On Norm-Dependent Positive Definite Functions

By

Yasuo Yamasaki*

Summary: Any norm-dependent positive definite function on an infinite dimensional normed space can be written as a superposition of \( \exp(-c\|x\|^2) \). Conversely, for a Hilbert space, any superposition of \( \exp(-c\|x\|^p) \) is positive definite. A norm-dependent positive definite function exists only if the norm is of cotype 2. If \( \exp(-\|x\|^2) \) is positive definite for some \( c > 0 \), such \( c \) form an interval \((0, a_0]\) where \( a_0 \geq 2 \). If \( a_0 = 2 \), then \( \|x\| \) is a Hilbertian norm. For \( \langle x, y \rangle \), \( 0 < p \leq 2 \), we have \( a_0 = p \). (Though \( \|x\| = \left( \sum |x_n|^2 \right)^{1/2} \) is not a norm for \( 0 < p < 1 \), the last statement remains valid).

In [1], Chapter 3, it was shown that on a Hilbert space, any positive definite function dependent only on the norm can be written in the form:

\[
(1) \quad 
\chi(\xi) = \int_{[0, \infty)} \exp(-c\xi^2) d\nu(c)
\]

where \( \nu \) is a finite measure on \([0, \infty)\). The proof is based on Bernstein's theorem, which claims:

**Proposition 1 (Bernstein’s Theorem).** Let \( f(t) \) be a function on \([0, \infty)\). If and only if \( f(t) \) is continuous and completely monotone, it is the Laplace transform of a positive measure on \([0, \infty)\), namely it can be written as

\[
(2) \quad f(t) = \int_{[0, \infty)} \exp(-st) d\nu(s).
\]

Here, complete monotonicity is defined as:

**Definition 1.** A function on \([0, \infty)\) is said to be completely monotone, if for any \( t, \tau > 0 \) and \( n = 0, 1, 2, \ldots \) we have

\[
(3) \quad (-1)^n \Delta^nf(t) \geq 0
\]

where

\[
(4) \quad \Delta f(t) = f(t+\tau) - f(t).
\]

Note that if \( f(t) \) is known to be infinitely differentiable, complete mono-
toneness is characterized by \((-1)^n(d^n/dt^n)f(t)\geq 0\).

The proof of Bernstein's theorem, omitted here, can be found for instance in [2], Chapter 4.

For a Hilbert space, if \(\chi(\xi) = \varphi(\|\xi\|^2)\) is positive definite, then \(\varphi\) must be completely monotone. The proof is given in [1] and also in [3] with some related discussions. But in favor of Dvoretzky's theorem, this statement is kept valid for any infinite dimensional normed space.

**Proposition 2 (Dvoretzky's theorem).** Let \(X\) be an infinite dimensional normed space. For any \(\epsilon > 0\) and positive integer \(n\), there exist an \(n\)-dimensional subspace \(R\) and a Hilbertian norm \(\|\cdot\|_H\) on \(R\) such that

\[
(1-\epsilon)\|\xi\|_H \leq \|\xi\| \leq (1+\epsilon)\|\xi\|_H \quad \text{for all } \xi \in R.
\]

This theorem appeared in [4], and arose many researcher's interest which led to more detailed discussions, for instance [5].

**Proposition 3.** Let \(X\) be an infinite dimensional normed space.

If \(\chi(\xi) = \varphi(\|\xi\|^2)\) is continuous and positive definite, \(\varphi\) must be completely monotone.

**Proof.** For given \(t_0 > 0\) and \(\eta > 0\), there exists an \(\epsilon > 0\) such that

\[
|\varphi(t) - \varphi(t_0)| \leq \eta \quad \text{for } (1-\epsilon)\epsilon t_0 \leq t \leq (1+\epsilon)\epsilon t_0.
\]

For this \(\epsilon\) and any given positive integer \(n\), there exist an \(n\)-dimensional subspace \(R\) of \(X\) and a Hilbertian norm \(\|\cdot\|_H\) on \(R\) which satisfies (5).

Let \(\{e_i\}_{i=1}^n\) be a CONS of \(R\) in \(\|\cdot\|_H\). Since \(\chi\) is positive definite, we have

\[
\sum_{i=1}^n \sum_{j=1}^n \chi\left(\frac{t_0}{2}(e_i - e_j)\right) = 0.
\]

For \(i \neq j\), we have

\[
\left\|\frac{t_0}{2}(e_i - e_j)\right\|_H = t_0, \quad \text{so that } \chi\left(\frac{t_0}{2}(e_i - e_j)\right) \leq \varphi(t_0) + \eta.
\]

Thus we get

\[
n\chi(0) + n(n-1)(\varphi(t_0) + \eta) \geq 0,
\]

hence \(\varphi(t_0) \geq -\chi(0)/n-1 - \eta\). Since \(n > 0\) and \(\eta > 0\) are arbitrary, we must have \(\varphi(t_0) \geq 0\).

Next, for given \(t_0 > 0\), \(\tau > 0\) and \(\eta > 0\), we assume that (6) holds also for \(t_0 + \tau\) instead of \(t_0\) and that \(R\) is \((n+1)\)-dimensional and \(\{e_i\}_{i=0}^n\) is its CONS in \(\|\cdot\|_H\). Put \(\xi_i = \sqrt{\frac{t_0}{2}} e_i, \alpha_i = 1\) for \(1 \leq i \leq n\) and \(\xi_i = \sqrt{\frac{t_0}{2}} e_{n+i} + \sqrt{\tau} e_0, \alpha_i = -1\) for \(n+1 \leq i \leq 2n\). Then, since \(\chi\) is positive definite, we have
Thus we get

\[ 0 \leq \sum_{i,j=1}^{n} e_{ij}(e_{ij} - e_{i,j}) \]

\[ = 2 \sum_{i,j=1}^{n} \left[ \chi \left( \frac{t_0}{2} (e_{ij} - e_{i,j}) \right) - \chi \left( \frac{t_0}{2} (e_{ij} - e_{i,j}) \pm \sqrt{t_0} e_{i,j} \right) \right]. \]

Since \( t_2 > 0 \) and \( t_7 > 0 \) are arbitrary, we must have \( \chi(0) = -n \). Hence, \( \varphi(t_0) = \chi(0) \).

In a similar way, we can prove \((-1)^m \varphi(t_0) = 0\) for any \( m \), hence \( \varphi \) is completely monotone. q.e.d.

Combining the above Proposition 3 with Proposition 1, we obtain the following result.

**Proposition 4.** Let \( X \) be an infinite dimensional normed space. If a positive definite function \( \chi(\xi) \) is continuous and depends only on the norm \( \|\xi\| \), it is written in the form of (1).

**Remark 1.** For a Hilbert space, any function \( \chi(\xi) \) in the form of (1) is positive definite, but for a general infinite dimensional normed space, the converse is false. Indeed, we know:

**Proposition 5.** If \( \chi(\xi) = \exp(-\|\xi\|^2) \) is positive definite on a normed space \( X \), then \( X \) must be a Hilbert space.

**Proof.** By (infinite dimensional) Bochner’s theorem (for instance, c.f. [6]), \( \chi \) corresponds to a \( \sigma \)-additive measure \( \mu \) on \( X^\alpha \), the algebraic dual space of \( X \). The correspondence is

\[ \chi(\xi) = \int \exp(it\xi)d\mu(t), \xi \in X, x \in X^\alpha. \]

For a fixed \( \xi \neq 0 \), the equality \( \chi(t\xi) = \exp(-t^2\|\xi\|^2) \) means that \( x(\xi) \) follows one-dimensional Gaussian distribution of the variance \( 2\|\xi\|^2 \). So that we have

\[ \|\xi\|^2 = \frac{1}{2} \int x(\xi)^2d\mu(\xi). \]

Thus, the function \( \Phi_\xi(x) = x(\xi) \) belongs to \( L^2(\mu) \), and the map \( \xi \mapsto \frac{1}{\sqrt{2}} \Phi_\xi \) becomes a norm-preserving embedding of \( X \) into \( L^2(\mu) \). Hence \( X \) is a Hilbert space as a subspace of \( L^2(\mu) \). q.e.d.

Next, we shall discuss about whether \( \exp(-\|\cdot\|^\alpha) \) is positive definite or not. The following results are essentially known ([7], [8]), but we shall formulate and prove them in our way. For a preparation, we state a lemma.

**Lemma.** On \([0, \infty)\), the function \( f_\alpha(t) = \exp(-t^\alpha) \) is completely monotone if and only if \( 0 \leq \alpha \leq 1 \).

**Proof.** Evidently \( f_\alpha(t) \) is not completely monotone for \( \alpha < 0 \). We shall check the sign of \( \frac{d^n f_\alpha}{dt^n} \).

\[
\frac{d}{dt} f_\alpha = -\alpha t^{\alpha-1} \exp(-t^\alpha) \leq 0
\]

is all right if \( \alpha \geq 0 \).

\[
\frac{d^2}{dt^2} f_\alpha = [-\alpha(\alpha-1)t^{\alpha-2} + \alpha^2 t^{\alpha-1}] \exp(-t^\alpha) \geq 0
\]

is true if \( 0 \leq \alpha \leq 1 \), but false for sufficiently small \( t \) if \( \alpha > 1 \). Suppose that

\[
\frac{d^n}{dt^n} f_\alpha = \sum_{k=1}^{n} a_k \alpha^n t^{\alpha-n} \exp(-t^\alpha) \text{ and } (-1)^n a_k \geq 0 \text{ for } 0 \leq \alpha \leq 1.
\]

Then we have

\[
\frac{d^{n+1}}{dt^{n+1}} f_\alpha = \sum_{k=1}^{n} \left[ a_k \alpha^n (k\alpha - n) t^{k\alpha-n-1} - a_{k+1} \alpha^{k+1} (k+1)\alpha-n-1 \right] \exp(-t^\alpha).
\]

This means that

\[
\begin{align*}
a_{1, n+1} &= a_{1n}(\alpha-n) \\
a_{k, n+1} &= a_{k,n}(k\alpha - n) - a_{k-1,n} \alpha (2 \leq k \leq n) \\
a_{n+1, n+1} &= -a_{nn} \alpha.
\end{align*}
\]

Thus, considering \( k \leq n \) and \( 0 \leq \alpha \leq 1 \), we get \( (-1)^{n+1} a_{k,n+1} \geq 0 \). This assures that \( f_\alpha(t) \) is completely monotone if \( 0 \leq \alpha \leq 1 \). q.e.d.

Proposition 6. If \( \exp(-\|\cdot\|^\alpha) \) is positive definite on a normed space \( X \), so is \( \exp(-\|\xi\|^\alpha) \) for \( 0 \leq \alpha \leq \alpha_0 \).

**Proof.** Since \( \exp(-t^{\alpha_0}) \) is completely monotone, from Bernstein’s theorem we have

\[
\exp(-\|\xi\|^\alpha) = \int_{[0,\infty)} \exp(-s\|\xi\|^\alpha) d\nu(s).
\]

(11)
Since positive definiteness is closed under pointwise convergence and linear combination with positive coefficients, (11) assures that $\exp(-\|\xi\|^\alpha)$ is positive definite.

Remark 3. The set $\{\alpha > 0; \exp(-\|\xi\|^\alpha) \text{ is positive definite}\}$ forms an interval, if not empty. This interval is closed at right, since positive definiteness is closed under pointwise convergence, so that it is of the form of $(0, \alpha_0]$.

We have $\alpha_0 \leq 2$, since every norm-dependent positive definite function is written in the form of (1), and $\exp(-t^{\alpha/T})$ is not completely monotone for $\alpha > 2$.

We have $\alpha_0 = 2$ if and only if $X$ is a Hilbert space.

Proposition 7. Let $\varphi(\xi)$ be a non-negative function on $X$. Suppose that for any $n, m$ and $t > 0, \tau > 0$, there exist $\xi_i, \xi_j (i = 1, 2, \ldots, n, j = 1, 2, \ldots, m)$ such that

\[ \varphi(\xi_i - \xi_j) = t \]

and

\[ \varphi(\pm \xi_{j_1} \pm \cdots \pm \xi_{j_k} \mid \xi_i - \xi_j) = t + k\tau \]

for $1 \leq i \neq j \leq n, 1 \leq j_1 < j_2 < \cdots < j_k \leq m$.

Then, every $\varphi(\cdot)$-dependent positive definite function $\chi(\xi) = F(\varphi(\xi))$ is written in the form of

\[ \chi(\xi) = \int_{(s, \infty)} \exp(-s\varphi(\xi))d\nu(s). \]

Proof is obtained similarly as the proof of Proposition 3. In this case $\chi(\xi) = F(\varphi(\xi))$ implies that $F$ is completely monotone.

Corollary. For the space $(l^P), 0 < p \leq 2$, every norm-dependent positive definite function $\chi(\xi)$ is written in the form of

\[ \chi(\xi) = \int_{(s, \infty)} \exp(-s\|\xi\|^p)d\nu(s). \]

Remark 4. Conversely, every $\chi(\xi)$ in the form of (13) is positive definite on $(l^P)$, because $\exp(-|t|^p)$ is positive definite on $R$ and $\exp(-\|\xi\|^p) = \prod_{k=1}^{\infty} \exp(-|\xi_k|^p)$.

Remark 5. The criterion of this corollary shows us that $\exp(-\|\xi\|^p)$ is positive definite if and only if $0 < p' \leq p$. Thus we have $\alpha_0 = p$ for $(l^P), 0 < p \leq 2$. ($\alpha_0$ is of the same meaning as in Remark 3).

The discussions in the proof of Proposition 7 do not require any norm. So, Corollary and Remarks 4 and 5 are valid also for $0 < p < 1$. ($\|\xi\| = (\sum_{k=1}^{\infty} |\xi_k|^p)^{1/p}$, whether it is a norm or not).
References


