Remark to the Ergodic Decomposition of Measures

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§ 1. Introduction

Let \((X, \mathcal{B}, \mu)\) be a measure space and \(\mathcal{A}\) be a sub-\(\sigma\)-field of \(\mathcal{B}\). A family \(\{\mu^x\}_{x \in X}\) of probability measures on \(\mathcal{B}\), indexed by \(x\) is called a system of conditional probabilities with respect to \(\mathcal{A}\) or a disintegration of \(\mu\) with respect to \(\mathcal{A}\) if it has the following properties, namely

(a) \(\forall B \in \mathcal{B}\), the function \(x \mapsto \mu^x(B)\) is \(\mathcal{A}\)-measurable and

(b) \(\forall B \in \mathcal{B}, \forall A \in \mathcal{A}, \mu(B \cap A) = \int_A \mu^x(B) d\mu(x)\).

In general, disintegrations of \(\mu\) with respect to \(\mathcal{A}\) do not exist. (See an example in the later discussions.) However, if \((X, \mathcal{B})\) is standard (that is, the measurable space \((X, \mathcal{B})\) is isomorphic to \((Y, \mathcal{B}_Y)\), where \(Y\) is a Polish space and \(\mathcal{B}_Y\) is the Borel \(\sigma\)-field of \(Y\)), then a disintegration of any probability measures on \(\mathcal{B}\) exists for all \(\mathcal{A}\) (\(\subset \mathcal{B}\)). (For example, see [1].) If a disintegration of \(\mu\) with respect to \(\mathcal{A}\) exists, then for any fixed \(A \in \mathcal{A}\), \(\mu^x(A) = \chi_A(x)\) holds for \(\mu\)-a.e.x, where \(\chi_A\) is the indicator function of \(A\). Especially for any fixed \(A \in \mathcal{A}\), \(\mu^x(A)\) takes only the values 0 or 1 for \(\mu\)-a.e.x. A strengthening form of this result is as follows.

(c) For \(\mu\)-a.e.x, \(\mu^x\) takes only the values 0 or 1 on \(\mathcal{A}\).

If a disintegration \(\{\mu^x\}_{x \in X}\) of \(\mu\) with respect to \(\mathcal{A}\) satisfies (c), then it is called an ergodic decomposition. The following fact is known for the ergodic decomposition.

**Theorem.** Let \((X, \mathcal{B})\) be a standard space, \(\{\mathcal{A}_n\}_{n=1, 2, \ldots}\) be a decreasing...
sequence of countably generated sub-\(\sigma\)-fields of \(\mathcal{B}\) and \(\mathcal{A} = \bigcap_{n=1}^{\infty} \mathcal{A}_n\). Then for any probability measure \(\mu\) on \(\mathcal{B}\), the disintegration of \(\mu\) with respect to \(\mathcal{A}\) is ergodic.

For the proof, see [2] or [3].

However even in a standard space, taking a suitable sub-\(\sigma\)-field \(\mathcal{A}\) there does exist a probability measure whose disintegration with respect to \(\mathcal{A}\) is non ergodic. The purpose of this note is to give such an example.

\section{Examples}

Let \(\mathbb{R}^\infty\) be the countable direct product of \(\mathbb{R}\), \(\mathcal{B}(\mathbb{R}^\infty)\) be the Borel \(\sigma\)-field on \(\mathbb{R}^\infty\) and \(\lambda\) be the standard Lebesgue measure on \((0, 1]\). Take \(0 < s < 1/2\), and using indicator function \(\chi_{a,k}(r)\) of the intervals \(((k-1)/n, k/n]\) \((n=1, 2, \ldots, k=1, 2, \ldots, n)\) define a map \(\phi(r) = (\phi_k(r))_k\) from \((0, 1]\) to \(\mathbb{R}^\infty\) such that \(\phi_k(r) = (n^s \chi_{a,k}(r)+1)\sqrt{r}\), if \(h = 2^{-1} n(n-1) + k\) \((1 \leq k \leq n)\). Then,

\(\int_0^1 \phi_k(r)^2 d\lambda(r) \leq 2(n^2/n+1) \leq 4\).

Hence for all \(a=(a_k)_k \in \mathbb{P}\), we have

\(|\sum_{k=1}^{\infty} a_k^2 \phi_k^2(r) < \infty\) for \(\lambda\)-a.e. \(r\).

However \(\{\phi_k(r)\}_k\) is not bounded for each \(r \in (0, 1]\), so

\(|\sum_{k=1}^{\infty} a_k^2 \phi_k^2(r) = \infty\) for \(\lambda\)-a.e. \(r\).

Now let \(g\) be the standard Gaussian measure with mean 0 and variance 1 on the usual Borel field \(\mathcal{B}(\mathbb{R}), dg(r) = (2\pi)^{-1/2} \exp(-r^2/2) dt\) and \(G\) be the product measure of \(g\), \(G = \prod_{n=1}^{\infty} g\). Using transformations \(T_r, S_r\) on \(\mathbb{R}^\infty\), \(T_r: x = (x_k) \in \mathbb{R}^\infty \mapsto (\phi_k(r) x_k) \in \mathbb{R}^\infty, S_r: x = (x_k) \in \mathbb{R}^\infty \mapsto \sqrt{r} (x_k) \in \mathbb{R}^\infty\), we put \(T_r G = G, S_r G = G_r\) and \(\mu^r = 2^{-1} (G^r + G_r)\). Then since

\(|\sum_{k=1}^{\infty} a_k^2 x_k^2 < \infty\) holds for \(G\)-a.e. \(x = (x_k)_k\) if and only if \(a = (a_k)_k \in P\), we have for the spaces \(H_a = \{x = (x_k)_k \in \mathbb{R}^\infty | \sum_{k=1}^{\infty} a_k^2 x_k^2 < \infty\}\) indexed by \(a = (a_k)_k \in P, \)

\(|\mu^r(H_a) = 1\) if \(|\sum_{k=1}^{\infty} a_k^2 \phi_k^2(r) < \infty\), and \(|\mu^r(H_a) = 1/2\) if \(|\sum_{k=1}^{\infty} a_k^2 \phi_k^2(r) = \infty\).

Next take \(r \in (0, 1]\) and fix it. Then for each \(n\) there exists unique \(1 \leq k_n \leq n\)
which satisfies $x_{n,k_n}(\tau) = 1$. Put $h_n = 2^{-1} n(n-1) + k_n$. Then in virtue of the law of large numbers,

$$\lim_{n \to \infty} \frac{x_1^2 + \cdots + x_{n-1}^2 n(n-1)}{2^{-1} n(n-1)} = 1 \quad \text{for G-a.e. \, x and}$$

$$\lim_{n \to \infty} \frac{x_1^2 + \cdots + x_{n}^2 n(n-1)}{n-1} = 1 \quad \text{for G-a.e. x.}$$

Consequently, it follows from $2s < 1$,

$$\lim_{n \to \infty} \frac{x_1^2 + \cdots + x_{n-1}^2 n(n-1)}{2^{-1} n(n-1)} = 0 \quad \text{for \, G-\, a.e. \, x.}$$

Hence,

$$\lim_{n \to \infty} \frac{x_1^2 + \cdots + x_{n-1}^2 n(n-1)}{2^{-1} n(n-1)} = \tau \quad \text{for \, G-a.e. \, x.}$$

On the other hand, it is easy to see that

$$\lim_{n \to \infty} \frac{x_1^2 + \cdots + x_{n-1}^2 n(n-1)}{2^{-1} n(n-1)} = \tau \quad \text{for G-\, a.e. \, x.}$$

Thus we have,

$$\lim_{n \to \infty} \frac{x_1^2 + \cdots + x_{n-1}^2 n(n-1)}{2^{-1} n(n-1)} = \tau \quad \text{for \, \mu^T-a.e. \, x.}$$

Define $p(x) = \lim_{n \to \infty} \frac{x_1^2 + \cdots + x_{n-1}^2 n(n-1)}{2^{-1} n(n-1)}$, if the limit exists and $p(x) = 0$, otherwise. Then it follows from (11) that $p(x) = \tau$ for $\mu^T$-a.e. $x$ and

$$\mu^\prime (p^{-1}(E)) = x_{\mu^\prime} (\tau) \quad \text{holds for all \, E \in \mathfrak{B}(\mathbb{R}).}$$

Now put $\mu(B) = \int_B \mu(B) \, d\lambda(x)$ for $B \in \mathfrak{B}(\mathbb{R})$. Then for all $B \in \mathfrak{B}(\mathbb{R})$ and for all $E \in \mathfrak{B}(\mathbb{R})$ we have $\mu(B \cap p^{-1}(E)) = \int_E \mu^\prime(B) \, d\mu(x)$. Especially,

$$p\mu = \lambda$$

and

$$\mu(B \cap p^{-1}(E)) = \int_{p^{-1}(E)} \mu^\prime(B) \, d\mu(x).$$

Further from (2) and (5) we have $\mu^\prime(H_a) = 1$ for $\lambda$-a.e. $\tau$ and therefore $\mu(H_a) = 1$. Thus,

$$\mu(B \cap H_a) = \int_{H_a} \mu^\prime(B) \, d\mu(x) \quad \text{for all \, B \in \mathfrak{B}(\mathbb{R}) \, and for all \, a = (a_k)_k \in \mathbb{P}.}$$
Let $\mathfrak{A}$ be a $\sigma$-field generated by $p^{-1}(\mathcal{B}(\mathbb{R}))$ and $H_a (a \in F)$. Then it is easy to see that

\[(16) \text{ for a fixed } B \in \mathcal{B}(\mathbb{R})^{\infty}, \mu^{p(a)}(B) \text{ is an } \mathfrak{A}\text{-measurable function of } x \text{ and} \]

\[(17) \mu(B \cap A) = \int_A \mu^{p(a)}(B) \, d\mu(x) \text{ for all } B \in \mathcal{B}(\mathbb{R})^{\infty} \text{ and for all } A \in \mathfrak{A}. \]

From (16) and (17) it follows that $\{\mu^{p(a)}\}_{x \in \mathbb{R}}$ is the disintegration of $\mu$ with respect to $\mathfrak{A}$. However for any $\tau$ there exists $b=(b_h)_{h \in P}$ which has property stated in (3). Consequently, $\mu^\tau(H_\tau)=1/2$ and therefore $\{\mu^{p(x)}\}_{x \in \mathbb{R}}$ is non ergodic decomposition.

Finally we will give a simple example of $(X, \mathcal{B})$ on which a probability measure $\mu$ does not admit any disintegration with respect to a sub-$\sigma$-field $\mathfrak{A}$.

Let $X=[0, 1]$, and consider a probability measure $\mu$ on $\mathcal{B}([0, 1])$ without atomic part. Let $\mathfrak{A}=\mathcal{B}([0, 1])$ and let $\mathcal{B}$ be the $\sigma$-field of all $\mathfrak{B}$-measurable sets. Suppose that there would exist some disintegration $\{\mu^x\}_{x \in \mathbb{R}}$ of $\mu$. Then for each $A \in \mathfrak{A}$, $\mu^x(A)=\chi_A(x)$ holds for $\mu$-a.e. $x$. Since $\mathfrak{A}$ is countably generated, there exists $\Omega \in \mathfrak{A}$ with $\mu(\Omega)=1$ such that $x \in \Omega$ implies $\mu^x=\delta_x$ on $\mathfrak{A}$, where $\delta_x$ is the Dirac measure at $x$. Especially we have $\mu^x(\{x\})=1$ for all $x \in \Omega$. Hence it holds $\mu^x=\delta_x$ on $\mathcal{B}$ for all $x \in \Omega$. Take any $B \in \mathcal{B}$ and put $C=\{x \in \mathbb{R} | \mu^x(B)=1\}$. Then $C \in \mathfrak{A}$ and $C \cap \Omega=B \cap \Omega$. Thus we have $B \cap \Omega \in \mathfrak{A}$ for all $B \in \mathcal{B}$. By the way the following lemma shows that there exists $N \in \mathfrak{A}$ such that $N \subset \Omega$, $\#N=\#x$ and $\mu(N)=0$. It follows from these facts that $\#\mathfrak{A}=2^\#x$. But it contradicts to $\#\mathfrak{A}=\#x$, since $\mathfrak{A}$ is countably generated.

**Lemma.** Let $\mu$ be a probability measure on $\mathcal{B}([0, 1])$ without atomic part and $\Omega$ be a $\mu$-measurable set with $\mu(\Omega)>0$. Then there exists Borel subset $N$ of $\Omega$ such that $\#N=\#x$ and $\mu(N)=0$.

**Proof.** Without loss of generality we may assume that $\Omega$ is a compact subset of $[0, 1]$. Put $f(t)=\mu(\Omega)^{-1} \mu(\Omega \cap [0, t])$ for $0 \leq t \leq 1$. By the assumption $f$ is continuous and it is easily checked that

\[(18) \mu(\Omega \cap f^{-1}(\alpha, \beta)) = (\beta - \alpha) \mu(\Omega) \text{ for } 0 \leq \alpha \leq \beta \leq 1. \]

Hence we have

\[(19) \mu(\Omega \cap f^{-1}(E)) = \mu(\Omega) \lambda(E) \text{ for all } E \in \mathcal{B}([0, 1]). \]

It follows from (18) that $\Omega \cap f^{-1}(\alpha, \beta) \neq \emptyset$ for $0 \leq \alpha \leq \beta \leq 1$. So using the complete intersection property of compact sets, $\Omega \cap f^{-1}(\alpha) \neq \emptyset$ holds for all
\[ \alpha \in [0, 1]. \] Now take Cantor's ternary set \( C \) and put \( N = \varnothing \cap f^{-1}(C) \). Then \( \mu(N) = 0 \) holds by (19) and \( \forall N = \mathbb{N} \) holds by the above arguments. Q.E.D.

References
