An Extension of Hodge Theory to Kähler Spaces with Isolated Singularities of Restricted Type

By

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Introduction

The present article is a continuation of the author's previous works [15] through [20], where degenerations of Hodge's spectral sequences have been observed on several non-compact Kähler manifolds. Here we shall be concerned with a problem related to a conjecture of Cheeger-Goreski-MacPherson on the coincidence of $L^2$ cohomology and intersection cohomology of projective varieties (cf. [C-G-M]). Let $X$ be a compact complex space of pure dimension $n$ equipped with a Kähler metric $ds^2$, let $\Sigma$ be the set of singular points of $X$ and let $X_{\#} := X \setminus \Sigma$.

We denote by $H^r(X_{\#}), H^r_0(X_{\#})$ and $H^r_2(X_{\#})$, respectively the $r$-th de Rham cohomology of $X_{\#}$, the $r$-th de Rham cohomology of $X_{\#}$ with compact support, and the $L^2$ de Rham cohomology of $X_{\#}$, all with coefficients in $\mathbb{C}$. Correspondingly $H^{p,q}(X_{\#}), H^p_0(X_{\#})$ and $H^p_2(X_{\#})$ shall denote the Dolbeault cohomologies of type $(p, q)$.

Our main result is stated as follows:

Theorem If $\dim \Sigma = 0$, then there exists a complete Kähler metric $ds^2_{\#}$ on $X_{\#}$ whose Kähler class is the same as that of $ds^2$, such that

$$
H^r_2(X_{\#}) \approx \begin{cases} 
H^r(X_{\#}) & \text{if } r < n \\
\text{Im}(H^r_0(X_{\#}) \to H^r(X_{\#})) & \text{if } r = n \\
H^r_0(X_{\#}) & \text{if } r > n
\end{cases}
$$

(1)

$$
H^p_2(X_{\#}) \approx \begin{cases} 
H^p,q(X_{\#}) & \text{if } p+q < n-1 \\
H^p_0,q(X_{\#}) & \text{if } p+q > n+1
\end{cases}
$$

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if the following condition (*) is satisfied.

(*) \( \dim \sum = 0 \) and there exists a desingularization \( \pi : \tilde{X} \to X \) by blowing up such that \( \pi^{-1}(\sum) \) is a disjoint union of nonsingular divisors.

**Corollary 1.** Under the above situation, the multiplication \( L \) by the Kahler class of \( ds^2 \) induces the bijections

\[
(3) \quad L^k : H^{n+k}(X_\#) \to H^{n+k}_0(X_\#), \quad k \neq 0.
\]

**Corollary 2.** Let \( X \) be as above, and let \( m \) be an odd integer. Then

\[
(4) \quad \dim_c IH^m(X) \equiv 0 \mod 2,
\]

where \( IH^m \) denotes the intersection cohomology in the sense of Goreski-MacPherson.

In case \( \dim X = 2 \), M. Nagase [13] has established the relation (1) for the original Kahler metric \( ds^2 \) (see also [12]).

For higher dimensional varieties with isolated singularities, L. Saper [22] establishes (1) for certain complete Kahler metrics which are nonequivalent to ours, although he did it under a very restrictive assumption on the singularity as we do here. The publication of the present paper might be admitted because we have proved (1) by a completely different method.

§ 1. \( L^2 \) Estimates for the Exterior Derivative

Let \( \Delta \subset C \) be the open unit disc centered at the origin, let \( \Delta_\# = \Delta \setminus \{0\} \), and let \((z_1, \cdots, z_{m+1})\) be the coordinate of \( C^{m+1} \). We put \( z = (z_1, \cdots, z_{m+1}) \), \( |z| = \max |z_j| \) and \( f(z) = z_{m+1} \). The modulus of the function \( f \) will be denoted by \( r \).

Let \( ds^2 \) be the euclidean metric on \( C^{m+1} \). We define a metric \( ds^2_f \) on \( \Delta^m \times \Delta_\# \) by

\[
ds^2_f = \frac{ds^2}{(1-\ln r) \ln^2 (2-\ln r)} + \frac{df d\bar{f}}{r^2 (1-\ln r)^2 \ln^2 (2-\ln r)}.
\]

We shall first ask for estimates of the solutions to the equations \( du = v \) on \( \Delta^m \times \Delta_\# \).

The \( L^2 \) norms with respect to \( ds^2 \) and \( ds^2_f \) will be denoted by \( || \||_e \) and \( || \||_f \), respectively.

**Lemma 1.** For any square integrable \((m+1)\)-form \( v \) on \( \Delta^m \times \Delta_\# \), satisfying \( dv = 0 \), and \( \text{supp } v \subset \Delta^m \times \{z \in \Delta_\# \mid |z| < \frac{1}{2} \} \), there exist a neighbourhood \( U \ni \Delta \) and a measurable \( m \)-form \( u \) on \( U \cap \Delta^m \times \Delta_\# \), square integrable with respect to \( ds^2_f \), such that \( du = v \) on \( U \cap \Delta^m \times \Delta_\# \).
Proof. We expand \( \nu \) as

\[
\nu = \sum_{\mu \in \mathbb{Z}} e^{i\mu \theta} (\nu_\mu' + d\theta \wedge \nu_\mu'),
\]

where \( \theta = \arg f \) and \( \nu_\mu', \nu_\mu'' \) are orthogonal to the forms \( d\theta \wedge \). Then we have \( dv_\mu' = 0 \) and \( i\mu \nu_\mu' - dv_\mu'' = 0 \) for all \( \mu \). Since \( \deg \nu_\mu' = m + 1 \) and \( \deg \nu_\mu'' = m \), we have

\[
(1) \quad ||\nu_\mu'||_{\nu_\nu} \leq ||\nu_\mu'||_{f,\nu},
\]

and

\[
(2) \quad ||\nu_\mu''||_{\nu_\nu} \leq ||d\theta \wedge \nu_\mu'||_{f,\nu}.
\]

Moreover, it is clear that

\[
(3) \quad ||\nu_\mu''||_{f,\nu} \leq ||d\theta \wedge \nu_\mu'||_{f,\nu}.
\]

From (2) and (3), the series \( u' := \sum_{\mu \in \mathbb{Z}} i\mu^{-1} e^{i\mu \theta} \nu_\mu' \) converges with respect \( || \cdot ||_{f,\nu} \) to and \( || \cdot ||_{\nu_\nu} \) so that \( ||u'||_{\nu_\nu} < \infty, ||u'||_{f,\nu} < \infty \) and

\[
\nu_0' + du' = \nu_0' + d\theta \wedge \nu_0''.
\]

We put

\[
\nu_0' = \alpha + dr \wedge \beta,
\]

where \( \alpha \) is orthogonal to the forms \( dr \wedge \nu_\nu \).

By assumption we have

\[
\alpha = dr \wedge \beta = 0 \quad \text{on} \quad r^{-1}\left(1, 1\right).
\]

Then, integrating \( dr \wedge \beta \) along the gradient vector field of \( r \), we obtain an \( m \)-form

\[
\xi = \xi(r) := \int_{1/2}^{r} dt \wedge \beta
\]

which is defined on \( U \cap (\mathbb{D}^m \times \mathbb{D}\ast) \) for some neighbourhood \( U \ni 0 \) and satisfies \( \text{supp} \xi \subset U \cap (\mathbb{D}^m \times \{z \in \mathbb{D}\ast \mid |z| < \frac{1}{2}\}) \).

Then we have

\[
d\xi = (d_r + d') \xi = dr \wedge \beta + \int_{1/2}^{r} dt \wedge d' \beta
\]
\[256\text{TAKEO OHSAWA}\]

\[d_r \alpha = dr \land \beta + \int_{1/2}^{r} d_r \alpha = dr \land \beta + \alpha,\]

where \(d_r\) denotes the exterior derivative with respect to \(r\) and we put \(d' = d - d_r\).

The square integrability with respect to \(ds^2\) follows immediately from the integral inequality

\[
\int_{r}^{s} \frac{1}{s} (2 - \ln s)^{-2} \ln s (2 - \ln s) \int_{1/2}^{s} g(t) dt |^2 ds \\
\leq \int_{r}^{s} \frac{1}{s} (2 - \ln s)^{-2} \int_{1/2}^{s} g(t) dt |^2 ds \\
\leq 4 \int_{r}^{s} |g(s)|^2 ds\]

for \(r \in \left(0, \frac{1}{2}\right)\),

which holds for any continuous function \(g: \left(0, \frac{1}{2}\right) \rightarrow \mathbb{C}\).

Thus it only remains to solve the equation \(du = d\theta \land \nu\).

But it is similar as above and left to the reader.

Lemma 1 shall be used to prove the vanishing of the middle \(L^2\) cohomology around isolated singular points. As we have shown in earlier papers \([18]-[20]\), the vanishing of higher \(L^2\) cohomology groups follows from a very general argument by applying an estimate on complete Kähler manifolds due to Donnelly and Fefferman \([7]\).

**Lemma 2**

Let \(u\) be any compactly supported \(C^\infty\) \(r\)-form on a Kähler manifold \((X, ds^2)\) of dimension \(n\) with a global potential function \(\varphi\) (i.e. \(ds^2 = dd\varphi\) on \(X\)). Then

\[||u|| \leq 4 \sup \{ |\partial \varphi|_p | p \in \text{supp } u\} (||du|| + ||d^*u||)\]

whenever \(r \neq n\). Here \(d^*\) denotes the adjoint of \(d\), and \(||u||\) denotes the \(L^r\)-norm of \(u\).

**Theorem 3.**

Let \((X, ds^2)\) be a complete Kähler manifold of dimension \(n\) and \(D \subset X\) an open subset. Suppose that there exists a proper \(C^\infty\) map \(\varphi: D \rightarrow (c_0, \infty)\) for some \(c_0 \in \mathbb{R} \cup \{ -\infty\}\), such that

1) The eigenvalues of \(i\partial \bar{\partial} \varphi\) are larger than a positive constant on \(D\).
2) \(\sup_{D} |\partial \varphi| < \infty\).

Then, for any non-critical value \(c\) of \(\varphi\) and for any square integrable \(k\)-form (resp. \((p, q)\)-form) \(\nu\) on \(D_c := \{x \in D; \varphi(x) < c\}\) with \(k > n\) (resp. \(p + q > n\) and \(d\nu \equiv 0\) (resp. \(\bar{\partial} \nu \equiv 0\)), there exists on \(D_c\), for any \(c' < c\), a square integrable \((k-1)\)-form
(resp. \((p, q-1)\)-form) \(u\) such that \(du=v\) (resp. \(\bar{\partial}u=v\)).

**Proof.** See Theorem 1.1 in [18].

**Remark.** A metric of type \(ds_f^2\) was first introduced by H. Grauert in [8] to show that every smooth (not necessarily compact) projective variety admits a complete Kahler metric. A remarkable property of \(ds_f^2\) is that it admits a bounded potential function.

§ 2. **Proof of Theorem**

Let \((X, ds^2)\) be a compact Kähler space of pure dimension \(n\) with isolated singular points \(\Sigma\), and let \(X_\ast=X\setminus\Sigma\). In virtue of Hironaka's desingularization theorem, there exists a Kähler manifold \(\tilde{X}\) and a proper holomorphic map \(\pi: \tilde{X} \to X\) which is a biholomorphism on \(\pi^{-1}(X_\ast)\). One can take \(\tilde{X}\) so that \(E:=\pi^{-1}(\Sigma)\) is supported on a divisor of simple normal crossings and there exists an effective divisor \(E_\ast\) on \(\tilde{X}\) supported on \(|E|\) such that \([-E_\ast]\) is very ample (cf. [10]). Similarly as in [17], we have then a positive \(C^\infty\) function \(\psi\) on \(X_\ast\) such that

1) \(\partial\bar{\partial}\ln\psi\) is extended smoothly along \(E\) as a metric on a neighbourhood \(W\supset E\) and \(-\ln\psi\mid W>1\).

2) \(\ln\psi-\ln|s|^2\) is \(C^\infty\) on \(\tilde{X}\), where \(s\) is a canonical section of \([E_\ast]\) and \(|s|\) denotes the length of \(s\) with respect to some \(C^\infty\) metric of the bundle.

Let \(\rho\) be a nonnegative \(C^\infty\) function such that \(\text{supp } \rho\subset W\) and \(\rho\equiv 1\) on a neighbourhood \(U\supset E\). We put

\[
ds_{\ast}^2 = N ds^2 + \partial\bar{\partial}(\rho \ln^{-1} \ln^2 \psi).
\]

Let \(\pi: \tilde{X} \to X\) be as in the hypothesis of Theorem to be proved, and fix a positive constant \(N\) so that \(ds_{\ast}^2\) is a complete Kähler metric on \(X_\ast\). Since \(ds_{\ast}^2\) is asymptotically equivalent to the metric of type \(ds_f^2\) near each \(p\in E\), we obtain the following.

**Lemma 4.** Let \(U\supset E\) be a neighbourhood and let \(u\) be a \(d\)-closed square integrable \(n\)-form on \(U\setminus E\) satisfying \(\text{supp } u \cup E \subset U\). Then there exists a square integrable \((n-1)\)-form \(u\) on \(U\setminus E\) satisfying \(du=u\).

Proof is similar as in Lemma 1.

For any open set \(V\subset X_\ast\), we denote by \(L^k(V)\) (resp. \(L^{k,q}(V)\)) the set of square integrable \(k\)-forms (resp. \((p, q)\)-forms) on \(V\) with respect to \(ds_{\ast}^2\).
Definition. Let \( V \) be as above. We put

\[
\begin{align*}
H^k_{(2)}(V) &= \{ f \in L^k(V); \ df = 0 \} / \{ g \in L^{k-1}(V) \ s.t. \ g = du \} \\
H^k_{(2)}(V) &= \{ f \in L^{k,q}(V); \ \bar{\partial} f = 0 \} / \{ g \in L^{k-1,q}(V) \ s.t. \ g = \bar{\partial}u \}.
\end{align*}
\]

Then we have the following exact sequences:

\[
\begin{align*}
(4) \quad \lim_k H^k_{(2)}(X_* \setminus K) &\to H^k_{(2)}(X_* \setminus \{ \} \setminus K) \to \lim_k H^k_{(2)}(X_* \setminus K) \\
(5) \quad \lim_k H^k_{(2)}(X_* \setminus K) &\to H^k_{(2)}(X_* \setminus \{ \} \setminus K) \to \lim_k H^k_{(2)}(X_* \setminus K),
\end{align*}
\]

where \( K \) runs through the compact subsets of \( X_* \). As a consequence we obtain

Lemma 5. If \( \lim_k H^k_{(2)}(X_* \setminus K) = \lim_k H^k_{(2)}(X_* \setminus K) = 0 \) (resp. \( \lim_k H^k_{(2)}(X_* \setminus K) = \lim_k H^k_{(2)}(X_* \setminus K) = 0 \)), then

\[
H^k_{(2)}(X_*) \simeq H^k_{(2)}(X_* \setminus K) \quad \text{(resp. } H^{k,q}_{(2)}(X_*) \simeq H^{k,q}_{(2)}(X_* \setminus K)\text{)}.
\]

Since the function \( \varphi = \ln^{-1} \ln^2 \varphi \) satisfies that

\[
\bar{\partial} \varphi \geq \partial \varphi \bar{\partial} \varphi
\]

on a neighbourhood of \( E \), combining Theorem 3 with Lemma 5 we obtain the isomorphisms

\[
H^k_{(2)}(X_*) \simeq H^k_{(2)}(X_*)
\]

and

\[
H^{k,q}_{(2)}(X_*) \simeq H^{k,q}_{(2)}(X_*)
\]

for \( k, p+q > n+1 \).

By Poincaré and Serre's duality, taking the finiteness of \( \dim H^{k,q}(X_*) \) \((p+q > n+1)\) and \( \dim H^{p,q}(X_*) \) \((p+q > n-1)\) into account (cf. [2]), we have

\[
H^k(X_*) \simeq H^k_{(2)}(X_*)
\]

and

\[
H^{p,q}(X_*) \simeq H^{p,q}_{(2)}(X_*)
\]

if \( k, p+q < n-1 \).

The proof of Theorem will be finished if we show the following.

Proposition 6.

\[
\lim_k H^k_{(2)}(X_* \setminus K) = 0.
\]
In fact, from the exact sequence (4) one has
\[ H^{n+1}_{(2)}(X_\ast) \cong H^{n+1}_0(X_\ast), \]
hence by the Poincaré duality
\[ H^{n-1}_{(2)}(X_\ast) \cong H^{n-1}(X_\ast). \]
As for the \( n \)-th \( L^2 \) cohomology, the map
\[ H^n_2(X_\ast) \to H^n_{(2)}(X_\ast) \]
is surjective, therefore the map
\[ H^n_{(2)}(X_\ast) \to H^n(X_\ast) \]
is injective. Hence we have
\[ H^n_{(2)}(X_\ast) \cong \text{Im}(H^n_2(X_\ast) \to H^n(X_\ast)). \]

Proof of Proposition 6: The \( L^2 \) vanishing shall be reduced to a vanishing theorem which has nothing to do with \( L^2 \) conditions. We note that the proof we give below does not use the assumption that \( \text{Sing} |E| = \phi \). Therefore, the following is valid for any compact complex space \( X \) with isolated singularities and any desingularization \( \pi: \tilde{X} \to X \) by blowing up.

**Lemma.** If \( \tilde{X} \backslash K \) is homotopically equivalent to \( E \), the homomorphism
\[ H^n(\tilde{X} \backslash K) \to H^n(X_\ast \backslash K) \]
is a zero map.

**Proof.** It suffices to show that the map
\[ \iota: H^n(\tilde{X} \backslash K) \to H^n(\tilde{X} \backslash K) \]
is surjective. Since \( \dim H^n(\tilde{X} \backslash K) = \dim H^n(\tilde{X} \backslash K), \) the surjectivity will follow from the injectivity of \( \iota \). Since we may assume that \( \tilde{X} \backslash K \) is an arbitrarily small neighbourhood of \( E \), we may assume that \( \tilde{X} \backslash K \) is biholomorphically equivalent to an open subset \( U \) of a nonsingular projective variety \( Y \) such that the image \( \tilde{Y} \) of \( Y \) under the blow down along \( E \) is projective algebraic. (Artin's theorem, cf. [4]). Let \( Z \subset Y \) be a nonsingular divisor which does not intersect with \( E \) and defines an ample divisor on \( \tilde{Y} \). Shrinking \( U \) if necessary, we may assume that \( Z \cap U = \phi \). Then, applying the Morse theory as in Andreotti-Frankel [1], we have
and that the restriction map
\[ H^n(Y \setminus Z) \to H^n(U) \]
is surjective.

**Sublemma.** The restriction map
\[ H^n(Y) \to H^n(U) \]
is surjective.

**Proof.** Note that the Green operator commutes with the complex exterior derivatives \( \partial \) and \( \bar{\partial} \). Since the Gysin map \( H^{n-1}(Z) \to H^{n+1}(Y) \) is of type \((1, 1)\), and \( Y \) is \( \text{Kählerian} \) the above property of the Green operator implies the following. Let \( v \) be any \( d \)-closed \( C^\infty \) \( n \)-form on \( Y \setminus Z \) with logarithmic poles along \( Z \). Let \( v = v_0 + \cdots + v_n \) be the decomposition into different types. Suppose that \( v_p = 0 \) for \( p \leq k \) for some integer \( k \). Then there exists a \( C^\infty \) \( d \)-closed \( n \)-form \( v' \) on \( Y \setminus Z \), with logarithmic poles along \( Z \), such that \( v'_p = 0 \) for \( p \leq k \), and \( \text{res}_Z (v - v')_{p+1} = 0 \). Here \( v'_p \) and \( (v - v')_p \) denote the \((p, q)\) components of \( v' \) and \( v - v' \), respectively, and \( \text{res}_Z \) denotes the residue along \( Z \). Combining this fact with the surjectivity of \( H^n(Y \setminus Z) \to H^n(U) \), the surjectivity of \( H^n(Y) \to H^n(U) \) follows immediately.

Now we proceed to prove the injectivity of \( \iota \).

Let \( v \) be a \( C^\infty \) compactly supported \( d \)-closed \( n \)-form on \( U \), and suppose that there exists a \( C^\infty \) \((n-1)\)-form \( u \) on \( U \) with \( du = v \). We shall show that the harmonic representative \( v_h \) of \( v \) as a cohomology class on \( Y \) is zero. By Sublemma, it will then follow that \( v \) represents zero in \( H^n(U) \). Let \( v_h = v_p + Lv_n \) be the decomposition into the primitive and nonprimitive parts. Then

\[
\int_Y v_h \wedge \bar{w} v_h = \int_Y v \wedge \bar{w} v_h = \int_U v \wedge \bar{w} v_p + \int_U v \wedge \bar{w} L v_n = \int_U v \wedge \bar{w} v + \int_U v \wedge L w.
\]

Here \( w \) is some \( C^\infty \) \( d \)-closed \((n-2)\)-form on \( Y \).
Since $Lw = \frac{i}{2} \partial \bar{\partial} \ln \phi \wedge w$ on $U \setminus E$ and $v$ is $d$-exact on $U$, we have
\[ \int_U v \wedge Lw = 0. \]

Since $v_p \perp Lv$, we have
\[ \int_Y v_h \wedge \bar{w} v = \int_Y v_p \wedge \bar{w} v. \]

Thus $Lv = 0$, which implies that $v_p$ is $d$-exact on $U$. Note that
\[ \int_U v \wedge \bar{v} = (-1)^{(n+1)/2} \int_U v \wedge C \bar{v}, \]

where $C$ denotes the Weil’s operator. Since $v_p |_U$ is $d$-exact, $Cv_p |_U$ must be $d$-exact, too, since $C$ is compatible with the canonical spectral sequence which abuts to $H^*(E)$ on the varieties with normal crossings.

Therefore,
\[ \int_U v \wedge C \bar{v} = 0. \]

Thus
\[ \int_Y v_h \wedge \bar{w} v = 0 \]

which implies that $v_h = 0$, and the proof of Lemma is completed.

Now we shall finish the proof of Proposition 6. By the Lemma, the image of the homomorphism
\[ \alpha: \lim_K H^p_\Phi(X_\Phi \setminus K) \to \lim_K H^p(X_\Phi \setminus K) \]

is zero. The injectivity of $\alpha$ follows immediately from Lemma 4. Therefore
\[ \lim_K H^p_\Phi(X_\Phi \setminus K) = 0. \] Q.E.D.

Corollaries 1 and 2 are straightforward applications of our theorem.

Remark. In case $X$ is projective algebraic, the corollaries have been obtained by Navaro Aznar [14] and Morihiko Saito [21] independently by
different methods. As for the basic results in this direction, see also [5] and [6].

References


Added in proof. The author must apologize to the reader that our result is not so satisfactory. He promises to give a complete result, i.e. one without any restriction on the singularity, in a forthcoming article.