Lifting Problem of $\eta$ and Mahowald's Element $\eta_j$

* Dedicated to Professor Shôrô Araki on his 60-th birthday

By

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§1. Introduction and Statements of Results

In this paper we consider the following lifting problem: for which $n$ and $k$ is there a lift $\tilde{\eta}$ making the following diagram (1) or (2) commute up to stable homotopy?

\[ \begin{array}{ccc}
\tilde{\eta} & \quad & Q_n^{n-k+1} \\
\downarrow & \quad & p \\
S^{4n} & \xrightarrow{\eta} & S^{4n-1} \\
\end{array} \]  \hspace{1cm} \begin{array}{ccc}
\tilde{\eta} & \quad & HP_n^{n-k+1} \\
\downarrow & \quad & p \\
S^{4n+1} & \xrightarrow{\eta} & S^{4n} \\
\end{array} \\

Here and throughout this paper we use the following notations;

Notation

$HP^n$: the quaternionic $n$-dimensional projective space.

$Q^n$: the quaternionic quasi-projective space of dimension $4n-1$.

$HP_{n-k+1}^n = HP^n / HP^{n-k}$.

$Q_{n-k+1}^n = Q^n / Q^{n-k}$.

$p$ is the canonical collapsing map.

$\eta$ is the non-trivial element of $\pi_1^S(S^0)$.

These problems are natural 'next' questions after the stable James number problem (For example, see [3]). Since $Q^n$ is a stable retract of $Sp(n)$ [5] and since $HP^n$ is a stable retract of $\Omega(U(2n + 2)/Sp(n + 1))$ [2], these problems are closely related to the unstable lifting problem of $\eta$ in the canonical Stiefel
bundles: the problem with respect to Diagram (1) is related to the lifting problem of $\eta$ to the quaternionic Stiefel bundle $X_{n,k} \to S^{4n-1}$, and the problem with respect to Diagram (2) is related to the relative complex-Quaternionic Stiefel fibration $[5] I_{n+1,k} \to \Omega S^{4n+1}$, where $I_{n,k}$ is the homotopy fiber of the inclusion $X_{n,k} \to W_{2n,2k}$. For the lifting problem of $\eta$ to the real Stiefel bundle $V_{n,k} \to S^n$, the complete answer is known by M.C.Crabb and W.A.Sutherland[4] and the complex case is easy.

**Theorem A.** If $n = 2^i$, then for any $k \leq n$ there exists a lift $\tilde{\eta}$ so that the diagram (1) commutes up to stable homotopy.

**Theorem B.** There exists a lift $\tilde{\eta}$ in diagram (2) if and only if one of the following conditions is satisfied;

1. $k = 1$ or $2$,
2. $k = 3$ or $4$, and $n \equiv 2 \mod 4$.

**Theorem C.** Let $i \geq 1$. Let $n = 2^i a$ for some odd integer $a > 1$. If there is a lift $\tilde{\eta}: S^{4n} \to Q_{n-2^i+1}^n$, then the composite

$$S^{4n} \overset{\tilde{\eta}}{\longrightarrow} Q_{n-2^i+1}^n \overset{\partial}{\longrightarrow} S^{4(n-2^i)}$$

is non-trivial, where the map $\partial$ is the usual one in the following usual cofiber sequence;

$$S^{4(n-2^i)-1} \overset{i}{\longrightarrow} Q_{n-2^i}^n \overset{p}{\longrightarrow} Q_{n-2^i+1}^n \overset{\partial}{\longrightarrow} S^{4(n-2^i)}.$$

Therefore there is no lift for $k = 2^i + 1$ when $n = 2^i a$ ($a$ is odd).

In fact the above composite is detected by the secondary operation associated to the following relation;

$$Sq^{2^i+1}Sq^1 + Sq^2Sq^{2^i+2} + Sq^4Sq^{2^i+2} + Sq^{2^{i+2}}Sq^2 = 0.$$

Therefore we have a family of the stable homotopy groups closely related to what Mahowald constructed in [8]. If we choose a specific lift, we get precisely Mahowald's element $\eta_{5,i+2}$ constructed in [9]. This fact follows from the construction and the result due to Mann-Miller[10] or Mann-Miller-Miller[11]. From Theorems A and C we get the following corollaries.

**Corollary D.** There exists a stable lift $\tilde{\eta}: S^{4n} \to Q^n$ if and only if $n = 2^t$ for some $t$.

**Remark.** There is no unstable lift of $\eta$ to the usual bundle projection $Sp(n) \to S^{4n-1}$, because $\pi_{4n}(Sp(n)) \cong \pi_{4n}(Sp)$ is $Z/2$ or 0 according as $n$ is odd or even and because the generator of $\pi_{8m+4}(Sp)$ comes from $Sp(1)$. 

Corollary E. Let \( i \geq 1 \). The Mahowald’s elements \( \eta_{5,i+2} \) as above referred are in the image of the \( S^3 \)-transfer homomorphism \( t: \pi_\ast(Q^\infty) \to \pi_\ast(S^0) \).

The following theorem is a partial result about the lifting problem in Diagram (1) in case that \( k \) is small.

Theorem F. Let \( k \leq 6 \). Then in Diagram (1) there exists a stable lift \( \tilde{\eta} \) for \( k \), if and only if one of the following conditions is satisfied.

1. \( k = 1 \) or \( 2 \),
2. \( k = 3 \) or \( 4 \) and \( n \equiv 0 \mod 4 \),
3. \( k = 5 \) or \( 6 \) and \( n \equiv 0 \mod 8 \).

§2. Proof of Theorem A

Throughout this paper, homology and cohomology are assumed to be with \( \mathbb{Z}/2 \)-coefficients.

For the proof of Theorem A we need the following lemmas;

Lemma 2.1.

(i) \( H_\ast(Q^2 S^5) = \mathbb{Z}/2[x_1, x_2, x_3, \ldots] \),
\( x_i = Q_1 Q_1 Q_1 \cdots Q_1(x_1) \) and the dimension of \( x_i = 2i + 1 - 1 \).

(ii) (S. Kochman[7]) In \( H_\ast(Sp) = \Lambda_{\mathbb{Z}/2}(\gamma_1, \gamma_2, \gamma_3, \ldots) \), \( Q_1(\gamma_n) = \gamma_{2n} \)
where \( Q_1 \) is the Dyer-Lashof (subscripted) homology operation.

Let \( x: S^3 \to Sp \) be the representative of a generator of \( \pi_3(Sp) \cong \mathbb{Z} \). Since \( Sp \) is an infinite loop space, we have a canonical extension \( \tilde{x}: \Omega^2 S^3 \to Sp \) of the map \( x \). Let \( \theta: Sp \to \Omega^\infty \Sigma^\infty Q^\infty \) be the James splitting[5]. Taking the adjoint of the composite \( \theta \circ \tilde{x} \), we have a stable map, say, \( g: \Omega^2 S^3 \to Q^\infty \).

Lemma 2.2. Let \( g_\ast: H_\ast(Q^2 S^5) \to H_\ast(Q^\infty) \) be the homology induced homomorphism of \( g \). Then,
\[ g_\ast(x_i) = \gamma_{2i - 1} \]
where \( \gamma_i \in H_{4i-1}(Q^\infty) \) is the standard generator.

Proof. Let \( \sigma: H_\ast(\Omega^\infty \Sigma^\infty Q^\infty) \to H_\ast(Q^\infty) \) be the homology suspension. Then \( \sigma \theta_\ast(\tilde{x}) = \gamma_i \) and \( \sigma \theta_\ast(\text{decomposables}) = 0[5] \). Now consider the following commutative diagram;

\[
\begin{array}{ccc}
H_\ast(Q^2 S^5) & \xrightarrow{\tilde{x}_\ast} & H_\ast(Sp) & \xrightarrow{\theta_\ast} & H_\ast(\Omega^\infty \Sigma^\infty Q^\infty) \\
\downarrow{g_\ast} & & & & \downarrow{\sigma} \\
& & & & H_\ast(Q^\infty).
\end{array}
\]
So it is enough to show that $\tilde{\alpha}_*(x_i) = \gamma_{2^i-1}$. When $i = 1$ it is obviously true. Since $\tilde{\alpha}$ is a double loop map, $\tilde{\alpha}_*$ commutes with $Q_i$-operations. Therefore the cases $i \geq 2$ follow from Lemma 2.1.

Recall, by Snaith decomposition [16], that the suspension spectrum of $\Omega^2 S^5$ is a wedge of spectra, say, $D_k$ for $k \geq 1$. Homologically, $H_*(D_k)$ corresponds to the submodule of height $k$ in $H_*(\Omega^2 S^5)$. Here the height $h$ is defined as $h(x_i) = 2^i-1$. Thus $D_2$ is stably $3\cdot 2^i - 1$ connected and of dimension $2^{i+2} - 1$ complex: the bottom cell corresponds to $x_1^2 \in H_3 \Omega^2 S^5 \cong \mathbb{Z}/2$ and the top to $x_{i+1} \in H_{2^{i+2} - 1} \Omega^2 S^5 \cong \mathbb{Z}/2$. According to Mahowald [8], Brown and Peterson [1], $D_k$ is homotopy equivalent to the Brown-Gitler spectrum $\Sigma^{3k} B \left[ \begin{array}{c} k \\ 2 \end{array} \right]$. Mahowald [8] proved that there is a stable map $g_i: S^{2i+2} \to D_2$, such that the composite:

$$S^{2i+2} \to D_2, \quad D_2(2^{i+2} - 2) = S^{2i+2} - 1$$

is $\eta$. Thus by Lemma 2.2 the stable map $g \circ g_i$ gives the desired lift of $\eta$. This completes the proof of Theorem A.

§ 3. Proof of Theorem C

Let $y_i \in H^{4i-1}(Q^\infty)$ be the dual basis of $\gamma_i \in H_{4i-1}(Q^\infty)$. The following lemma easily follows by using the cofiber sequence;

$$CP^\infty \longrightarrow HP^\infty \longrightarrow Q^\infty \longrightarrow \Sigma CP^\infty.$$

Lemma 3.1. $Sq^{4i}(y_i) = \left( \begin{array}{c} 2i-1 \\ 2j \end{array} \right) y_{i+j}$ where $Sq^k$ is the Steenrod operation.

Now the proof of Theorem C follows by standard arguments. However, for my own safety I give the details. Let $n = 2^i a$ for some odd integer $a > 1$. If there is a lift $\tilde{\eta}: S^{4n} \to Q^n_{-2^i+1}$, then we denote the composite

$$S^{4n} \xrightarrow{\tilde{\eta}} Q^n_{-2^i+1} \xrightarrow{\delta} S^{4(n-2^i)}$$

by $h_i \in \pi^2_{2^i-1}(S^0)$. For convenience we denote the normalized spectrum of the mapping cone of $h_i$, say $e_{hi}$, by $X_i \cong S^0 \cup_{hi} e^{4-2^i+1}$. Let $u \in H^0(X_i)$ be the bottom generator. All we have to do is to calculate the secondary composition associated to the following sequence;

$$X_i \xrightarrow{u} K(0) \xrightarrow{f} K(1) \times K(2^{i+2}) \times K(2^{i+2} - 2) \times K(2) \xrightarrow{g} K(2^{i+2} + 2),$$

where $f = Sq^1 \times Sq^{2i+2} \times Sq^{2^{i+2} - 2} \times Sq^2$, $g = Sq^{2i+2+1} + Sq^2 + Sq^4 + Sq^{2^{i+2}}$ and $K(m)$ is the $m$-fold suspension of the Eilenberg-MacLane spectrum.
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$HZ/2$. By the definition there is a cofibration:

$$C_\eta \to C_h \xrightarrow{w} \Sigma Q_{n-2}^n.$$

Let $v \in H^0(\Sigma^{4(2^i-n)+1}Q_{n-2}^n)$ be the bottom generator. Then there is a commutative (up to stable homotopy) diagram;

$$\begin{array}{cccc}
X_{l+1} & \xrightarrow{u} & K(0) & \xrightarrow{f} K(1) \times K(2^i+2) \times K(2^{i+2}-2) \times K(2) \xrightarrow{g} K(2^{i+2}+2) \\
\uparrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \downarrow v \\
X_{l+1} \xrightarrow{w} \Sigma^{4(2^i-n)+1}Q_{n-2}^n, & \xrightarrow{f \circ v} K(1) \times K(2^i+2) \times K(2^{i+2}-2) \times K(2) \xrightarrow{g} K(2^{i+2}+2).
\end{array}$$

So it is enough to compute the bracket $\langle g, f, u \rangle$. From Lemma 3.1 it is easy to see that $\langle g, f, u \rangle = \langle g, f \circ v, \omega \rangle \neq 0$ without indeterminacy. This completes the proof of Theorem C.

Now we shall prove Corollaries. First, let $M_k = \text{the order of } J(\xi_k), \text{where } \xi_k \text{ is the canonical symplectic line bundle over } HP^{k-1} \text{ and } J \text{ is the classical } J-\text{homomorphism.}$ Then by James periodicity and by Theorem A, we see that there is a lift $\tilde{\eta}$ in diagram (1) for $n = 2^i + M_2$, and $k = 2^i$. In this case, since $n = 2^i a \text{ for some odd integer } a \text{ (see Sigrist and Suter [15]), by Theorem C we get a non-trivial family } h_{i \in \pi_{2i+1}^3(S^0)}. \text{ Now according to B. M. Mann and E. Y. Miller [10] or B. M. Mann, E. Y. Miller and H. Miller [11], there is a commutative diagram up to homotopy;}

$$\begin{array}{ccc}
Sp & \xrightarrow{\theta} & \Omega^\infty \Sigma^\infty Q^\infty \\
\downarrow & & \downarrow t \\
SO & \xrightarrow{J} & \Omega^\infty S^\infty.
\end{array}$$

Here $t$ is the representative as an infinite loop map of the $S^3$-transfer homomorphism. Note [6][14] that the $S^3$-transfer homomorphism $t : \pi_k^3(Q^\infty) \to \pi_k^3(S^0)$ for $k \leq 4l+1$ is induced by the map $\tilde{c} : Q_{M_l+1}^{M_{l+1}} \to S^{kM_l}$ using James periodicity [5]. Thus if we take the lift as in the proof of Theorem A, then from the constructions of Mahowald’s element $\eta_{5,i+2}[8][9]$ and our element $h_0$, we see that our $h_i$ coincides to the Mahowald element $\eta_{5,i+2}$. This proves Corollary E.

§4. Proof of Theorem B and F

First we prove Theorem B. For $k = 1$ or 2, it is trivial. So there is a lift $\tilde{\eta}$: $S^{4n+1} \to HP_{n-1}^n$. Consider the cofibration;

$$\begin{array}{cccc}
S^{4(n-k)} & \xrightarrow{i_k} & HP_{n-k}^n & \xrightarrow{p_k} HP_{n-k+1}^n \xrightarrow{\tilde{c}_k} S^{4(n-k)+1},
\end{array}$$
where $i_k$ is the bottom inclusion and $p_k$ is the collapsing map. Let $k = 2$.
Consider the composite $\partial_2 \circ \tilde{\eta}$. Then we have

**Lemma 4.1.**

$$
\partial_2 \circ \tilde{\eta} = \begin{cases} 
\tilde{v} & \text{if } n \equiv 0 \mod 4 \\
\eta\sigma & \text{if } n \equiv 1 \mod 4 \\
0 & \text{if } n \equiv 2 \mod 4 \\
\varepsilon & \text{if } n \equiv 3 \mod 4.
\end{cases}
$$

The above lemma has been known [12][13], but here we give a very simple (at least, theoretically) proof. Recall $\pi^s_0(S^0) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ generated by $\tilde{v}$ and $\varepsilon$. Put

$$
\partial_2 \circ \tilde{\eta} = av + be.
$$

Note that the integer $a$ and $b$ are independent of choice of $\tilde{\eta}$. By using $e$-invariant methods or the Hurewicz homomorphism of $\text{Im} J$ theory, occasionally denoted by $h^4$, we see that $b = 0$ if and only if $n$ is even. Here the symbol $A$ means $A$-theory, which is defined as the fiber spectrum of $\Phi^3 - 1: ko \to k\text{spin}$, where $ko$ (resp. $k\text{spin}$) is the connective (resp. 2-connected) cover of $(2)$-localized $KO$-theory. On the other hand, by using the well-known structure of $H^*(HP^n_{n-k})$ as a module over the Steenrod algebra, we see that $\partial_2 \circ \tilde{\eta}$ is detected by the secondary operation cited ($i = 1$) in §2 if and only if $n \equiv 0$ or $1 \mod 4$. This implies that $a \neq 0$ if and only if $n \equiv 0$ or $1 \mod 4$. This proves Lemma 4.1.

Thus from the above lemma we see that for $k = 3$ there is a lift of $\eta$ if and only if $n \equiv 2 \mod 4$. Since $\pi^s_1(S^0) = 0$, we see that for $k = 4$ there is a lift of $\eta$ if and only if $n \equiv 2 \mod 4$. Now we shall prove that there is no lift of $\eta$ for $k \geq 5$. For this purpose we use $KO$ theory and Adams operation. Assume that there exists a map $f: S^{4n+1} \to HP^n_{n-k+1}$ such that the following diagram commutes;

$$
\begin{array}{ccc}
KO^*(S^{4n+1}) & \leftarrow & KO^*(S^{4n}) \\
\downarrow f^* & & \downarrow p^* \\
KO^*(HP^n_{n-k+1}) & &
\end{array}
$$

Recall that $KO^*(HP^n_{n-k+1}) \cong KO^*(S^0) \{x^s | n - k + 1 \leq s \leq n\}$, where $x^s \in KO^s(HP^n_{n-k+1})$. Let $\alpha_k \in KO^{-4k-1}(S^0)$ be the element such that

$$
f^*(x^s) = \alpha_{n-s} \cdot l_{4n+1},
$$

where $\alpha_k$ is the bottom inclusion and $p_k$ is the collapsing map. Let $k = 2$. Consider the composite $\partial_2 \circ \tilde{\eta}$. Then we have

**Lemma 4.1.**

$$
\partial_2 \circ \tilde{\eta} = \begin{cases} 
\tilde{v} & \text{if } n \equiv 0 \mod 4 \\
\eta\sigma & \text{if } n \equiv 1 \mod 4 \\
0 & \text{if } n \equiv 2 \mod 4 \\
\varepsilon & \text{if } n \equiv 3 \mod 4.
\end{cases}
$$

The above lemma has been known [12][13], but here we give a very simple (at least, theoretically) proof. Recall $\pi^s_0(S^0) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ generated by $\tilde{v}$ and $\varepsilon$. Put

$$
\partial_2 \circ \tilde{\eta} = av + be.
$$

Note that the integer $a$ and $b$ are independent of choice of $\tilde{\eta}$. By using $e$-invariant methods or the Hurewicz homomorphism of $\text{Im} J$ theory, occasionally denoted by $h^4$, we see that $b = 0$ if and only if $n$ is even. Here the symbol $A$ means $A$-theory, which is defined as the fiber spectrum of $\Phi^3 - 1: ko \to k\text{spin}$, where $ko$ (resp. $k\text{spin}$) is the connective (resp. 2-connected) cover of $(2)$-localized $KO$-theory. On the other hand, by using the well-known structure of $H^*(HP^n_{n-k})$ as a module over the Steenrod algebra, we see that $\partial_2 \circ \tilde{\eta}$ is detected by the secondary operation cited ($i = 1$) in §2 if and only if $n \equiv 0$ or $1 \mod 4$. This implies that $a \neq 0$ if and only if $n \equiv 0$ or $1 \mod 4$. This proves Lemma 4.1.

Thus from the above lemma we see that for $k = 3$ there is a lift of $\eta$ if and only if $n \equiv 2 \mod 4$. Since $\pi^s_1(S^0) = 0$, we see that for $k = 4$ there is a lift of $\eta$ if and only if $n \equiv 2 \mod 4$. Now we shall prove that there is no lift of $\eta$ for $k \geq 5$. For this purpose we use $KO$ theory and Adams operation. Assume that there exists a map $f: S^{4n+1} \to HP^n_{n-k+1}$ such that the following diagram commutes;

$$
\begin{array}{ccc}
KO^*(S^{4n+1}) & \leftarrow & KO^*(S^{4n}) \\
\downarrow f^* & & \downarrow p^* \\
KO^*(HP^n_{n-k+1}) & &
\end{array}
$$

Recall that $KO^*(HP^n_{n-k+1}) \cong KO^*(S^0) \{x^s | n - k + 1 \leq s \leq n\}$, where $x^s \in KO^s(HP^n_{n-k+1})$. Let $\alpha_k \in KO^{-4k-1}(S^0)$ be the element such that

$$
f^*(x^s) = \alpha_{n-s} \cdot l_{4n+1},\]
where \( t_m \in KO^m(S^m) \) is the standard generator. Note that \( p^*(x^n) = \tau_{4n} \) and that \( x_0 \neq 0 \). Let \( \Phi^3 \) be the stable Adams operation in \( KO \)-theory. It is not difficult to show that

\[
\Phi^3(x^n) \equiv \sum_{i=0}^{\lfloor \frac{n-s}{2} \rfloor} \binom{s}{i} x^{2i+s} \mod 2,
\]

in \( KO^*(HP_{n-k+1}) \), where \( y \in KO^{-n}(S^0) \) is the standard generator. From the commutativity between Adams operation and an induced homomorphism, we see that, for any \( s \) such that \( n - k + 1 \leq s \leq n \), the following relations hold

\[
\sum_{i=1}^{\lfloor \frac{n-s}{2} \rfloor} \binom{s}{i} y^i x^{2i-s} = 0.
\]

Also note that \( x_{odd} = 0 \). Let \( k = 5 \). Then, applying the above equation, we have

\[(n - 4)y x_2 + \binom{n-4}{2} y^2 x_0 = 0.\]

Since \( n \) must be even if \( k \geq 3 \), we get that \( \binom{n-4}{2} \equiv 0 \mod 2 \). Thus we see that \( n \equiv 0 \mod 4 \). But this contradicts the condition that \( n \equiv 2 \mod 4 \) for \( k = 4 \). Therefore there is no lift of \( \eta \) for \( k = 5 \). This completes the proof of Theorem B.

Now we shall study necessary conditions for the existence of a lift of \( \eta \) with respect to Diagram (1). For convenience, we take the S-dual of Diagram (1). Then we get the following diagram for some integer \( m \);

Diagram (3)

Recall that \( A \)-theory is defined as the fiber spectrum of \( \Phi^3 - 1: k_0 \to k_{spin} \), where \( k_0 \) (resp. \( k_{spin} \)) is the connective (resp. 2-connected) cover of \( KO \). Then by similar consideration, using \( A \)-theory, as in the proof of Theorem B, we get the following necessary condition;

\[
\left\lfloor \frac{k-1}{2} \right\rfloor < 2^{v_2(m)}
\]

where \( v_2(m) \) is the exponent of 2 in the prime decomposition of \( m \). Thus taking S-dual again, we see that the following condition is necessary for the existence of a lift in Diagram (1);
Remark that the condition obtained from Theorem C is more restrictive than this condition. This implies that the essential obstruction of co-extending $\eta$ is not in the image of the classical $J$-homomorphism. So the problem does not seem to be solved by $e$-invariant methods. However, for the case that $k$ is small, by using both $e$-invariant and secondary operation in §2, we can solve the problem. Thus we obtain Theorem F. Details are omitted.

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References