Hodge Spectral Sequence and Symmetry on Compact Kähler Spaces

By

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Introduction

For every complex manifold $M$, there exists a canonical spectral sequence which abuts to the de Rham cohomology of $M$. It consists of the set of $C^\infty$ differential forms on $M$, and the complex exterior derivatives $\partial$ and $\bar{\partial}$ of type $(0,1)$ and $(1,0)$, respectively, and its $E_1$-term is defined to be $\text{Ker} \, \partial/\text{Im} \, \bar{\partial}$. This will be referred to as the Hodge spectral sequence on $M$, after the celebrated result of W. Hodge [4].

Hodge's theorem states that the Hodge spectral sequence degenerates at $E_1$ and that $E_1^{pq}(M) \cong E_1^{qp}(M)$ if $M$ is a compact Kähler manifold. Here $E_1^{pq}(M)$ denotes the $(p, q)$-component of the $E_1$-term.

The purpose of the present note is to study an analogue of Hodge spectral sequences on compact complex spaces within the spirit of the previous note [7], where we considered the spaces which admit only isolated singularities.

Our main result is as follows.

**Theorem 1** Let $X$ be a compact Kähler space of pure dimension and let $Y$ be an analytic subset of $X$ containing the singular locus of $X$. Then, the Hodge spectral sequence on $X \setminus Y$ degenerates for the total degrees less than $\text{codim} \, Y - 1$ at the $E_1$-term. Moreover, $E_1^{pq}(X \setminus Y) \cong E_1^{qp}(X \setminus Y)$ for $p + q < \text{codim} \, Y - 1$.

In order to understand the symmetry $E_1^{pq}(X \setminus Y) \cong E_1^{qp}(X \setminus Y)$, we shall also prove the following.

Received December 24, 1986.

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Theorem 2 Let $X$ and $Y$ be as above, and let $(E, h)$ be a flat Hermitian vector bundle over $X \setminus Y$. Then, $H^{p,q}(X \setminus Y, E) \cong H^{q,p}(X \setminus Y, E^*)$, for $p + q < \text{codim } Y - 1$. Here $H^{p,q}$ denotes the cohomology of type $(p, q)$ in the sense of Dolbeault and $E^*$ denotes the dual bundle of $E$.

For the proof of the above mentioned results, an $L^2$-version of Andreotti-Grauert's vanishing theorem on $q$-complete spaces is necessary which is to be proved in §2 by using a new $L^2$-estimate obtained in [8].

The author expresses his hearty thanks to Professors H. Flenner and S. Tsuyumine for stimulating discussions during the author's stay at Mathematisches Institut of the University of Göttingen.

§ 1. Preliminaries

Definition A (reduced) complex space $X$ together with the following data $\{U_j, \varphi_j\}_{j \in A}$ is called a Kähler space.

1) $A$ is a set of indices.
2) $\{U_j\}_{j \in A}$ is an open covering of $X$.
3) $\varphi_j$ is a $C^\infty$ strictly plurisubharmonic function on $U_j$.
4) $\varphi_j - \varphi_k$ is pluriharmonic on $U_j \cap U_k$.

Given a Kähler space $X$, one attaches a Kähler metric on the complement of the singular locus by $\partial \bar{\partial} \varphi_j$, which is globally well defined by condition 4).

Let $X$ be a compact Kähler space of pure dimension $n$ with singular locus $Z$, and let $Y$ be an analytic subset of $X$ containing $Z$. We shall denote by $ds^2$ the prescribed Kähler metric on $X \setminus Z$.

Let $\{Y_a\}_{a=0}^m$ be a partition of $Y$ into subsets satisfying the following properties.

i) $Y_a$ are pure dimensional analytic subsets of $Y$.
i) $Y_{a+1} \subseteq \overline{Y}_a$.
i) $\text{dim } Y_a = m - a$.
i) The reduced structures of $Y_a$ are regular.
Such a partition (a stratification of $Y$) always exists, since the singular loci of complex analytic spaces are analytic subsets.

As a complex manifold $Y_a$ has a holomorphic coordinate patch. In other words, for each point $y \in Y_a$ one can find a Stein open neighbourhood $U$ in $Y_a$ and a biholomorphic map from $U$ onto a domain in $\mathbb{C}^{m-a}$. Since every holomorphic function on $U$ is holomorphically extendable to a neighbourhood of $U$ in $X$, it follows that $U$ is a holomorphic neighbourhood retract in $X$. Therefore, $Y_a$ can be covered by Stein open subsets, each of which has a Stein neighbourhood, say $V$, with a holomorphic embedding into a domain of some complex number space $\mathbb{C}^N$ such that the image of $V \cap Y_a$ is contained in a linear subspace of dimension $m-a$. Identifying $V$ as a subspace of $\mathbb{C}^N$, one sees that the restrictions of linear functions vanishing on $V \cap Y_a$, say $z_1, \ldots, z_{N-m+a}$, generate the ideal of holomorphic functions vanishing on $V \cap Y_a$ in the ring of holomorphic functions on $V$. One associates to $V$ a possibly smaller Stein open set

$$W := \left\{ x \in V; \sum_{\nu=1}^{N-m+a} |z_\nu(x)|^2 < \frac{1}{2} \right\}$$

and define a plurisubharmonic function $\phi_W$ on $W$ by

$$\phi_W(x) := -\ln(-\ln|z'(x)|^p).$$

Here we put $z' := (z_1, \ldots, z_{N-m+a})$ and $||z'(x)||^p := \sum_{\nu=1}^{N-m+a} |z_\nu(x)|^2$.

Suppose that a point $y$ in $Y_a$ belongs to the polar sets of two such functions $\phi_W$ and $\phi_{W'}$ (i.e. $\phi_W(y) = \phi_{W'}(y) = -\infty$). Then, there exists a neighbourhood $\mathcal{O} \ni y$ and a constant $C$ such that

$$(1) \quad |\exp(-\phi_W) - \exp(-\phi_{W'})| < C \quad \text{on} \quad \mathcal{O} \setminus Y.$$

In fact, this follows from that $z_i$ are generators of the ideal sheaf of $V \cap Y_a$.

Now let $\mathcal{J}_a$ be the ideal sheaf of $\bar{Y}_a$ in the structure sheaf $\mathcal{O}_X$ of $X$. Then, for each point $y \in \bar{Y}_a$ there exists a neighbourhood $U_y$ in $X$ and finitely many holomorphic functions $f_1, \ldots, f_m (m = m(y))$ which generate the stalks of $\mathcal{J}_a$ at every point of $U_y$ (cf. [3]). Then we put

$$W_y := \left\{ x \in U_y; \sum_{i=1}^{m} |f_i|^2 < \frac{1}{2} \right\}$$

and $\phi_y := -\ln(-\ln||f||^p)$, where $||f||^p := \sum_{i=1}^{m} |f_i|^2$. 
Let \( \{W_k\} \) be a finite system of such Stein open subsets of \( X \) whose union contains \( \overline{Y}_a \), where we put \( W_k = W_{k,2} \) and let \( \phi_k \) be the associated plurisubharmonic functions on \( W_k \) defined as above. Such a system \( \{W_k, \phi_k\} \) shall be referred to as a polarized cover along \( \overline{Y}_a \). Suppose that \( y \in W_k \cap W_i \). Then, by the same reasoning as above, one sees that there exists a neighbourhood \( \Omega \ni y \) and a constant \( C \) such that

\[
|\exp(-\phi_k) - \exp(-\phi_i)| < C \quad \text{on } \Omega \setminus \overline{Y}_a.
\]

Let \( \{W_k, \phi_k\} \) be a polarized cover along \( \overline{Y}_a \) and let \( \{\rho_k, \rho\} \) be a \( C^\infty \) partition of unity associated to the covering \( \{W_k, X \setminus \overline{Y}_a\} \) of \( X \) such that \( \rho_k \geq 0 \). Namely, \( \rho_k \) is a system of nonnegative \( C^\infty \) functions on \( X \) such that \( \supp \rho_k \subseteq W_k \) and \( \sum \rho_k = 1 \) on a neighbourhood of \( \overline{Y}_a \), say \( W_a \), and \( \rho = 1 - \sum \rho_k \).

We put \( \phi_k^* = \sum \rho_k \phi_k \). Then we have

\[
\begin{align*}
\partial \bar{\partial} \phi_k &= \sum \partial \rho_k \partial \phi_k + \sum \bar{\partial} \phi_k \bar{\partial} \rho_k + \sum \bar{\partial} \rho_k \partial \phi_k - \sum \rho_k \partial \bar{\partial} \phi_k \\
&= \sum \partial \rho_k \partial \phi_k - \sum (\partial \sum \rho_i \partial \phi_k + \sum \partial \phi_k \bar{\partial} \rho_k - \sum \partial \phi_k (\partial \sum \rho_i) \\
&\quad + \sum \bar{\partial} \rho_k \partial \phi_k - \sum \bar{\partial} \sum \rho_i \partial \phi_k + \sum \rho_k \partial \bar{\partial} \phi_k \\
&= \sum_k \partial \rho_k (\partial \phi_k - \partial \phi_i) + \sum_{k,i} (\partial \phi_k - \partial \phi_i) \bar{\partial} \rho_k + \sum_k (\rho_k - \rho_i) \partial \bar{\partial} \rho_k \\
&\quad + \sum \rho_k \partial \bar{\partial} \phi_k,
\end{align*}
\]

on \( W_a \setminus \overline{Y}_a \).

We are going to estimate the eigenvalues of \( \partial \bar{\partial} \phi_k \).

Once for all, let \( |.|_k \) denote the length of the differential forms measured by \( ds^2 + \partial \bar{\partial} \phi_k \). Then we have \( |\partial \phi_k|_k \leq \sqrt{2} \), since \( \phi_k = -\ln(-\ln||f_k||^2) \) for some vector \( f_k \) of holomorphic functions and

\[
\partial \bar{\partial} \phi_k = \frac{-\partial \bar{\partial} \ln||f_k||^2}{\ln||f_k||^2} + \frac{\partial \ln||f_k||^2 \partial \ln||f_k||^2}{(\ln||f_k||^2)^2} \geq \partial \bar{\partial} \phi_k \partial \phi_k.
\]

Let \( K_{kl} \subseteq W_k \cap W_l \) be any compact subset. Then,

\[
C_{kl} (ds^2 + \partial \bar{\partial} \phi_k) \leq ds^2 + \partial \bar{\partial} \phi_l \leq C_{kl} (ds^2 + \partial \bar{\partial} \phi_k)
\]

on \( K_{kl} \setminus \overline{Y}_a \), where \( C_{kl} \) is a constant depending on \( K_{kl} \). In particular we have

\[
|\partial \phi_k|_k \leq \sqrt{2} C_{kl} \quad \text{on } K_{kl} \setminus \overline{Y}_a.
\]

Proof of (3): We put \( f_k = (a_1, \ldots, a_m) \).
Then
\[ \partial \bar{\partial} \phi_k = \frac{\sum_{\mu < \nu} (a_{\mu} \partial a_{\nu} - a_{\nu} \partial a_{\mu}) (a_{\mu} \partial a_{\nu} - a_{\nu} \partial a_{\mu})}{(-\ln || f_k ||^2 || f_k ||^4) + \frac{\sum_{\mu} a_{\mu} \partial a_{\mu}}{\ln || f_k ||^2 || f_k ||^4}}. \]

Let \( \phi_i = -\ln (-\ln || f_i ||^2) \) and \( f_i = (b_1, \ldots, b_{m_i}) \). Then
\[ a_{\mu} = \sum_{j=1}^{m_i} u_{\mu} b_j, \quad 1 \leq \mu \leq m_k \]
for some holomorphic functions \( u_{\mu} \) on \( W_k \cap W_i \).

Substituting (6) into (5) and applying the Cauchy-Schwarz inequality, we have
\[ \partial \bar{\partial} \phi_k \geq \sum_{i \leq j} (\partial b_i - \partial b_j) (\partial b_i - \partial b_j) \]
\[ \quad \cdot \frac{(-\ln || f_k ||^2 || f_k ||^4) + \frac{\sum_{\mu} u_{\mu} b_j \partial b_j}{\ln || f_k ||^2 || f_k ||^4}} + O_{k_i}, \]
on \( K_{k_i} \setminus \bar{V}_a \) for some constant \( C_{k_i} \). Here \( O_{k_i} \) has bounded length with respect to \( ds^2 \).

Note that \( \sum_{i=1}^{m} |\xi_i|^2 (\sum_{j=1}^{m} |\eta_j|^2) = \sum_{i \neq j} |\xi_i \eta_j - \xi_j \eta_i|^2 + |\sum_{i=1}^{m} \xi_i \eta_i|^2 \), for any complex numbers \( \xi_i \) and \( \eta_j \), \( 1 \leq i, j \leq m \) (Lagrange's equality). Applying this equality to (7), we have
\[ \frac{(\sum_{\mu} \sum_{j} u_{\mu} b_j \partial b_j) (\sum_{\mu} \sum_{j} u_{\mu} b_j \partial b_j)}{\ln || f_k ||^2 || f_k ||^4} \]
\[ \leq C_{i,j} \sum_{i,j} (b_i \partial b_j - b_j \partial b_i) (b_i \partial b_j - b_j \partial b_i) + (\sum_{i} \partial b_i) (\sum_{j} \partial b_j) \]
on \( K_{k_i} \setminus \bar{V}_a \), for some constant \( C \).

Since we have chosen \( W_k \) so that \( \ln || f_k ||^2 < -\ln 2 \) on \( W_k \), we have
\[ \partial \bar{\partial} \phi_k \leq C' \partial \bar{\partial} \phi_i + O_{k_i} \quad \text{on} \quad K_{k_i} \setminus \bar{V}_a, \]
where \( C' \) is a constant and \( O_{k_i} \) is bounded with respect to \( ds^2 \). (3) follows from (9) immediately.

From (1'), (2), (3) and (4), we obtain
\[ -A_k ds^2 + \frac{1}{2} \sum_{\rho_k} \partial \bar{\partial} \phi_k \leq \partial \bar{\partial} \phi_{c_3}, \]
for sufficiently large $A_\alpha \geq 1$.

Thus we know that $Ad_\alpha^2 + \partial \partial \phi_\alpha$ is a metric on $X \setminus Y$ for any $A > A_\alpha$. Furthermore, let $\lambda_1^\alpha \geq \ldots \geq \lambda_n^\alpha$ be the eigenvalues of $\partial \partial \phi_\alpha$ with respect to the metric $Ad_\alpha^2 + \partial \partial \phi_\alpha (A > A_\alpha)$. Then, from (10) one immediately sees that, for any $\varepsilon > 0$, there exists an $A > A_\alpha$ such that $\lambda_j^\alpha > -\varepsilon$ for $n - \alpha < j$ on $X \setminus Y$. Moreover, (10) implies that at least $n - \alpha$ eigenvalues of $\partial \partial \phi_\alpha$ with respect to $d\alpha^2$ tend to $+\infty$ as one approaches to a point in $Y_\alpha$ (see (5) and recall Courant's mini-max principle). Hence, for any point $y \in Y_\alpha$ and $\varepsilon > 0$, one can choose a neighbourhood $\Omega \ni y$ in $X$ so that $1 - \varepsilon < \lambda_j^\alpha < 1 + \varepsilon$ for $1 \leq j \leq n - \alpha$ on $\Omega \setminus Y$.

Note that (4) implies $\partial \phi_\alpha \bar{\partial} \phi_\alpha < C(Ad_\alpha^2 + \partial \partial \phi_\alpha)$ for some $C > 0$.

For any positive number $u$ we put $\phi_u := u \sum_{\alpha=0}^m \phi_\alpha$ and $d\alpha_{u,\alpha} := Ad_\alpha^2 + \partial \partial \phi_\alpha$. Then, $d\alpha_{u,\alpha}$ is a complete Kähler metric on $X \setminus Y$ whenever $A > u \sum_{\alpha=0}^m A_\alpha$.

Now we have the following.

**Proposition 1.1** Let $(X, d\alpha^2)$ be a compact Kähler space of pure dimension $n$ and $Y$ an analytic subset containing the singular locus of $X$. Then, for any $\varepsilon > 0$, there exist a complete Kähler metric $d\alpha_Y$ on $X \setminus Y$, a proper $C^\infty$ map $\phi : X \setminus Y \to (-\infty, 0]$ and a neighbourhood $W \ni Y$ such that,

\begin{align}
\text{(*)} & \quad |\partial \phi|^2 < \varepsilon, \\
\text{(**)} & \quad |\partial \partial \phi|^2_Y < 2n, \\
\text{(***)} & \quad \text{The eigenvalues } \lambda_1 \geq \ldots \geq \lambda_n \text{ of } \partial \partial \phi \text{ with respect to } d\alpha_Y \text{ satisfy} \\
& \quad 1 - \varepsilon < \lambda_j < 1 + \varepsilon \quad \text{for } 1 \leq j \leq \text{codim } Y \text{ on } W \setminus Y, \\
& \quad -\varepsilon < \lambda_j < 1 \quad \text{for } j > \text{codim } Y \text{ on } X \setminus Y.
\end{align}

Here $|_Y$ denotes the length with respect to the metric $d\alpha_Y$.

**Proof** Let $A \gg 0$, $u \ll \frac{1}{A}$, and put $\phi = \phi_u$, $d\alpha_y = d\alpha_{u,\alpha}^2$.

**§ 2. Vanishing of the Local $L^2$–Cohomology**

Let $(M, d\alpha_y)$ be a Hermitian manifold of dimension $n$, and let $(E, h)$ be a Hermitian holomorphic vector bundle over $M$. For any $C^\infty (1, 1)$–form $G = i \sum A_{\alpha\beta} dz_\alpha \wedge d\bar{z_\beta}$ with $A_{\alpha\beta} = A_{\bar{\beta}\bar{\alpha}}$ on $M$, we define real–valued functions $\Gamma_{A,\alpha}[G]$ by
\[ \Gamma_{h,t}[G](x) := \min \left\{ \sum_{a=1}^{\beta} \lambda_{i_a}(x) + \sum_{b=1}^{\gamma} \lambda_{j_b}(x) - \sum_{k=1}^{n} \lambda_k(x) : \right. \\
\lambda_k(x) (1 \leq k \leq n) \text{ are the eigenvalues of } G \text{ at } x, \\
1 \leq i_1 < \ldots < i_\beta \leq \eta \text{ and } 1 \leq j_1 < \ldots < j_\gamma \leq n \}. \]

In terms of \( \Gamma_{h,t} \) we shall state a sufficient condition for an à priori estimate for the operator \( \bar{\delta} \). The \( L^2 \)-norm for \( E \)-valued forms will be denoted by \( \| \cdot \|_h \).

Let \( \omega \) be the fundamental form of \( ds^2 \) and \( A \) the adjoint of the multiplication \( w \mapsto \omega \wedge u \). We denote by \( \bar{\partial}_h^* \) the \( (L^2-)\) adjoint of the operator \( \bar{\delta} \) with respect to the metrics \( ds^2 \) and \( h \). The operator \( \bar{\partial}^* := -\star \bar{\partial}_h^* \) (\( \star \) : the conjugate after the Hodge's star) acts on \( E \)-valued forms and we denote by \( \bar{\partial}_h \) the adjoint of \( \bar{\partial}^* \) with respect to \( ds^2 \) and \( h \). Then we have \([\bar{\partial}_h, A] = -i \bar{\partial}_h^* + T_1 \) and \([\partial_h, A] = -i \partial_h^* + T_2 \), where \([\cdot, \cdot]\) denotes the Poisson bracket and \( T_j (j = 1, 2) \) contain no differentiation (i.e. \( T_j \) are function-linear).

Let \( \langle T_j \rangle \) denote the \( (L^2-) \) operator norms of \( T_j \). Then, from the explicit expression of the operator \( T_1 + T_2 \) in terms of \( dw \) and other elementary operators like \( \bar{\omega}, A \), etc. (cf. [5] appendix), we see that there exists a positive number \( \beta_n \) depending only on \( n \) such that \( 3\langle T_1 \rangle^2 + \langle T_2 \rangle^2 \leq \beta_n |dw|^2 \). In what follows we fix such \( \beta_n \).

**Proposition 2.1** Let \( F_1 \) be a \( C^\infty \) real-valued function on \( M \) and \( h_1 := h \exp(-F_1) \). Let \( \Theta \) be the curvature form of \( h \). Suppose that there exists a \( C^\infty \) real-valued function \( F \) satisfying
\[(11) \quad \Gamma_{h,t}[\partial \partial (F + F_1)] \geq n |\Theta| + \beta_n |dw|^2 + 3 |\partial F|^2 + \varepsilon \]
for some \( \varepsilon > 0 \). Then
\[ \| \partial u \|^2_h + \| \bar{\partial}_h^* u \|^2_h \geq \varepsilon \| u \|^2_h, \]
for any compactly supported \( E \)-valued \( C^\infty (p, q) \)-form \( u \) on \( M \).

For the proof, see [8], Corollary 1.7.

**Definition** A Hermitian vector bundle \((E, h)\) is said to be flat, if the operator \((\bar{\delta} + \partial_h) \circ (\bar{\delta} + \partial_h)\) is identically zero.

By the above definition, \((E, h)\) is flat if and only if \( \Theta \equiv 0 \).
In §1 we have constructed a metric $ds^2$ and a function $\phi$ satisfying several properties, from which we shall produce the functions $F_1$ and $F$ as above. In particular, for flat vector bundles we have the following.

**Proposition 2.2** Let $(N, ds_N^2)$ be a Kähler manifold of dimension $n$ and let $(E, h)$ be a flat Hermitian vector bundle over $N$. Suppose that there exist a positive integer $r$ and a $C^\infty$ real-valued function $\phi$ on $N$ such that

(i) $|\partial \phi|^2 < 1/12.$
(ii) $|\partial \partial \phi| < 2n.$
(iii) The eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$ of $\partial \partial \phi$ satisfy

$$\lambda_j^{1/2} < 1 + \frac{1}{4n} \quad \text{for } 1 \leq j \leq r$$
$$-\frac{1}{4n} < \lambda_j \quad \text{for } r < j.$$

Then, for any $A > 2^{16} \beta_n^2 n^4$ and $c \in \mathbb{R}$, the inequality (11) is satisfied by $M = \{ x \in N ; \phi(x) < c \}$, $ds^2 = (A(c-\phi)^{-2} + 1) ds_N^2 + 2A(c-\phi)^{-3} \partial \phi \partial \phi$, $F_1 = A(c-\phi)^{-1}$ and $\varepsilon = 1/8$, for $p + q > 2n - r$.

**Proof** Let $| \cdot |_A$ denote the length of the forms with respect to the metric $ds^2$. Let $\omega_N$ and $\omega$ be the fundamental forms of $ds_N^2$ and $ds^2$, respectively. Then, $d\omega = A(c-\phi)^{-3} d\phi \wedge (2\omega_N - i \partial \partial \phi)$. We estimate $|d\omega|_A$ as follows.

First, from the definition of $ds^2$, $|d\phi|_A < 2A^{-1/2}(c-\phi)^{3/2}$ and $|\omega_N|_A < 2n(A(c-\phi)^{-2} + 1)^{-1}$. Secondly, from (ii), $|\partial \partial \phi|_A < 2n(A(c-\phi)^{-2} + 1)^{-1}$.

Therefore,

$$|d\omega|_A \leq A(c-\phi)^{-3} |d\phi|_A (2 |\omega_N|_A + |\partial \partial \phi|_A) < 6nA^{1/2}(c-\phi)^{-3/2}(A(c-\phi)^{-2} + 1)^{-1}.$$

Hence,

$$|d\omega|_A < 6nA^{1/2}(c-\phi)^{-3/2} < 6nA^{-1/4} \quad \text{if } A < (c-\phi)^2,$$

and

$$|d\omega|_A < 6nA^{-1/2}(c-\phi)^{1/2} \leq 6nA^{-1/4} \quad \text{if } A \geq (c-\phi)^2.$$

Thus, $\beta_n |d\omega|_A^2 \leq 36 \beta_n n^2 A^{-1/2}$, so that
(12) \[ \beta_n |\omega|^2 < \frac{1}{4} \quad \text{if } A > 2^{15} \beta_n^4. \]

To estimate the left hand side of the inequality, let \( x \in M \) be any point and let \( L \) be the subspace of the complex tangent space of \( M \) at \( x \) spanned by the eigenvectors corresponding to \( \lambda_1(x), \ldots, \lambda_r(x) \). Then, for any vector \( v \in L \), one has, for \( F = \phi \) and \( F_1^1 = A(e - \phi)^{-1} \),

\[ 1 - \frac{1}{4n} < \left\langle \partial \bar{\partial} (F + F_1^1), v, \bar{v} \right\rangle < 1 + \frac{1}{4n}, \]

from (iii). Here \( |v|_A \) denotes the length of \( v \) with respect to \( ds^2 \).

Similarly, for any unit tangent vector \( w \) at \( x \),

\[ < \partial \bar{\partial} (F + F_1^1), w, \bar{w} > - \frac{1}{4n}. \]

Combining (13) and (14), we have

\[ \Gamma_{k,q} < \partial \bar{\partial} (F + F_1^1) > - \frac{3}{4}, \quad \text{if } p + q > 2n - r. \]

From (i) we have

\[ 3 |\partial F|_A^2 < \frac{1}{4}. \]

Combining (12), (15) and (16), we obtain the desired inequality for the flat bundle \((E, h)\).

Applying Proposition 2.2 to the Kähler manifold \((W' \setminus Y, ds_Y^2)\) described in Proposition 1.1, we obtain the following.

**Proposition 2.3** Let \((X, ds^2)\) be a compact Kähler space of pure dimension \( n \) and \( Y \) an analytic subset containing the singular locus of \( X \). Then, there exists a \( C^0 \) proper map \( \phi : X \setminus Y \to (-\infty, 0] \) and \( c \in \mathbb{R} \) (\( c \); arbitrarily small) such that, for any compactly supported \( C^0 \) \((p, q)\)-form \( u \) on \( W := \{ x \in X \setminus Y ; \phi(x) < c \} \) with values in a flat vector bundle \((E, h)\) over \( X \setminus Y \), the estimate

\[ ||\partial u||_{W^*}^2 + ||\partial_{W^*} u||_{W^*}^2 \geq \frac{1}{4} ||u||_{W^*}^2 \]

holds for \( p + q > 2n - \text{codim} \ Y \) with respect to the metrics

\[ ds_Y^2 = (A(e - \phi)^{-2} + 1) ds^2 + 2A (e - \phi)^{-2} \partial \bar{\partial} \phi \]

and
and \( h_w = h \exp(-A(c-\phi)^{-1}) \), where \( A > \frac{16}{\ell^2} n^4 \) and \( ds_Y^2 \) is some (i.e., not arbitrary) complete Kähler metric on \( X \setminus Y \).

Since the above \( (W, ds_Y^2) \) is a complete Hermitian manifold, Proposition 2.3 implies that the Hermitian bundle \( (E|_W, h_w) \) is \( W^{p,q} \)-elliptic in the sense of Andreotti–Vesentini [2], if \( p + q > 2n - \text{codim } Y \).

Thus, in virtue of Andreotti–Vesentini’s theorem, we have the following corollary to Proposition 2.3.

**Corollary 2.4** Under the above situation, let \( f \) be any \( E \)-valued \((p,q)\)-form on \( W \) which is square integrable with respect to \( ds_Y^2 \) and \( h_w \) and \( \bar{\partial} f = 0 \) in the sense of distribution. If \( p + q > 2n - \text{codim } Y \), then there exists an \( E \)-valued \((p, q-1)\)-form \( g \) on \( W \), square integrable with respect to \( ds_Y^2 \) and \( h_w \) such that \( \bar{\partial} g = f \) and \( \|g\|_{h_w} \leq 2\|f\|_{h_w} \).

### §3. \( L^2 \) Cohomology and Harmonic Forms

Let \( (M, ds_M^2) \) be a Hermitian manifold of dimension \( n \), and let \((E, h)\) be a Hermitian vector bundle over \( M \). We denote by \( L^{p,q}(M, E)_h \) the set of square integrable \( E \)-valued \((p, q)\)-forms on \( M \) with respect to \( ds_M^2 \) and \( h \), and put

\[
H^{p,q}_{\text{loc}}(M, E)_h := \{ f \in L^{p,q}(M, E)_h ; \bar{\partial} f = 0 \} / \{ g \in L^{p,q}(M, E)_h ; \exists u \in L^{p,q-1}(M, E)_h \text{ such that } g = \bar{\partial} u \}.
\]

Here the derivatives are taken in the distribution sense.

Let \( L^{p,q}_{\text{loc}}(M, E) \) be the set of locally square integrable \( E \)-valued \((p, q)\)-forms on \( M \). We put

\[
H^{p,q}(M, E)_h := \{ f \in L^{p,q}_{\text{loc}}(M, E)_h ; \bar{\partial} f = 0 \} / \{ g \in L^{p,q}_{\text{loc}}(M, E)_h ; \exists u \in L^{p,q-1}_{\text{loc}}(M, E)_h \text{ such that } \bar{\partial} u = g \}.
\]

Since the \( L^2 \)-version of Dolbeault’s Lemma is valid (cf. [6] or [9]), \( H^{p,q}(M, E)_h \) is canonically isomorphic to the \( E \)-valued Dolbeault cohomology of type \((p,q)\).

We put \( \square_h := \partial \bar{\partial} + \partial \bar{\partial} \) and \( \square_{h^*} = \partial_h \bar{\partial}^* + \bar{\partial}^* \partial_h \). Clearly, \( \square_{h^*} = (\square_h)^* \).

We put \( \mathfrak{H}^{p,q}(E)_h := \{ f \in L^{p,q}(M, E)_h ; \square_h f = 0 \} \).

If the metric \( ds_M^2 \) is Kählerian, one has \([\partial_h, A] = -i \partial_h^* \) and \([\bar{\partial} , A] = i \partial_h^* \). Hence \( i(\bar{\partial} + \partial_h)(\bar{\partial} + \partial_h) A = \partial \cdot i[\partial_h, A] + i[\partial_h, A] \bar{\partial} + \partial_h \cdot i[\partial, A] + \)
If the bundle \((E, h)\) is flat, then we have \(\partial h = \overline{\partial h}\). Thus we obtain

**Lemma 3.1** Let \((M, ds^2_M)\) be a Kähler manifold and \((E, h)\) a flat Hermitian vector bundle over \(M\). Then, \(\mathcal{H}^{p,q}(E)_h \cong \mathcal{H}^{p-q,q}(E)_h\). Here the isomorphism is given by \(f \mapsto hf\).

Identifying \(h\) as a \(C^\infty\) section of \(\text{Hom}(E, \overline{E})\), we have \(\partial h = h^{-1} \partial h\). Therefore we obtain

**Lemma 3.2** Under the situation of Lemma 3.1, \(\mathcal{H}^{p,q}(E)_h \cong \mathcal{H}^{q,p}(E^*)_h\). Here the isomorphism is given by \(f \mapsto \overline{hf}\). (\(h^* := h^{-1}\)).

The following is fundamental.

**Proposition 3.3** Let \((M, ds^2_M)\) be a complete Hermitian manifold and \((E, h)\) a Hermitian vector bundle over \(M\). Then

\[
\mathcal{H}^{p,q}(E)_h = \{ f \in L^{p,q}(M, E)_h; \quad \partial f = 0, \quad \overline{\partial} f = 0 \}.
\]

**Proof.** See Andreotti–Vesentini [2].

Thus, if the metric \(ds^2_M\) is complete, then we have an orthogonal decomposition:

\[
L^{p,q}(M, E)_h = \mathcal{H}^{p,q}(E)_h \oplus R^p_q(E)_h \oplus \overline{R}^p_q(E).
\]

Here \(R^p_q(E)\) (resp. \(\overline{R}^p_q(E)\)) denotes the range of \(\partial\) (resp. \(\overline{\partial}\)), and \(R^p_q(E)\) (resp. \(\overline{R}^p_q(E)\)) its closure.

From the above decomposition we obtain

\[
H^{p,q}_{\partial}(M, E)_h = \mathcal{H}^{p,q}(E)_h,
\]

if \(R^p_q(E)\) is closed (for instance it is the case when \(H^{p,q}_{\partial}(M, E)_h\) is finite dimensional).

Combining Lemma 3.2 with (17), we have

**Proposition 3.4** Let \((M, ds^2_M)\) be a complete Kähler manifold and \((E, h)\) a flat Hermitian vector bundle over \(M\). Suppose that \(\dim H^{p,q}_{\overline{\partial}}(M, E)_h < \infty\) and \(\dim H^{p,q}_{\partial}(M, E^*)_h < \infty\). Then \(H^{p,q}_{\overline{\partial}}(M, E)_h \cong H^{p,q}_{\overline{\partial}}(M, E^*)_h\).
§ 4. Proof of Theorems

First we shall prove Theorem 2.

Let \( X, Y, (E, h) \), etc. be as in Proposition 2.3. We shall show that the natural homomorphism \( \tau: H^q_{\text{c}}(X \setminus Y, E) \to H^q_{\text{c}}(X \setminus Y, E)_h \) is isomorphism if \( p + q > 2n - \text{codim } Y + 1 \). Here \( H^q_{\text{c}} \) denotes the cohomology with compact support and the \( L^2 \) cohomology \( H^q_{\text{c}} \) is with respect to \( ds_Y^2 \).

Surjectivity: Let \([u] \in H^q_{\text{c}}(X \setminus Y, E)_h\), where \( u \in L^p(X \setminus Y, E)_h \) and \( \bar{\partial} u = 0 \). Clearly, \( u \rvert_W \) is square integrable, for any choice of \( W(\text{or } \omega) \), with respect to \( ds_Y^p \) and \( h_W \). Hence, by Corollary 2.4, one can find a \( v \in L^q_{\text{c}}(W, E) \), square integrable with respect to \( ds_Y^q \) and \( h_w \), such that \( \bar{\partial} v = u \). Since \( ds_Y^p \) is quasi-isometric to \( ds_Y^q \) on a neighbourhood of \( Y \), it follows immediately that \( u \) is represented by a compactly supported form, which completes the proof of the surjectivity.

Injectivity: Let \([w] \in H^p_{\text{c}}(X\setminus Y, E)\). If \( \tau([w]) = 0 \), then there exists an \( f \in L^{p+q-1}(X \setminus Y, E)_h \) such that \( \bar{\partial} f = w \). Since the support of \( w \) is compact, \( \bar{\partial} f = 0 \) near \( Y \). Hence, applying Corollary 2.4, one can find a \( v \in L^q_{\text{c}}(W, E) \), square integrable with respect to \( ds_Y^q \) and \( h_w \), such that \( \bar{\partial} v = u \). Since \( ds_Y^p \) is quasi-isometric to \( ds_Y^q \) on a neighbourhood of \( Y \), it follows immediately that \( u \) is represented by a compactly supported form, which completes the proof of the injectivity.

In virtue of Andreotti-Grauert’s finiteness theorem (cf. [1]), \( \dim H^p_{\text{c}}(X\setminus Y, E) < \infty \) for \( p + q < \text{codim } Y - 1 \). Hence, by Serre-Malgrange’s duality

\[
\text{(18)} \quad \dim H^p_{\text{c}}(X\setminus Y, E^*) < \infty, \quad \text{for } p + q > 2n - \text{codim } Y + 1.
\]

Similarly, we have

\[
\text{(19)} \quad \dim H^q_{\text{c}}(X\setminus Y, E) < \infty, \quad \text{for } p + q > 2n - \text{codim } Y + 1.
\]

In view of the above isomorphism, we obtain the finite dimensionality of \( H^p_{\text{c}}(X\setminus Y, E)_h \) and \( H^q_{\text{c}}(X\setminus Y, E^*)_h \) for \( p + q > 2n - \text{codim } Y + 1 \). Thus, by Proposition 3.4, we have \( H^p_{\text{c}}(X\setminus Y, E)_h \cong H^q_{\text{c}}(X\setminus Y, E^*)_h \) for \( p + q > 2n - \text{codim } Y + 1 \), so that \( H^p_{\text{c}}(X\setminus Y, E) \cong H^q_{\text{c}}(X\setminus Y, E^*) \) for \( p + q > 2n - \text{codim } Y + 1 \).

Hence, by the duality again we obtain

\[ H^{p, q}(X\setminus Y, E) \cong H^{q, p}(X\setminus Y, E^*), \quad \text{for } p + q < \text{codim } Y - 1, \]

which completes the proof of Theorem 2.

Proof of Theorem 1 \( E^p_{\text{c}}(X\setminus Y) = E^q_{\text{c}}(X\setminus Y) \) if every cohomology
class in $H^{p,q}(X \setminus Y)$ and $H^{p-1,q}(X \setminus Y)$ is represented by a $d$-closed form. This can be shown for $p+q < \text{codim } Y - 1$ as follows.

First, taking the dual of the isomorphism $\tau: H^p_0(X \setminus Y) \to H^p_{\mathbb{C}}(X \setminus Y)$ we have $H^p_{\mathbb{C}}(X \setminus Y) \cong H^p(X \setminus Y)$ for $p+q < \text{codim } Y - 1$. (For the trivial bundle, $(E, h)$ is not referred to.)

Therefore, from (17) $H^p(X \setminus Y) \cong \mathcal{H}^{p,q}$ for $p+q < \text{codim } Y - 1$.

Since by the equality $\square = \square$ combined with Proposition 3.3, every form in $\mathcal{H}^{p,q}$ is $\mathcal{H}$-closed, the assertion is proved.

That $E^1_{p,q}(X \setminus Y) \cong E^{p,1}_1(X \setminus Y)$ for $p+q < \text{codim } Y - 1$ is a corollary of Theorem 2.

References
