A Study on New Muller’s Method

By

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Abstract

When we compute a root of equation \( F(X) = 0 \), Muller's Method uses three initial approximations \( X_0, X_1 \), and \( X_2 \) and determines the next approximation \( X_3 \) by the intersection of the X-axis with the parabola through \( (X_0, F(X_0)), (X_1, F(X_1)) \), and \( (X_2, F(X_2)) \). The procedure is repeated successively to improve the approximate solution of an equation \( F(X) = 0 \).

Suppose a continuous function \( F \), defined on the interval \([X_0, X_1]\) is given, with \( F(X_0) \) and \( F(X_1) \) being opposite signs. In our New-Muller’s Method we choose \( X_0 \) and \( X_1 \) as the ends of the interval and take another initial approximation \( X_2 \) as the mid-point of \( X_0, X_1 \) and new approximation \( X_3 \) is the intersection of the X-axis with a quadratic curve through \( (X_0, F(X_0)), (X_2, F(X_2)), \) and \( (X_1, F(X_1)) \). This method is proposed to improve the rate of convergence and calculate faster for reducing the interval. Let us call this method New-Muller’s Method in this paper.

§1. New-Muller’s Method

Let \( F(X) \) be continuous function that has a root in an interval \([X_0, X_1]\).

Beginning with the initial approximations \( X_0 \) and \( X_1 \) under the condition \( F(X_0) \cdot F(X_1) < 0 \) an intermediate initial approximation is taken as \( X_2 = (X_0 + X_1)/2 \).

Then \( X_3 \) is the intersection of the X-axis with a quadratic curve \( G(X) = 0 \) through three point \( (X_0, F(X_0)), (X_2, F(X_2)), \) and \( (X_1, F(X_1)) \).

Next we determine the closed interval \( I \) which includes the solution of \( F(X) = 0 \), as follow:

1) If \( F(X_3) F(X_0) < 0 \longrightarrow I_0 = [X_0, X_3] \)

2) If \( F(X_3) F(X_1) < 0 \longrightarrow I_0 = [X_3, X_1] \)

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1) If $F(X_0)F(X_2)<0$
   
   a) $X_2<X_2 \quad \Rightarrow I_0=[X_2, X_2]$
   
   b) $X_2>X_2 \quad \Rightarrow I_0=[X_2, X_2]$

   The closed interval $I_0$ is determined as above. Note that the length of $I_0$ is diminished less than half of the original interval in any case. To find the second approximation, we put newly the lower bound of $I_0$ as $X_0$, and upper bound of $I_0$ as $X_1$. We repeat above New-Muller's Method to find the nearer root of the quadratic equation whose curve passes through the last three points, and this root of the quadratic equation is included in closed interval. Now we select a contracted closed interval $I$ described as above.

   We continue the process and have a nested closed intervals $I_0, I_1, \ldots$. We repeat it until the interval shrinks sufficiently near the solution.

**Definition 1-1)** We take $X_0$ and $X_1$ such that $F(X_0)$ and $F(X_1)$ have the opposite signs. $X_2$ is another intermediate initial approximation determined by the initial approximations $X_0, X_1$; that is, $X_2=(X_0+X_1)/2$.

**Definition 1-2)** The closed interval determined by New-Muller's Method is called New-Muller's restricted closed interval.

Although there may occur several cases according to the signatures of $F(X_n)$, it is not difficult to show that the following theorem holds in any cases.

**Theorem 1-1)** Let $F(X)$ be continuous on the closed interval $[X_n, X_{n+1}]$. We apply the above procedure to have the sequence of intervals $I_n$, where $n=0, 1, 2, \ldots$. Then New-Muller's Method gives a Cantor's sequence of the nested closed interval, and if $n \to \infty$ then $I_n \to R$. (R is a root of $F(X)=0$.)

§ 2. Method of Calculation

When the initial approximations $X_0$ and $X_1$ satisfies $F(X_0) \cdot F(X_1)<0$ another initial value is $X_2=(X_0+X_1)/2$. Let us put the quadratic polynomial that passes through the three points $(X_0, F(X_0))$ $(X_2, F(X_2))$, and $(X_1, F(X_1))$ to be
New Muller’s Method

\[ G(X) = a(X - X_1)^2 + b(X - X_1) + c. \]

From the above conditions we have
\[
F(X_0) = a(X_0 - X_1)^2 + b(X_0 - X_1) + c
\]
\[
F(X_2) = a(X_2 - X_1)^2 + b(X_2 - X_1) + c
\]
\[
F(X_1) = a_0 + b_0 + c
\]
and we obtain
\[
a = \frac{2[F(X_0) - F(X_1)] - 4[F(X_2) - F(X_1)]}{(X_0 - X_1)^2}
\]
\[
b = \frac{4[F(X_2) - F(X_1)] - [F(X_0) - F(X_1)]}{X_0 - X_1}
\]
\[
c = F(X_1)
\]

To find out the root \( X \) which is \( G(X) = 0 \), we use the formula as follows to avoid the error arised by subtracting two close numbers:
\[
X_3 = X_1 - \frac{2c}{b + \text{sign}(b) \sqrt{b^2 - 4ac}}
\]

Here \( \text{sign}(b) \) is determined by the same method as Muller’s Method.

Next initial approximation \( X \) are obtained as described in §1. This process is repeated continuously until a satisfactory solution is found.

§ 3. New-Muller’s Algorithm

Suppose a continuous function \( F \) defined in the interval \( [X_0, X_1] \), is given with \( F(X_0) \) and \( F(X_1) \) being opposite signs. This produces uses following algorithm.

Step 1 set \( i = 1 \)
Step 2 while \( i \leq N \) do step 3-9
Step 3 set \( X_2 = (X_0 + X_1)/2 \) (compute \( X \))
Step 4 If \( F(X_2) = 0 \) or \( (X_1 - X_0)/2 < \text{EPS} \) then
  OUTPUT \( X_2 \) : (Procedure completed successfully)
  STOP
Step 5 set \( h = X_1 - X_0 \);
  \( S = F(X_0) - F(X_1) \);
  \( S = F(X_2) - F(X_1) \);
  \( a = (2S_1 - 4S_2)/h \);
  \( b = (4S_2 - S_1)/h \);
$c = F(X_1)$

Step 6 \[ D = (b - 4ac)^{1/2} \]

Step 7 If \[ |b - D| < |b + D| \] then set \[ E = b + D \]
else set \[ E = b - D \]

Step 8 Set \[ h = -2c/E \]

Step 9 If \[ |h| < \text{EPS} \] then

\[
\text{OUTPUT}(X_3); \quad \text{(Procedure completed successfully)}
\]

STOP

Step 10 If \[ F(X_2)F(X_0) < 0 \]
then set \[ X_1 = X_2 \] GO TO Step 2

Step 11 If \[ F(X_2)F(X_3) < 0 \] and \[ X_2 < X_3 \]
then set \[ X_0 = X_2, \quad X_1 = X_3 \] GO TO Step 2

Step 12 If \[ F(X_2)F(X_3) < 0 \]
then set \[ X_0 = X_3 \] GO TO Step 2

Step 13 If \[ F(X_2)F(X_0) < 0 \]
then set \[ X_1 = X_3 \] GO TO Step 2

Step 14 If \[ F(X_2)F(X_3) < 0 \] and \[ X_1 < X_2 \]
then set \[ X_0 = X_3, \quad X_1 = X_2 \] GO TO Step 2

Step 15 If \[ F(X_2)F(X_3) < 0 \]
then set \[ X_0 = X_2 \] GO TO Step 2

**Examples:**

1) \[ X^3 - X - 1 = 0 \]
2) \[ X^4 - 3X^3 - X^2 + 2X + 3 = 0 \]
3) \[ X^5 - 2X^4 - 4X^3 + X^2 + 5X + 3 = 0 \]
4) \[ X^6 - 8X^4 - 4X^3 + 7X^2 + 13X + 6 = 0 \]
5) \[ X^7 + X^6 - 12X^5 + 2X^4 - 3X^3 + 20X^2 + 19X + 6 = 0 \]

The results are as follows by **Muller's Method** and **New Muller's**

<table>
<thead>
<tr>
<th>Degree of Equation</th>
<th>Accuracy of last Root to be found</th>
<th>Times of Repeating</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Muller</td>
<td>New Muller</td>
</tr>
<tr>
<td>3</td>
<td>$10^{-12}$</td>
<td>$10^{-12}$</td>
</tr>
<tr>
<td>4</td>
<td>$10^{-12}$</td>
<td>$10^{-12}$</td>
</tr>
<tr>
<td>5</td>
<td>$10^{-12}$</td>
<td>$10^{-12}$</td>
</tr>
<tr>
<td>6</td>
<td>$10^{-12}$</td>
<td>$10^{-12}$</td>
</tr>
</tbody>
</table>

(Table a)
Method for above.

We have failed in being computed by Muller’s Method as below Table b. (This Table b is a case of Example 5.)

<table>
<thead>
<tr>
<th>Solved case with needed root</th>
<th>failed case</th>
</tr>
</thead>
<tbody>
<tr>
<td>RUN-NUMBER=20</td>
<td>RUN-NUMBER=20</td>
</tr>
<tr>
<td>S(1) = 1.000</td>
<td>S(1) = 1.000</td>
</tr>
<tr>
<td>S(2) = 1.000</td>
<td>S(2) = 1.000</td>
</tr>
<tr>
<td>S(3) = -8.000</td>
<td>S(3) = -8.000</td>
</tr>
<tr>
<td>S(4) = -12.000</td>
<td>S(4) = -12.000</td>
</tr>
<tr>
<td>S(5) = 3.000</td>
<td>S(5) = 3.000</td>
</tr>
<tr>
<td>S(6) = 20.000</td>
<td>S(6) = 20.000</td>
</tr>
<tr>
<td>S(7) = 19.000</td>
<td>S(7) = 19.000</td>
</tr>
<tr>
<td>S(8) = 6.000</td>
<td>S(8) = 6.000</td>
</tr>
<tr>
<td>X0=1.500000000000000</td>
<td>X0=0.</td>
</tr>
<tr>
<td>X1=2.000000000000000</td>
<td>X1=0.500000000000000</td>
</tr>
<tr>
<td>X2=2.500000000000000</td>
<td>X2=1.000000000000000</td>
</tr>
<tr>
<td>XANS = 1.486557539197503663430666165</td>
<td>XANS = -0.181953492716743103763121781</td>
</tr>
<tr>
<td>XANS = 1.4803699439312102958787355183</td>
<td>XANS = -0.595205607486539223227595130</td>
</tr>
<tr>
<td>XANS = 1.475050976250605636616616861</td>
<td>XANS = -0.812730423617922199706949969</td>
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<tr>
<td>XANS = 1.474989038025216081528867562</td>
<td>XANS = -0.6802508288747008857307846</td>
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<td>XANS = -0.686026232904809407653345943</td>
</tr>
<tr>
<td>XANS = 1.474989038334796698226369926</td>
<td>XANS = -0.6860029460265930893635171</td>
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<tr>
<td>1.474989038334796698226369926</td>
<td>XANS = -0.6860029460265930893635171</td>
</tr>
<tr>
<td>-1.77635683940039d-15</td>
<td>-0.6860029460265930893635171</td>
</tr>
</tbody>
</table>

|(Table b)|

§ 4. Conclusion

New Muller’s Method is different from Muller’s Method in the following points:

1) New Muller’s Method starts from two initial approximations \(X_0, X_1\) and \(X_2=(X_0+X_1)/2\) is used as an intermediate initial approximation.

2) As being shown at the Table b, when the initial approximation was not almost the approached value of the root we failed. But New Muller’s Method is a otherwise. (By the intermediate value theorem, it is explicit.)
References

