \partial^z y, \eta) \bigg|_{t=0} \partial^i \partial^z y^{\alpha'} (y)^{\alpha' + 1} \]
where $C(r, s) := (r!)/|s!(r-s)!|$. We classify the proof of the assertion 3) into the following four cases:

\[
\begin{align*}
\text{Case 3.1.} & \quad j \geq 2, s \geq 1. \\
\text{Case 3.2.} & \quad j \geq 2, s = 0. \\
\text{Case 3.3.} & \quad j = 1, s \geq 1. \\
\text{Case 3.4.} & \quad j = 1, s = 0.
\end{align*}
\]

Case 3.1. Since $s \leq i - 1$, the inductive assumption yields
\[N[(y)^{\ell-1} \partial^z y, \eta)] = N[(y)^{\ell-1} (y)^{\alpha'} \partial^i \partial^z y, \eta)] \subset N[(y)^{\ell} \partial^i \partial^z y, \eta)] \subset N(f). \]
Case 3.2. Since $j = 0$ for $j \geq 2$,
$$N[\gamma_j(0, y, \eta)] = N(0) = \varnothing \subset N(f).$$

Case 3.3. Note that the inductive assumption yields
$$N[(\gamma)^{i-1}(\partial \gamma_1(0, y, \eta))\partial_{\xi, k} F(X; \gamma, Z)] \subset N(f).$$
Since $N((\gamma)^{i-1}) \subset N(f)$ and since
$$N[(\gamma)^{i-1}(\partial \gamma_1(0, y, \eta))] = N[(\gamma)^{i-1}(\partial \gamma_1 X_1(0, y, \eta))] \subset N(f),$$
we get
$$N[(\gamma)^{-1}(\partial \gamma_1 X_1(0, y, \eta))] \subset N(f).$$
in the case 3.3.

Case 3.4. Note that the assertion 1) of Lemma 7.5 has been already shown for $i$ (Step 1). Thus we have
$$N[(\gamma)^{-1}(\partial \gamma_1 X_1(0, y, \eta))] = N(f).$$

By the cases 3.1-3.4, we get the assertion 3) of Lemma 7.5.

Step 4. Proof of the assertions 4) and 5).
Since the assertion 4) is trivial for $i \geq 2$, we only have to prove the assertion 5). We may assume $q \geq 2$, since if $q = 1$ then the assertion 5) is trivial.

We use Notation 12.3. By virtue of Lemma 12.1 we have:
$$(1/i)\partial \gamma_1(0, y, \eta)$$
$$= (1/i)\sum_{\beta, \gamma} (\alpha ! \beta ! k !)^{-1} \partial \gamma_1 \beta F F \partial x_1 F (x, y, \eta)$$
$$= (1/i) \sum_{\beta, \gamma} K(\alpha, \beta, k; \{ u(j, \lambda), d(j, \lambda) \})$$
where $\{ u(j, \lambda), d(j, \lambda) \}$ and $K(\alpha, \beta, k; \{ u(j, \lambda), d(j, \lambda) \})$ are defined by (12.7) and (12.6) respectively.

By the Leibniz's rule it follows
$$\partial \gamma_1 \beta F F \partial x_1 F (x, y, \eta)$$
$$= -\eta F (\alpha \beta, k+1) - F(\alpha \beta+1, k) - F(\alpha \beta, k+1).$$
Remark 12.5. The following inclusion of ideals holds:
\[
(y)^{i-1}\left\{-\eta F_{\{\alpha, \beta, k; \{u(j, \lambda), d(j, \lambda)\}\}} - F_{\{\alpha, \beta, k; \{u(j, \lambda), d(j, \lambda)\}\}}\right\} \bigcup_{(j, \lambda) \in \{u(j, \lambda), d(j, \lambda)\}} K(\alpha, \beta, k; \{u(j, \lambda), d(j, \lambda)\}) \subset (y)^{i-1} + (\eta).
\]

Proof. We only have to verify
\[
(y)^{i-1}\left\{-F_{\{\alpha, \beta, k; \{u(j, \lambda), d(j, \lambda)\}\}}\right\} \bigcup_{(j, \lambda) \in \{u(j, \lambda), d(j, \lambda)\}} K(\alpha, \beta, k; \{u(j, \lambda), d(j, \lambda)\}) \subset (y)^{i-1} + (\eta).
\]

Note that this term appears only if \(\beta_1 \geq 1\). Hence the left hand side of (12.24) is contained in the following ideal \(\mathcal{J}\):

\[
\mathcal{J} := (y)^{i-1}\left\{-F_{\{\alpha, \beta, k; \{u(j, \lambda), d(j, \lambda)\}\}}\right\} \bigcup_{(j, \lambda) \in \{u(j, \lambda), d(j, \lambda)\}} K(\alpha, \beta, k; \{u(j, \lambda), d(j, \lambda)\}) \subset (y)^{i-1} + (\eta).
\]

Since \(d(j, \lambda) \leq i-1\), the inductive assumption yields
\[
\mathcal{J} := (y)^{i-d(1, k)} \{y)^{i(1, k)}\} \bigcup_{(j, \lambda) \in \{u(j, \lambda), d(j, \lambda)\}} K(\alpha, \beta, k; \{u(j, \lambda), d(j, \lambda)\}) \subset (y)^{i-1} + (\eta).
\]
Thus (12.24) follows.

By virtue of Remark 12.5, it suffices for the assertion 5) to show
\[
(y)^{i-1}\left\{-F_{\{\alpha, \beta, k; \{u(j, \lambda), d(j, \lambda)\}\}}\right\} \bigcup_{(j, \lambda) \in \{u(j, \lambda), d(j, \lambda)\}} K(\alpha, \beta, k; \{u(j, \lambda), d(j, \lambda)\}) \subset (y)^{i-1} + (\eta).
\]

We classify the proof of (12.25) into following three cases:

- Case 4.1. \(|\beta'| + k \geq 1\).
- Case 4.2. \(|\beta'| + k = 0, \beta_1 \geq 1\).
- Case 4.3. \(|\beta'| + k + \beta_1 = 0\).

Case 4.1. In this case we have
\[
K(\alpha, \beta, k; \{u(j, \lambda), d(j, \lambda)\}) \subset \sum_{j=0}^{n} (\partial^{(1, k)} Z(0, y, \eta)) \bigcup (\partial^{(n+1, 1)} Z(0, y, \eta))
\]
which implies that the left hand side of (12.25) is contained in
\[
(y)^{i-1}\left\{\sum_{j=0}^{n} (\partial^{(1, k)} Z(0, y, \eta)) \bigcup (\partial^{(n+1, 1)} Z(0, y, \eta))\right\}
\]
\[
= \sum_{j=0}^{n} (y)^{i-1-u(1, k)} [y^{u(1, k)}(\partial^{(1, k)} Z(0, y, \eta))]_{(j)} + (y)^{i-u(n+1, 1)} [y^{u(n+1, 1)}(\partial^{(n+1, 1)} Z(0, y, \eta))]_{(j)}.
\]

Note that the Newton polygons of the blackets \([\cdots]_{(j)} (1 \leq j \leq n)\) are contained in \(N(f)\) by the inductive assumption, and that
\[
N(f) \subset \mathbb{N}_{q-1, 1} := \{(s, t); (s/(q-1)) + t \geq 1\}.
\]
Hence we get (12.25) in the case 4.1.
Case 4.2. In this case we have
\[ K(\alpha, \beta, k; \{u(j, \lambda), d(j, \lambda)\}) \subset (\partial^{(1,1)}_d \mathcal{E}_1(0, y, \eta)). \]
Thus the left hand side of (12.25) is contained in
\[ (y)^{\alpha^{-1} + \eta} \]
which is a consequence of the inductive assumption.

Case 4.3. In this case we have the left hand side of (12.25) is contained in
\[ (y)^{\alpha^{-1} + \eta} \]
since the assumption \([B.4]\) implies
\[ F_{(\mathbb{A}^1; \mathbb{A}^1, 0, 0)} \subseteq (y)^{\alpha^{-1} + \eta}. \]
Hence the following inequality
\[ i-1+q-1-\alpha_1-|\alpha'| \geq q-1 \]
yields (12.25) as desired.

The assertion 5) in Lemma 7.5 is proved.

The proof of Lemma 7.5 is complete. Q.E.D.

§13. Summary of Local Dimension Theory

In this section we give a summary of local dimension theory of analytic sets. Our summary starts from a review of a way of regarding an analytic set \(X\) of a domain \(D\) in \(\mathbb{C}^n\) as a reduced complex space \((X, \mathcal{O}_X)\). We only give outlines of this way (for its detail, see \([Gr-Re]\)).

Let \(X\) be an analytic set of a domain \(D\), that is, \(X\) can be defined locally as a common zero set of finitely many holomorphic germs of functions on \(D\).

**Definition 13.1.** We define the *ideal sheaf* \(i(X)\) of \(X\) as the sheafication of the following presheaf \(\{(U, i(U))\}\) of ideals of \(\mathcal{O}_D\):
\[(13.1) \quad U : \text{open in } D \mapsto i(U) := \{f \in \mathcal{O}_D(U); f|_{X \cap U} \equiv 0\}.\]

**Definition 13.2.** We define a *structure sheaf* \(\mathcal{O}_X\) on \(X\) by
\[(8.29) \quad \mathcal{O}_X := (\mathcal{O}_D/i(X))|_X\]
that is, the restriction on \(X \subset D\) of the sheaf \(\mathcal{O}_D/i(X)\) on \(D\), where \(\mathcal{O}_D/i(X)\) is defined by the sheafication of the presheaf on \(D\) which is determined by the following data:
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U: open in \( D \rightarrow \mathcal{O}_D(U)/\mathfrak{i}(X)(U) \).

Note that each stalk ideal \( \mathfrak{i}(X)_x := \lim_{\mathfrak{i}(X)(U)} \) is a reduced ideal of \( \mathcal{O}_{D, x} \), that is, if \( f^m \in \mathfrak{i}(X)_x(\exists \mathfrak{m}) \) then \( f \in \mathfrak{i}(X)_x \) follows. Thus the stalk \( \mathcal{O}_{X, x} \) of the structure sheaf \( \mathcal{O}_X \) has no non-trivial nilpotent elements, that is, the ringed space \( (X, \mathcal{O}_X) \) is a reduced ringed space.

We regard an analytic set \( X \) in \( D \) as a complex space by this way, where we use the terminology of "complex space" in the sense of [Gr-Re], that is, a ringed space \( (X, \mathcal{O}) \) is called a complex space if it is a \( C \)-ringed space with a coherent structure sheaf \( \mathcal{O} \), and with a Hausdorff topological space \( X \).

Remark 13.3. It needs more work to show that our reduced ringed space \( (X, \mathcal{O}_X) \) forms a complex space. This justification is based on the famous Cartan's coherence theorem [Gr-Re: Fundamental Theorem 4.2, p. 84], which says that the ideal sheaf \( \mathfrak{i}(X) \) is a coherent \( \mathcal{O}_D \)-sheaf.

Definition 13.4. Let \( (X, \mathcal{O}_X) \) be a complex space. We say \( (X, \mathcal{O}_X) \) is locally irreducible at \( x \in X \), if the stalk \( \mathcal{O}_{X, x} \) is an integral domain.

Next we explain a way of regarding holomorphic maps in the sense of Definition 3.1 as morphisms in the category of complex spaces.

Let \( X \) [or \( Y \), resp.] be an analytic set of a domain \( D[D'] \), and let \( f: X \rightarrow Y \) be a holomorphic map in the sense of Definition 3.1. Recall the (0-th) direct image sheaf \( f_*\mathcal{O}_X \) which is defined as the sheafification of the following presheaf \( \{(17, (f_*\mathcal{O}_X)(U))\} \):

\[
U: \text{open in } Y \rightarrow (f_*\mathcal{O}_X)(U) := \mathcal{O}_X(f^{-1}(U)).
\]

We want to construct a morphism of the form

\[
(f, f^\sim): (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)
\]

where \( f^\sim: \mathcal{O}_Y \rightarrow f_*\mathcal{O}_X \) is a sheaf map on \( Y \), and where \( (X, \mathcal{O}_X), (Y, \mathcal{O}_Y) \) are the reduced complex spaces constructed by the way of Definition 13.2.

Since \( f: X \rightarrow Y \) is a holomorphic map in the sense of Definition 3.1, there exists a holomorphic map \( g: D \rightarrow D' \) such that \( f \) is induced by \( g \):

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
D & \xrightarrow{g} & D'
\end{array}
\]

We consider the pull-back \( g^*: \mathcal{O}_{D'} \rightarrow f_*\mathcal{O}_D \) and show the

Lemma 13.5. There exists a canonical sheaf map \( f^\sim: \mathcal{O}_Y \rightarrow f_*\mathcal{O}_X \) on \( Y \) which
is induced by $g^*$.

**Proof.** Let $U$ be an open set in $Y$, and let $u \in \mathcal{O}_Y(U)$. Recall that $\mathcal{O}_Y$ is defined by the restriction on $Y \subset D'$ of the sheaf $\mathcal{O}_{D'}/\iota(Y)$ on $D'$. Hence we have the following 1) and 2):

1) $u = (u_y) \in \prod_{y \in U} \mathcal{O}_{Y,y}$.

2) For any $y \in U$, there exists an open neighborhood $E'$ in $D'$ and a section $v \in \mathcal{O}_{D'}(E')/\iota(Y)(E')$ such that

$$u_y = s_{E',x}(v) \quad \text{for any } z \in E' \cap Y$$

where $s_{E',x} : \mathcal{O}_{D'}(E')/\iota(Y)(E') \rightarrow \mathcal{O}_{D',x}/\iota(Y)_x$ is the canonical map.

We need the following simple

**Claim 13.6.** If $\varphi \in \iota(Y)(E')$ then $g^*\varphi \in \iota(X)(g^{-1}(E'))$. Hence there exist a canonical map

$$[g^*] : \mathcal{O}_{D'}(E')/\iota(Y)(E') \rightarrow \mathcal{O}_{D'}(g^{-1}(E'))/\iota(X)(g^{-1}(E')).$$ 

**Proof.** This claim is a direct consequence of Definition 13.1 and (13.3). Indeed, we have

$$\varphi \in \iota(Y)(E') \iff \varphi|_{E' \cap Y} \equiv 0$$

$$\iff \varphi(g(x)) = 0 \quad \forall x \in g^{-1}(E') \cap X$$

$$\iff g^*\varphi \in \iota(X)(g^{-1}(E')).$$

Hence Claim 13.6 follows. Q.E.D.

Using the map $[g^*]$ given by (13.5), we define a section

$$w := f^{-1}(u) \in \mathcal{O}_X(f^{-1}(U)) = (f_*\mathcal{O}_X)(U)$$

as follows:

$$w = (w_x) \in \prod_{x \in f^{-1}(U)} \mathcal{O}_{X,x}$$

$$w_x := s_{g^{-1}(E'),x}(v) \in \mathcal{O}_{D,x}/\iota(X)_x$$

where $s_{g^{-1}(E'),x} : \mathcal{O}_{D'}(g^{-1}(E'))/\iota(X)(g^{-1}(E')) \rightarrow \mathcal{O}_{D,x}/\iota(X)_x$ is the canonical map.

We must verify that the definition (13.6) is well-defined. Let $y \in U \subset Y$, let $E_i$ ($i = 1, 2$) be open neighborhoods of $y$ in $D'$, and let

$$v_i \in \mathcal{O}_{D'}(E'_i)/\iota(Y)(E'_i) \quad (i = 1, 2)$$

be sections satisfying
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\[ u_z = s_{E_1}(v_1) = s_{E_2}(v_2) \text{ for any } z \in E_1 \cap E_2 \cap Y. \]

Putting \( E^z := E_1 \cap E_2 \), we have

\[ (v_1 - v_2)|_{E^z} = 0 \]

as an element of \( \mathcal{O}_{E^z}/(Y)(E^z) \).

Thus we get

\[ s_{E^z}(E^z)(v_1 - v_2) = 0 \in \mathcal{O}_{X,z} \text{ for any } x \in \mathcal{E}^{-1}(E^z) \]

which shows that the section \( w = (w_x) \in \prod_{x \in \mathcal{E}^{-1}(E^z)} \mathcal{O}_{X,x} \) given by (13.6) is determined independently of the choice of \( v \) satisfying (13.4). Hence the sheaf map \( f^* : \mathcal{O}_{Y \to u} \to w = f^*(u) \in f_* \mathcal{O}_X \) is well-defined.

The proof of Lemma 13.5 is complete. Q.E.D.

**Definition 13.7.** Let \( (f, f^*) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y) \) be a morphism of complex spaces. We call \( (f, f^*) \) is a finite morphism if the underlying map \( f : X \to Y \) is a finite holomorphic map in the sense of Definition 3.3.

Now we recall a definition of analytic subsets in a complex space, and their local dimensions.

**Definition 13.8.** Let \( (X, \mathcal{O}_X) \) be a complex space. We call a subset \( Z \) of \( X \) at a point \( x \in X \) if there exist finitely many germs \( f_1, \ldots, f_k \in \mathcal{O}_{X,x} \) such that the germ \( (Z, x) \) of a subset \( Z \) at \( x \) is given by the common zero set of \( f_1, \ldots, f_k \). We denote this by \( (Z, x) = \text{Null}(f_1, \ldots, f_k) \).

**Definition 13.9.** Let \( (X, \mathcal{O}_X) \) be a complex space. We define its local dimension \( \dim_x(X, \mathcal{O}_X) \) at a point \( x \in X \) by

\[ \dim_x(X, \mathcal{O}_X) := \min\{k ; \exists f_1, \ldots, f_k \in \mathcal{O}_{X,x} \text{ such that } \text{Null}(f_1, \ldots, f_k) \cap X = \{x\}\} \]

that is, the minimum integer of such numbers \( k \) of germs \( f_1, \ldots, f_k \in \mathcal{O}_{X,x} \) which make the point \( x \) be isolated in \( \text{Null}(f_1, \ldots, f_k) \).

**Definition 13.10.** Let \( Z \) be an analytic set of \( (X, \mathcal{O}_X) \) at \( x \in Z \subset X \). We define its local dimension \( \dim_xZ \) at \( x \) by

\[ \dim_xZ := \dim_x(Z, \mathcal{O}_Z) \]

where \( (Z, \mathcal{O}_Z) \) is the closed reduced complex subspace of \( (X, \mathcal{O}_X) \) defined by

\[ \mathcal{O}_Z := (\mathcal{O}_X/(Z))|_Z \]

as similar as Definition 13.2.

Now we quote several propositions which are used in §§8 and 14.
Proposition 13.11 [Gr-Re: Remark 5.1.1, p. 94]. Let 
\((f, f^\sim):(X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)\)
be a finite morphism of complex spaces. Then
\[ \dim_x(X, \mathcal{O}_X) \leq \dim_{f(x)}(Y, \mathcal{O}_Y). \]

Proposition 13.12 [Gr-Re: Active Lemma 5.2.4, p. 100]. Let \((X, \mathcal{O}_X)\) be a complex space and let \(g \in \mathcal{O}_{x,X}\) be a germ. If the zero set \(\text{Null}(g)\) of \(g\) is nowhere dense in \((X, \mathcal{O}_X)\), then
\[ \dim_x\text{Null}(g) = \dim_x(X, \mathcal{O}_X) - 1. \]

Proposition 13.13 [Gr-Re: Proposition 5.3.2, p. 103]. Let \((X, \mathcal{O}_X)\) be a complex space which is locally irreducible at \(x \in X\). Let \(Z\) be an analytic subset at \(x\) of \((X, \mathcal{O}_X)\). If
\[ \dim_x Z = \dim_x(X, \mathcal{O}_X) \]
then there exists an open neighborhood \(U\) of \(x\) in \(X\) such that
\[ Z \cap U = X \cap U. \]

We also use the following theorem in §§8 and 14:

Theorem 13.14 [Gr-Re: Finite Mapping Theorem 3.1.3, p. 64]. Let 
\((f, f^\sim):(X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)\) be a finite morphism. Then the image \(f(X)\) is an analytic set in \((Y, \mathcal{O}_Y)\).

The following famous theorem is used in §§4, 8, and 10:

Rückert’s Nullstellensatz [Gr-Re: Theorem 4.1.5, p. 82]. Let \((X, \mathcal{O}_X)\) be a complex space, and let \(\mathcal{J} \subset \mathcal{O}_X\) be a coherent sheaf of ideals with zero set \(\text{Null}(\mathcal{J})\). Let \(\mathcal{I}(\text{Null}(\mathcal{J}))\) denote the ideal sheaf of the analytic set \(\text{Null}(\mathcal{J})\), and let \(\text{Rad}(\mathcal{J})\) denote the radical of \(\mathcal{J}\), that is, the sheafication of the following presheaf \(\{(U, \text{Rad}(\mathcal{J})(U))\}\) of ideals of \(\mathcal{O}_X\):
\[ U: \text{open in } X \rightarrow \text{Rad}(\mathcal{J})(U) := \{f \in \mathcal{O}_X(U) : \exists m \in \mathbb{N}, f^m \in \mathcal{J}(U)\}. \]
Then it follows that
\[ \mathcal{I}(\text{Null}(\mathcal{J})) = \text{Rad}(\mathcal{J}). \]

§14. Proof of Lemma 9.5

In this last section we prove the following Theorem 14.1 which contains Lemma 9.5 as a special case.

Let us consider a non-zero germ \(f \in \mathcal{O}_{x,0}\) of two independent variables. For a local coordinate system \((x, y)\) at the origin satisfying
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(14.1) \[ f(x, 0)f(0, y) \neq 0, \]

we define the Newton polygon \( N(f) \) of \( f \) with respect to the coordinate system by \[
f(x, y) = \sum_{j, k} c_{jk} x^j y^k \quad \rightarrow \quad N(f) := \text{ch}\left[ \bigcup_{j + k \neq 0} (j, k) + R^2_+ \right]
\]
where \( \text{ch}[A] \) denotes the convex hull of \( A \subset \mathbb{R}^2 \), and we set \( R^2_+ := \{ t \in \mathbb{R}; t \geq 0 \} \).

Since we assume (14.1), we can find positive integers \( p(\mu), q(\mu) \) where \( 1 \leq \mu \leq m := \# \text{Seg} N(f) \) such that \( N(f) \) can be written as

(14.2) \[ N(f) = \sum_{\mu=1}^{m} N_{q(\mu), p(\mu)} \]

where we set \( N_{a, b} := \{ (s, t) \in R^2_+ ; (s/a) + (t/b) \geq 1 \} \).

We may assume that \( p(\mu), q(\mu) \) are arranged as

(14.3) \[ p(1)/q(1) > p(2)/q(2) > \cdots > p(m)/q(m). \]

In this situation we have the

**Theorem 14.1** There exist positive integers \( i(\mu), c(\mu, i), \nu(\mu, i) \) and non-unit germs \( f_\mu \in \mathcal{O}_{\mathcal{C}^2, 0} \) such that the following 1)-3) hold:

1) The following decomposition

(14.4) \[ f(x, y) = \prod_{\mu=1}^{m} \prod_{i=1}^{i(\mu)} f_\mu(x, y)^{c(\mu, i)} \]
is an irreducible decomposition of \( f(x, y) \) in the ring \( \mathcal{O}_{\mathcal{C}^2, 0} \).

2) Each Newton polygon of \( f_\mu(x, y) \) is given by

(14.5) \[ N(f_\mu) = c(\mu, i)N_{q(\mu), p(\mu)} \quad \text{for} \quad 1 \leq \mu \leq m, \quad 1 \leq i \leq i(\mu). \]

In (14.5), the integers \( \hat{p}(\mu), \hat{q}(\mu) \) are defined by

\[ \hat{p}(\mu) := p(\mu)/(p(\mu), q(\mu)), \quad \hat{q}(\mu) := q(\mu)/(p(\mu), q(\mu)) \]

where \( (a, b) \) denotes the greatest common divisor of \( a, b \in \mathbb{Z} \).

3) For any \( \mu, \{ c(\mu, i), \nu(\mu, i); 1 \leq i \leq i(\mu) \} \) satisfies

(14.6) \[ \sum_{i=1}^{i(\mu)} c(\mu, i)\nu(\mu, i) = (p(\mu), q(\mu)). \]

**Remark 14.2.** Theorem 14.1 contains Lemma 9.5 as the special case that the following condition holds:

(14.7) \[ (p(\mu), q(\mu)) = 1 \quad \text{for all} \quad 1 \leq \mu \leq m. \]

**Proof.** The conditions (14.6), (14.7) imply

\[ i(\mu) = 1, \quad c(\mu, 1) = \nu(\mu, 1) = 1, \quad \text{and} \]

...
Hence Theorem 14.1 yields an irreducible decomposition

\[ f(x, y) = \prod_{\mu=1}^{m} f_{\mu}(x, y) \]

with the conditions

\[ N(f_{\mu}) = N_{q(x), p(y)} \quad \text{for} \quad 1 \leq \mu \leq m. \]

Thus Lemma 9.5 follows from Theorem 14.1. Q.E.D.

Now we prove Theorem 14.1. We derive this theorem from the following

**Proposition 14.3.** Let \( g \in \mathcal{O}_{\mathbb{C}^2, 0} - \{0\} \) be a non-unit germ. Assume that the complex curve \( \text{Null}(g) \) has only one irreducible component locally at the origin. Then, for any local coordinate system \( (x, y) \) at the origin satisfying

\[ (14.1)' \quad g(x, 0)g(0, y) \neq 0, \]

it follows that \( N(g) \) has only one segment, that is,

\[ \# \text{Seg } N(g) = 1. \]

**Proof of "Proposition 14.3 \( \Rightarrow \) Theorem 14.1".** Let

\[ f = \prod_{j=1}^{k} g_{j}^{*} \]

be an irreducible decomposition of \( f \) in the ring \( \mathcal{O}_{\mathbb{C}^2, 0} \), and let \( (x, y) \) be a local coordinate system at the origin satisfying (14.1). Since this coordinate system satisfies (14.1)' for all \( g_j \), Proposition 14.3 yields

\[ N(g_j) = N_{a(j), b(j)} \]

for suitable positive integers \( a(j), b(j) \). With no loss of generality we may assume

\[ b(1)/a(1) \geq b(2)/a(2) \geq \cdots \geq b(k)/a(k). \]

We take integers

\[ 0 = j_s < j_1 < \cdots < j_s = k \]

such as

\[ \begin{align*}
\frac{b(1+j_{\lambda-1})}{a(1+j_{\lambda-1})} &= b(j_1)/a(j_1) \quad \text{and} \\
\frac{b(j_{\lambda-1})}{a(j_{\lambda-1})} &= b(j_1)/a(j_1) \quad \text{for all } 1 \leq \lambda \leq s.
\end{align*} \]

We set \( a_\lambda, b_\lambda \) \( (1 \leq \lambda \leq s) \) by the following conditions:

\[ \begin{align*}
\frac{b_{\lambda}}{a_{\lambda}} &= b(j_1)/a(j_1) \quad \text{and} \\
(a_{\lambda}, b_{\lambda}) &= 1.
\end{align*} \]
Then, for $1 \leq j \leq k$, we can find an integer $d(j) > 0$ such that
\[
\begin{align*}
    a(j) &= d(j) a_1 \\
    b(j) &= d(j) b_1
\end{align*}
\] for $1 + j_{1-1} \leq j \leq j_1$.

Thus, by the additivity of Newton polygons, (14.9)-(14.12) yield
\[
N(f) = \sum_{j=1}^{k} \nu_{\gamma}(j) N_{\lambda,(j)\beta(j)} = \sum_{j=1}^{k} \left\{ \sum_{j_{1-1} \leq j \leq j_1} \nu_{\gamma}(j) d(j) \right\} N_{\lambda,\beta_2}.
\]

Now, comparing (14.13) with (14.2), we get
\[
\begin{align*}
    m &:= \# \{ \mu \} = \# \{ \lambda \} = s \\
    q(\mu) &= \left\{ \sum_{j_{1-1} \leq j \leq j_1} \nu_{\gamma}(j) a(j) \right\} a_1, \\
    \nu(\mu) &= \left\{ \sum_{j_{1-1} \leq j \leq j_1} \nu_{\gamma}(j) d(j) \right\} b_1.
\end{align*}
\]

We define $i(\mu), c(\mu, i), \nu(\mu, i), (1 \leq i \leq \nu(\mu))$ and $f_{\mu}(x, y)$ as follows:
\[
\begin{align*}
    i(\mu) &:= j_{1-1} - j_{1-1} \\
    c(\mu, i) &:= d(i - j_{1-1}) \\
    \nu(\mu, i) &:= \nu_{\gamma}(i - j_{1-1}) \\
    f_{\mu}(x, y) &:= g + \sum_{j_{1-1} \leq j \leq j_1} \nu_{\gamma}(j) d(j)
\end{align*}
\]

Then, from (14.14), (14.15), we have the irreducible decomposition
\[
f = \prod_{j=1}^{k} f_{\mu}(x, y) = \prod_{\mu=1}^{m} \prod_{i=1}^{\nu(\mu)} g_{\mu}(x, y) = \prod_{\mu=1}^{m} \prod_{i=1}^{\nu(\mu)} f_{\mu}(x, y)
\]
with the conditions
\[
\begin{align*}
    N(f_{\mu}) &= N(g_{\mu}) = N_{\lambda,(i+j_{1-1})} \quad \text{and} \\
    \sum_{\mu=1}^{\nu(\mu)} c(\mu, i) \nu(\mu, i) &= \sum_{\mu=1}^{\nu(\mu)} d(i + j_{1-1}) \nu_{\gamma}(i + j_{1-1}) \\
    &= \sum_{j_{1-1} \leq j \leq j_1} d(j) \nu_{\gamma}(j) = (q(\mu), \nu(\mu)).
\end{align*}
\]

Thus Theorem 14.1 follows if Proposition 14.3 is established. Q.E.D.

From now on we prove Proposition 14.3. We shall prove the following contrapositive proposition of Proposition 14.3:

**Proposition 14.4.** Let $g \in \mathcal{O}_{C^r, \omega} - \{0\}$ be a non-unit germ. If there exists a
local coordinate system \((x, y)\) at the origin such that
\[
\begin{align*}
\text{(14.16)} & \quad g(x, 0)g(0, y) \neq 0 \quad \text{and} \\
\text{(14.17)} & \quad m := \# \text{Seg } N(g) > 1
\end{align*}
\]
then the complex curve \(\text{Null}(g)\) has at least two irreducible components locally at the origin.

We prove Proposition 14.4 by induction on \(\nu := \text{ord}_{g} g\). Since \(\nu = 1\) immediately yields \(m = 1\), Proposition 14.4 is trivial in the case \(\nu = 1\). Let \(\nu \geq 2\), and assume that Proposition 14.4 holds for any germs with order \(< \nu\).

We first show the

**Lemma 14.5.** Let \(g(x, y)\) be a germ with a vanishing order \(\nu \geq 2\), and let \((x, y)\) be a local coordinate system satisfying (14.16) and (14.17). Then there exists a local coordinate system \((x^\sim, y^\sim)\) at the origin such that
\[
\begin{align*}
\text{(14.16\sim)} & \quad g^\sim(x^\sim, 0)g^\sim(0, y^\sim) \neq 0 \\
\text{(14.17\sim)} & \quad m^\sim := \# \text{Seg } N(g^\sim) > 1 \quad \text{and} \\
\text{(14.18)} & \quad \text{ord}[g^\sim(x^\sim, y^\sim)]_{y^\sim=0} = \nu
\end{align*}
\]
where \(g^\sim(x^\sim, y^\sim)\) denotes the expression of \(g\) by the coordinate system \((x^\sim, y^\sim)\), and \(N(g^\sim)\) denotes the Newton polygon of \(g^\sim\) with respect to \((x^\sim, y^\sim)\).

**Proof.** If either \(\text{ord}[g(x, 0)] = \nu\) or \(\text{ord}[g(0, y)] = \nu\), then we can take a local coordinate system \((x^\sim, y^\sim)\) as
\[
(x^\sim, y^\sim) := \text{either } (x, y) \text{ or } (y, x).
\]
Thus we may assume
\[
\nu < \min\{\text{ord}[g(x, 0)], \text{ord}[g(0, y)]\}.
\]
Using the identification
\[
T \mathbb{C}^* \cong X \partial_x + Y \partial_y \rightarrow (X, Y) \in \mathbb{C}^*
\]
we write \(g(x, y)\) as the form
\[
\text{(14.20)} \quad g(x, y) = \text{Loc}[g](x, y) + h(x, y)
\]
where \(\text{ord}[h] > \nu\), and \(\text{Loc}[g]\) is a homogeneous polynomial of degree \(\nu\). Note that \(\text{Loc}[g]\) can be written as the form
\[
\text{(14.21)} \quad \text{Loc}[g](x, y) = c x^i y^j \prod_{k=1}^{n} (x - e_k y)^{\nu(k)}
\]
where \(i, j, n \geq 0, \nu(k) \geq 1\) satisfy the relation
\[ i+j+\sum_{k=1}^{n} \nu(k) = \nu \]

and where \( c, e \in C - \{0\} \). Moreover the condition (14.19) yields

\[ i > 0 \quad \text{and} \quad j > 0. \]

Taking a linear coordinate transformation

\[ x^\sim = x, \quad y^\sim = x - ey \quad (e \neq 0) \]

we can write \( \text{Loc}[g] \) as

\[ \text{Loc}[g^\sim](x^\sim, 0) \neq 0, \quad \text{and} \quad \text{Loc}[g^\sim](0, y^\sim) = 0. \]

Since (14.22), (14.23) yield

\[ (\nu, 0), (i, \nu - i) \in \mathbb{N}(g^\sim) \quad \text{with} \quad (\nu, 0) \neq (i, \nu - i) \]

(0, \nu) \notin \mathbb{N}(g^\sim)

we get (14.17) and (14.18). Note that \( \{x = 0\} = \{x^\sim = 0\} \) which yields

\[ \text{ord}[g^\sim(0, y^\sim)] = \text{ord}[g(0, y)] < \infty. \]

Hence (14.16) also follows. The proof of Lemma 14.15 is complete. Q.E.D.

Now we prove Proposition 14.4 for the case \( \nu \geq 2 \). By virtue of Lemma 14.5, we may assume

\[ 2 \leq \nu := \text{ord}[g] = \text{ord}[g(x, 0)] < \text{ord}[g(0, y)] < \infty. \]

Hence we have the following expression:

\[ \text{Loc}[g](x, y) = cx^i \prod_{k=1}^{n} (x - e_k y)^{\nu(k)} \]

where \( i \geq 1, \ n \geq 0, \ \nu(k) \geq 1 \) with the relation

\[ i + \sum_{k=1}^{n} \nu(k) = \nu \]

and where \( c, e_k \in C - \{0\} \) with \( e_k \neq e_k \) if \( k \neq k^\sim \).

In order to prove Proposition 14.4 by induction, we use the notion of blowing ups of the complex curve \( \text{Null}(g) \) with center a point.
**Definition 14.6** (see, for example, [Hi: Lecture 1]). Let $Z:=\mathbb{C}^2$, let 
\[ \pi: \mathbb{C}^2\setminus\{0\} \rightarrow \mathbb{P}^1 := (\mathbb{C}^2\setminus\{0\})/(\mathbb{C}\setminus\{0\}) \]
be the natural map. Let $Z'$ be the closure of 
\[ \text{graph}(\pi) = \{(x, y ; [\xi : \eta]) \in (\mathbb{C}^2\setminus\{0\}) \times \mathbb{P}^1 ; x\eta = y\xi \} \]
in $\mathbb{C}^2 \times \mathbb{P}^1$, and let $\pi: Z'\rightarrow Z$ be the map induced by the following diagram:

\[
\begin{array}{ccc}
Z' & \xrightarrow{\pi} & \mathbb{C}^2 \times \mathbb{P}^1 \\
\downarrow & & \downarrow \text{projection} \\
Z = \mathbb{C}^2 & \xrightarrow{\pi} & \mathbb{C}^2 
\end{array}
\]

This map $\pi: Z'\rightarrow Z$ is called the **blowing up** (or the **quadratic transformation**) of $Z$ with center $\{0\}$.

**Note 14.7.** The blowing up $\pi: Z'\rightarrow Z$ has the following properties:

(14.26) 
\[ Z' = \text{graph}(\pi) \cup (\{0\} \times \mathbb{P}^1) \]

(14.27) 
\[ \pi^{-1}(0) = \{0\} \times \mathbb{P}^1 \]
and the map $\pi: Z'\rightarrow Z$ induces an isomorphism 

(14.28) 
\[ \pi|_{Z'\setminus\{0\}}: Z'\setminus\{0\} \sim \mathbb{C}^2 \]

**Proof.** Since (14.27), (14.28) easily follow from (14.26), we only have to verify (14.26). Let \( \{(x_n, y_n ; [\xi_n : \eta_n])\}_{n=1,2,...} \) be a sequence in graph($\pi_0$) which converges to a point $\( (x, y ; [\xi : \eta]) \) in $\mathbb{C}^2 \times \mathbb{P}^1$. Then the following two cases occur.

\[
\begin{cases} 
\text{Case 1. When } (x, y) \neq 0. \\
\text{Case 2. When } (x, y) = 0.
\end{cases}
\]

In the case 1, with no loss of generality, we may assume $x \neq 0$. Then we have $x_n \neq 0$ for $n \gg 1$. Hence the condition $(x_n, y_n ; [\xi_n : \eta_n]) \in \text{graph}(\pi_0)$ can be written as 

\[ [\xi_n : \eta_n] = [1 : y_n/x_n] \quad \text{for } n \gg 1. \]

Thus, taking the limit $n\rightarrow\infty$, we have 

\[ (x, y ; [\xi : \eta]) = (x, y ; [1 : y/x]) \in \text{graph}(\pi_0). \]

In the case 2, for any $[\xi : \eta] \in \mathbb{P}^1$, we can choose a sequence in graph($\pi_0$) which converges to $(0, 0 ; [\xi : \eta])$ as follows: we set 

\[
(x_n, y_n ; [\xi_n : \eta_n]) := \begin{cases} 
(1/n, (\eta/\xi)/n ; [\xi : \eta]) & \text{if } \xi \neq 0 \\
((\xi/\eta)/n, 1/n ; [\xi : \eta]) & \text{if } \eta \neq 0.
\end{cases}
\]
Then the sequence satisfies the desired property.

Hence we get (14.26) as desired. The proof of Note 14.7 is complete.

Q.E.D.

We remark that $Z'$ is equipped with a structure of complex manifold of dimension 2. Indeed, we define two maps $w_x, w_y$ by

$$
\begin{align*}
  w_x: C^2 &\ni (x, y) \mapsto (x, xy; [1: y_1]) \in Z' \\
  w_y: C^2 &\ni (x, y) \mapsto (yx_1, y; [x_1: 1]) \in Z'
\end{align*}
$$

and set

$$
\Omega_x := w_x(C^2), \quad \Omega_y := w_y(C^2).
$$

Then they have the following properties (14.29)-(14.31):

(14.29) $Z' = \Omega_x \cup \Omega_y$. 

(14.30) $C^2 \xrightarrow{w_x} \Omega_x \subset Z', \quad C^2 \xrightarrow{w_y} \Omega_y \subset Z'$.

(14.31) $w_x(x, y_1) = w_y(x_1, y) \in \Omega_x \cap \Omega_y$ \quad \text{(14.3)} \quad \text{or} \quad "xy \neq 0 \text{ and } x_1 = x/y, y_1 = y/x" \quad \text{or} \quad "x = y = 0, x_1y_1 = 1".$

**Notation 14.8.** We denote the variable $x_1$ [or $y_1$, resp.] by $x/y$ [$y/x$].

According with this notation, we have the following 1)-3):

1) The coordinate neighborhoods $\Omega_x$ and $\Omega_y$ can be written as follows:

$$
\Omega_x := \{(x, y(y/x); [1: y/x]) \in C^2 \times \mathbb{P}^1; (x, y/x) \in C^2\}.
\Omega_y := \{(y(y/x), y; [x/y: 1]) \in C^2 \times \mathbb{P}^1; (x/y, y) \in C^2\}.
$$

2) The blowing up $\pi: Z' \rightarrow Z$ can be represented on the coordinate neighborhoods $\Omega_x, \Omega_y$ as follows:

$$
\pi|_{\Omega_x}: \Omega_x \approx C^2 \ni (x, y(x/y)) \mapsto (x, x(y/x)) \in Z.
\pi|_{\Omega_y}: \Omega_y \approx C^2 \ni (x/y, y) \mapsto (y(x/y), y) \in Z.
$$

3) In particular it follows that

$$
\begin{align*}
  \Omega_x \cap \pi^{-1}(0) &\approx \{(x, y/x); x = 0\}.
  \Omega_y \cap \pi^{-1}(0) &\approx \{(x/y, y); y = 0\}.
\end{align*}
$$

By virtue of the above 1)-3) we get the following figure (14.35) of $Z'$:

$$
\begin{align*}
Z' &= \Omega_x \cup \Omega_y \\
\pi^{-1}(0) &\approx \mathbb{P}^1
\end{align*}
$$

**FIGURE 14.35**

the origin of $\Omega_y$ at $x/y$ \\
the origin of $\Omega_x$ at $x$
Definition 14.9. Let $X$ be a complex curve defined by $f(x, y)=0$ in a neighborhood of the origin of $Z$. Let $\nu=\text{ord}_0[f]$ be the (vanishing) order of $f$ at the origin. We can write the pull-back $f^* \pi$ as

$$
(14.36) \begin{cases}
 f(x, x(y/x))=x^*f'(x, y/x) & \text{on } (O_x, O_x \cap \pi^{-1}(0)) \\
 f(y(x/y), y)=y^*f'(x/y, y) & \text{on } (O_y, O_y \cap \pi^{-1}(0))
\end{cases}
$$

where $f'_1$ [or $f'_2$ resp.] is a holomorphic germ defined in a neighborhood of $O_x \cap \pi^{-1}(0)$ [of $O_y \cap \pi^{-1}(0)$]. We set

$$
(14.37) \begin{cases}
 X'_1:=w_x(((x, y/x); f((x, y/x)=0)) \subset O_x \\
 X'_2:=w_y(((x/y, y); f'(x/y, y)=0)) \subset O_y.
\end{cases}
$$

Then $X'_1 \cap X'_2$ determines a complex curve $X'$ in a neighborhood of $\pi^{-1}(0)$ in $Z'$. We call $X'$ the strict transform of $X$ by the blowing up $\pi$. Note that the blowing up $\pi: Z' \to Z$ induces a holomorphic map

$$
(14.38) p: X' \to X
$$

This map $p$ is called the strict transformation of $X$ with center $\{0\}$.

Proof of the well-definedness of $X'$. We have

$$
x=y(x/y), \quad y=x(y/x) \quad \text{and} \quad (x/y)(y/x)=1 \quad \text{on } O_x \cap O_y
$$

which yield

$$
x^*f'(x, y/x)=y^*(x/y)^\nu f'(y(x/y), (x/y)^{-1}).
$$

Thus it follows that

$$
(14.39) f'(x/y, y)=(x/y)^\nu f'(y(x/y), (x/y)^{-1}).
$$

Since $x/y$ does not vanish on $O_x \cap O_y$, (14.39) implies

$$
X'_1=X'_2 \quad \text{on } O_x \cap O_y
$$
as desired. Q.E.D.

Now we return the proof of Proposition 14.4. Let $g(x, y)$ be a germ of the order $\nu \geq 2$, and let $(x, y)$ be the local coordinate system satisfying (14.24) and (14.17). Recall the expression of the localization $\text{Loc}[g]$ at the origin:

$$
(14.25) \begin{cases}
 \text{Loc}[g](x, y)=c x^i \prod_{k=1}^{n} (x-e_k y)^{e_k} \\
 \text{where } i \geq 1, \quad n \geq 0, \quad \text{and } c, e_k \in \mathbb{C}-\{0\} \quad \text{with } e_k \neq e_{k'} (k \neq k').
\end{cases}
$$

Lemma 14.10. Set $X:=\{(x, y); g(x, y)=0\} = \text{Null}(g)$, and let $X'$ be the strict transform of $X$. Then

1) The pre-image $p^{-1}(0)=X' \cap \pi^{-1}(0)$ consists of the following finite points

$$
(14.40) \{(x/y, y) \in O_y; y=0, x/y=0, e_1, \ldots, e_n\}.
$$
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(14.41) \[ \text{ord}_{(a,0)}[g_1^1] \leq i \]
(14.42) \[ \text{ord}_{(e,0)}[g_1^1] \leq \nu(k) \quad \text{for } k = 1, \ldots, n. \]

Proof. We calculate the defining germs \( g_1^1(x, y/x, y) \) as follows.
Since \( g(x, y) \) can be written as
\[ g = \text{Loc}[g] + h \quad (\text{where } h(x, y) = \sum_{q + r \geq 0} c_{q,r} x^q y^r) \]
the definition (14.36) and the expression (14.25) lead us to
\[ (14.43) \quad g_1^1(x, y/x, y) = y^{-v} \{ \text{Loc}[g](y(x/y), y) + \sum_{q + r \geq 0} c_{q,r} (x/y)^q y^{q+r} \} \]
\[ = \text{Loc}[g](x/y, 1) + \sum_{q + r \geq 0} c_{q,r} (x/y)^q y^{q+r-v} \]
\[ \quad = c(x/y)^t \prod_{k=1}^n ((x/y) - e_k)^{\nu(k)} + y^3 h_3(x/y, y). \]
Since \( \Omega_y \cap \pi^{-1}(0) = \{(x/y, y); y = 0\} \), we get
\[ (14.44) \quad p^{-1}(0) \cap \Omega_y = \{(x/y, y); (x, 0) \in \Omega_y \}; 1 \leq k \leq n}. \]

We similarly have
\[ g_1^1(x, y/x) = \text{Loc}[g](1, y/x) + \sum_{q + r \geq 0} c_{q,r} x^q y^{q-r}(y/x)^r \]
\[ \quad = c(x/y)^t \prod_{k=1}^n (1 - e_k (y/x))^{\nu(k)} + x^3 h_3(x, y/x). \]
Thus \( (x, y/x) = (0, 0) \notin \{g_1^1 = 0\} \) which shows that
\[ (14.45) \quad (\Omega_x - \Omega_y) \cap p^{-1}(0) = \emptyset. \]
Hence the assertion 1) of Lemma follows from (14.44) and (14.45).
The assertion 2) is a direct consequence of the expression (14.43): Indeed, we have
\[ \text{ord}_{(a,0)}[g_1^1] \leq \text{ord}_{0}[g_1^1(x/y, 0)] = i \quad \text{and} \]
\[ \text{ord}_{(e,0)}[g_1^1] \leq \text{ord}_{0}[g_1^1(x/y, 0)] = \nu(k). \]
Hence the proof of Lemma 14.10 is complete. Q.E.D.

Corollary 14.11. The strict transformation \( p : X' \rightarrow X \) determines the finite holomorphic map germs
\[ (14.46) \quad p^{(k)} : (X', (e_k, 0)) \rightarrow (X, (0, 0)) \quad k = 0, 1, \ldots, n \]
where we set \( e_0 := 0 \).

We classify the proof of Proposition 14.4 for \( \nu \geq 2 \) into the following two
cases:

\[
\begin{aligned}
\text{Case 1.} & \quad \text{When } n \geq 1, \text{ that is, } \# p^{-1}(0) = n + 1 \geq 2. \\
\text{Case 2.} & \quad \text{When } n = 0, \text{ that is, } \# p^{-1}(0) = 1.
\end{aligned}
\]

**Proof of Proposition 14.4 in the case 1.** Since \( p^{(k)}(k = 0, 1, \ldots, n) \) are finite map germs, we can regard them as finite morphisms by the way given in § 13. Thus, by virtue of Proposition 13.14, the images \( p^{(k)}(X') \) are analytic sets of \( X, \) for \( 0 \leq k \leq n. \) Then Propositions 13.11 and 13.12 yield

\[
1 = \dim((e, 0), X') \leq \dim((e, 0), [p^{(k)}(X')]) \leq \dim((e, 0), X) = 1.
\]

Hence we conclude that \( p^{(k)}(X') \) are complex curves at \((0, 0)\) contained in \( X.\) On the other hand the blowing up \( \pi : Z' \to Z \) induces the isomorphism (14.28). Thus we get \( p^{(k)}(X') \neq p^{(k-1)}(X') \) as germs of curves at \((0, 0)\) if \( k \neq k'. \) Consequently we get \( X = \{g = 0\} \) has at least two irreducible components locally at the origin in the case 1, as desired. Q. E. D.

It remains the proof of Proposition 14.4 in the case 2.

We observe the effect of the strict transformation \( p : X' \to X \) to the Newton polygon \( N(g). \) Since we assume (14.17), (14.24), and the case 2, there exist positive integers \( m \geq 2 \) and \( a(\mu), b(\mu) (1 \leq \mu \leq m) \) such that

\[
(14.47) \quad N(g) = \sum_{\mu=1}^{m} N_{a(\mu), b(\mu)} \quad \text{where } N_{a, b} := \{(s, t); (s/a) + (t/b) \leq 1\}
\]

\[
(14.48) \quad b(1)/a(1) > \cdots > b(m-1)/a(m-1) > b(m)/a(m) > 1.
\]

The effect of \( p : X' \to X \) to \( N(g) \) is given by the

**Lemma 14.12.** Let \( g_2(x/y, y) \) be the defining germ of the strict transform \( X' \) on the coordinate neighborhood \( \Omega_y. \) Assume the case 2. Then it follows

\[
(14.49) \quad N(g_2) = \sum_{\mu=1}^{m} N_{a(\mu), b(\mu), a(\mu)}
\]

where we denote by \( N(g_2) \) the Newton polygon of \( g_2 \) at the unique pre-image \( (0, 0) \) of \( (0, 0) \in X \) by \( p, \) with respect to the coordinate system \( (x/y, y). \)

**Proof.** Since we assume the case 2, the expression (14.43) yields

\[
(14.50) \quad g_2(x/y, y) = c(x/y)^r + \sum_{q+r \geq 0} c_{qr}(x/y)^q y^{q+r}.
\]

In particular we have

\[
\ord_{c}[g_2(x/y, 0)] = \nu, \quad \ord_{c}[g_2(0, y)] = \min\{r - \nu; c_{vr} \neq 0\}.
\]

which yield
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\begin{equation}
(\nu, 0) \in \partial N(g_1) \subseteq N(g_1)
\end{equation}

and

\begin{equation}
(0, b(1) + \cdots + b(m) - a(1) - \cdots - a(m))
\end{equation}

\(= (0, b(1) + \cdots + b(m) - \nu) \in \partial N(g_1) \subseteq N(g_1)\).

On the other hand, since \((q, r) \in N(g)\) if \(c_{aq} \neq 0\), it follows

\begin{equation}
r \geq -(b(\mu)/a(\mu))[q - \{a(1) + \cdots + a(\mu)\}]
\end{equation}

\(+ b(\mu + 1) + \cdots + b(m)\) for \(1 \leq \mu \leq m\).

Indeed, Lemma 0.2 leads us to

\(V{(g)} = \{(a(1) + \cdots + a(\mu), b(\mu + 1) + \cdots + b(m)) ; 0 \leq \mu \leq m\}\).

By the inequality (14.52), we have

\[ q + r - \nu \geq -\{(b(\mu)/a(\mu)) - 1\}[q - \{a(1) + \cdots + a(\mu)\}]
\end{equation}

\[ + a(1) + \cdots + a(\mu) + b(\mu + 1) + \cdots + b(m) - \nu.\]

Thus, from \(\nu = a(1) + \cdots + a(m)\), we get

\begin{equation}
q + r - \nu \geq -\{(b(\mu)/a(\mu)) - 1\}[q - \{a(1) + \cdots + a(\mu)\}]
\end{equation}

\[ + \{b(\mu + 1) - a(\mu + 1)\} + \cdots + \{b(m) - a(m)\}.\]

From (14.53) and (14.51) we conclude

\begin{equation}
N(g_1) \subseteq \sum_{\mu=1}^{m} N(a(\mu), b(\mu) - a(\mu))\).
\end{equation}

Let us fix \(1 \leq \mu < m\). Note that the equalities hold in (14.52) simultaneously for \(\mu\) and for \(\mu + 1\) if and only if

\((q, r) = (a(1) + \cdots + a(\mu), b(\mu + 1) + \cdots + b(m))\).

Thus we get

\begin{equation}
(a(1) + \cdots + a(\mu), b(\mu + 1) + \cdots + b(m) - a(\mu + 1) + \cdots - a(m)) \in N(g_1)
\end{equation}

for \(1 \leq \mu < m\).

By (14.51), (14.54) and (14.55), we conclude the equality (14.49) as desired.

The proof of Lemma 14.12 is complete. Q. E. D.

**Proof of Proposition 14.4 in the case 2.** Let us divide \(b(m)\) by \(a(m)\):

\begin{equation}
\begin{cases}
b(m) = a(m)d + c \\
0 \leq c < a(m), \text{ and } d \geq 1 (c, d \in \mathbb{Z}).
\end{cases}
\end{equation}

We classify the proof as follows:

- **Case 2a.** When \(c = 0\).
- **Case 2b.** When \(c > 0\).
First we prove Proposition 14.4 in the case 2a). We consider the following sequence of blowing ups:

\[
Z = C^2 \leftarrow \pi_1 Z' \leftarrow \cdots \leftarrow \pi_{d-1} Z^{(d-1)}
\]

\[
X = g^{-1}(0) \leftarrow p_1 X' \leftarrow \cdots \leftarrow p_{d-1} X^{(d-1)}
\]

where \( \pi_j : Z^{(j)} \to Z^{(j-1)} \) is the blowing up of \( Z^{(j-1)} \) with center \( \{ x_{j-1} \} \), and \( p_j : X^{(j)} \to X^{(j-1)} \) is the strict transformation of \( X^{(j-1)} \) induced by \( \pi_j \) such that \( x_j \in X^{(\omega)} \) satisfies

\[
p_j(x_j) = x_{j-1} \quad \text{for } j \geq 1.
\]

Note that such a sequence (14.57) is determined uniquely if we give \( x_0 \) by

\[
x_0 := (0, 0) \in X = \text{Null}(g)
\]

since the germ \( (X^{(j)}, x_{j-1}) \) lies in the case 2, for \( 1 \leq j \leq d-1 \).

By virtue of Lemma 14.12, we have

\[
N(g^{(j)}) = \sum_{\mu=1}^{m} N_{a(\mu), b(\mu)-a(\mu)} + N_{a(m), a(m)}
\]

at \( x_j \in X^{(\omega)} \) \( 0 \leq j \leq d-1 \) with

\[
\begin{align*}
&\{ b(\mu)-(d-1)a(\mu) \}/a(\mu) = \{ b(\mu)/a(\mu) \}-(d-1) \\
&\quad \geq \{ b(m)/a(m) \} -(d-1) = 1 \quad \text{for } \mu < m.
\end{align*}
\]

Thus \( X^{(d-1)} \) has at least two irreducible components locally at \( x_{d-1} \) (note that the expression (14.60)' yields \( \# \text{ Seg } N(g^{(d-1)}) = \# \text{ Seg } N(g) > 1 \)).

Since the composite map germ

\[
p_1 \circ p_2 \circ \cdots \circ p_{d-1} : (X^{(d-1)}, x_{d-1}) \to (X, x_0)
\]

is a finite map germ which induces an isomorphic map germ

\[
(X^{(d-1)} - \{ x_{d-1} \}, x_{d-1}) \sim (X - \{ x_0 \}, x_0),
\]

we conclude that \( X = \{ g = 0 \} \) also has at least two irreducible components at \( x_0 = (0, 0) \) as desired.

The proof of Proposition 14.4 in the case 2a) is complete.

It only remains the case 2b). As similar as the case 2a), we consider the following sequence of blowing ups:
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(14.57)

\[ Z = C^2 \leftarrow \pi_1 \leftarrow \cdots \leftarrow \pi_{d-1} \leftarrow \pi_d \leftarrow Z^{(d)} \]

\[ X = g^{-1}(0) \leftarrow X' \leftarrow \cdots \leftarrow \rho_{d-1} \leftarrow \rho_d \leftarrow X^{(d)} \]

where \( \pi_j \) and \( \rho_j \) are as same as in the above proof of the case 2a).

Note that \( c>0 \) implies

(14.61)

\((X^{(d)}, x_{j-1}) \) lies in the case 2 for \( 1 \leq j \leq d \), and

(14.62)

\[ \text{ord}_{x_d}[g^{(d)}] < \nu = \text{ord}_{x_{d-1}}[g^{(d-1)}] = \cdots = \text{ord}_{x_0}[g]. \]

Indeed, Lemma 14.12 yields that the Newton polygon \( N(g^{(d)}) \) is given by the expression (14.60) with the inequalities

\[ \{ b(m) - ja(m) \} / a(m) \begin{cases} \geq d + \{ c/a(m) \} - (d-1) > 1 & \text{for } j \leq d-1, \\ = d + \{ c/a(m) \} - d < 1 & \text{for } j = d. \end{cases} \]

Thus we have the assertions (14.61) and (14.62).

Note that the expression (14.60) yields \( \# \text{Seg } N[g^{(d)}] = \# \text{Seg } N(g) > 1 \). Hence, by virtue of (14.62), we can apply the inductive assumption to the curve \( X^{(d)} \), which says that \( X^{(d)} \) has at least two irreducible components at \( x_d \).

Thus, the finiteness of the composite map germ

\[ \rho_1 \circ \rho_2 \circ \cdots \circ \rho_d : (X^{(d)}, x_d) \longrightarrow (X, x_0) \]

and the isomorphism of the induced map germ

\[ (X^{(d)} - \{ x_d \}, x_d) \sim (X - \{ x_0 \}, x_0) \]

yield that \( X = \{ g = 0 \} \) also has at least two irreducible components at \( x_0 = (0, 0) \) as desired.

The proof of Proposition 14.4 is complete. Q. E. D.

References
