Lower \(L^p\)-Bounds for Scattering Solutions of the Schrödinger Equations

By

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Abstract

In this article, the asymptotic behavior in time of scattering solutions to the Schrödinger equation

\[ i\partial_t u = Hu, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n, \quad H = -\Delta + V, \]

\[ u(0, x) = \phi(x), \quad x \in \mathbb{R}^n, \]

is investigated.

Under rather natural assumptions, \(L^p(\mathbb{R}^n)\)-lower bound estimates of the form

\[ \liminf_{t \to +\infty} |t|^{\frac{n}{p} - \frac{n}{p'}} \| e^{-itH} \phi \|_{L^p(\mathbb{R}^n)} > 0 \quad (1 \leq p \leq +\infty) \]

for \(\phi \in \mathcal{S}_{\text{cont}}(H)\) with \(\phi \neq 0\) are established, where \(\mathcal{S}_{\text{cont}}(H)\) denotes the continuous spectral subspace of \(H\).

This shows that the estimates obtained by the author in [10] are optimal.

Introduction

We study the asymptotic behavior in time of scattering \(L^p\)-solutions to the Cauchy problem for the equation

\[ i\partial_t u = Hu, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n \quad (n \geq 1), \]

\[ u(0, x) = \phi(x), \quad x \in \mathbb{R}^n, \]

where \(H\) is a self-adjoint operator in the Hilbert space \(L^2(\mathbb{R}^n)\). A scattering solution means a solution of the form \(e^{-itH} \phi\) with \(\phi\) in the continuous spectral subspace \(\mathcal{S}_{\text{cont}} = \mathcal{S}_{\text{cont}}(H)\) of \(H\). Our main attention is on the case where \(H\) takes the form

\[ H = H_0 + V, \quad H_0 = -\Delta \]

with a potential \(V\) which should satisfy some conditions specified later. Roughly speaking, for a short-range potential \(V\) satisfying the conditions

\[ |V(x)| \leq C(1 + |x|)^{-\alpha} \quad \text{and} \quad V(x) + (1/2)x \cdot \nabla V(x) \leq C(1 + |x|)^{-\alpha} \]


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for almost every \( x \in \mathbb{R}^n \), where \( \alpha > \max(2, 4-n) \) and \( C > 0 \), it is known that every scattering solution \( e^{-itH}\phi \) with a nice initial datum \( \phi \) decays in the \( L^p \)-norm if \( p > 2 \). In fact, the following estimate holds:

\[
\|e^{-itH}\phi\|_{L^p(\mathbb{R}^n)} \leq C |t|^{-\delta_n(p)}, \quad |t| \geq 1,
\]

where \( \delta_n(p) = n/2 - n/p \). For \( 1 \leq p \leq 2 \), we also have the same estimate as (0.1) with more general potentials. But they do not imply the decay of solutions.

The above results have been obtained by the author in [10].

We shall show that these estimates are really optimal. To be more precise, under certain hypotheses it is shown that for any \( \phi \in \mathcal{H} \setminus \{0\} \),

\[
\liminf_{t \to \pm \infty} |t|^{\delta_n(p)} \|e^{-itH}\phi\|_{L^p(\mathbb{R}^n)} > 0 \quad (1 \leq p \leq \infty)
\]

holds. We deduce from this lower bound estimate the result that any scattering solution with its \( L^p \)-norm decaying faster than \( O(|t|^{-\delta_n(p)}) \) \((2 \leq p \leq \infty)\) or growing slower than \( O(|t|^{-\delta_n(p)}) \) \((1 \leq p < 2)\) vanishes. We shall prove these facts by mainly using Strauss’ argument in [13].

\section{1. Preliminaries}

We use the following notations. For an open subset \( \mathcal{O} \subset \mathbb{R}^n \) and \( p \in [1, \infty] \), we denote by \( L^p(\mathcal{O}) \) the usual Lebesgue space of \( p \)-th integrable functions on \( \mathcal{O} \). The associated norm is denoted by \( \| \cdot \|_{L^p(\mathcal{O})} \). We abbreviate \( L^p(\mathbb{R}^n) \) by \( L^p \). \( L^p_{\text{loc}} \) denotes the space of locally \( p \)-th integrable functions on \( \mathbb{R}^n \). For \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \), we write \( |x| = \left( \sum_{j=1}^n x_j^2 \right)^{1/2} \), \( a(x) = (1+|x|^2)^{1/2} \). For \( m, s \in \mathbb{R} \), the weighted Sobolev spaces \( H^{m,s} \) and \( \tilde{H}^{m,s} \) are defined respectively by

\[
H^{m,s} := \{ \phi \in \mathcal{S}' ; \| \phi \|_{m,s} := \| (1-\Delta)^{m/2} \phi \|_2 + \| a^s \phi \|_2 < \infty \}
\]

and

\[
\tilde{H}^{m,s} := \{ \phi \in \mathcal{S}' ; \| \phi \|_{m,s} := \| a^s(1-\Delta)^{m/2} \phi \|_2 < \infty \}.
\]

For an operator \( T \) in \( L^p \), we denote by \( D(T) \) its domain. \( A \) denotes the generator of dilations: \( A = (1/2i)(x \cdot \nabla + \nabla \cdot x) \). Strauss’ lemma will be used in the following form:

**Lemma** (Strauss [13]). For any \( \phi \in L^8 \setminus \{0\} \), there exist \( k, k' > 0, t_0 > 0 \), and \( C_0 > 0 \) such that

\[
\int_{k t_0 < |x| < k't_0} |(e^{-itH}\phi)(x)|^3 \, dx \geq C_0 \quad \text{for all} \quad t > t_0.
\]

**Proof.** Although the lemma in [13] is stated in a rather restrictive form, we see that the proof in [13] with some modifications shows the lemma above. So we omit the details.

Q. E. D.
§ 2. The Case $2 \leq p \leq \infty$

We consider the following hypotheses on a self-adjoint operator $H$ in $L^2$:

(H0) $H$ has no singular continuous spectrum.

(H1) The wave operator

$$W_+ = \text{s-lim}_{t \to +\infty} e^{itH} e^{-itH_0}$$

exists and is complete.

For sufficient conditions of (H0) and (H1), see, e.g., [1], [2], [3], [8], [9], [11] and [12].

In this section we prove:

Theorem 2.1. Suppose that (H0) and (H1) hold. Let $2 \leq p \leq \infty$ and $\phi \in \mathcal{H}_{\text{cont}} \setminus \{0\}$. Assume that there exists $t_0 > 0$ such that

$$e^{-itH}\phi \in L^p_{\text{loc}} \quad \text{for all } t > t_0.$$  

Then, there exist $k > 0$ and $k' > k$ such that

$$\lim_{t \to +\infty} t^{\delta_{\text{cont}}(p)} \|e^{-itH}\phi\|_{L^p(\{x \in \mathbb{R}^n; \ k < |x| < k' t\})} > 0.$$ 

Proof. Let $2 \leq p \leq \infty$ and let $\phi \in \mathcal{H}_{\text{cont}}$ satisfy (2.1). Assume $\phi \neq 0$. It suffices to prove that there exist $t_1 > 0$, $k' > k > 0$ and $C_1 > 0$ such that

$$\|e^{-itH}\phi\|_{L^p(\{x \in \mathbb{R}^n; \ k < |x| < k' t\})} \geq C_1 t^{-\delta_{\text{cont}}(p)} \quad \text{for all } t > t_1.$$ 

From the assumptions on $H$, we conclude that

$$\text{Range}(W_+) = \mathcal{H}_{\text{cont}}$$

and that

$$\lim_{t \to +\infty} e^{itH_0} e^{-itH} P_{\text{cont}} W_+ = W_+^*$$

where $P_{\text{cont}}$ is the orthogonal projection on $\mathcal{H}_{\text{cont}}$ and $W_+^*$ is the adjoint of $W_+$. Consequently,

$$W_+^* \phi \neq 0.$$ 

By virtue of Strauss’ lemma, there exist $t_2 > 0$, $k' > k > 0$ and $C_2 > 0$ such that

$$\int_{k t < |x| < k' t} |(e^{-itH} W_+^* \phi)(x)|^2 \, dx \geq C_2 \quad \text{for all } t > t_2.$$ 

By H"older’s inequality, we see that

$$\left( \int_{k t < |x| < k' t} |(e^{-itH} \phi)(x)|^p \, dx \right)^{1/p} \leq (\omega_n (k^{tn} - k^n)^{n/p})^{1/p-1/p} \|e^{-itH}\phi\|_{L^p(\{x \in \mathbb{R}^n; \ k < |x| < k' t\})}$$
for any $t>0$, where $\omega_t = \pi^{n/2}/\Gamma(n/2+1)$ and $\Gamma$ denotes the gamma function.

We estimate the L.H.S. of (2.8) from below as follows:

$$\left(\int_{k\leq|x|<k+t} \left| (e^{-itH}\phi(x)) |^2 \right| dx \right)^{1/2}$$

$$\geq \left(\int_{k\leq|x|<k+t} \left| (e^{-itH_0}\phi(x)) |^2 \right| dx \right)^{1/2}$$

$$- \left(\int_{k\leq|x|<k+t} \left| (e^{-itH_0}\phi - e^{-itH_0}W_+\phi(x)) |^2 \right| dx \right)^{1/2}$$

$$\geq C_{\phi}^{1/2} - \|e^{-itH_0}\phi - e^{-itH_0}W_+\phi\|_2$$

$$= C_{\phi}^{1/2} - \|e^{-itH_0}\phi - W_+\phi\|_2$$

for any $t>t_3$.

Hence, there exist $t_3>0$ and $C_3>0$ such that

$$\left(\int_{k\leq|x|<k+t} \left| (e^{-itH}\phi(x)) |^2 \right| dx \right)^{1/2} \geq C_3$$

for any $t>t_3$.

(2.8) and (2.10) give

$$\left(\int_{k\leq|x|<k+t} \left| (e^{-itH}\phi(x)) |^2 \right| dx \right)^{1/2} \geq C_3$$

for any $t>t_3$,

as required.

Q. E. D.

**Corollary 2.1.** Suppose that (H0) and (H1)$_+$ hold. Let $2 \leq p \leq \infty$ and $\phi \in \mathcal{H}_{\text{cont}} \setminus \{0\}$. Assume that there exists $t_0>0$ such that

$$e^{-itH}\phi \in L^p(\mathbb{R}^n)$$

for all $t>t_0$.

Then we have

$$\liminf_{t \to +\infty} t^\delta_n(p) \|e^{-itH}\phi\|_p > 0.$$

**Remark 2.1** For sufficient conditions of (2.12), see [10].

**Corollary 2.2.** Suppose that (H0) and (H1)$_+$ hold. Let $2 \leq p \leq \infty$ and let $\phi \in \mathcal{H}_{\text{cont}}$. Assume that for each $k'>k>0$,

$$\liminf_{t \to +\infty} t^\delta_n(p) \|e^{-itH}\phi\|_{L^p(\mathbb{R}^n; \mathbb{R}^n)} = 0.$$

Then we have $\phi=0$.

**Remark 2.2** (H0) and (H1)$_-$:

(H1)$_-$

$\mathcal{H}_{\text{cont}} \setminus \{0\}$ satisfying (2.12), we have
Remark 2.3. The asymptotic behavior as $t \to \pm \infty$ of the L.H.S. of (2.10) has been studied in numerous articles by different approaches (see, e.g., [3], [4], [9] and [11]).

Remark 2.4. When $H=H_0$ and $p=\infty$, some classes of initial data actually give $\lim$ instead of $\liminf$ in (2.2). Such examples can be found in [11].

§ 3. The Case $1 \leq p < 2$

In the case $1 \leq p < 2$, we need some additional assumptions:

Theorem 3.1. Suppose that (H0) and (H1)_+ hold. Let $1 \leq p < 2$ and let $\phi \in \mathcal{K}_{\text{cont}} \setminus \{0\}$. Assume that there exist $t_0 > 0$ such that

$$e^{-itH}\phi \in L^\infty_{loc}(\mathbb{R}^n) \quad \text{for all } t > t_0.$$ 

Assume in addition that there exists an increasing function $\alpha : \mathbb{R} \to [0, \infty)$ such that for each $k' > k > 0$,

$$\sup_{t > t_0} \alpha(t) \left\| e^{-itH} \phi \right\|_{L^\infty(x \in \mathbb{R}^n; k \leq |x| < k' t)} < \infty$$

and $\alpha(t_0) > 0$. Then, there exist $h' > h > 0$ such that

$$\liminf_{t \to +\infty} \alpha(t)^{-1/p} \left\| e^{-itH} \phi \right\|_{L^p(x \in \mathbb{R}^n; k \leq |x| < k' t)} > 0.$$ 

Proof. Put

$$C_{k, h'} := \sup_{t > t_0} \alpha(t) \left\| e^{-itH} \phi \right\|_{L^\infty(x \in \mathbb{R}^n; k \leq |x| < k' t)}.$$ 

Since

$$\left( \int_{k < |x| < k' t} |(e^{-itH} \phi)(x)|^\frac{1}{p} \, dx \right)^{1/p}$$ 

$$\leq \left\| e^{-itH} \phi \right\|_{L^p(x \in \mathbb{R}^n; k \leq |x| < k' t)} \left\| e^{-itH} \phi \right\|_{L^p(x \in \mathbb{R}^n; k \leq |x| < k' t)}^{1/p}$$ 

$$\leq C_{k, h'} \alpha(t)^{-1/p} \left\| e^{-itH} \phi \right\|_{L^p(x \in \mathbb{R}^n; k \leq |x| < k' t)},$$

a slight modification of the preceding proof works. We omit the details.

Q.E.D.

In view of (2.10) and (3.4), we easily obtain:

Corollary 3.1. Suppose that (H0) and (H1)_+ hold. Let $1 \leq p < 2$, and let $\phi \in \mathcal{K}_{\text{cont}}$. Assume that there exists $t_0 > 0$ satisfying (3.1). Then, we have $\phi = 0$ if either (A) or (B) holds:

(A) There exists an increasing function $\alpha : \mathbb{R} \to [0, \infty)$ such that $\alpha(t_0) > 0$ and
that for each $k'>k>0$,
\[
\sup_{t,t_0} \alpha(t) \| e^{-itH} \phi \|_{L^\infty(\mathbb{R}^n; \chi(t') \chi(t'))} < \infty
\]
and
\[
\lim \inf_{t \rightarrow +\infty} \alpha(t)^{1-\frac{2}{p}} \| e^{-itH} \phi \|_{L^p(\mathbb{R}^n; \chi(t') \chi(t'))} = 0.
\]

(B) There exists an increasing function $\alpha: \mathbb{R} \rightarrow [0, \infty)$ such that $\alpha(t_0)>0$ and that for each $k'>k>0$,
\[
\sup_{t,t_0} \alpha(t)^{1-\frac{k}{p}} \| e^{-itH} \phi \|_{L^p(\mathbb{R}^n; \chi(t') \chi(t'))} < \infty
\]
and
\[
\lim \inf_{t \rightarrow +\infty} \alpha(t)^{1-\frac{k}{p}} \| e^{-itH} \phi \|_{L^\infty(\mathbb{R}^n; \chi(t') \chi(t'))} = 0.
\]

§ 4. Lower Bounds of Growth Order in Time
for Scattering $L^p$-Solutions ($1 \leq p < 2$)

We consider the following class of potentials $V$, which is identical to that of [10; Theorem 6.1].

(H2) The form $i[A, V]$ on $D(A) \cap D(H)$, defined by
\[
(i[A, V] \phi, \psi) = i(V \phi, A \psi) - i(A \phi, V \psi), \phi, \psi \in D(A) \cap D(H),
\]
extends to a bounded operator $V^* \in \mathcal{L}(H^{1,0}; H^{-1,0})$.

(H3) There exist $\alpha > \max(2, 4-n)$ and $C>0$ such that $\alpha^a V \in L^\infty$ and $V + (1/2)V^* \leq C \alpha^{-a}$ as forms on $H^{2,0}$.

In order to describe lower bounds for growth order of scattering $L^p$-solutions for $1 \leq p < 2$, we put
\[
\beta(t) = |t|^{1-\frac{1}{p}}
\]
for $n=1$,
\[
\beta(t) = |t|^{1-\frac{2}{p}} \log |t|^{1-\frac{1}{p}}
\]
for $n=2$,
\[
\beta(t) = |t|^{1-\frac{n}{p}}
\]
for $n \geq 3$.

Theorem 4.1. Suppose that (H2) and (H3) hold. Let $\max(2, 4-n) < \rho < \alpha$.
When $n \leq 4$, suppose in addition that the generalized eigenspace for zero (see [10; Theorem A]) equals $\{0\}$. Let $1 \leq p < 2$ and let $\phi \in \mathcal{M}_{\text{cont}} \setminus \{0\}$ satisfy the following regularity assumptions:

(1) $\phi \in H^{0, p/2}$ for $n=1$.
(2) $\phi \in H^{2, p/2}$ for $n=2$.
(3) $\phi \in \tilde{H}^{(n-1)/2, p/2}$ for $n \geq 3$.

Then there exists $k'>k>0$ such that
\[
\liminf_{t \to \infty} \beta(t) \|e^{-itH} \phi\|_{L^p(B(x_0; \delta))} > 0.
\]

**Proof.** All we have to do is to determine the \(L^\infty\)-decay rate \(\alpha(t)\) in (3.1) of Theorem 3.1. See [10] for details. Q.E.D.

§ 5. Remarks on the Non-Linear Schrödinger Equations

In this section, we shall give some comments on lower bounds for solutions to the non-linear Schrödinger equations. We restrict our attention to the following non-linear Schrödinger equation with a single power interaction:

\[
\begin{align*}
\frac{iu}{t} & = -\Delta u + |u|^{p-1} u, \\
\phi(0, x) & = \delta(x), \\
x & \in \mathbb{R}^n,
\end{align*}
\]

where \(1 + \frac{2}{n} < p < \alpha(n)\) with \(\alpha(n) = \infty\) for \(n \leq 2\) and \(\alpha(n) = (n+2)/(n-2)\) for \(n \geq 3\).

We recall the following theorem of Y. Tsutsumi & K. Yajima [14]:

*For any \(\phi \in H^{1,1}\), there exist \(u_\pm \in L^1\) such that*

\[
\lim_{t \to \infty} \|e^{-itH_0} u_\pm - u(t)\|_2 = 0
\]

where \(u \in C(\mathbb{R}; H^{1,1})\) is the unique solution of the integral equation

\[
\begin{align*}
u(t) & = e^{-itH_0} \phi - \int_0^t e^{-i(t-\tau)H_0} |u|^{p-1} u(\tau) d\tau.
\end{align*}
\]

We note that for any \(\phi \in H^{1,1}\), the solution \(u\) in the theorem satisfies \(u(t) \in L^q(\mathbb{R}^n)\) for any \(t \in \mathbb{R}\) provided \(2 \leq q < \alpha(n)+1\).

Now we have:

**Theorem 5.1.** Let \(2 \leq q < \alpha(n)+1\) and let \(\phi \in H^{1,1}\setminus \{0\}\). Then, the unique solution of (5.3) has the following estimate:

\[
\liminf_{t \to \infty} \|\hat{\phi}(\tau)\|_{L^q(\mathbb{R}^n; \delta(t) < \delta')} > 0
\]

for some \(k' > k > 0\).

**Proof.** We note that \(\phi=0\) implies \(u_\pm=0\) in (5.2). By virtue of the theorem above, almost the same argument as in the proof of Theorem 2.1 yields Theorem 5.1. Q.E.D.

**Remark 5.1.** For the decay estimates (from above) of the solutions of (5.3), see [5] and [6]. Theorem 5.1 tells that their results are best possible with respect to the decay order in time. See also [7] for detailed analysis of \(L^\infty(\mathbb{R}^n)\)-decay for the classical solutions of (5.1).
References


