Type I Orbits in the Pure States of a C*-Dynamical System. II

By

Akitaka KISHIMOTO*

Abstract

For a C*-algebra $A$ with an action of a locally compact abelian group $G$, one considers the pure states of $A$ with the associated action. Type I orbits are defined and studied in the previous paper [4]. We continue this study; in particular, we shall show that if $A$ is separable and simple and if there is a type I orbit through a pure state $\phi$ with trivial stabilizer $\{t^G : x_f^* \alpha_t \sim x_f \} = \{0\}$, then there is a type I orbit with stabilizer equal to any given closed subgroup of $G$.

Let $A$ be a separable C*-algebra and let $\alpha$ be a continuous action of a separable locally compact abelian group $G$ on $A$. Let $P(A)$ denote the set of pure states of $A$ and let $P^*(A)$. We call the orbit $o_f = \{f \circ \alpha_t : t^G \}$ through $f$ in $P(A)$ type I if the representation $\rho_f \equiv \int_G \pi_f \alpha_t dt$ of $A$ on $L^h(G, \mathcal{H}_f)$ is of type I.

We denote by $\tilde{\alpha}$ the extension of $\alpha$ to an action on $\rho_f(A)^\sigma$; in other words, $\tilde{\alpha}_t \rho_f = \rho_f \alpha_t$, $t \in G$. Since $\tilde{\alpha}$ is ergodic on the center $Z$ of $\rho_f(A)^\sigma$, $\text{Sp}(\tilde{\alpha}|Z)$ is a closed subgroup of $\hat{G}$, which we denote by $\Delta(\pi_f)$ or $\Delta(\pi_f, \alpha)$. Let $G_f$ be the set of $s \in G$ such that $\pi_f^* \alpha_s$ is equivalent to $\pi_f$. If $o_f$ is type I, then $G_f = \Delta(\pi_f)^\lambda$ (see 0.1 in [4]).

In this case there is a weakly continuous action $\beta$ of $G_f$ on $\pi_f(A)^\sigma = B(\mathcal{H}_f)$ such that $\beta_t \pi_f = \pi_f \alpha_t$, $t \in G_f$, but in general $\pi_f$ may not be $\alpha|G_f$-covariant, i.e., $\beta$ may not be implemented by a unitary representation of $G_f$. If in addition $\pi_f$ is $\alpha|G_f$-covariant, we call the orbit $o_f$ regular type I.

For the system $(A, G, \alpha)$ we defined $\Gamma_i(\alpha)$, a subset of $\hat{G}$, in [4] as follows: $p \in \Gamma_i(\alpha)$ if for any non-zero $x \in A$, any compact neighbourhood $U$ of $p$, and 

Communicated by H. Araki, November 25, 1986.
* Department of Mathematics, College of General Education, Tohoku University, Sendai, Japan.
any $\varepsilon > 0$, there is an $a \in A^*(U)$ such that $\|a\| = 1$ and $\|xax^*\| \geq (1-\varepsilon)\|x\|^2$. Let us now define another technical spectrum $\Gamma_s(\alpha)$ as follows: $p \in \Gamma_s(\alpha)$ if for any non-zero $x \in A$, any compact neighbourhood $U$ of $p$, and any $\varepsilon > 0$, there is an $a \in A^*(U)$ such that $\|a\| = 1$ and $\|x(a+a^*)x^*\| \geq 2(1-\varepsilon)\|x\|^2$.

Now our results are as follows when the $C^*$-algebra $A$ is simple and unital. If there is a regular type I orbit of such that the Connes spectrum $\Gamma(\alpha|G_f)$ equals $\hat{G}$, then for any closed subgroup $H$ of $G$ there is a regular type I orbit $\alpha_f$ with $G_f = H$ (Theorem 2). In particular if $(A, G, \alpha)$ is asymptotically abelian, there is always a covariant irreducible representation (see 2.3 and 3.1 in [4]). If there is a covariant irreducible representation, then $\Gamma_s(\alpha) = \Gamma(\alpha)$ (Theorem 7). When $G$ is a connected Lie group and $\alpha$ is not uniformly continuous, there is always a non-type I orbit (Theorem 9).

We first prove the following properties of $\Gamma_s(\alpha)$.

1. **Proposition.** Let $A$ be a separable $C^*$-algebra and let $\alpha$ be a continuous action of a separable locally compact abelian group $G$ on $A$. Then the following properties hold:

   (i) For any faithful family $F$ of irreducible representations of $A$, $\Gamma_s(\alpha)$ includes $\bigcap_{\pi \in F} \Delta(\pi)$.

   (ii) There exists a faithful family $F$ of irreducible representations of $A$ such that $\Gamma_s(\alpha) = \bigcap_{\pi \in F} \Delta(\pi)$. In particular, $\Gamma_s(\alpha)$ is a closed subgroup of $\hat{G}$.

   (iii) $\Gamma_s(\alpha) \subseteq \Gamma_1(\alpha)$, and if $\Gamma_1(\alpha) = \hat{G}$ then $\Gamma_s(\alpha) = \hat{G}$.

**Proof.** If $p \in \Delta(\pi)$, there is a sequence $\{x_n\}$ in $A$ of spectrum $p$ such that $\|x_n\| \leq 1$ and $\lim \pi(x_n) = 1$ ([4]). Here $\{x_n\}$ is of spectrum $p$ if for any neighbourhood $U$ of $p$ there is an $N$ such that $x_n \in A^*(U)$ for any $n \geq N$. This immediately implies (i).

To prove (ii) we adapt the proof of (3) $\Rightarrow$ (5) in 3.1 in [4]. We take for $\{U_n\}$ the subsequence of $\{U_n\}$ consisting of those which intersect $\Gamma_s(\alpha)$. Since we are not assuming the primeness of $A$ here, we simply take for $\{I_n\}$ a constant sequence consisting of a non-zero ideal of $A$. By the same procedure as in [4] in this setting we obtain a pure state $\pi$ of $A$ such that $\|f\| = 1$, and for any $p \in \Gamma_s(\alpha)$ and any unit vector $\xi \in \mathcal{M}_f$ there is a $Q \in \mathcal{M}(p)$ such that $\|Q\| \leq 1$ and $\Re\langle Q\xi, \xi \rangle \geq 1$, or in fact $\|Q\| = 1 = \langle Q\xi, \xi \rangle$. (See Section 1 of [4] for the definition of $\mathcal{M}(p)$.) Then by an argument given in the proof of 3.1 in [4] one can conclude that $\mathcal{M}(p) \equiv f$ for $p \in \Gamma_s(\alpha)$, or equivalent $\Delta(\pi_f) \supseteq \Gamma_s(\alpha)$. Taking for $F$ the set of $\pi_f$ for all non-zero ideals of $A$, one obtains that $\Gamma_s(\alpha) \subseteq \bigcap_{\pi \in F(\pi)}$. Hence the equality follows by (i). Since each $\Delta(\pi)$ is a closed subgroup of $\hat{G}$, so is $\Gamma_s(\alpha)$.

As for (iii), it is obvious that $\Gamma_s(\alpha) \subseteq \Gamma_1(\alpha)$. Suppose $\Gamma_1(\alpha) = \hat{G}$. Then since the same procedure as above applies, one can conclude that $\Gamma_s(\alpha) = \hat{G}$ (see 3.1
Now we state our first main result:

2. Theorem. Let $A$ be a separable prime C*-algebra and let $\alpha$ be a continuous action of a separable locally compact abelian group $G$ on $A$. Let $H$ be an arbitrary closed subgroup of $G$. Then the following conditions are equivalent:

(I) $\Gamma(\alpha)=G$ and there exists an $\alpha$-covariant irreducible representation of $A$ such that the corresponding representation of the crossed product $A\times_\alpha G$ is faithful.

(i') $\Gamma(\alpha)=\hat{G}$ and there exists a family $F$ of irreducible representations of $A$ such that $\cap_{\pi\in F}\ker\pi=(0)$ and $\pi$ is $\alpha$-covariant for $\pi \in F$.

(ii) $\Gamma(\alpha)=\hat{G}$ and $\Gamma(\alpha)=G$.

(iii) $\Gamma(\alpha)=\hat{G}$.

(iv) $\Gamma(\alpha\mid H)=\hat{H}$ and there exists an $\alpha\mid H$-covariant irreducible representation $\pi$ of $A$ such that the corresponding representation of $A\times_{\alpha\mid H} H$ is faithful and $\Delta(\pi)=H^\perp$.

(iv') $\Gamma(\alpha\mid H)=\hat{H}$ and there exists a family $F$ of irreducible representations of $A$ such that $\cap_{\pi\in F}\ker\pi=(0)$, and $\pi$ is $\alpha\mid H$-covariant and $\Delta(\pi)=H^\perp$ for $\pi \in F$.

Moreover, if $G$ is discrete the above conditions are equivalent to

(v) $\alpha_i$ is properly outer for each $i \in G \setminus \{0\}$.

Proof. It is trivial that (i) implies (i') and (iv) does (iv'). Let $F$ be as in (i') and for each $\pi \in F$ let $u$ be the unitary representation of $G$ which implements $\alpha$, so that $\hat{\pi}=\pi \times u$ is the corresponding representation of $A\times_\alpha G$. Since $\pi(A)^*=\hat{\pi}(A\times_\alpha G)^*$ and the spectrum of the action on the quotient $A\times_\alpha G/(\ker\pi) \times_\alpha G$ induced by $\hat{\alpha}$ is $G$, 1.3 in [4] implies that $\Delta(\hat{\pi}, \hat{\alpha})=G$ (note that the faithfulness assumption in 1.3 in [4] was needed only for $\rho$ instead of $\pi$; in this case this amounts to the property that $\cap_{\pi\in F}\ker\pi\times \hat{\alpha}(\ker\pi)\times_\alpha G\times_\alpha G$. Hence (i') implies (ii) by 1(i).

To prove (ii)$\Rightarrow$(iii) we first give the following result:

3. Proposition. Let $A$ be a separable prime C*-algebra and let $\alpha$ be a continuous action of a separable locally compact abelian group $G$ on $A$. Suppose that $\Gamma(\alpha)=\hat{G}$. Then there exists an $\alpha$-covariant irreducible representation of $A$ such that the corresponding representation of $A\times_\alpha G$ is faithful.

Proof. To prove this it suffices to show that $\Gamma(\alpha)=\hat{G}$ and $\Gamma(\alpha|=\hat{G}$. Because, if this is the case, $A\times_\alpha G$ is separable and prime ([5]), and hence due to 3.1 in [4] and the lemma below there exists a faithful irreducible representation $\pi$ of $A\times_\alpha G$ such that $\pi(A\times_\alpha G)^*=\pi(A)^*$, where $\pi$ is the extension of $\pi$ to the multiplier algebra $M(A\times_\alpha G)$. Thus $\pi|A$ has the desired properties.
4. Lemma. If $\pi$ is a representation of $A \times a G$ such that $\Delta(\pi, \alpha) = G$, then $\pi(A)^* = \pi(A \times a G)^*$.

Proof. Define a representation $\rho$ of $A \times a G$ by

$$\rho = \int_{\mathcal{A}} \pi^* \alpha_d d\mu$$

on $L^2(\hat{G}, \mathcal{A}_x) = L^2(\hat{G}) \otimes \mathcal{A}_x$ and let $\beta$ be the extension of $\alpha$ to an action on $\mathcal{M} = \rho(A \times a G)^*$. It suffices to prove that $\mathcal{M} = p(A)^*$ (see 1.1 in [4]). It is obvious that $\mathcal{M} \supseteq p(A)^*$. For $f \in L^1(G) \cap L^2(G)$ it is known ([5]) that $\beta(p(\lambda(f))Qp(\lambda(f)^*))$ is $\beta$-integrable for $Q \in \mathcal{M}$ and

$$\int \beta_{p(\lambda(f))} \beta_{p(\lambda(f)^*)} d\mu \in p(A)^*.$$ 

By a limiting procedure in $\mathcal{M}$ one can conclude that $Q \in p(A)^*$ for $Q \in \mathcal{M}$.

Going back to the proof of the proposition, it is obvious that $\Gamma(\alpha) = \hat{G}$. To prove $\Gamma(\alpha) = G$ we first note

5. Lemma. Under the assumption of the above proposition let $H$ be a closed subgroup of $G$ and suppose that there exists a faithful irreducible representation $\pi$ of $A$ such that $\pi$ is $\alpha | H$-covariant and $\Delta(\pi) = H^\perp$. Then there exists an irreducible representation $\Phi$ of $A \times a G$ such that $\Phi$ is $\alpha | H^\perp$-covariant, $\Delta(\Phi, \alpha) = H$, and $\bigcap_{\mu} \ker \Phi \approx = (0)$. In particular, $\Gamma(\alpha) = H$.

Proof. Let $\varphi$ be a measurable function of $G/H$ into $G$ such that $\varphi(t) + H = t$, $t \in G/H$, and let $\nu$ be a continuous unitary representation of $H$ such that $\pi^* \alpha_t = \text{Ad} \nu(t) \cdot \pi$. Define a representation $\Phi$ of $A \times a G$ on $L^2(G/H, \mathcal{A}_x)$ by

$$\Phi(\alpha) = \int_{\mathcal{G}_H} \alpha \varphi(\alpha) d\mu, \quad a \in A,$$

$$\Phi(\lambda(g)) = \left( \int_{\mathcal{G}_H} \nu(g + \varphi(s) - \varphi(s + \tilde{g})) ds \right) u_g, \quad g \in G,$$

where $\Phi$ is the extension of $\Phi$ to $M(A \times a G)$, $\lambda$ is the canonical unitary representation of $G$ in $M(A \times a G)$, $\tilde{g} = g + H \in G/H$, and $u$ is the unitary representation of $G/H$ defined by

$$(u, \xi)(t) = \xi(t + s), \quad \xi \in L^2(G/H, \mathcal{A}_x).$$

As is easily shown, $\Phi$ is in fact well defined, and since $\Phi(A)^* = L^2(G/H) \otimes B(\mathcal{A}_x)$ and $u \in \Phi(A \times a G)^*$, $\Phi$ is irreducible. Define a unitary representation $\varphi$ of $(G/H)^\perp$ on $L^2(G/H, \mathcal{A}_x)$ by
Then since $\Phi\ast \hat{\alpha}_p = \text{Ad} w_p \Phi$, $p \in H\hat{1}$, $\Phi$ is $a|H\hat{1}$-covariant. Since $\Phi(\lambda(g)) = \Phi(A)^g$ for $g \in H$, one obtains that $\Phi(A\lambda(g))\equiv 1$, for $g \in H$, i.e., $\Delta(\Phi, \hat{\alpha}) = H$. Since $\bigcap_{p \in \ker \Phi} \Phi \ast \hat{\alpha}_p = (0)$ as $\pi$ is faithful, this implies that $\Gamma(\hat{\alpha}) \supset H$.

6. Lemma. Under the assumption of the above proposition let $H$ be a compact subgroup of $G$. Then there exists an $a|H$-covariant irreducible representation $\pi$ of $A$ such that the corresponding representation of $A \times a|H$ is faithful and $\Delta(\pi) = H\hat{1}$.

Proof. Let $\hat{\beta} = a|H$. Then $\Gamma(\hat{\beta}) = \hat{H} = \Gamma(\hat{\beta})$, and so $A \times \hat{\beta} H$ is prime. By the proof of 3.3 in [4], it suffices to show that the following two conditions are satisfied: For any neighbourhood $U$ of any $p \in H\hat{1}$, and any $x \in A \times \hat{\beta} H$,

$$\sup\{\|x(a + a^*)x^*\| ; a \in A^a(U)_1\} = 2\|x\|^a,$$

where $A^a(U)_1$ denotes the unit ball of $A^a(U)$; for any neighbourhood $U$ of any $s \in H$, and any $x \in A \times \hat{\beta} H$,

$$\sup\{\|x(a + a^*)x^*\| ; a \in (A \times \hat{\beta} H)^\hat{\beta}(U)_1\} = 2\|x\|^a$$

or equivalently $\Gamma(\hat{\beta}) = H$. The former can be proved as in 3.3, [4], and the latter can be proved by using the fact that there is a $\beta$-covariant faithful irreducible representation of $A$ (see [1], [4]).

To complete the proof of Proposition 3, we have to show that $\Gamma(\hat{\alpha}) = \hat{G}$. By the previous two lemmas and 3.3 in [4], it follows that $\Gamma(\hat{\alpha}) \supset H$ for any compact or discrete closed subgroup $H$ of $G$. Since any compactly generated subgroup of $G$ is of the form $K \times Z^l \times R^m$ (where $K$ is a compact group and $l, m$ are non-negative integers) and $\Gamma(\hat{\alpha})$ is a closed subgroup of $G$, it easily follows that $\Gamma(\hat{\alpha}) = \hat{G}$.

Proof of Theorem 2. To prove (ii)$\Rightarrow$(iii) we apply Proposition 3 to $(A \times aG, \hat{G}, \hat{\alpha})$ to yield an $a$-covariant faithful irreducible representation of $A \times aG$, which in turn gives a faithful irreducible representation $\pi$ of $A$ with $\Delta(\pi) = \hat{G}$. This implies that $\Gamma(\hat{\alpha}) = \hat{G}$.

The proof of (iii)$\Rightarrow$(iv) goes in exactly the same way as the proof of Lemma 6 or 3.3 in [4] as we know by Proposition 3 that there is an $a$-covariant faithful irreducible representation of $A$.

Suppose that (iv') holds. Then applying (i')$\Rightarrow$(iii) for $\hat{\beta} = a|H$ one obtains that $\Gamma(\hat{\beta}) = \hat{H} = \hat{G}/H\hat{1}$. One also knows that $\Gamma(\hat{\alpha}) \supset H\hat{1}$. Using these two properties, as in the proof of 3.3 in [4], one obtains a faithful irreducible representation $\pi$ of $A$ such that $\Delta(\pi, \beta) = \hat{G}/H\hat{1}$ and $\Delta(\pi, \alpha) \supset H\hat{1}$. From this
follows that $\Delta(\pi, \alpha) = \hat{G}$ or (iii) $\Gamma_\beta(\alpha) = \hat{G}$. (By considering the representation $\rho$ of $A$ defined by

$$\rho = \int_0^\infty \pi\cdot \alpha_t dt \otimes \int_0^\infty \pi\cdot \alpha_t \vert_f \alpha_t dt ds$$

with $f$ a measurable section of $G/H$ into $H$, one can conclude that both the integrals are central and so the center of $\rho(A)^*$ equals $L^\infty(G) \otimes 1$.) Then by taking $G$ for $H$ in the implication (iii) $\Rightarrow$ (iv), one obtains (i).

In general (i) implies (v) via (iv) with $H = (0)$. If $G$ is discrete, (v) $\Rightarrow$ (i) follows from 3.4 in [4] by using [1]. This completes the proof of Theorem 2.

7. Theorem. Let $A$ be a separable prime C*-algebra and let $\alpha$ be a continuous action of a separable locally compact abelian group $G$ on $A$. If there is a faithful family of $\alpha$-covariant irreducible representations of $A$, then for any closed subgroup $H$ of $G$ with $H \supseteq \Gamma(\alpha)^+$, there exists a faithful irreducible representation $\pi$ of $A$ such that $\pi$ is $\alpha \vert H$-covariant and $\Delta(\pi, \alpha) = H^\perp$. In particular, $\Gamma_\beta(\alpha) = \Gamma(\alpha)$.

Proof. Let $F$ be the family of irreducible representations in the theorem. For $\pi \in F$ let $u$ be the implementing unitary representation of $G$ and let $\pi = \pi \cdot u$ be the corresponding representation of $A \times_a G$. Then $\Delta(\pi \cdot \hat{\alpha}_p, \hat{\alpha}) = G$ for each $\pi \in F$ and $p \in \hat{G}$, and the set $F_1$ of $\pi \cdot \hat{\alpha}_p, \pi \in F, p \in \hat{G}$, is a faithful family of irreducible representations of $A \times_a G$. Thus $\Gamma_\beta(\alpha) = \hat{G}$, by 1(i).

Let $\beta = \alpha \vert H^\perp$. Since $H^\perp \subseteq \Gamma(\alpha)$, one has that $I \cap \beta_t(I) \neq (0)$ for any $t \in H^\perp$ and for any non-zero ideal $I$ of $A \times_a G$. We assert:

8. Lemma. For each $\rho \in F_1$, there is a primitive ideal $P$ of $A \times_a G$ such that $P$ is $\beta$-invariant, $P \subset \ker \rho$ and $\Gamma_\beta(\beta/P) = (H^\perp)^\perp = G/H$, where $\beta/P$ denotes the action on the quotient $A \times_a G/P$ induced by $\beta$.

Proof. Let $\mathcal{P}$ be the set of primitive ideals $P$ of $A \times_a G$ such that $P \subseteq \ker \rho$ and there is an irreducible representation $\pi$ of $A \times_a G$ satisfying $\ker \pi = P$ and $\Delta(\pi, \hat{\alpha}) = G$. We define an order on $\mathcal{P}$ by inclusion.

For a totally ordered set $\{P_\nu\}$ in $\mathcal{P}$ we claim that there is a $P_1$ in $\mathcal{P}$ such that $P_1 \subseteq \cap_{\nu \in H^\perp} \beta_\nu(P_\nu)$ for all $\nu$. Once this is proved, we simply take a minimal one in $\mathcal{P}$ for $P$ in the lemma.

Let $\{P_\nu\}$ be as above and let $P_\infty = \bigcap_\nu P_\nu$. Since $A \times_a G$ is separable we may assume that the index set $\{\nu\}$ is the positive integers. (For example, for the primitive ideal $P = P_\nu$ of $A \times_a G$, let $\{x_n\}$ be a dense sequence in $A \times_a G$ and let $x_n$ be such that $\|x_n + P_{\nu_1}\| \geq \|x_n + P_{\nu_0}\|/2$ and $\nu_0 \geq \nu_{n-1}$, and set $P_{\nu} = P_{\nu_0}$. For each $n$ let $\{I_{nk}\}$ be a decreasing sequence of ideals such that $I_{nk}$ is not contained in $P_\nu$ and for any non-zero ideal $J$ not contained in $P_\nu$ there is an $I_{nk}$ with $J \supseteq I_{nk}$. (For example, for the primitive ideal $P = P_\nu$ of $A \times_a G$, let $\{x_k\}$ be a dense sequence in $A \times_a G \setminus P$ and let $J_k$ be
the smallest ideal of \( A \times_\alpha G \) such that \( \|x_k + J_k\| \leq \|x + P\|/2 \), and set \( I_{nk} = J_1 \cap \cdots \cap J_k \).

Let \( \{p_i\} \) be a dense sequence in \( H^\perp \). We consider the set \( S = \{ \beta_{p_i}(I_{nk}) : n, k, \ell = 1, 2, \ldots \} \).

We want to prove that if \( J_i \in S, i = 1, \ldots, m \), then \( \bigcap_{i=1}^m J_i \) is essential in \( J \).

First, if \( J_i = I_{nk} \) and \( J_k = I_{nk'} \), then we may assume that \( J_i \subseteq J_k \) or \( J_i \supseteq J_k \) and if \( J_i \supseteq J_k \) and \( J_k \) is essential in \( J \), because \( P_{nk} \) is primitive and \( J_i \supseteq J_k \subseteq P_{nk} \). Second, if \( J_i = I_{nk} \) and \( J_k = \beta_{p_i}(I_{nk}) \), and if \( J \subseteq J_k \) is an ideal orthogonal to \( J_i \cap J_k \), one must have that \( J \cap \beta_{p_i}(J) = (0) \) which contradicts that \( I'(a) \supseteq H^\perp \) unless \( J = (0) \). Thus \( J_i \cap J_k \) is essential in \( J \). Since if \( J_i \subseteq J_k \) and \( J_k \) is essential in \( J \), then \( J_i \cap J \) is essential in \( J \) for any ideal \( J \), combining these two cases we get the assertion.

Let \( \{J_n\} \) be an enumeration of \( S \); we may assume that \( \{J_n\} \) is decreasing, replacing \( J_n \) by \( J_\ell \sim J_n \).

From what we have proved above it follows that \( J_m \) is essential in \( J_n \) for \( m \geq n \).

Now we use the procedure in the proof of 3.3 in [4] for \( (A \times_\alpha G, \hat{G}, \hat{a}) \) with \( \{J_n\} \) instead of \( \{I_n\} \). Then we obtain an irreducible representation \( \pi \) of \( A \times_\alpha G \) such that \( \Delta(\pi, \hat{a}) = G \) and \( \pi | J_n = (0) \) for any \( n \). If \( \ker \pi \subseteq \beta_{p_l}(P_0) \), then \( \ker \pi \subseteq \beta_{p_l}(P_n) \) for large \( n \) and then \( \ker \pi \supseteq \beta_{p_l}(I_{nk}) \) for large \( k \), a contradiction. Thus \( \ker \pi \supseteq \beta_{p_l}(P_0) \) for any \( l \) and so \( \ker \pi \supseteq \beta_{p_l}(P_0) : \pi \in H^\perp \).

Now we resume the proof of Theorem 7. Let \( \mathcal{P} \) be the set of primitive ideals \( P \) of \( A \times_\alpha G \) such that \( P \) is \( \beta \)-invariant and \( \Gamma_*'(\beta/P) = G/H \). For each \( P \in \mathcal{P} \), Theorem 2 is applicable to the system \( (A \times_\alpha G/P, H^\perp, \hat{\beta}) \). Thus \( \bar{P} = P \times_\beta H^\perp \) is a primitive ideal of \( A \times_\alpha G \times_\beta H^\perp \), and for the quotient system \( (A \times_\alpha G \times_\beta H^\perp/\bar{P}, G/H, \hat{\beta}/\bar{P}) \), \( \Gamma_*(\hat{\beta}/\bar{P}) = H^\perp \) follows. Since \( \cap \{\bar{P}, P \in \mathcal{P}\} = (0) \), this implies that \( \Gamma_*(\hat{\beta}) = H^\perp \).

On the other hand, by using \( \Delta(\pi, \hat{a}) = G \) for \( \pi \in F_1 \) as in the beginning of the proof, we obtain that for any \( t \in H \), any neighbourhood \( U \) of \( t \), and any \( x \in A \times_\alpha G \times_\beta H^\perp \),

\[
\sup \{ \|x(a + a^*)x^*\| : a \in (A \times_\alpha G) \hat{\delta}(U) \} = 2\|x\|^2.
\]

By using this with \( \Gamma_*(\hat{\beta}) = H^\perp \) we obtain an irreducible representation \( \rho \) of \( A \times_\alpha G \times_\beta H^\perp \) such that \( \beta(A \times_\alpha G)^* = \rho(A \times_\alpha G \times_\beta H^\perp)^* \) and \( \Delta(\pi, \hat{a}) = H \) for \( \pi = \rho| A \times_\alpha G \). Moreover we can easily assume that \( \pi | A \) is faithful. By Lemma 5 or its proof we obtain an irreducible representation \( \Phi \) of \( A \times_\alpha G \times_\beta G \cong A \times K \), where \( K \) is the compact operators on \( L^2(G) \), such that \( \Phi \) is \( \hat{\delta} | H \)-covariant, \( \Delta(\Phi, \hat{\delta}) = H^\perp \), and \( \Phi | A \) is faithful. This completes the proof by the duality for crossed products.

For a C*-algebra \( A \) we denote by \( P_f(A) \) the set of pure states \( \varphi \) of \( A \) with \( \ker \pi \varphi = (0) \).
9. **Theorem.** Let $A$ be a separable prime $C^*$-algebra and let $\alpha$ be a continuous action of an abelian Lie group $G$ on $A$. Then the following conditions are equivalent:

(i) $\alpha^*$ on $P_f(A)$ is strongly continuous.

(ii) For each $\varphi \in P_f(A)$, $\alpha|G_0$ extends to a $\sigma$-weakly continuous action on $\pi_{\varphi}(A)^* = B(\mathcal{A}_{\varphi})$, where $G_0$ is the connected component of the identity in $G$.

(iii) For each $\varphi \in P_f(A)$, $\rho_{\varphi}(A)^*$ is of type I with atomic center, where $\rho_{\varphi}$ is defined by $\rho_{\varphi} = \int_{G_0} \pi_{\varphi} \cdot \alpha_t dt$.

(iv) Every orbit in $P_f(A)$ is of type I.

(v) $\alpha_t$ is not properly outer for any $t \in G_0$, where $G_0$ is defined in (ii).

10. **Remark.** In the above theorem if in addition $A$ is simple and unital, $\alpha$ is uniformly continuous (cf. [3]).

**Proof.** Suppose that (i) holds. Then for $\varphi \in P_f(A)$ there is an open neighbourhood $U$ of $0 \in G$ such that

$$\|\varphi \cdot \alpha_t - \varphi\| < 2, \quad t \in U,$$

which implies that $\alpha_t$ is weakly inner in $\pi_{\varphi}$ for $t \in U$. Since $G_0$ is generated by $G_0 \cap U$, it follows that $\alpha_t$ is weakly inner in $\pi_{\varphi}$ for any $t \in G_0$. Hence there is an action $\beta$ on $\pi_{\varphi}(A)^*$ such that $\beta_t \cdot \pi_{\varphi} = \pi_{\varphi} \cdot \alpha_t$, $t \in G_0$. Since $\beta$ is automatically $\sigma$-weakly continuous (e.g. [4]), $\beta$ is the desired action on $\pi_{\varphi}(A)^*$ in (ii).

Suppose that (ii) holds. For $\varphi \in P_f(A)$

$$\rho_{\varphi} = \int_{G_0} \pi_{\varphi} \cdot \alpha_t dt$$

is quasi-equivalent to $\pi_{\varphi}$ and so $\rho_{\varphi}(A)^*$ is a type I factor. Since $G/G_0$ is discrete, (iii) follows immediately.

(iii) $\Rightarrow$ (iv) is trivial.

Suppose that (iv) holds and set

$$H = \{ s \in G : \alpha_s \text{ is not properly outer} \}.$$

If $H$ is not closed, let $s \in \overline{H} \setminus H$. Then there is a $\varphi \in P_f(A)$ such that $\pi_{\varphi} \cdot \alpha_s \uparrow \pi_{\varphi}$ (cf. [2]). Then the orbit $\alpha_{\varphi}$ through $\varphi$ cannot be of type I. Because if $\alpha_{\varphi}$ were of type I, then $G_{\varphi} = \{ t \in G : \pi_{\varphi} \cdot \alpha_t \sim \pi_{\varphi} \}$ would be closed, but $G_{\varphi} \supset H$ and $G_{\varphi} \cap \overline{H}$. Hence $H$ must be closed.

If $H$ does not include $G_0$, then $G/H$ has a closed subgroup which is isomorphic to $T$ or $R$. By using this fact one can easily find subgroups $D_1$, $D_2$ of $G/H$ such that $D_1 \cong \mathbb{Z} \oplus \mathbb{Z} \cong D_2$, $D_1 \cap D_2 = \{0\}$, and $\overline{D_1} = \overline{D_2}$, and then subgroups $D'_i$, $D''_i$ of $G$ such that $D'_i \cap H = \{0\} = D''_i \cap H$, $q(D'_i) = D_i$, $i = 1, 2$, where $q$ is the
quotient map of \( G \) onto \( G/H \). By Theorem 2 applied to the discrete subgroup \( D_1 + D_2 \), there is a \( \varphi \in P_f(A) \) such that \( \pi_{\varphi} \cdot \alpha_t \sim \pi_{\varphi} \) for \( t \in D_1 \), and \( \pi_{\varphi} \cdot \alpha_t \perp \pi_{\varphi} \) for \( t \in D_2 \). Thus \( G_{\varphi} \supseteq q^{-1}(D_1) \) and \( G_{\varphi} \supseteq q^{-1}(D_2) \). Since \( \bar{D}_1 = \bar{D}_2 \), it follows that \( q^{-1}(D_1) = q^{-1}(D_2) \). This implies that \( G_{\varphi} \) is not closed and hence the orbit \( a_{\varphi} \) is not type I.

Suppose that \((v)\) holds. Then for \( \varphi \in P_f(A) \), \( \alpha_t \) is weakly inner in \( \pi_{\varphi} \) for \( t \in G_\varphi \). Then as in the proof of \((i) \Rightarrow (ii)\), one can show that the action \( \beta \) of \( G_\varphi \) defined by \( \beta_t \cdot \pi_{\varphi} = \pi_{\varphi} \cdot \alpha_t \) is continuous. Hence \( t \mapsto a_{\varphi} \varphi \) is norm continuous.

Let \( P \) be a primitive ideal of \( A \). Then a pure state of the quotient \( C^*-\text{algebra} A/P \) is naturally regarded as a pure state of \( A \). Thus \( P(A) \) is regarded as the disjoint union of \( P_f(A/P) \) with \( P \) running over the primitive ideals of \( A \).

11. Corollary. Let \( A \) be a separable \( C^*-\text{algebra} \) and let \( \alpha \) be a continuous action of an abelian Lie group \( G \) on \( A \). Then the following conditions are equivalent:

(i) Every orbit in \( P(A) \) is of type I.

(ii) For any primitive ideal \( P \) of \( A \) the induced action \( (\alpha | G_P)_\pi \) on \( P_f(A/P) \) is strongly continuous, where \( G_P = \{ t \in G : \alpha_t(P) = P \} \).

Proof. Suppose that \((ii)\) holds. Let \( \varphi \in P(A) \). With \( P = \ker \pi_{\varphi} \), \( \varphi \) belongs to \( P_f(A/P) \). By the previous theorem

\[
\rho_{\varphi} = \int_{G/P} \pi_{\varphi} \cdot \alpha_t dt
\]

is of type I. Let \( f \) be a measurable function of \( G/G_P \) into \( G \) such that \( f(s) \cdot G_P = s \), \( s \in G/G_P \). Then \( \rho_{\varphi} \) is equivalent to

\[
\int_{G/G_P} \rho_1 \cdot \alpha_f(s) ds
\]

and we assert that

\[
\rho_{\varphi}(A)^\pi = L^\pi(G/G_P) \otimes \rho_1(A)^\pi,
\]

which implies \((i)\). To prove this it suffices to show that \( N = \text{Sp}(\alpha | Z) \supseteq G_P \) where \( \tilde{\alpha} \) is the extension of \( \alpha \) to an action on \( \rho_{\varphi}(A)^\pi \) and \( Z \) is the center of \( \rho_{\varphi}(A)^\pi \). If \( s \in N \), then it follows that for any neighbourhood \( U \) of \( 0 \in G \)

\[
\int_U \pi_{\varphi} \cdot \alpha_t dt \quad \text{and} \quad \int_U \pi_{\varphi} \cdot \alpha_{t+1} dt
\]

are mutually quasi-equivalent. In particular the kernels of these representations are equal:
Since this is true for any neighbourhood $U$ of $0 \in G$, one obtains that $P = a_s(P)$, i.e., $s \in G_P$. Hence $N^+ \subseteq G_P$.

Suppose that (ii) does not hold. Then there is a primitive ideal $P$ of $A$ such that $(a|G_P)^*$ on $P$ is not strongly continuous. Then by Theorem 9 there is a $\varphi \in P_f(A/P)$ such that the orbit through $\varphi$ under $(a|G_P)^*$ is non-type I. Then as in the proof of (i)$\Rightarrow$(ii), $\rho_\varphi(A)^* = L^\infty(G/G_P) \otimes \rho_\varphi(A)^*$ and hence the orbit through $\varphi$ under $a^*$ is non-type I.

References


