Asymptotic Expansions of Distribution Solutions of Some Fuchsian Hyperbolic Equations

By

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Introduction

In [3] a class of Fuchsian hyperbolic operators has been considered and a general result was given concerning the structure of distribution solutions defined in a full neighborhood of a point of the characteristic hypersurface $t=0$. The operators treated in [3] are strictly hyperbolic for $t \neq 0$.

Our aim is to consider in this paper the more general case where the operators are strictly hyperbolic only for $t > 0$. For some results in this direction, see Bernardi [1]. In this Introduction we state our main result in a particular, but typical, example.

Consider the following operator:

\[(0.1) \quad P = (t \partial_t)^2 - t \sum_{j=1}^n \partial_{x_j}^2 + \alpha(t, x) t \partial_t + \sum_{j=1}^n \beta_j(t, x) t \partial_{x_j} + \gamma(t, x)\]

defined on some neighborhood $\mathcal{O}$ of the origin in $\mathbb{R}_t \times \mathbb{R}_x$ (the coefficients are supposed to be in $C^\infty(\mathcal{O})$).

We explicitly remark that the results contained in [3] cannot be applied directly to the operator (0.1), since $P$ is hyperbolic only for $t > 0$. Denote by $\rho_1(x)$, $\rho_2(x)$ the roots of the indicial equation

\[(0.2) \quad \rho^2 + \alpha(0, x) \rho + \gamma(0, x) = 0\]

and denote by $\mathcal{D}^\prime_+$ the set of all germs of distributions $u(t, x)$ defined on some $\mathcal{O}' \cap \{t > 0\}$, with $\mathcal{O}'$ an open neighborhood of $(t=0, x=0)$. Then we have the following result.

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Theorem. Suppose that:
\[ \rho_1(0), \rho_2(0) \notin \frac{1}{2} \mathbb{Z}; \rho_1(0) - \rho_2(0) \notin \frac{1}{2} \mathbb{Z}. \]

Then:

i) Every \( u \in D^\prime_+ \) for which \( Pu = 0 \) has the asymptotic expansion as \( t \to 0^+ \):
\[
(0.3) \quad u(t, x) \sim \sum_{j=1,2} \left[ \varphi_j(x) t^{\rho_j(x)} + \sum_{k=1}^\infty \sum_{h=0}^k \left( L_{j,k,h}(x, \partial_x) \varphi_j(x) \right) \times t^{\rho_j(x)+k/2} (\log t)^k \right]
\]

for some unique germs of distributions \( \varphi_1(x), \varphi_2(x) \) defined near \( x = 0 \), where the \( L_{j,k,h}(x, \partial_x) \) are linear differential operators (with smooth coefficients) depending only on \( P \).

ii) Conversely, for every germs \( \varphi_1(x), \varphi_2(x) \) there exists a unique germ \( u \in D^\prime_+ \) satisfying \( Pu = 0 \) and having the asymptotic expansion \( (0.3) \) as \( t \to 0^+ \).

The precise meaning of the expansion \( (0.3) \) will be defined in Sect. 3.

The example \( (0.1) \) is a particular case of the class of operators we will consider. Actually we shall prove that asymptotic expansions as \( (0.3) \) hold for solutions of equations of the following form:

\[ Pu = \sum_{j+|\alpha| \leq m} a_{j,\alpha}(t, x) (t \partial_t)^j (t^{1/k} \partial_x)^{\alpha} u = 0, \quad t > 0, \]

where \( k \) is a positive integer (in the case \( (0.1) k = 2 \)). For precise definitions, see Sect. 1.

It is worth to mention that asymptotic expansions like \( (0.3) \) were established first in \([5]\) for \( C^\infty \)-solutions on \( t > 0 \). The possibility of passing from \( C^\infty \) to distribution solutions relies on two essential results: an extendability theorem, which is proved in Sect. 2, and the local representation formula proved in \([3]\).

§ 1. Class of Operators

We consider operators \( P \) of the form:
\[
(1.1) \quad P = \sum_{j+|\alpha| \leq m} a_{j,\alpha}(t, x) (t \partial_t)^j (t^{1/k} \partial_x)^\alpha
\]
defined on some box \([0, T[ \times U \subseteq \mathbb{R}^+_t \times \mathbb{R}^n_x\), where \( U \) is a neighborhood
of \( x = 0 \) and \( 0 < T \leq +\infty \); \( m \) and \( k \) are positive integers.

We shall make the following assumptions on \( P \):

i) \( a_{m,0} \neq 0 \) on \([0, T[ \times U\).

ii) For every \( j, \alpha \), \( a_{j,\alpha} \in C^m([0, T[ \times U) \cap C^0([0, T[ \times U) \)

and has the following expansion

\[
(1.2) \quad a_{j,\alpha} \sim \sum_{i=0}^{\infty} a_{j,\alpha,i}(x) t^{i/k}, \text{ as } t \to 0^+,
\]

for some \( a_{j,\alpha,i} \in C^m(U), l \geq 0 \). The expansion (1.2) means that for every \( N \in \mathbb{Z}^+ \) and for every \( h \in \mathbb{Z}^+ \) we have:

\[
(1.3) \quad t^{k-N/h} \frac{\partial^N}{\partial t^N} \left[ a_{j,\alpha}(t, x) - \sum_{i=0}^{N} a_{j,\alpha,i}(x) t^{i/k} \right] \to 0
\]

in \( \mathcal{C}^0(U) \) as \( t \to 0^+ \).

When \( k = 1 \), the expansion (1.2) is equivalent to say that \( a_{j,\alpha} \in C^m([0, T[ \times U) \).

iii) For every \( (t, x, \xi) \in [0, T[ \times U \times (\mathbb{R}^n \setminus 0) \), the polynomial

\[
\lambda \mapsto \sum_{j+|\alpha|=m} a_{j,\alpha}(t, x) \lambda^{j,\alpha}
\]

has \( m \) real distinct roots \( \lambda_1(t, x, \xi), \ldots, \lambda_m(t, x, \xi) \).

We denote by \( \Phi^m_T([0, T[ \times U) \) the class of all operators as \( P \).

For any \( P \in \Phi^m_T([0, T[ \times U) \) we define the indicial polynomial by

\[
(1.4) \quad I_P(x; \rho) = \sum_{j=0}^{m} a_{j,0}(0, x) \rho^j.
\]

The roots of \( I_P(x; \rho) \) will be denoted by \( \rho_1(x), \ldots, \rho_m(x) \). We observe that the change of variables \( (t, x) \to (t^{1/k}, x) \) transforms operators \( P \) in the class \( \Phi^m_T \) into operators \( \tilde{P} \in \Phi^m_T \) and we have the identity

\[
(1.5) \quad I_P(x; \rho) = I_{\tilde{P}}(x; \rho/k).
\]

In Sect. 2 we shall consider only the case \( k = 1 \) and we need operators defined in a full box neighborhood \([-T, T[ \times U \) of \((t=0, x=0)\) and satisfying conditions i) \sim iii) \) on \([-T, T[ \times U \). We denote by \( \Phi^m_T([-T, T[ \times U) \) the class of such operators and remark that every \( P \in \Phi^m_T([0, T[ \times U) \) has an extension \( \tilde{P} \in \Phi^m_T([-T, T[ \times U) \).

\section*{§ 2. Extendability Results}

In this Section we shall prove some preliminary results to our
Theorem 1. Let \( P \in \Phi^*_T(-T, T[x \times U]) \) and suppose that the roots \( \rho_j(x), j = 1, \ldots, m \), of the indicial polynomial satisfy the condition:

\[
\rho_j(0) \notin \{-1, -2, \ldots, -n, \ldots\}, \quad j = 1, \ldots, m.
\]

Then, for every distribution \( u \in \mathcal{D}'([0, e[\times \omega]), \) with \( \omega \) a neighborhood of \( x = 0 \) and \( ]0, e[\times \omega \subset ]0, T[x \times U, \) for which \( Pu = 0 \) on \( ]0, e[\times \omega, \) there exists a distribution \( v \) such that:

1) \( v \) is defined on some neighborhood \( ]-e', e'[\times \omega' \subset ]-T, T[x \times U \) of the origin and \( Pv = 0 \) on \( ]-e', e'[\times \omega'.

2) \( v \ |_{0, e'[\times \omega} = u \ |_{0, e'[\times \omega}. \)

The proof will follow from some lemmas.

Lemma 1. Let \( P \in \Phi^*_T(-T, T[x \times U]) \) and let \( u \in \mathcal{D}'([0, e[\times \omega), \) be as in the statement of Theorem 1. Then there is a distribution \( \omega \in \mathcal{D}'(\mathbb{R}_\times \mathbb{R}^n) \) such that:

i) \( \text{supp}(\omega) \subset \overline{\mathbb{R}_\times \mathbb{R}^n}. \)

ii) \( \omega \ |_{0, e'[\times \omega} = u \ |_{0, e'[\times \omega} \) for some neighborhood \( ]0, e'[\times \omega \subset ]0, e[\times \omega. \)

Lemma 2. Let \( P \in \Phi^*_T(-T, T[x \times U]) \) and suppose that the roots \( \rho_j(x), j = 1, \ldots, m \), of the indicial polynomial satisfy the condition:

\[
\rho_j(x) \notin \{-1, -2, \ldots, -n, \ldots\}, \quad j = 1, \ldots, m, \quad x \in U.
\]

Then, for every \( f \in \mathcal{D}'(-T, T[x \times U]) \) with \( \text{supp}(f) \subset \{t = 0\} \), there exists a unique \( g \in \mathcal{D}'(-T, T[x \times U]) \) with \( \text{supp}(g) \subset \{t = 0\} \) such that \( Pg = f \) on \( ]-T, T[x \times U. \)

We now show how the two lemmas imply Theorem 1.

Proof of Theorem 1. Given \( u \) as in the statement, let \( \omega \in \mathcal{D}'(\mathbb{R}_\times \mathbb{R}^n) \) be as in Lemma 1. Put \( \omega = \omega \ |_{-e', e'[\times \omega} \) and let \( f = Pu. \) Then \( f \in \mathcal{D}'(-e', e'[\times \omega') \) with \( \text{supp}(f) \subset \{t = 0\}. \) By shrinking \( \omega' \) and taking into account (2.1), we can suppose that condition (2.2) holds for every \( x \in \omega'. \) Application of Lemma 2 yields a distribution \( g \in \mathcal{D}'(-e', e'[\times \omega') \) with \( \text{supp}(g) \subset \{t = 0\} \) and \( Pg = f. \) By defining
v = u - g, the theorem is proved.

We now prove the lemmas.

Proof of Lemma 1. By a modification of P outside a neighborhood of the origin we can suppose that \( P \in \mathcal{D}'(\mathbb{R}^n) - T, T[ \times \mathbb{R}^n] \) with constant coefficients \( a_{i,a} \) for \( |t| + |x| \) large enough. By using a bounded domain of dependence argument and partial hypoellipticity of \( P \) for \( t \neq 0 \) we can suppose that \( u \in C^\infty(]0, \varepsilon[ ; \mathcal{D}'(\mathbb{R}^n)) \) and \( Pu = 0 \) on \( ]0, \varepsilon[ \times \mathbb{R}^n \).

We now prove that for every \( K \subset \mathbb{R}^n \) there exists a positive number \( a \) such that:

\[
(2.3) \quad t^a u(t, \cdot), \varphi(\cdot) \mathcal{D}(\mathbb{R}^n) = O(1), \text{ as } t \to 0^+,
\]

uniformly with respect to bounded sets of \( \varphi \in \mathcal{D}(K) \).

It is easy to show that property (2.3) implies the extendability of \( u \) and hence the lemma.

Now let us fix \( K \subset \mathbb{R}^n \). By a cut-off argument we can find a distribution \( u \in C^\infty(]0, \varepsilon[ ; \mathcal{D}'(\mathbb{R}^n)) \) such that \( Pu = 0 \) on \( ]0, \varepsilon[ \times \mathbb{R}^n \) and \( u = v \) on \( ]0, \varepsilon[ \times K \).

Now, following [2; p. 185], define

\[
(2.4) \quad v^{(b)} = (tA)^{m-h-i+1}(tD_x)^{i-1}v, \quad h = 1, \ldots, m, \quad j = 1, \ldots, m - h + 1,
\]

where \( A = (1 + |D_x|^2)^{1/2} \).

The vector \( \vec{v} = (v^{(1)}, \ldots, v^{(m)}, v^{(m+1)}, \ldots, v^{(2m-1)}, \ldots, v^{(m)}) \in C^\infty(]0, \varepsilon[ ; H^\infty(\mathbb{R}^n)^N), N = m(m+1)/2, \) satisfies a first order system on \( ]0, \varepsilon[ \times \mathbb{R}^n \):

\[
(2.5) \quad I_N t \partial \vec{v} = tA(t, x, D_x) \vec{v} + B(t, x, D_x) \vec{v},
\]

where:

i) \( A(t, x, D_x) \) is an \( N \times N \) matrix of classical pseudodifferential operators of order 1 (depending smoothly on \( t \in [-T, T] \) and satisfying uniform estimates on \( (x, \xi) \)). The principal symbol of \( A \) has the following form

\[
(2.6) \quad \sigma_1(A)(t, x, \xi) = \left[ \begin{array}{c} m \\ \hline \end{array} \right] \left[ \begin{array}{c} A(t, x, \xi) \\ \hline \end{array} \right] \left[ \begin{array}{c} \vline \\ \hline \end{array} \right] N - m,
\]

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where the matrix $A'(t, x, \xi)$ has the roots $\lambda_j(t, x, \xi), j = 1, \ldots, m,$ as eigenvalues.

ii) $B(t, x, D_\sigma)$ is an $N \times N$ matrix of classical pseudo differential operators of order 0 (depending smoothly on $t \in [-T, T]$ and satisfying uniform estimates on $(x, \xi)$).

For any $a > 0$, define $\tilde{v}_a = t^a \tilde{v}$; then $\tilde{v}_a$ satisfies the system:

\begin{equation}
\begin{cases}
\mathcal{P}_{a\tilde{v}_a} = 0 & \text{on } ]0, \epsilon[ \times \mathbb{R}^n, \\
\mathcal{P}_a = I_N t \partial_t - t A - (B + a I_N).
\end{cases}
\end{equation}

To prove (2.3) it will be enough to show that there exists $a_0 > 0$ such that:

\begin{equation}
t < \tilde{v}_a(t, \xi), \tilde{\phi}(\cdot) >_{H^\infty \times H^\infty} = O(1), \quad t \to 0 +,
\end{equation}

uniformly with respect to bounded sets of $\tilde{\phi} \in H^\infty(\mathbb{R}^n)$ and $a \geq a_0$.

For the adjoint system $\mathcal{P}_a^* = - I_N t \partial_t - t A^* - (B^* + (a + 1) I_N)$ and for every $s \in ]0, \epsilon[,$ consider the following Cauchy problem:

\begin{equation}
\begin{cases}
\mathcal{P}_a^* \tilde{w}_s = 0 & \text{on } ]0, \epsilon[ \times \mathbb{R}^n, \\
\tilde{w}_s |_{t=s} = \tilde{\phi} \in H^\infty(\mathbb{R}^n).
\end{cases}
\end{equation}

Since for $t > 0$ $\mathcal{P}_a^*$ is a symmetrizable hyperbolic system (see e.g. [6]), we know that (2.9) has a unique solution $\tilde{w}_s \in C^\infty(]0, \epsilon[; H^\infty(\mathbb{R}^n))^N$.

For every $(s, S) \in \mathcal{D} = \{(s, S) | 0 < s < S \leq \epsilon\}$ from the identity

\begin{align}
0 &= \int_s^S \langle \mathcal{P}_{a\tilde{v}_a}(t), \tilde{\phi}_s(t) \rangle dt - \int_s^S \langle \tilde{v}_a(t), \mathcal{P}_a^* \tilde{w}_s(t) \rangle dt
\end{align}

we get the relation:

\begin{equation}
s < \tilde{v}_a(s), \tilde{\phi} > = S < \tilde{v}_a(S), \tilde{\phi}_s(S) >.
\end{equation}

To prove (2.8) it is enough to show that for any $k \in \mathbb{Z}_+$ and any bounded subset $\mathcal{B} \subset H^\infty(\mathbb{R}^n)$ we have

\begin{equation}
\sup_{(s, S) \in \mathcal{D}, \tilde{\phi} \in \mathcal{B}} \| A^{2k} \tilde{\phi}_s(S) \| < \infty
\end{equation}

($\| \cdot \|$ means here the $L^2$-norm).

We prove (2.11) by induction on $k$. Denote by $R(t, x, D_\sigma)$ an $N \times N$ matrix of classical pseudodifferential operators of order 0 (depending smoothly on $t$ and satisfying uniform estimates on $(x, \xi)$) such that:
i) \( \sigma_0(R)(t, x, \xi) \) is a symmetrizer for \( \sigma_1(A^*)(t, x, \xi) \).

ii) \( R = R^* \).

iii) There exists a \( \gamma > 0 \) for which \( (R\tilde{\phi}, \tilde{\phi}) \geq \gamma ||\tilde{\phi}||^2 \) for every \( \tilde{\phi} \in L^2(R^n)^N \) and any \( t \in [0, \varepsilon] \).

(For the existence of \( R \) see e.g. [6]).

To simplify notation we write \( \tilde{\phi} \) instead of \( \tilde{\phi}_s \). We have:

\[
(2.12) \quad t \frac{d}{dt}(R\tilde{\phi}, \tilde{\phi}) = -t((RA^* + AR)\tilde{\phi}, \tilde{\phi}) - ((RB^* + BR)\tilde{\phi}, \tilde{\phi})
- 2(a+1)(R\tilde{\phi}, \tilde{\phi}) + t\left(\frac{d}{dt}R\right)\tilde{\phi}, \tilde{\phi} \leq C||\tilde{\phi}||^2 + 2||RB^*|| ||\tilde{\phi}||^2 - 2(a+1)(R\tilde{\phi}, \tilde{\phi})
\]

for some \( C > 0 \) independent of \( \tilde{\phi} \) and \( t \in ]0, \varepsilon[ \). Taking into account property iii) of \( R \), from (2.12) we obtain

\[
(2.13) \quad t \frac{d}{dt}(R\tilde{\phi}, \tilde{\phi}) \leq \frac{C}{\gamma} t(R\tilde{\phi}, \tilde{\phi}) + \frac{2}{\gamma} ||RB^*|| ||R\tilde{\phi}, \tilde{\phi}|| - 2(a+1)(R\tilde{\phi}, \tilde{\phi}).
\]

Choose \( a_0 > 0 \) such that

\[
(2.14) \quad a_0 + 1 - \frac{1}{\gamma} \sup_{t \in [0, \varepsilon]} ||RB^*(t)|| > 0.
\]

Then for every \( a \geq a_0 \) we obtain from (2.13)

\[
(2.15) \quad \left(t \frac{d}{dt} - \alpha t + \beta\right)(R\tilde{\phi}, \tilde{\phi}) \leq 0
\]

with \( \alpha = C/\gamma, \beta = 2(a+1) - \frac{2}{\gamma} \sup_{t \in [0, \varepsilon]} ||RB^*(t)|| > 0 \). Inequality (2.15) is equivalent to

\[
(2.16) \quad \frac{d}{dt}(e^{-\alpha t} \beta(R\tilde{\phi}, \tilde{\phi})) \leq 0.
\]

Integrating from \( s \) to \( S \) we get

\[
(R(S)\tilde{\phi}_s(S), \tilde{\phi}_s(S)) \leq e^{-\alpha(S-s)} \left(\frac{s}{S}\right)^\beta(R(s)\tilde{\phi}, \tilde{\phi})
\]

and finally, by iii),

\[
||\tilde{\phi}_s(S)||^2 \leq \frac{1}{\gamma} e^{\alpha \varepsilon} \sup_{t \in [0, \varepsilon]} ||R(t)|| ||\tilde{\phi}||^2
\]

which proves (2.11) for \( k = 0 \).
Suppose now that (2.11) is proved up to $k-1$. Then it is easy to show that $A^k \tilde{\eta}$ satisfies the following Cauchy problem:

\begin{equation}
(\phi^* + t[A^*, A^k]A^{-k}) A^k \tilde{\eta} = -[B^*, A^k]A^{-k+1}A^{-1} \tilde{\eta}, \quad A^k \tilde{\eta}|_{t=0} = A^k \tilde{\phi}.
\end{equation}

Since $[A^*, A^k]A^{-k}$ and $[B^*, A^k]A^{-k+1}$ are of order 0, proceeding as above we obtain

\begin{equation}
\frac{d}{dt}(RA^k \tilde{\phi}, A^k \tilde{\phi}) \leq C_k ||A^k \tilde{\phi}||^2 + 2||RB^*|| ||A^k \tilde{\phi}||^2 \\
- 2(a+1) (RA^k \tilde{\phi}, A^k \tilde{\phi}) + C'_k ||A^k \tilde{\phi}||^2 \\
\leq C_k ||A^k \tilde{\phi}||^2 + (2||RB^*|| + \delta C'_k) ||A^k \tilde{\phi}||^2 \\
- 2(a+1) (RA^k \tilde{\phi}, A^k \tilde{\phi}) + \frac{C'_k}{\delta} ||A^{k-1} \tilde{\phi}||^2
\end{equation}

for some $C_k, C'_k > 0$ (independent of $\tilde{\phi}$ and $t \in [0, \varepsilon]$) and for every $\delta > 0$. By choosing $\delta$ small enough and $a \geq a_0$ we get

\begin{equation}
\left( \frac{d}{dt} - \alpha_k t + \beta_k \right)(RA^k \tilde{\phi}, A^k \tilde{\phi}) \leq \frac{C'_k}{\delta} ||A^{k-1} \tilde{\phi}||^2
\end{equation}

for some $\alpha_k, \beta_k > 0$.

By multiplying both sides of (2.19) for $e^{-a_k t} \beta_k$ and integrating from 0 to $S$ we get

\begin{equation}
(R(S) A^k \tilde{\phi}_s(S), A^k \tilde{\phi}_s(S)) \leq e^{-a_k (S-t)} \left( \frac{s}{S} \right)^\beta_k (R(S) A^k \tilde{\phi}(S)) \\
+ \frac{C'_k}{\delta} S^\beta_k \int_0^S e^{-a_k (S-\sigma)} \sigma^\beta_k - 1 ||A^{k-1} \tilde{\phi}_s(\sigma)||^2 d\sigma.
\end{equation}

Since

\[ \frac{1}{S^\beta_k} \int_s^S e^{-a_k \sigma} d\sigma \leq \frac{1}{\beta_k}, \]

from (2.20) we obtain

\[ (R(S) A^k \tilde{\phi}_s(S), A^k \tilde{\phi}_s(S)) \leq e^{a_k S} \left( \sup_{t \in [0, S]} ||R(t)|| ||A^k \tilde{\phi}||^2 + \frac{C'_k}{\delta \beta_k} \sup_{s \in [0, S]} ||A^{k-1} \tilde{\phi}_s(S)||^2 \right). \]

By induction the above inequality implies (2.11). The proof of Lemma 1 is completed.
Proof of Lemma 2. Let \( f = \sum_{j=0}^{N} f_j(x) \otimes \partial \delta^j \) and \( g = \sum_{i=0}^{N} g_i(x) \otimes \partial \delta^i \) be two distributions with \( f_j, g_i \in \mathcal{D}'(\omega), \ j=0, \ldots, N, \ N \in \mathbb{Z}_+, \ \omega \subset U. \) We remark that the operator \( P \) can be decomposed as
\[
P = I_P(x; t \partial_x) - tR(t, x, t \partial_x, \partial_x)
\]
for some differential operator \( R \) with smooth coefficients.

Taking into account the identities
\[
(t \partial_x)^j \partial \delta^j = (-1)^j (1 + l)^j \partial \delta^j, \quad j, l = 0, 1, \ldots,
\]
it is easy to see that the equation \( Pg = f \) is equivalent to the following triangular system:
\[
\begin{cases}
I_P(x; -N+1) g_N = f_N, \\
I_P(x; -N) g_{N-1} = f_{N-1} + L_{N-1}(g_N), \\
& \quad \vdots \\
I_P(x; -1) g_0 = f_0 + L_0(g_N, g_{N-1}, \ldots, g_1),
\end{cases}
\]
where \( L_j, j \geq 0, \) are linear differential operators depending only on \( P. \)

Under condition (2.2) system (2.23) is uniquely solvable in \( \mathcal{D}'(\omega). \) The proof of the lemma is now a trivial consequence of this remark.

§ 3. Asymptotic Expansions

In this Section we prove the main result of this paper.

Let \( P \in \mathcal{D}'_{\mathbb{K}}([0, T[ \times U) \) and denote by \( \mathcal{D}'_+ \) the set of all distributions defined on some open subset \( ]0, \epsilon[ \times \omega \subset ]0, T[ \times U, \ \omega \) being a neighborhood of \( x=0. \) Then we have the following theorem.

Theorem 2. Suppose that the roots of the indicial polynomial of \( P \) satisfy the condition:
\[
\begin{cases}
\rho_i(0) \in \frac{1}{k} \mathbb{Z}, & i, j = 1, \ldots, m, \\
\rho_i(0) - \rho_j(0) \in \frac{1}{k} \mathbb{Z}, & i \neq j.
\end{cases}
\]

Then:

i) For every \( u \in \mathcal{D}'_+ \) with \( Pu = 0 \) there exist uniquely determined germs
of distributions $\varphi_j(x)$, $j=1,\ldots,m$, defined near $x=0$, for which the following asymptotic expansion holds as $t\to 0+$:

$$u(t, x) \sim \sum_{j=1}^{m} \varphi_j(x) t^{\rho_j(x)} + \sum_{j=1}^{m} \sum_{k=0}^{\infty} (L_{j,1,k}(x, \partial_x) \varphi_j(x)) t^{\rho_j(x) + 1/k} (\log t)^k,$$

where the $L_{j,1,k}(x, \partial_x)$ are linear differential operators (with smooth coefficients) depending only on $P$.

ii) Conversely, for every germs $\varphi_j(x)$, $j=1,\ldots,m$, there exists a unique germ $u \in \mathcal{D}'$ satisfying $Pu=0$ and having the asymptotic expansion (3.2) as $t\to 0+$.

Before proving the Theorem we make precise the meaning of the asymptotic expansion (3.2).

If $u \in \mathcal{D}'(]0, \varepsilon] \times \omega)$, we can suppose that $\varphi_j \in \mathcal{D}'(\omega)$, $j=1,\ldots,m$, for some neighborhood of the origin $\omega \subset \omega$. Since $Pu=0$ on $]0, \varepsilon] \times \omega$, by partial hypoellipticity we have $u \in C^\infty(]0, \varepsilon[; \mathcal{D}'(\omega))$. Moreover, by condition (3.1) and shrinking $\omega$ if necessary we can suppose that the roots $\rho_j(x)$ are smooth functions of $x \in \omega$.

Now the definition of (3.2) is the following one. For every $a>0$ there exists $N_0>0$ such that: for every $N>N_0$ and for every $p \in \mathbb{Z}^+$ we have

$$t^{-a} (t \partial_t)^b [u(t, x) - \sum_{j=1}^{m} \varphi_j(x) t^{\rho_j(x)} + \sum_{j=1}^{m} \sum_{k=0}^{N} (L_{j,1,k}(x, \partial_x) \varphi_j(x)) t^{\rho_j(x) + 1/k} (\log t)^k] \rightarrow 0$$
in $\mathcal{D}'(\omega)$ as $t\to 0+$.

**Proof of Theorem 2.** Let $u \in \mathcal{D}'(]0, \varepsilon] \times \omega)$ satisfy $Pu=0$ in $]0, \varepsilon[ \times \omega$. Consider the change of variables $\chi(s, x) = (t=s^a, x)$, $s>0$. By the remark in Sect. 1 the operator $P$ is transformed to $\tilde{P} \in \Phi^1_T$ and we can actually suppose that $\tilde{P} \in \Phi^1_T(]T, T[ \times \omega)$, $T=\varepsilon^{1/k}$. Then the distribution $\tilde{u}(s, x) = \chi^*(u)$ satisfies $\tilde{P}\tilde{u}=0$ on $]0, T[ \times \omega$. Application of Theorem 1 yields the existence of a distribution $v \in \mathcal{D}'(]T-T', T'[ \times \omega')$ such that $\tilde{P}v=0$ on $]T-T', T'[ \times \omega'$ and $v=\tilde{u}$ on $]0, T'[ \times \omega' \subset ]0, T[ \times \omega$.

Now we use Theorem 2 in [3] and can represent $v$ in the following form:

$$v(s, x) = \sum_{j=1}^{m} \int_{s}^{s+1/k} \int_{T-T'} r_j(s, x,y) \varphi_j(y) dy + \int_{s}^{s+1/k} r_j(s, x,y) \varphi_j(y) dy$$
for some germs \( \varphi_j, \psi_j, j = 1, \ldots, m, \) of distributions defined near \( x = 0, \) uniquely determined by \( \nu. \) To obtain (3.3) from Theorem 2 in [3] we use the fact that \( k\rho_j, j = 1, \ldots, m, \) are the roots of the indicial polynomial of \( \tilde{P} \) (see Sect. 1) and note that hypothesis (3.1) is equivalent to the hypothesis in Theorem 2 in [3]. The kernels \( r_j(s, x, y) \) are suitable distributions defined near \( s = 0, x = y = 0, \) satisfying the following conditions:

\[
\begin{align*}
(3.4) & \quad 1) \text{supp}(r_j) \subseteq \{(s, x, y) \mid |x - y| \leq M |s|\}, \\
& \quad 2) \text{WF}(r_j) \subseteq \{(s, x, \sigma, \xi), (y, \eta) \mid \eta \neq 0, \|x - y\| \leq M |s|, |\sigma| \leq M |\eta|, |\xi + \eta| \leq M |s| |\eta|\}
\end{align*}
\]

for some \( M > 0. \)

Furthermore, by the construction performed in [3] it follows that every \( r_j \) has an asymptotic expansion of the following form as \( s \to 0: \)

\[
(3.5) \quad r_j(s, x, y) \sim \delta(x - y) + \sum_{l=1}^{\infty} \sum_{|\alpha|=0}^l (c_{j, l, \alpha}(x) \partial_x^\alpha \delta(x - y)) s^l
\]

for some \( C^\infty \)-functions \( c_{j, l, \alpha}(x) \) defined in a common neighborhood \( \omega \subset \omega \) of \( x = 0. \) The meaning of the expansion (3.5) is the following (noting that \( r_j \in C^\infty(]-T', T'[ ; \mathcal{D}'(\omega \times \omega)) \) as a consequence of (3.4), 2)):

For every \( N, h \in \mathbb{Z}_+ \) we have

\[
s^{-N}(s\partial_x)^h[r_j(s, x, y) - \delta(x - y) - \sum_{l=1}^{\infty} \sum_{|\alpha|=0}^l (c_{j, l, \alpha}(x) \partial_x^\alpha \delta(x - y)) s^l] \to 0
\]

in \( \mathcal{D}'(\omega \times \omega) \) as \( |s| \to 0. \)

By restriction to \( s > 0 \) we obtain from (3.3)

\[
(3.6) \quad \bar{u}(s, x) = \sum_{j=1}^m \int_3^{k\rho_j(y)} r_j(s, x, y) \varphi_j(y) dy.
\]

By using \( \chi^{-1} \) and the expansions (3.5) we get (3.2) for \( u(t, x). \)

The uniqueness of the \( \varphi_j \) in (3.2) is proved as follows.

Suppose that for some distributions \( \varphi_j(x), j = 1, \ldots, m, \) defined on some neighborhood \( \omega' \subset \omega \) of \( x = 0 \) we have

\[
(3.7) \quad 0 \sim \sum_{j=1}^m [\varphi_j(x) t^{\rho_j(x)} + \sum_{l=1}^{\infty} \sum_{h=0}^l (L_{j, l, h}(x, \partial_x) \varphi_j(x)) t^{\rho_j(x) + l/h}(\log t)^h].
\]

We have to show that \( \varphi_j = 0 \) near \( x = 0 \) for every \( j. \)

We can obviously rearrange the \( \rho_j(x) \) in such a way that

\[
(3.8) \quad \text{Re } \rho_1(0) = \cdots = \text{Re } \rho_k(0) < \text{Re } \rho_{k+1}(0) = \cdots = \text{Re } \rho_{k+t_2}(0) < \cdots < \text{Re } \rho_{k+\cdots+t_p}(0) = \cdots = \text{Re } \rho_m(0)
\]
and decompose accordingly \( \{1, \ldots, m\} = I_1 \cup I_2 \cup \ldots \cup I_\nu \) (disjoint union). We may assume that there exist real numbers \( m_1 < m_2 < \cdots < m_\nu - 1 \) such that

\[
\begin{align*}
\sup_{j \in I_h} \Re \rho_j(x) &< m_h < \inf_{j \in I_{h+1}} \Re \rho_j(x) \quad j \in I_h, \quad x \in \omega', \quad h = 1, \ldots, \nu - 1. \\
\inf_{j \in I_h} \Re \rho_j(x) + 1/k &> m_h \quad j \in I_h.
\end{align*}
\]

From (3.7) we obtain

\[
\lim_{t \to 0^+} t^{-m_1}(t \partial_\nu)^{\rho_1} \phi_1(x) t^{\rho_1(x)} + \cdots + \phi_k(x) t^{\rho_k(x)} = 0
\]

in \( \mathcal{D}'(\omega') \) as \( t \to 0^+ \) for any \( \rho \in \mathbb{Z}_+ \).

By taking \( \rho = 0, 1, \ldots, k_1 - 1 \) we get

\[
\lim_{t \to 0^+} t^{-m_1}(t \partial_\nu)^{\rho_1} \phi_1(x) t^{\rho_1(x)} + \cdots + \phi_k(x) t^{\rho_k(x)} = 0
\]

in \( \mathcal{D}'(\omega') \) as \( t \to 0^+ \). Since the matrix in (3.10) is invertible (by (3.1)) we get \( \phi_j \sim_{t \to 0^+} 0 \) in \( \mathcal{D}'(\omega') \) as \( t \to 0^+ \) for \( j = 1, \ldots, k_1 \). By condition (3.9) we conclude that \( \phi_j = 0 \) on \( \omega' \) for \( j = 1, \ldots, k_1 \). As a consequence, \( L_{j, l, h}(x, \partial_\nu) \phi_j = 0 \) on \( \omega' \) for every \( l \) and \( h \) and for \( j = 1, \ldots, k_1 \). Hence (3.7) is reduced to

\[
0 \sim \sum_{j=1}^{k_1} [\phi_j(x) t^{\rho_j(x)} + \sum_{l=1}^{k_2} (L_{j, l, h}(x, \partial_\nu) \phi_j(x)) t^{\rho_j(x)+1/k} (\log t)^k].
\]

Using the same procedure as above we conclude that \( \phi_j = 0 \) on \( \omega' \) for \( j = k_1 + 1, \ldots, k_1 + k_2 \), and so on. Thus, part i) in Theorem 2 is proved.

To prove ii), let \( \varphi_j(x) \in \mathcal{D}'(\omega), j = 1, \ldots, m \), and define

\[
v(s, x) = \sum_{j=1}^{m} \left[ \int_{s_+}^{k_2(x)_+} r_j(s, x, y) \varphi_j(y) dy \right].
\]

By Theorem 2 in [3] it follows that \( \tilde{P}u = 0 \) on some box \( ] - T' \\times \omega' \). By defining \( u = (x' - t)^* u \) we obtain \( u \in \mathcal{D}'_+ \), with \( Pu = 0 \), having the asymptotic expansion (3.2).

To prove uniqueness we observe that if two distributions \( u_1, u_2 \in \mathcal{D}'_+ \) satisfy \( Pu_1 = Pu_2 = 0 \) on some box \( ]0, s[ \times \omega \) and if they have
the same asymptotic expansion (3.2), then $u_1 - u_2$ satisfies $P(u_1 - u_2) = 0$ on $]0, \varepsilon[ \times \omega$ and it is extendable as a $C^\infty$ function in $t$ up to $t = 0$, i.e. $u_1 - u_2 \in C^\infty([0, \varepsilon[; \mathcal{D}'(\omega']]$ for some box $[0, \varepsilon[ \times \omega' \subset [0, \varepsilon[ \times \omega$. Furthermore, $u_1 - u_2$ is flat at $t = 0$. Hence, application of the local uniqueness results of [2, 4] yields that $u_1 = u_2$ in a smaller box.

Thus, Theorem 2 is proved.

§ 4. Examples and Remarks

(1) The result stated in the Introduction is a consequence of Theorem 2.

(2) Let $P \in \Phi_{h}^{-k}(\mathbb{R}_t^+ \times \mathbb{R}_x^k)$. A consequence of our proof in Sect. 2 is that any distribution $u$ defined in an open subset of $\mathbb{R}_t^+ \times \mathbb{R}_x^k$ near a point $(0, x_0)$ of $\partial(\mathbb{R}_t^+ \times \mathbb{R}_x^k)$ which satisfies $Pu = 0$ is extendable as a distribution in a full neighborhood of $(0, x_0)$. Moreover, if the coefficients of $P$ are smooth up to $t = 0$ and if the roots $\rho_j(x)$ of the indicial polynomial satisfy the condition $\rho_j(x_0) \not\in \{-1, -2, \ldots\}$, $j = 1, \ldots, m$, then $u$ can be extended as a distribution solution $\bar{u}$ of $P\bar{u} = 0$.

(3) Under the hypotheses of Theorem 2 we can define "boundary values" of a solution $u$ of $Pu = 0$ by taking the leading coefficients $\varphi_1, \ldots, \varphi_m$ of the asymptotic expansion (3.2) of $u$.

(4) Let us consider the Fuchsian hyperbolic operators of weight $m - h \geq 0$ considered in [3]:

$$P = t^h P_m + t^{h-1} P_{m-1} + \cdots + P_{m-h}.$$ 

By combining the results in [3] with the arguments in Theorem 2 one can prove that every local distribution solution $u$ of $Pu = 0$, defined in some box $]0, \varepsilon[ \times \omega$, has an asymptotic expansion of the form:

$$u \sim \sum_{j=0}^{m-h-1} \left[ \phi_j(x) t^j + \sum_{i=1}^\infty \left( L_{i,j}(x, \partial_2) \phi_j(x) \right) t^{i+j} \right] + \sum_{j=1}^h \left[ \varphi_j(x) t^{\rho_j(x)} + \sum_{i=0}^\infty \left( L_{i,j}(x, \partial_2) \varphi_j(x) \right) t^{\rho_j(x)+i} (\log t)^i \right],$$

as $t \to 0^+$, for some germs of distributions $\phi_0, \ldots, \phi_{m-h-1}$, $\varphi_1, \ldots, \varphi_h$, provided the non trivial roots of the indicial polynomial $\rho_1(x), \ldots, \rho_h(x)$ satisfy the conditions: $\rho_j(0) \not\in \mathbb{Z}$, $j = 1, \ldots, h$ and $\rho_j(0) - \rho_{j'}(0) \not\in \mathbb{Z}$ for every $j, j', j \neq j'.$
(5) The example \( P = t \partial_t^2 - \Delta_x + \alpha \partial_t + \sum_{j=1}^{n} \beta_j \partial_{x_j} + \gamma \) is not included in our classes and has been already treated by Bernardi [1].

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**References**