On the Extension of $L^2$ Holomorphic Functions II

By

Takeo Ohsawa*

Introduction

Let $(X, ds^2)$ be a complete Hermitian manifold of dimension $n$, and let $\varphi$ be a real-valued $C^\infty$ function on $X$. By the theory of L. Hörmander [H], a $\bar{\partial}$-closed form $u$ on $X$ is $\bar{\partial}$-exact if it satisfies the estimate

$$|(u, v)_\varphi| \leq C_u(||\bar{\partial}v||_\varphi + ||\bar{\partial}^*v||_\varphi)$$

for any compactly supported $C^\infty$ form $v$, where $C_u$ is a number independent of $v$, and in many cases the estimate is true for $C_u=\text{const}. ||u||_\varphi$. In our previous work [O-T], we have established a new $L^2$-inequality involving the $\bar{\partial}$ operator, in which the estimation for $C_u$ is more elaborate. As a consequence, it enabled us to prove the following.

Theorem. Let $D$ be a bounded pseudoconvex domain in $\mathbb{C}^n$ and let $H \subset \mathbb{C}^n$ be a complex hyperplane. Then, every $L^2$ holomorphic function on $D \cap H$ has an $L^2$ holomorphic extension to $D$.

The purpose of the present paper is to formulate and prove a generalized $L^2$ extension theorem from higher codimensional submanifolds which includes our previous result as a special case, by using our new $L^2$ inequality.

Our main result is as follows.

Theorem. Let $X$ be a Stein manifold of dimension $n$, $Y \subset X$ a closed complex submanifold of codimension $m$, and $(E, h)$ a Nakano-semipositive vector bundle over $X$. Let $\varphi$ be any plurisubharmonic function on $X$ and let $s_1, \cdots, s_m$ be holomorphic functions on $X$ vanishing on $Y$. Then, given a holomorphic $E$-valued $(n-m)$-form $g$ on $Y$ with

Received August 1, 1987.

* Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606, Japan.
\[
\left| \int_{Y} e^{-\varphi} h(g) \wedge \bar{g} \right| < \infty, \\
\]there exists for any \( \varepsilon > 0 \), a holomorphic \( E \)-valued \( n \)-form \( G_{\varepsilon} \) on \( X \) which coincides with \( g \wedge ds_{1} \wedge \cdots \wedge ds_{m} \) on \( Y \) and satisfies

\[
\left| \int_{X} e^{-\varphi}(1 + \{s\}^{2})^{-m-\varepsilon} h(G_{\varepsilon}) \wedge \bar{G}_{\varepsilon} \right| \leq \varepsilon^{-1} C_{m} \int_{Y} e^{-\varphi} h(g) \wedge \bar{g},
\]

where \( \{s\}^{2} = \sum_{i=1}^{m} |s_{i}|^{2} \) and \( C_{m} \) is a positive number which depends only on \( m \).

To see that one cannot drop \( \varepsilon \) in (1), it suffices to consider the case \( X = C \), \( Y = \{0\} \), \( \varphi = 0 \) and \( g = 1 \).

Among a few direct consequences of Theorem, the following two observations might be of interest.

**Corollary 1.** Let \( X \) be a weakly 1-complete manifold of dimension \( n \) which admits a positive line bundle, let \( s \) be a holomorphic function on \( X \) such that \( ds \neq 0 \) on \( Y := s^{-1}(0) \), and let \( (E, h) \) be a Nakano semipositive vector bundle over \( X \). Then the restriction map

\[
\Gamma(X, \mathcal{O}_{X}(K_{X} \otimes E)) \to \Gamma(Y, \mathcal{O}_{Y}(K_{X} \otimes E))
\]

is surjective.

**Corollary 2.** Let \( Y \) be a pure dimensional closed complex submanifold of \( \mathbb{C}^{N} \), let \( \mathcal{Q} \) be a bounded domain of holomorphy, and \( \varphi \) a plurisubharmonic function on \( \mathcal{Q} \). Then, for any holomorphic function \( f \) on \( \mathcal{Q} \cap Y \) with

\[
\int_{\mathcal{Q} \cap Y} e^{-\varphi} |f|^{2} dV_{Y} < \infty
\]

there exists a holomorphic extension \( F \) to \( \mathcal{Q} \) such that

\[
\int_{\mathcal{Q}} e^{-\varphi} |F|^{2} dV \leq A \int_{\mathcal{Q} \cap Y} e^{-\varphi} |f|^{2} dV_{Y}.
\]

Here \( A \) depends on \( Y \) and \( \sup \{ ||z|| ; z \in \mathcal{Q} \} \), but does not depend on \( f \).

In §1 we improve the estimates shown in [O–T] and [O–2], so that one can dispense with auxiliary complete Kähler metrics which we needed before. We shall prove Theorem in §2 by solving \( \bar{\partial} \)-equations on a family of strongly pseudoconvex domains and taking a limit of solutions. In case \( m = 1 \), it amounts to solve the equation

\[
\bar{\partial} u = 2\pi i g \wedge [Y]
\]
with an appropriate $L^2$ estimate, where $[Y]$ denotes the $(1, 1)$-current associated to $Y$. For $m \geq 2$ the limit equation is hard to describe in the framework of distributions, and it might be interesting to know its legitimate description.

§1. Notations and Preliminaries

Let $(X, ds^2)$ be a Kähler manifold of dimension $n$, $(E, h)$ a Hermitian vector bundle over $X$ and $\phi$ a $C^\infty$ real valued function on $X$. We shall use the following notations.

$$
C^{p,q}(X, E) = \{C^\infty E\text{-valued (p, q)-forms on } X\}
$$

$$
C^p_\phi(X, E) = \{f \in C^{p,q}(X, E); \text{supp } f \subset X\}
$$

$$
L^p_\phi(X, E) = \{\text{measurable } E\text{-valued (p, q)-forms } f \text{ on } X \text{ satisfying}
\int_X e^{-\phi} |f|^2 \, dV_X < \infty\},
$$

where $|f|$ denotes the length of $f$ and $dV_X$ denotes the volume form.

We denote by $\bar{\partial}: L^p_\phi(X, E) \rightarrow L^p_\phi(X, E)$ the complex exterior derivative of type $(0, 1)$ defined on

$$
\text{Dom } \bar{\partial} = \{f \in L^p_\phi(X, E); \bar{\partial} f \in L^p_\phi(0,1)(X, E)\}.
$$

The adjoint of $\bar{\partial}$ will be denoted by $\bar{\partial}^\ast$. For a $(p, q)$-form $f$ on $X$ we denote by $e(f)$ the left multiplication by $f$ in the exterior algebra of differential forms on $X$. The (pointwise) adjoint of $e(f)$ is denoted by $e(f)^\ast$. We shall denote by $\omega$ the fundamental form of $ds^2$, and put

$$
\omega = e(\omega)^\ast.
$$

The curvature form of $\omega$ will be denoted by $\Theta = \sum_{\alpha, \beta} \Theta_{\alpha \beta} \, dz^\alpha \wedge d\bar{z}^\beta$. The left multiplication by $\Theta$ to $E$-valued forms is well-defined and denoted by $e(\Theta)$. $(E, h)$ is said to be Nakano semipositive if the Hermitian form $\sum_{\alpha, \beta} \Theta_{\alpha \beta} \, \xi^\alpha \bar{\xi}^\beta$ is semipositive. We note that $(f, g)_\phi = i^p(-1)^{n(a-1)/2} \int_X e^{-\phi} \, h(f) \wedge \bar{g}$ if $f, g \in L^p_\phi(X, E)$. In particular $L^p_\phi(0)(X, E)$ does not depend on the choice of $ds^2$. We say $X$ is a weakly 1-complete manifold if there exists a $C^\infty$ plurisubharmonic function $\phi: X \rightarrow \mathbb{R}$ such that $X_c := \{x \in X; \phi(x) < c\}$ is relatively compact for any $c \in \mathbb{R}$. For the basic materials on weakly 1-complete manifolds, see [O–1].
Lemma 1. Let $D \subseteq X$ be a strongly pseudoconvex domain with $C^\infty$ smooth boundary, let $\varphi_0$ be a $C^\infty$ defining function of $D$, and let $\psi$ be a nonnegative $C^\infty$ function defined on $\overline{D}$. Then, for any $\epsilon > 0$ and a compact subset $K \subseteq D$, there exists a $C^\infty$ function $\psi_K$ on $D$ satisfying the following properties.

(i) $\psi \geq \psi_K$ and $\inf_D \psi_K = 0$ on $K$.
(ii) $\inf_D \psi_K = 0$ and $D^\epsilon := \{ x \in D; \psi_K(x) \geq \epsilon \}$ is compact for all $\epsilon \in (0, \infty)$.
(iii) $|d\psi_K \wedge d\varphi_0| \leq C$, where $C$ does not depend on the choice of $K$.
(iv) $\psi - \psi_K$ is plurisubharmonic.

Proof. Since $D$ is strongly pseudoconvex, we may assume that $\varphi_0$ is strictly plurisubharmonic on a neighbourhood of $\partial D$. Let $\delta$ be any positive number satisfying $\delta < -\sup_{K} \varphi_0$ and let $\lambda_\delta$ be a $C^\infty$ function on $(-\infty, 0)$ such that $\lambda_\delta(t), \lambda_\delta'(t), \lambda_\delta''(t) \geq 0$ for all $t$, $\lambda_\delta(-t) = -t^{-1} \in (-\delta, 0)$, and $\lambda_\delta(t) = -t^{-1}$ on $(-\infty, -\delta)$. Then, for any $\epsilon > 0$ and $\tau > 0$ there exists a $\delta_0 > 0$ such that the function $\Phi_\delta := \tau \lambda_\delta(\varphi_0) - \psi_\tau$ satisfies

$$\frac{\partial \bar{\partial} \Phi_\delta}{2\varphi_0} \geq \frac{1}{4\tau} \partial \bar{\partial} \varphi_0 + \frac{1}{4\tau} \partial \bar{\partial} \Phi_\delta$$

for $0 < \delta < \delta_0$.

Note that $\Phi_\delta \leq -\epsilon/2$ on $D_{-\delta/2}$ if $\tau < -\frac{\epsilon}{2} \log 2^{-1}$. Let $X : R \to R$ be a $C^\infty$ increasing function such that $X(t) = t$ on $(-\infty, -\epsilon/2)$ and $X(t) = -t^{-1}$ on $(2, \infty)$. If we put $\psi_K = -X(\Phi_\delta)$ for $\delta \leq -\epsilon/2 \log 2^{-1}$, then $\psi_K$ satisfies (i) through (iv). In fact, (i), (ii), (iii) are trivial and (iv) follows from (*)

Proposition 2. Let $D \subseteq X$ be a strongly pseudoconvex domain with $C^\infty$ boundary, and let $\psi$ be a nonnegative $C^\infty$ function defined on $\overline{D}$. Then, for any $C^\infty$ function $\varphi$ on $\overline{D}$ and a $C^\infty$ $E$-valued $(n, q)$-form $u$ on $D$ with $\bar{\Psi} u |_{\partial D} = 0$,

$$||\sqrt{\delta} \varphi \bar{\partial} u||^{2}_{E,D} + ||\sqrt{\delta} \bar{\partial} u||^{2}_{E,D}$$

$$\geq (\epsilon \varphi (\delta \bar{\partial} \varphi + \Theta) - \delta \bar{\partial} \psi) Au, u |_{E,D} + 2 \Re (\epsilon (\delta \varphi) \delta \bar{\partial} u, u)_{E,D},$$

where $\star$ denotes the Hodge's star operator,

$$||\sqrt{\delta} \varphi \bar{\partial} u||^{2}_{E,D} = \int_{D} \varphi e^{-\psi} |\delta \bar{\partial} u|^{2} dV_{X}, \quad etc.$$
(**) \[ \| \sqrt{\psi} \partial^*_\phi u \|_{\varphi,D} + \| \sqrt{\psi} \partial u \|_{\varphi,D} \geq (ie(\varphi(\partial \overline{\phi} + \Theta) - \partial \overline{\psi}) \Lambda u, u)_{\varphi,D} + (e(\partial \overline{\psi}) \partial^*_\phi u, u)_{\varphi,D} \]

(cf. [O-2] §1, (6)).

Since \( \overline{\psi} u |_{\vartheta D} = 0 \),

\[ (\partial e(\partial \overline{\psi})^a u, u)_{\varphi,D} = (u, \partial e(\partial \overline{\psi})^a u)_{\varphi,D} \]

By (i) and (iii),

\[ \| e(\partial \overline{\psi})^a u - e(\partial \overline{\psi})^a u - \|_{\varphi,D} \leq \text{const.} \| u \|_{\varphi,D,K} \]

By (iv),

\[ (ie(\varphi(\partial \overline{\phi} + \Theta) - \partial \overline{\psi}) \Lambda u, u)_{\varphi,D} \geq (ie(\psi \partial \overline{\phi} - \partial \overline{\psi}) \Lambda u, u)_{\varphi,D} \]

Thus, taking the limit of the inequality (**) we obtain the desired estimate.

In order to apply the estimate (2) effectively we have to digress a bit into linear algebra.

Let \( V \) be a complex vector space of dimension \( n \) and let \( s_1 \) be a Hermitian form on \( V \). Let \( V^* \otimes C = V^* \oplus V^* \) be the decomposition into the \( \pm \sqrt{-1} \)-eigenspaces of the complex structure and let \( V^*_{s_1} \) be the subspace of \( (\bigwedge V^*_{s_1}) \otimes (\bigwedge V^* \otimes C) \) spanned by the vectors \( u \wedge (u_1 \wedge \cdots \wedge u_q) \), where \( u \in \bigwedge V^*_{s_1} \) and \( u_k(\xi) = 0 \) for \( 1 \leq k \leq q \) on \( \{ \xi \in V \otimes C ; s_1(\xi, \xi) = 0 \} \). Let \( \{ v_1, \ldots, v_n \} \) be a basis of \( V^*_{s_1} \) such that \( s_1 = \sum v_\beta \otimes \overline{v}_\beta - \sum v_\beta \otimes \overline{v}_\beta \). Then \( V^*_{s_1} \) is spanned by \( u \wedge (\overline{v}_1 \wedge \cdots \wedge \overline{v}_q) \), where \( u \in \bigwedge V^*_{s_1} \) and \( 1 \leq i_1 < \cdots < i_q \leq m \). The star operator \( *_{s_1} : V^*_{s_1} \rightarrow \bigwedge (V^* \otimes C) \)

is defined as a uniquely determined linear map which satisfy

\[ *_{s_1}(v_1 \wedge \cdots \wedge v_n \wedge \overline{v}_j_1 \wedge \cdots \wedge \overline{v}_j_q) = v_{j_1+1} \wedge \cdots \wedge v_{j_q} \times s_{n+1} \cdots s_{j_q} \]

Here \( s_j = 1 \) if \( 1 \leq j \leq l, \ s_j = -1 \) if \( l+1 \leq j \leq m \) and \( s_j = 0 \) if \( m < j \leq n \).

Then we have a nondegenerate pairing
Let $\omega_{s_1}$ be the imaginary part of $s_1$ and denote by $e(\omega_{s_1})$ the multiplication by $\omega_{s_1}$. Let $s_2$ be any positive Hermitian form on $V$. We denote by $e(\omega_{s_2})^*$ the adjoint of $e(\omega_{s_1})$ with respect to $s_2$. We put

$$\langle u, v \rangle_{s_2} = \frac{\omega_{s_2}^*}{n!} v$$

Then we have

$$\langle v, v \rangle_{s_1} = \langle e(\omega_{s_2}) e(\omega_{s_1})^{-1} v, v \rangle_{s_2} \quad \text{for } v \in V_{s_1}^{*,1}.$$ 

Here $e(\omega_{s_2})^{-1}$ denotes the inverse map of $e(\omega_{s_2}) : \bigwedge^n V_{s_1}^{*,1} \to \bigwedge^n V_{s_1}^{*,1}$. If $s_1$ is semi-positive, then

$$(3) \quad |\langle u, v \rangle_{s_2}|^2 \leq \langle e(\omega_{s_1}) e(\omega_{s_2})^* u, u \rangle_{s_2} \langle v, v \rangle_{s_1}$$

for any $u \in \bigwedge^n (V^* \otimes \mathbb{C})$ and $v \in V_{s_1}^{*,1}$. Let $(W, h_1)$ be another Hermitian vector space. Then the inner product $\langle v, v \rangle_{s_1}$ is naturally extended to $W \otimes V_{s_1}^{*,1}$, which will be also denoted by $\langle v, v \rangle_{s_1}$. We have similar estimates as (3) for the elements of $W \otimes (V_{s_1}^{*,1} \otimes \mathbb{C})$ and $W \otimes V_{s_1}^{*,1}$. Thus we have the following inequality for the bundle valued forms.

**Proposition 3.** Let $\alpha$ be a semipositive $(1, 1)$-form on $X$ and let $u, v \in L^{n,1}_\phi(X, E)$. If $v(x) \in E_x \otimes (T_x, x)^{*,1}$ for any $x \in X$, then

$$|\langle u, v \rangle_\phi|^2 \leq (e(\alpha) Au, u)_\phi \int_X e^{-\phi} \langle v, v \rangle_\sigma \quad dV_x$$

and

$$\int_X e^{-\phi} \langle v, v \rangle_\sigma \, dV_x = (e(\omega) e(\sigma)^{-1} v, v)_\phi.$$

To simplify the notation we set

$$L^{n,1}_\phi(X, E)_\sigma = \{ v \in L^{n,1}_\phi(X, E); v(x) \in (T_x, x)^{*,1} \text{ for any } x \in X \}.$$

**Proposition 4.** Let $D \subseteq X$ be a strongly pseudoconvex domain with $C^\infty$-smooth
boundary, \( \psi \) a nonnegative \( C^m \) function on \( \tilde{D} \), and \( \varphi \) a \( C^m \) function on \( \bar{D} \). Suppose that \((E, h)\) is Nakano semipositive and there exists a positive locally bounded function \( \eta \) on \( D \) such that

\[
\sigma(\eta) := \psi \bar{\partial} \partial \varphi - \partial \bar{\partial} \varphi - \eta^2 |\partial \psi|^2 \partial \bar{\partial} \psi
\]

is semipositive. Then, for any \( C^m \) \( E \)-valued \((n, 1)\)-form \( u \) on \( \tilde{D} \) with \( \bar{\partial} u \mid_{\bar{D}} = 0 \) and \( v \in L^2_\psi(D, E) \psi(\varphi) \),

\[
|\langle u, v \rangle_{\psi, D}|^2 \leq (e(\omega) e(\sigma(\eta))^{-1} v, v)_{\psi, D} \\
\times (||\sqrt{\psi + \eta} |\partial \psi|\d \psi_u||^2_{\psi, D} + ||\sqrt{\psi} \bar{\partial} u||^2_{\psi, D}).
\]

\( \S 2 \). Proof of Theorem

Let the notations be as in the introduction. Since \( X \) is a Stein manifold one can find a decreasing sequence of \( C^m \) plurisubharmonic functions \( \{\varphi_k\}_{k=1}^\infty \) which converges to \( \varphi \) almost everywhere. Hence it suffices to prove Theorem in case \( \varphi \) is \( C^m \). Moreover we may assume that \( ds_1 \wedge \cdots \wedge ds_m = 0 \) everywhere. In fact, take an analytic subset \( Z \subset X \) of codimension one such that

\[
Z \supset \{x; ds_1 \wedge \cdots \wedge ds_m \mid_x = 0\}.
\]

Then it suffices to show the extendability of \( g \wedge ds_1 \wedge \cdots \wedge ds_m \) to \( X \setminus Z \), since the apparent singularity along \( Z \) is improper in virtue of the \( L^2 \) condition.

As in [O–T] we fix an increasing family of strongly pseudoconvex domains \( X_1 \subset X_2 \subset \cdots \subset X_\mu \subset \cdots \) with \( C^m \) smooth boundaries such that

\[
X = \bigcup_{\mu=1}^\infty X_\mu.
\]

Then it suffices to find the extensions to \( X_\mu \), since one obtains a desired extension as a weak limit of a subsequence of the extensions to \( X_\mu \) \((\mu \to \infty)\).

Let \( G \) be an arbitrary holomorphic extension of \( g \wedge ds_1 \wedge \cdots \wedge ds_m \) to \( X \). It certainly exists since \( X \) is a Stein manifold. Let \( \chi: \mathbb{R} \to \mathbb{R} \) be a \( C^m \) function satisfying \( \chi(t) = 1 \) on \((-\infty, 1/2)\) and \( \chi(t) = 0 \) on \((1, \infty)\). For any \( \delta > 0 \) we put

\[
G^{[\delta]} = \begin{cases}
\chi(|s|^2/\delta^2) G & \text{on } \{x \in X; |s(x)| < \delta\} \\
0 & \text{otherwise}.
\end{cases}
\]

Then \( G^{[\delta]} \) is a \( C^m \) extension of \( g \wedge ds_1 \wedge \cdots \wedge ds_m \). We put

\[
v^\delta = \bar{\partial} G^{[\delta]}.
\]
Note that $v^4 = 0$ on a neighbourhood of $Y$. Taking an arbitrary Kähler metric $ds^2$ of $X$, we fix a metric $ds^2$ of $X$ by

$$ds^2 = ds^2 + 2\partial \bar{\partial} \log (1 + |s|^2).$$

Then, for any $\mu$ one can find a sufficiently small $\delta_\mu$ such that

(4) $$\int_{\mathcal{X}_\mu} e^{-\varphi} |s|^{-2m} \partial^2 |v^4|^2 dV_x \leq C_m \int_Y e^{-\varphi} |g|^2 dV_Y$$

if $\delta < \delta_\mu$.

Here $C_m$ depends only on $m$. Let $\lambda: \mathbb{R} \to \mathbb{R}$ be a $C^\infty$ function satisfying $\lambda'(t) \geq 0$, $\lambda''(t) \geq 0$, $\sup \lambda(t) \leq 1$, $\sup \lambda''(t) < 2/3$, and

$$\lambda(t) = \begin{cases} 0 & \text{on } (-\infty, 0), \\ t - 1 & \text{on } (2, \infty). \end{cases}$$

We put $\psi_0 = \lambda(-\log(|s|^2 + \delta^2))$. Then $\partial \bar{\partial} \psi_0 = -\partial \bar{\partial} \log(|s|^2 + \delta^2)$ if $|s|^2 + \delta^2 < e^{-2}$, $\partial \bar{\partial} \psi_0 = 0$ if $|s|^2 + \delta^2 > 1$, and

$$\partial \bar{\partial} \psi_0 \leq \frac{8e^4}{3} \partial \bar{\partial} \log(|s|^2 + 1)$$

if $e^{-2} \leq |s|^2 + \delta^2 \leq 1$.

On the other hand

$$\partial \psi_0 \bar{\partial} \psi_0 \leq (|s|^2 + 1)^2 (|s|^2 + \delta^2)^{-1} \partial \bar{\partial} \log(|s|^2 + 1).$$

Thus $|\partial \psi_0|^2$ is estimated from above by $(|s|^2 + 1)^2 (|s|^2 + \delta^2)^{-1}$. For any $\varepsilon > 0$ we put

$$\psi_\varepsilon = \psi_0 + \varepsilon^{-1} \left( \frac{8e^4}{3} + 4 \right).$$

Then $|\partial \psi_\varepsilon|^2 \leq (|s|^2 + 1)^2 (|s|^2 + \delta^2)^{-1}$ and

$$\psi_\varepsilon \leq \alpha_{\varepsilon} (|s|^2 + \delta^2)^{-1},$$

where

$$\alpha_{\varepsilon} = \sup_{|s| < 1} (|s|^2 + \delta^2) \left( -\log(|s|^2 + \delta^2) + \varepsilon^{-1} \left( \frac{8e^4}{3} + 4 \right) \right).$$

We put

$$\varphi_\varepsilon = \varphi + 2m \log |s| + \varepsilon \log(|s|^2 + 1).$$

Then
is semipositive and
\[ \sigma_x \geq \partial \overline{\partial} \log (|s|^2 + \delta^2) \]
on \{x; |s(x)| < \delta\}.

Combining it with (4) we see that there exists a \( \delta'_\mu > 0 \) such that
\[ (e(\omega) e(\sigma_x)^{-1} v^3, v^3)_{\theta e, X\mu} \leq 4C_m \int_Y e^{-\psi} |g|^2 d\mu \]
if \( \delta < \delta'_\mu \).

Therefore by Proposition \( \psi \), if \( \delta < \delta'_\mu \)
\[ |(u, v^3)_{\theta e, X\mu}|^2 \leq 4C_m \int_Y e^{-\psi} |g|^2 d\mu \]
\[ \times \{||\sqrt{\psi} + |\partial \psi|\overline{\partial} u||_{\theta e, X\mu}^2 + \sqrt{\psi} |\partial u||_{\theta e, X\mu}^2\} \]
for any \( C^\infty \) \( E \)-valued \((n, 1)\)-form \( u \) on \( X\mu \) with \( \partial u|_{\partial X\mu} = 0 \). Since the same estimate also holds for \( u \in \text{Dom} \overline{\partial} \cap L^\infty_y(X\mu, E) \), there exists a solution \( b^\delta \) to the equation \( \overline{\partial}(\sqrt{\psi} + |\partial \psi|) b^\delta = v^3 \) with
\[ ||b^\delta||_{\theta e, X\mu}^2 \leq 4C_m \int_Y e^{-\psi} |g|^2 d\mu \]
(cf. [H]).

We put
\[ G^\delta : = \Phi(\sqrt{\psi} + |\partial \psi|) b^\delta. \]

Then \( G^\delta \) is a holomorphic extension of \( g \wedge ds_1 \wedge \cdots \wedge ds_m \) to \( X\mu \). The verification of the \( L^2 \) estimate is left to the reader.

**Proof of Corollary 1.** Let \( \varphi : X \to \mathbb{R} \) be any \( C^\infty \) plurisubharmonic exhaustion function and let \( (B, a) \) be a positive line bundle over \( X \). Then, for any \( c \in \mathbb{R}, X_c := \{x; \varphi(x) < c\} \) is embeddable into a projective space by holomorphic sections of \( B^m(m = m(c) \gg 0) \). In particular there exists a proper analytic subset \( Z_c \subset X_c \) such that \( Z_c \supset Y \cap X_c \setminus Z_c \) is a Stein manifold. Let \( g \wedge ds \in \Gamma(Y, \mathcal{O}_Y (K_X \otimes E)) \) and choose a convex increasing \( C^\infty \) function \( \lambda \) such that
\[ g \in L^\lambda_{\text{hol}}(Y, E). \]

Applying Theorem to the manifolds \( X_c \setminus Z_c \supset Y \cap X_c \setminus Z_c \) and \( g | Y \cap X_c \setminus Z_c \), we...
have extensions of $g \wedge ds$ to $X_\epsilon \setminus Z_\epsilon$ whose norms in $L^{n,0}_{\mathcal{A}(\phi)}(X_\epsilon \setminus Z_\epsilon, E)$ are dominated by const. $\|g\|_{\mathcal{A}(\phi)}$. Since the singularities along $Z_\epsilon$ are improper, by taking a weak limit of these extended forms in $L^{n,0}_{\mathcal{A}(\phi)}(X, E)$ we obtain a holomorphic extension of $g \wedge ds$ to $X$.

**Remark 1.** The assumption that $X$ admits a positive line bundle was only used to ensure the existence of the divisors $Z_\epsilon$. Hence one can replace the existence of a positive bundle in the hypothesis by the existence of a Zariski dense Stein open subset $\mathcal{O} \subset X$ such that $X \setminus \mathcal{O}$ does not contain any connected component of $Y$.

**Remark 2.** Corollary 1 may be regarded as an extension of Kazama-Nakano's vanishing theorem on weakly 1-complete manifolds (cf. [0–1]). In fact, if $E$ is Nakano-positive then $H^1(X, \mathcal{O}(K_X \otimes E)) = 0$ so that the surjectivity of the above map follows immediately. H. Skoda [S–2] has established a similar surjectivity theorem on weakly 1-complete manifolds as a generalization of his $L^2$ corona theorem on pseudoconvex domains in $\mathbb{C}^n$ (cf. [S–1]).

**Proof of Corollary 2.** Let $w_1, \cdots, w_k$ be holomorphic functions on $\mathbb{C}^N$ which generate the stalks of the ideal sheaf of $Y$ at each point of $\mathcal{O}$. Let $m = \text{codim } Y$. Then for each $m$-tuple $(w_{i_1}, \cdots, w_{i_m})$ we apply Theorem as follows. Let $\Sigma_I \subset \mathbb{C}^N$ ($I = (i_1, \cdots, i_m)$) be an analytic subset of codimension one which contains the set $\{x \in Y; dw_1, \cdots, dw_m(x) = 0\}$, and let $\sigma_I$ be a defining function of $\Sigma_I$. Then, by Rückert's theorem there exists a $p \in \mathcal{N}$ such that for all $I$

$$\sigma_I^p \cdot dz_1 \wedge \cdots \wedge dz_N = g_I \wedge dw_{i_1} \wedge \cdots \wedge dw_{i_m} \text{ on } Y$$

for some holomorphic $(N-m)$-form $g_I$ on $Y$. Here $(z_1, \cdots, z_N)$ denotes the coordinate of $\mathbb{C}^N$. Then $f \sigma_I^p \cdot dz_1 \wedge \cdots \wedge dz_N$ has an extension $G_I$ to $\mathcal{O}$ with

$$\int_{\Omega} e^{-\varphi} |G_I \wedge \overline{G}_I| \leq C_I \int_{Y \cap \Omega} e^{-\varphi} |f| |g_I \wedge \overline{g}_I|,$$

where $C_I$ does not depend on $f$. Let $\eta_I$ be holomorphic functions on $\mathbb{C}^N$ satisfying

$$\sum_{I \in \{i_1, \cdots, k\}} \sigma_I^p \cdot \eta_I = 1 \text{ on } Y.$$

Then we define a function $F$ by

$$F \cdot dz_1 \wedge \cdots \wedge dz_n = \sum_{I \in \{i_1, \cdots, k\}} \eta_I \cdot G_I.$$
Clearly $F$ is an extension of $f$ with desired properties.

Remark. Corollary 2 is easily generalized to relatively compact pseudo-convex domains of Stein manifolds. The detail is left to the reader.

References


