Conditions for Well-posedness in Gevrey Classes of the Cauchy Problems for Fuchsian Hyperbolic Operators II

By

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 Detected to Professor Shoji IRIE on his sixtieth birthday

Introduction

In this article we shall present a sufficient condition for well-posedness in Gevrey classes of some Fuchsian hyperbolic Cauchy problems. Namely we show that we can determine a function space in which the Cauchy problem for a given Fuchsian hyperbolic operator is well-posed.

In the case that the initial surface is non-characteristic, there are many results.

The results independent of the lower order terms were obtained by Ohya [12], Leray-Ohya [8], Steinberg [13], Ivrii [5], Trepreau [15], Bronstein [2], Kajitani [7] and Nishitani [11], which show that the multiplicity of the characteristic roots determines the well-posed class.

On the other hand, it is an interesting problem to study how the lower order terms have an effect on the well-posed class. Ivrii showed the following in [6].

(I) Let \( P = \partial_t^2 - t^2 \partial_x^2 + at \partial_x \), where \( \ell \) and \( s \) are non-negative integers and \( a \) is a non-zero constant. When \( 0 \leq s < \ell - 1 \), the Cauchy problem for \( P \) is \( r^{(\ell)}_{\text{loc}} \)-well-posed if and only if \( 1 \leq \epsilon < (2 \ell - s)/(\ell - s - 1) \).

(II) Let \( P = \partial_t^2 - x^2 \partial_x^2 + ax^2 \partial_x \), where \( \mu \) and \( \nu \) are non-negative integers and \( a \) is a non-zero constant. When \( 0 \leq \nu < \mu \), the Cauchy problem for \( P \) is \( r^{(\mu)}_{\text{loc}} \)-well-posed if and only if \( 1 \leq \epsilon < (2 \mu - \nu)/\mu \).
These examples are extended for more general operators by Igari [3], Uryu [17] and Tahara [14] concerning (I) and Uryu-Itoh [18] and Itoh [4] concerning (II).

Furthermore we propose the following operator.

\[(III)
P = \partial_x^2 - t^2 x^{2\mu} \partial_x^2 + at^x \partial_x, \text{ where } \mu, s \text{ and } \nu \text{ are non-negative integers and } a \text{ is a non-zero constant.}
\]

In this paper we consider the Cauchy problem for the operators which are the most general extension of (III), noting that Fuchsian partial differential operators introduced by Baouendi-Goulaouic [1] are the natural extension of non-characteristic operators.

§1. Main Result and Remarks

Let \((x, t) \in \mathbb{R}^n \times [0, T]\) and \((D_x, D_t) = (D_{x_1}, \ldots, D_{x_n}, D_t) = (-\sqrt{-1} \partial / \partial x_1, \ldots, -\sqrt{-1} \partial / \partial x_n, -\sqrt{-1} \partial / \partial t)\). Let us denote by \((\xi, \tau)\) the dual variable of \((x, t)\).

Now we shall define the Gevrey classes.

**Definition 1.1.** \((\tau^{(\kappa)}) \cap \tau^{(\kappa)}\); \(\kappa \geq 1\) implies that \(f(x) \in C^\kappa(\mathbb{R}^n)\) nad for any compact set \(K \subset \mathbb{R}^n\), there exist constants \(c, R > 0\) such that

\[
|D^\alpha f(x)| \leq c R^|\alpha| |\alpha|^\kappa, \quad x \in K, \quad \text{for any } \alpha.
\]

\(f(x) \in \tau^{(\kappa)}\) implies that \(f(x) \in C^\kappa(\mathbb{R}^n)\) and (1.1) holds for any \(x \in \mathbb{R}^n\).

Next we shall define Fuchsian partial differential operators according to Baouendi-Goulaouic [1].

Let

\[
L = L(x, t, D_x, D_t)
= t^k D_t^k + L_1(x, t, D_x) t^{k-1} D_t^{m-1} + \cdots + L_m(x, t, D_x) D_t^{m-k} + L_{m+1}(x, t, D_x) D_t^{m-k}.\]

Then \(L\) is said to be of Fuchsian type with weight \(m-k\) with respect to \(t\) when it has the following properties:

(A-1) \(k \in \mathbb{Z}, 0 \leq k \leq m,\)

(A-2) \(\text{ord } L_j(x, t, D_x) \leq j,\)

(A-3) \(\text{ord } L_j(x, 0, D_x) = 0 \text{ for } 1 \leq j \leq k.\)

From (A-3), we can set \(L_j(x, 0, D_x) = a_j(x)\) for \(1 \leq j \leq k.\)

A characteristic polynomial associated with \(L\) is
(1.2) \[ C(\lambda, x) = \lambda(\lambda-1)\cdots(\lambda-m+1) + \sqrt{-1}a_1(x)\lambda(\lambda-1)\cdots(\lambda-m+2) + \cdots + \sqrt{-1}a_k(x)\lambda(\lambda-1)\cdots(\lambda-m+k+1). \]

It's roots, called characteristic exponents, are denoted by 0, 1, \ldots, m-k-1, \lambda_1(x), \ldots, \lambda_k(x).

(A-4) there exists a constant \(c>0\) such that
\[ |(\lambda-\lambda_1(x))\cdots(\lambda-\lambda_k(x))| \geq c/\lambda(\lambda-1)\cdots(\lambda-m+k+1) \text{ for } \lambda \in \mathbb{Z}, \lambda \geq m-k. \]

In this paper we deal with the following Fuchsian partial differential operator. Let
\[ t^{-m}L = \tilde{L}(x, t, D_x, D_t) = \tilde{L}_0(x, t, D_x, D_t) + \tilde{L}_1(x, t, D_x, D_t), \]
where
\[ \tilde{L}_0(x, t, D_x, D_t) = t^m D_t^m + \sum_{|\alpha| + j = m} t^{|\alpha|+j}a(x)^{|\alpha|}a_{\alpha,j}(x, t)D_x^\alpha D_t^j, \]
and
\[ \tilde{L}_1(x, t, D_x, D_t) = \sum_{|\alpha| + j \leq m-1} t^{|\alpha|+j}a(x)^{|\alpha|}a_{\alpha,j}(x, t)D_x^\alpha D_t^j. \]

We assume the following conditions on \(L\).

(A-5) \( \lambda\)-roots of \(\lambda^m + \sum_{|\alpha| + j = m} a_{\alpha,j}(x, t)\xi^\alpha \lambda^j = 0\) are real and distinct.

(A-6) \( a_{\alpha,j}(x, t) \in \mathcal{B}([0, T], \tau^{(\alpha)}). \)

(A-7) \( \sigma(x) \in \tau^{(\sigma)} \) and is a real-valued function.

(A-8) \( \iota \) is a positive rational number and \( \mu, s(\alpha, j) \) and \( \nu(\alpha, j) \) are integers such that \( \mu \geq 1, s(\alpha, j) \geq 0 \) and \( \nu(\alpha, j) \geq 0. \)

We define \( \rho \) as follows:
\[ \rho = \max_{|\alpha| + j \leq m-1} \{(m-j-s(\alpha, j)/\iota)/(m-j-|\alpha|), (m-j-\nu(\alpha, j)/\mu)/(m-j-|\alpha|), 1\} . \]

Then we have

**Theorem 1.1.** Under (A-1)\(\sim\)(A-8), if \(1 \leq \varepsilon < \rho/(\rho-1)\), the Cauchy problem for \(L\):
\[
\begin{align*}
\left\{ \begin{array}{ll}
Lu(x, t) = f(x, t) & \text{in } \mathbb{R}^n \times (0, T) \\
D_t^i u(x, t) |_{t=0} = u'(x), & 0 \leq i \leq m-k-1 \text{ on } \mathbb{R}^n
\end{array} \right.
\end{align*}
\]

is \(\tau^{(\rho)}_\text{loc}\)-well-posed, i.e. for any \(f(x, t) \in \mathcal{B}([0, T], \tau^{(\rho)}_\text{loc})\) and any \(u'(x) \in \mathcal{B}([0, T], \tau^{(\rho)}_\text{loc})\), \(0 \leq i \leq m-k-1\), there exists a unique solution \(u(x, t) \in \mathcal{B}([0, T], \tau^{(\rho)}_\text{loc})\) of (1.6).
Remark 1.1. From the definition of $p$, we may only consider the case that $s(\alpha, j) \leq |\alpha| |l|$ and $v(\alpha, j) \leq |\alpha| |\mu|$.

Remark 1.2. From (A-3), $s(\alpha, j) > 0$ if $|\alpha| > 0$.

Remark 1.3. In the case that $k=0$, $\sigma(x)$ is a polynomial and $a_{\alpha,j}(x, t) \in \mathcal{B}([0, T], \tau^{(s)})$, Ivrii showed in [6] that if (1.6) is locally $\tau^{(s)}_{loc}$-well-posed, then $1 \leq \varepsilon \leq \rho/(\rho-1)$.

§2. Proof of Theorem 1.1

In this section we shall reduce Theorem 1.1 to Theorem 2.1.

Definition 2.1. We say that $f(x) \in H^m(\mathbb{R}^n)$ belongs to $\Gamma(\varepsilon)$ if there exist constants $c, R > 0$ such that

\begin{equation}
\|D^a_x f(x)\| \leq c R^{[a]}|\alpha|^\varepsilon \quad \text{for any } \alpha,
\end{equation}

where $\| \cdot \|$ denotes $L^2$-norm with respect to $x$.

Theorem 2.1. Under $(A-1) \sim (A-8)$, if $1 \leq \varepsilon < \rho/(\rho-1)$, then the assertions (1°) and (2°) hold.

(1°) (1.6) is $\Gamma(\varepsilon)$-well-posed.

(2°) If $\text{supp } u(x) \subset K$, $0 \leq i \leq m-k-1$ and $\text{supp } f(x, t) \subset C_d(K)$ for any compact set $K \subset \mathbb{R}^n$, then $\text{supp } u(x, t) \subset C_d(K)$. Here

$$C_d(K) = \{(x, t) \in \mathbb{R}^n \times [0, T]; \min_{\rho \in K} |x-y| \leq \lambda_{\max} |t|^{1/4}\},$$

where $\lambda_{\max} = \max_{1 \leq i \leq m} \sup_{x, t \in \mathbb{R}^n \times [0, T], |t|^{1/4}} |\sigma(x)^i \lambda_j(x, t, \xi)|$ and $\lambda_j(x, t, \xi)$ are $\lambda$-roots in (A-5).

Lemma 2.1. Theorem 1.1 follows from Theorem 2.1.

Proof. (1°; the case that $\varepsilon > 1$) First we shall show the existence of a solution of (1.6). Let $\{\phi_p(x)\}$ be a partition of unity. Namely $\phi_p(x)$ are compactly supported $\tau^{(s)}$-functions satisfying the following three conditions: (i) $0 \leq \phi_p(x) \leq 1$, (ii) $\sum \phi_p(x)$ is locally finite and (iii) $\sum \phi_p(x) \equiv 1$ on $\mathbb{R}^n$. For any $u(p)(x) \in \tau^{(s)}_{loc}$, $0 \leq i \leq m-k-1$ and any $f(x, t) \in \mathcal{B}([0, T], \tau^{(s)}_{loc})$, we set $u_p(x)(x) = \phi_p(x) u(x) \in \Gamma(\varepsilon)$ and $f_p(x, t) = \phi_p(x) f(x, t) \in \mathcal{B}([0, T], \Gamma(\varepsilon))$. Then from (1°) in Theorem 2.1, there exists a unique solution $u_p(x, t) \in \mathcal{B}([0, T], \Gamma(\varepsilon))$ of the Cauchy problem:

\begin{align*}
\begin{cases}
Lu_p(x, t) = f_p(x, t) \\
D_i u_p(x, t)|_{t=0} = u_p(x), \quad 0 \leq i \leq m-k-1.
\end{cases}
\end{align*}
We note that \( T^{(\kappa)} \subset \gamma^{(\kappa)} \) by Sobolev's lemma. Therefore \( u_\rho(x, t) \in \mathcal{B}([0, T], \gamma^{(\kappa)}) \). Furthermore since the summation \( \sum_{\rho} u_\rho(x, t) \) is locally finite, then \( u(x, t) = \sum_{\rho} u_\rho(x, t) \) belongs to \( \mathcal{B}([0, T], \gamma^{(\kappa)}_{1, \infty}) \) and is a solution of (1.6).

Next we shall show the uniqueness of solutions. For any \((x_0, t_0) \in \mathbb{R}^n \times (0, T]\), we set

\[
D_0(x_0, t_0) = \{ (x, t) \in \mathbb{R}^n \times [0, T] ; \ |x-x_0| < \lambda_{\max}(t_0-t^2)/4 \} \quad \text{and} \quad K = D_0(x_0, t_0) \cap \{(x, 0); x \in \mathbb{R}^n \} .
\]

Let \( \phi(x) \) be a compactly supported \( \gamma^{(\kappa)} \)-function such that \( \phi(x) = 1 \) on \( K \). Let us assume that \( u(x, t) \in \mathcal{B}([0, T], \gamma^{(\kappa)}_{1, \infty}) \) satisfies the following equation:

\[
\begin{cases}
Lu(x, t) = 0 & \text{in } \mathbb{R}^n \times (0, T] \\
D^1_i u(x, t) \big|_{t=0} = 0 , \quad 0 \leq i \leq m-k-1 & \text{on } \mathbb{R}^n .
\end{cases}
\]

Since \( L(\phi u) = \phi L u + [L, \phi] u = [L, \phi] u = J^t(x, t) \) and \( L \) is a differential operator, we get that \( \mathcal{J}^t(x, t) \subset C_\kappa(K^\kappa) \). Here \([\cdot, \cdot]\) is the commutator. Therefore from (2.3) in Theorem 2.1, we find that \( \text{supp } \phi u \subset C_\kappa(K^\kappa) \). Then \( u \equiv 0 \) on \( D_0(x_0, t_0) \). Hence \( u(x_0, t_0) = 0 \).

(II; the case that \( \kappa = 1 \)) In (I), we have already showed that if \( 1 < \kappa < \rho/\rho - 1 \), there exists a unique solution \( u(x, t) \in \mathcal{B}([0, T], \gamma^{(\kappa)}_{1, \infty}) \) of (1.6). Therefore it is sufficient to show the analyticity of the solution. If we refer to the method of Mizohata [9] and §5 in this paper, we can easily see this fact. Q.E.D.

We shall prove Theorem 2.1 by the method of successive approximations. Therefore we decompose \( \bar{L} \) as follows and consider the following scheme.

\[
(2.2) \quad \bar{L} = Q_0(x, t, D_x, D_t) + Q_1(x, t, D_x, D_t) .
\]

For \( \alpha, j \) such that \( s(\alpha, j) = |\alpha| \ell \) and \( \nu(\alpha, j) = |\alpha| \mu \), we set

\[
(2.3) \quad Q_0(x, t, D_x, D_t) = \bar{L}_0(x, t, D_x, D_t) + \sum_{|\alpha| + j \leq m-1} t^{s(\alpha, j)+j} \sigma(x)^{\nu(\alpha, j)} a_{\alpha, j}(x, t) D_x^\alpha D_t^j
\]

and for \( \alpha, j \) such that \( s(\alpha, j) < |\alpha| \ell \) or \( \nu(\alpha, j) < |\alpha| \mu \), we set

\[
(2.4) \quad Q_1(x, t, D_x, D_t) = \sum_{|\alpha| + j \leq m-1} t^{s(\alpha, j)+j} \sigma(x)^{\nu(\alpha, j)} a_{\alpha, j}(x, t) D_x^\alpha D_t^j .
\]

\[
(2.5) \quad \begin{cases}
Q_0 u_0(x, t) = t^{m-k} f(x, t) & \text{in } \mathbb{R}^n \times (0, T] \\
D^1_i u_0(x, t) \big|_{t=0} = u^0(x) , \quad 0 \leq i \leq m-k-1 & \text{on } \mathbb{R}^n
\end{cases}
\]

and for \( j \geq 1 \)
The following proposition will be proved in §3.

**Proposition 2.1.** Under (A–1)–(A–8), (1°) and (2°) hold.

(1°) The Cauchy problem for $Q_0$

\[
\begin{aligned}
Q_0v(x, t) &= t^{m-k}v(x, t) \quad \text{in } \mathbb{R}^n \times (0, T] \\
D_i^j v(x, t) &\big|_{t=0} = \nu^j(x), \quad 0 \leq i \leq m-k-1 \quad \text{on } \mathbb{R}^n
\end{aligned}
\]

is $H^\infty$-well-posed.

(2°) If $\text{supp } v(x) \subset K, 0 \leq i \leq m-k-1$ and $\text{supp } f(x, t) \subset C(K)$ for any compact set $K \subset \mathbb{R}^n$, then $\text{supp } u(x, t) \subset C(K)$.

**Corollary 2.1.** When $\rho = 1$, (1.6) is $C^\infty$-well-posed.

If we note that $Q_0$ is a differential operator and $H^{(\rho)} \subset H^\infty$ and use Proposition 2.1, then we find that $u_j(x, t) \in \mathcal{B}([0, T], H^\infty)$ for any $j \geq 0$. Therefore our aim is to show the formal solution

\[
u(x, t) = \sum_{j=0}^\infty u_j(x, t)
\]

converges in $\mathcal{B}([0, T], H^{(\rho)})$.

Our plan is as follows. In §4, we shall get an energy inequality for $Q_0$. In §5, we shall estimate derivatives of a solution of the Cauchy problem:

\[
\begin{aligned}
Q_0v(x, t) &= g(x, t) \\
D_i^j v(x, t) &\big|_{t=0} = 0, \quad 0 \leq i \leq m-k-1,
\end{aligned}
\]

where $g(x, t) \in \mathcal{B}([0, T], H^{(\rho)})$ such that for any sufficiently large fixed integer $s, D_i g(x, t) |_{t=0} = 0, 0 \leq i \leq s-1$. And in §6, we shall obtain an estimate of $Q_0v(x, t)$. Using the consequence in §5 and §6, we shall prove Theorem 2.1 in §7.

### §3. Proof of Proposition 2.1

Let us note that

\[L_0(x, t, \xi, \tau) = \prod_{j=1}^n (\tau - t^4\sigma(x)^j \lambda_j(x, t, \xi)),\]

where $\lambda_j(x, t, \xi)$ are $\lambda$-roots in (A–5). And modifying $\lambda_j(x, t, \xi)$ near $\xi = 0$, we
may assume that if $i \neq j$, there exists a constant $\delta > 0$ such that $|\lambda_i - \lambda_j| < 2\delta$, where $\lambda_i(x, t, \xi) \in \mathcal{B}([0, T], S^k)$ and $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$. Here for real $k$, $S^k$ is the symbol class of classical pseudo-differential operators.

We shall define the modules $W_k$, $0 \leq k \leq m - 1$, over the ring of pseudo-differential operators in $x$ of order zero.

Let $\partial_j := tD_i - t' = \lambda_j(x, t, D_x)$ and $\Pi_m = \partial_1 \cdots \partial_m$. Let $W_{m-1}$ be the module generated by the monomial operators $\Pi_m/\partial_i = \partial_1 \cdots \partial_i \cdots \partial_m$ of order $m - 1$ and let $W_{m-2}$ be the module generated by the operators $\Pi_m/\partial_i \partial_j$, $i \neq j$, of order $m - 2$ and so on.

**Lemma 3.1.** For any $i, j$, there exist pseudo-differential operators $A_{ij}$, $B_{ij}$ and $C_{ij} \in \mathcal{B}([0, T], S^0)$ such that

$$[\partial_i, \partial_j] = A_{ij} \partial_i + B_{ij} \partial_j + C_{ij},$$

where $[\cdot, \cdot]$ is the commutator.

**Proof.** Let $\sigma_0([\partial_i, \partial_j])$ be the principal symbol of $[\partial_i, \partial_j]$. Then by the product formula of pseudo-differential operators, we get

$$\sigma_0([\partial_i, \partial_j]) = \partial_i (t \tau - t\sigma(x)^n \lambda_i) D_j (t \tau - t\sigma(x)^n \lambda_j)$$

$$- \partial_j (t \tau - t\sigma(x)^n \lambda_j) D_i (t \tau - t\sigma(x)^n \lambda_i)$$

$$+ \sum_{k=1}^n \{ \partial_{\xi_k} (t \tau - t\sigma(x)^n \lambda_i) D_{\xi_k} (t \tau - t\sigma(x)^n \lambda_j)$$

$$- \partial_{\xi_k} (t \tau - t\sigma(x)^n \lambda_j) D_{\xi_k} (t \tau - t\sigma(x)^n \lambda_i) \}$$

$$= t\sigma(x)^n D_{ij} (x, t, \xi),$$

where $D_{ij} \in \mathcal{B}([0, T], S^1)$.

If we set $A_{ij} = D_{ij}/(\lambda_i - \lambda_j)$ and $B_{ij} = D_{ij}/(\lambda_i - \lambda_j)$, then $A_{ij}$, $B_{ij} \in \mathcal{B}([0, T], S^0)$ and $A_{ij}(x, t, \xi) (t \tau - t\sigma(x)^n \lambda_i) + B_{ij}(x, t, \xi) (t \tau - t\sigma(x)^n \lambda_j) = t\sigma(x)^n D_{ij}(x, t, \xi)$.

**Q.E.D.**

**Lemma 3.2.** For any monomial $\omega_k^i \in W_k$, $0 \leq k \leq m - 1$, there exist $\partial_i$ and $\omega_{k+1}^i \in W_{k+1}$ such that

$$\partial_i \omega_k^i = \omega_{k+1}^i + \sum_{j=1}^{k+1} \sum_{(j)} C_{ij} \omega_{k+1-j}^i,$$

where $C_{ij} \in \mathcal{B}([0, T], S^0)$.

**Proof.** For any $\omega_k^i = \partial_{j_1} \cdots \partial_{j_k}$, $1 \leq j_1 < \cdots < j_k \leq m$, there exists some $i \in \{j_1, \cdots, j_k\}$ with $1 \leq i \leq m$. Hence if we use Lemma 3.1, we easily obtain (3.2).

**Q.E.D.**

**Lemma 3.3.** Let
\[ \mathcal{V}(t) = \sum_{k=0}^{m-1} \sum_{\alpha} ||\omega_{k}^\alpha u||, \]

then there exists a constant \( c_1 > 0 \) such that

\[ t \frac{d}{dt} \mathcal{V}(t) \leq c_1 \{ ||\Pi_m u|| + \mathcal{V}(t) \}, \]

for \( u(x, t) \in \mathcal{B}([0, T], H^m) \).

**Proof.** From Lemma 3.2 and Lemma A.2 in Appendix, we get that for any \( k \) with \( 0 \leq k \leq m-1 \),

\[ t \frac{d}{dt} ||\omega_{k}^\alpha u||^2 = 2 \Re (\sqrt{-1} t^\alpha \sigma(x)^\alpha \omega_{k+1}^\alpha u + \omega_{k+1}^\alpha u + \sum_{j=1}^{k+1} \sum_{\gamma} C_{\gamma,j} \omega_{k+1-j}^\gamma u, \omega_{k}^\alpha u) \]

\[ \leq c_2 ||\omega_{k}^\alpha u|| + ||\omega_{k+1}^\alpha u|| + \sum_{j=1}^{k+1} \sum_{\gamma} ||\omega_{k+1-j}^\gamma u|| ||\omega_{k}^\alpha u||. \]

Therefore we obtain (3.3). Q.E.D.

**Lemma 3.4.** Let \( \Pi_s = \partial_{i_1} \cdots \partial_{i_s}, 1 \leq i_1 < \cdots < i_s \leq m \). Then \( \Pi_s \), the symbol of \( \Pi_s \), is expressed in the form:

\[ \sigma(\Pi_s) = \prod_{j=1}^s (t \tau - t^\alpha \sigma(x)^\alpha \lambda_{i_j}) + R_{s-1} + \cdots + R_0, \]

where \( R_{s-j} = \sum_{p + q = s-j} t^\beta \sigma(x)^\beta b_{ij}(x, t, \xi) \tau^\delta \) for some \( b_{ij} \in \mathcal{B}([0, T], S^p) \).

**Proof.** We carry out the proof by induction on \( s \). When \( s = 1 \), (3.4) is trivial. Suppose (3.4) holds for \( s \). Since \( \Pi_{s+1} = \Pi_s \partial_{i_{s+1}} \),

\[ \sigma(\Pi_{s+1}) = \sigma(\Pi_s) (t \tau - t^\alpha \sigma(x)^\alpha \lambda_{i_{s+1}}) + \sum_{\alpha} \partial_{i_{s+1}}^{\alpha} \sigma(\Pi_s) D_{s+1}^\alpha (t \tau - t^\alpha \sigma(x)^\alpha \lambda_{i_{s+1}}). \]

Substituting the right hand side of (3.4) for \( \sigma(\Pi_s) \), we have (3.4) with \( s+1 \).

Q.E.D.

**Lemma 3.5.** There exist \( A_j(x, t, \xi) \in \mathcal{B}([0, T], S^0) \) such that for \( i' + j' = m-k \), \( 1 \leq k \leq m \),

\[ t^{i'\alpha+j'\beta} \sigma(x)^{i'\alpha} b_{i'j'}(x, t, \xi) \tau^{j'} \]

\[ = \sum_{j=k}^{m} A_j(x, t, \xi) \prod_{i \leq j, i \neq k} (t \tau - t^\alpha \sigma(x)^\alpha \lambda_i(x, t, \xi)), \]

where \( b_{ij} \in \mathcal{B}([0, T], S^i) \).

**Proof.** Substituting \( t^\alpha \sigma(x)^\alpha \lambda_i(x, t, \xi) \) for \( t \tau \), then we obtain
\[
I_t^{(m-k)} \sigma(x)^{(m-k)\mu} K_j(x, t, \xi) = A_j(x, t, \xi) I_t^{(m-k)} \sigma(x)^{(m-k)\mu} \prod_{i+j, j \in \mathbb{Z}^n} (\lambda_j - \lambda_i),
\]
where \( K_j(x, t, \xi) \in \mathcal{B}[[0, T], S^{m-k}] \). Therefore if we set \( A_j(x, t, \xi) = K_j(x, t, \xi) \times \{ \sum_{i+j, j \in \mathbb{Z}^n} (\lambda_j - \lambda_i) \}^{-1} \), (3.5) is realized. Q.E.D.

**Corollary 3.1.** There exist pseudo-differential operators \( C_k(x, t, D_x) \in \mathcal{B}[[0, T], S^0] \) such that

\[
Q_0 - \Pi_m = \sum_{k=0}^{m-1} \sum_{a} C_k(x, t, D_x) \omega_x^a.
\]

**Proof.** From (3.4) with \( s = m \),

\[
\sigma(Q_0 - \Pi_m) = \sum_{j=1}^{m} \sum_{p+q = m-j} t^{\beta+q} \sigma(x)^{\mu} b_{p_j}(x, t, \xi) \tau^q,
\]
where \( b_{p_j}(x, t, \xi) \in \mathcal{B}[[0, T], S^0] \). Using Lemma 3.5, the principal symbol of \( Q_0 - \Pi_m \) is

\[
\sum_{j=1}^{m} A_j(x, t, \xi) \prod_{i+j, j \in \mathbb{Z}^n} (\tau-t^\ell \sigma(x)^{\mu} \lambda_j(x, t, \xi)),
\]
where \( A_j(x, t, \xi) \in \mathcal{B}[[0, T], S^0] \). Applying (3.4) for \( s = m-1 \),

\[
\sigma(Q_0 - \Pi_m - \sum_{j=1}^{m} A_j \prod_{i+j, j \in \mathbb{Z}^n} \partial \mu \sigma(x)^{\mu} b_{p_j}(x, t, \xi) \tau^q,
\]
where \( b_{p_j}(x, t, \xi) \in \mathcal{B}[[0, T], S^0] \). Repeating these steps, (3.6) is verified. Q.E.D.

**Lemma 3.6.** There exists a constant \( c_3 > 0 \) such that

\[
it^\frac{d}{dt} \Psi(t) \leq c_3 \{ ||Q_0 u|| + \Psi(t) \}.
\]

**Proof.** Using Lemma 3.3 and Corollary 3.1, we obtain that

\[
it^\frac{d}{dt} \Psi(t) \leq c_3 \{ ||Q_0 u|| + \Psi(t) \}
\]
\[
\leq c_3 \{ ||Q_0 u|| + ||(Q_0 - \Pi_m) u|| + \Psi(t) \} \leq c_3 \{ ||Q_0 u|| + \Psi(t) \}.
\]
Q.E.D.

For a sufficiently large integer \( N \), we put

\[
u_N(x, t) = u(x, t) - \sum_{j=0}^{m+1} \frac{t^j}{j!} \partial^j u(x, 0).
\]

Then \( u_N(x, t) \) satisfies the equation:
Q_0 u_N(x, t) = f(x, t) - Q_0\left(\sum_{i=0}^{m-k} \partial^i u(x, 0)\right) = f_N(x, t).

Here we note that from (A-4), for any i ≥ 0, D_i u(x, 0) is represented by f(x, t) and u^i(x), 0 ≤ i ≤ m - k - 1 (cf. Baouendi-Goulaouic [1]).

Lemma 3.7. For sufficiently large N, the following energy estimate holds.

(3.8) \( ||u(\cdot, t)||_s \lesssim \text{const.} \left\{ \sum_{i=0}^{m-k} \frac{t^i}{i!} \left\| \partial^i u(\cdot, 0) \right\|_s + t^N \int_0^t \| D_t^{N+1} f_N(\cdot, \tau) \|_s d\tau \right\} , \)

where \( \| \cdot \|_s \) denotes \( H^s \)-norm with respect to x.

Proof. If we redefine \( \Psi(t) \) replacing \( u(x, t) \) by \( u_N(x, t) \), then from Lemma 3.6,

\[ \frac{d}{dt} \left( t^{-s} \Psi(t) \right) \leq c_3 t^{-s-1} \| f_N(\cdot, t) \| . \]

We can choose \( N \) such that \( t^{-s} \Psi(t) \big|_{t=0} = 0 \). Then

\[ \Psi(t) \leq c_3 t^s \int_0^t \tau^{-s-1} \| f_N(\cdot, \tau) \| d\tau . \]

On the other hand, since \( D_i f_N(x, 0) = 0 \) for \( 0 \leq i \leq N \),

\[ f_N(x, t) = \frac{1}{N!} \int_0^t (t-\tau)^N \partial_t^{N+1} f_N(x, \tau) d\tau . \]

Thus

\[ ||u_N(\cdot, t)|| \leq \text{const.} \cdot t^N \int_0^t || D_t^{N+1} f_N(\cdot, \tau) || d\tau . \]

Similarly we get that for real \( s \),

\[ ||u_N(\cdot, t)||_s \leq \text{const.} \cdot t^N \int_0^t || D_t^{N+1} f_N(\cdot, \tau) ||_s d\tau . \]

Therefore we can obtain the desired estimate. Q.E.D.

Proof of Proposition 2.1. For any \( i \) with \( m - k \leq i \leq m - 1 \), we calculate \( D_i v(x, 0) \) and let them \( v^i(x) \), \( m - k \leq i \leq m - 1 \). Next we define the \( \delta \)-translation \( Q_0^{\delta} \) of \( Q_0 \) by

(3.9) \[ Q_0^{\delta}(x, t, D_x, D_t) = Q_0(x, t+\delta, D_x, D_t) \quad \text{for} \quad 0 \leq \delta \leq 1 . \]

Now we consider the following non-characteristic Cauchy problem:

(3.10) \[ \begin{align*}
Q_0^{\delta} v_0(x, t) &= t^{m-k} f(x, t) \quad \text{in} \quad \mathbb{R}^n \times (0, T] \\
D_i v_0(x, t) \big|_{t=0} &= v^i(x) , \quad 0 \leq i \leq m - 1 \quad \text{on} \quad \mathbb{R}^n .
\end{align*} \]
For $\delta > 0$, (3.10) is $H^\infty$-well-posed (cf. Uryu [16]). Further from Lemma 3.7, the following energy estimate holds uniformly in $\delta$:

$$||v_{\delta}(\cdot, t)||_s \leq \text{const.} \left\{ \sum_{j=0}^{m+N} t^j \left( \partial_j^0 v_{\delta}(\cdot, 0) \right) ||_{s + t^N} \int_0^t ||D_{N+1}^j f_{\delta}(\cdot, \tau)||_{s} d\tau \right\}.$$ 

Therefore there exists a subsequence $\{v_{\delta_j}\}$ which converges weakly in $\mathcal{D}([0, T], H^s)$ as $\delta_j \to 0$. This limit function $v$ is a unique solution of (2.6). Hence (1°) has proved.

In order to prove (2°), we note the following fact. For $\delta > 0$, initial surface $\{t = 0\}$ is non-characteristic with respect to $Q^\delta_0$ and $Q^\delta_5$ is invariant under the Holmgren transformation:

$$\begin{cases} x' = x \\ t' = t + |x|^2 \end{cases}.$$ 

Thus by the well-known method (for example, see Mizohata [10]), we find that the domain of dependence is finite, i.e. for any $(x_0, t_0) \in \mathbb{R}^n \times (0, T)$, if $f(x, t) \equiv 0$ in $D_0$ and $v^i(x) \equiv 0$ on $D_0 \cap \{(x, 0); x \in \mathbb{R}^n\}$, then $v_{\delta}(x, t) \equiv 0$ in $D_0$, where $D_0 = \{(x, t) \in \mathbb{R}^n \times (0, T); |x-x_0| < \lambda_{\max} \{(t_0+\delta)^{1-(t+\delta)^{1/2}}\}/\delta\}$.

Then the following fact holds for limit function $v(x, t)$. If $f(x, t) \equiv 0$ in $D$ and $v^i(x) \equiv 0$ on $D \cap \{(x, 0); x \in \mathbb{R}^n\}$, then $v(x, t) \equiv 0$ in $D$, where $D = \bigcap_{\delta > 0} D_\delta$. Since we can easily see that $D = D_{\delta_0}$ (2°) is verified.

This completes the proof. Q.E.D.

§ 4. Energy Inequality for $Q_0$

The aim of this section is to show the following lemma.

**Lemma 4.1.** Let

$$W_r(t) = \sum_{k=0}^{m-1} \sum_a ||A^a \phi_k u||,$$

where $A$ is the pseudo-differential operator with symbol $\langle \xi \rangle$. Then there exist constants $c_4$, $R > 0$ such that

$$\left(4.1\right) \quad t \frac{d}{dt} W_r(t) \leq c_4 \left( ||A^r Q^\delta u|| + W_r(t) + t^s \sum_{j=1}^{r-1} \hat{R}^{j-1} (j-1)! \left( r^s \right) W_{r+1-j}(t) \right)$$

$$+ \sum_{j=1}^{r-1} \hat{R}^j j! \left( r^s \right) W_{r-j}(t) + \hat{R}^r t^s W_0(t).$$

**Proof.** For $r > 0$, operating $A^r$ on both sides of (3.2), we get that
\[ \partial_t A^r u = [\partial_t, A^r] \omega^* u + A^r \omega^*_{k+1} u + \sum_{j=1}^{k+1} \sum_{r} (C_{\gamma j} A^r \omega^*_{k+1-j} u + [A^r, C_{\gamma j}] \omega^*_{k+1-j} u). \]

Similar to the proof of Lemma 3.3, we have that for any \( k \) with \( 0 \leq k \leq m-1 \),

\[
t \frac{d}{dt} \| A^r \omega^* u \| \leq c \{ \| A^r \omega^* u \| + \| [A^r, \partial_t] \omega^* u \| + \| A^r \omega^*_{k+1} u \| + \sum_{j=1}^{k+1} \sum_{r} (\| A^r \omega^*_{k+1-j} u \| + \| [A^r, C_{\gamma j}] \omega^*_{k+1-j} u \|) \}. \]

It follows from Lemma A.3 in Appendix that

\[
\| [A^r, \partial_t] \omega^* u \| \leq \alpha t^d \sum_{j=1}^{r^*} \hat{\rho} t^{j-1/2} (r^*)^j \| A^{r+j} \omega^* u \| + t^e \hat{\rho} r^{e} \| \omega^* u \|
\]

and

\[
\| [A^r, C_{\gamma}] \omega^*_{k+1-i} u \| \leq \sum_{j=1}^{r^*} \hat{\rho} t^{j-1/2} (r^*)^j \| A^{r-j} \omega^*_{k+1-i} u \| + t^e \hat{\rho} r^{e} \| \omega^*_{k+1-i} u \| .
\]

Therefore we obtain that

\[
t \frac{d}{dt} \| v(t) \| \leq c \{ \| A^{r+1} u \| + \| v(t) \| + \| \sum_{j=1}^{r^*} \hat{\rho} t^{j-1/2} (r^*)^j \| \}
\]

where \( r^* = \Gamma(r+1) \) and \( r^* \) is the lowest integer greater than or equal to \( r \),

\[Q.E.D.\]

Here \( r^* = \Gamma(r+1) \) and \( r^* \) is the lowest integer greater than or equal to \( r \),
where \( \Gamma(\cdot) \) is the gamma function.

\[\S 5. \ \text{Estimate of } A^r v(x, t)\]

We assume the existence of solutions of the following Cauchy problem:

\[
\begin{align*}
Q_0 v(x, t) &= g(x, t) \\
D_t^i v(x, t) \big|_{t=0} &= 0, \quad 0 \leq i \leq m-k-1,
\end{align*}
\]

where \( g(x, t) \in C([0, T], \Gamma^{(s)}) \) such that for any sufficiently large fixed integer \( s, D_t^i g(x, t) \big|_{t=0} = 0, \quad 0 \leq i \leq s-1. \)

Therefore we may assume that for any \( r \geq 0 \), there exist constants \( c, R, M > 0 \) such that

\[
\| A^r g(x, t) \| \leq c R^r t^s \exp (M R^* t^s).
\]

For simplification we use the notation
\[ w_r(s, t, R) = R^r t^s \exp \left( M r^* t^\ell \right). \]

**Lemma 5.1.** For any \( r \geq 0 \), there exists a constant \( A' > 0 \) such that for sufficiently large \( R, M, s \),

\[ \Psi_r(t) \leq c A' s^{-1} w_r(s, t, R). \]

**Proof.** We carry out the proof by induction on \( r \).

When \( r = 0 \), it follows from Lemma 3.6 and (5.1) that

\[ t \frac{d}{dt} \Psi_0(t) \leq c_1 \{ c w_0(s, t, R) + \Psi_0(t) \}. \]

From this inequality,

\[ \frac{d}{dt} (t^{-c_5} \Psi_0(t)) \leq c c_3 t^{-c_5-1} w_0(s, t, R). \]

If we note that \( s \) is sufficiently large,

\[ \Psi_0(t) \leq t^s \int_0^t c c_5 t^{-c_5-1} d\tau = c c_3 t^s (s - c_3)^{-1} t^{-c_5} \leq c A' s^{-1} w_0(s, t, R), \]

if we choose \( A' \) such that \( A' \geq 2c_3 \).

We assume (5.2) is valid for any \( r \) such that \( 0 \leq r \leq n \). Let us show that (5.2) is valid for \( n < r \leq n + 1 \). It follows from Lemma 4.1 that

\[ \frac{d}{dt} \left\{ t^{-c_4} \exp \left( -c_4 r^* t^{\ell} / \ell \right) \Psi_r(t) \right\} \leq c_4 t^{-c_4-1} \exp \left( -c_4 r^* t^{\ell} / \ell \right) \{ ||A' Q_0|| \}
\]

\[ + t^\ell \sum_{j=2}^{r} \hat{R}^{j-1}(j-1)! e_{r-j}(r^*) \Psi_{r+1-j}(t) + \sum_{j=1}^{r-1} \hat{R}^j \Psi_{r-j}(t) + \hat{K}^r \Psi_0(t) \}. \]

Hence we get that

\[ \Psi_r(t) \leq c_4 t^s \exp \left( c_4 r^* t^{\ell} / \ell \right) \int_0^t t^{-c_4-1} \exp \left( -c_4 r^* t^{\ell} / \ell \right) \{ ||A' Q_0|| \}
\]

\[ + t^\ell \sum_{j=2}^{r} \hat{R}^{j-1}(j-1)! e_{r-j}(r^*) \Psi_{r+1-j}(t) + \sum_{j=1}^{r-1} \hat{R}^j \Psi_{r-j}(t) + \hat{K}^r \Psi_0(t) \} d\tau \]

\[ \leq c_4 t^s \exp \left( c_4 r^* t^{\ell} / \ell \right) \int_0^t t^{-c_4-1} \exp \left( -c_4 r^* t^{\ell} / \ell \right) \{ c w_r(s, \tau, R) \}
\]

\[ + t^\ell \sum_{j=2}^{r} \hat{R}^{j-1}(j-1)! e_{r-j}(r^*) c A' s^{-1} w_{r+1-j}(s, \tau, R) + \sum_{j=1}^{r-1} \hat{R}^j c A' s^{-1} w_{r-j}(s, \tau, R) \} d\tau \]

\[ \leq c_4 t^s \exp \left( c_4 r^* t^{\ell} / \ell \right) \int_0^t t^{-c_4-1} \exp \left( -c_4 r^* t^{\ell} / \ell \right) \]

\[ \times \{c \omega \rho (s, r, \tau) + \tau^j \sum_{j=1}^J (\hat{R}/R)^j \left( \frac{r^*}{j} \right)^{j-1} cA^s w_r(s, \tau, R) \] 
\[ + \sum_{j=1}^J (\hat{R}/R)^j \left( \frac{r^*}{j} \right)^{j-1} cA^s w_r(s, \tau, R) + (\hat{R}/R)^j cA^s w_r(s, \tau, R) \} \, d\tau. \]

Let \( R \geq 2 \hat{R} \), then
\[ \Psi_r(t) \leq c \tau^j \exp \left( c \tau^j t^j / j \right) \int_0^t \tau^{-c^*} \exp \left( -c \tau^j t^j / j \right) \times \{c \omega \rho (s, r, \tau) + \tau^j \tau^j cA^s w_r(s, \tau, R) + cA^s w_r(s, \tau, R) \} \, d\tau \]
\[ \leq c \tau^j \exp \left( c \tau^j t^j / j \right) R^r t^j \exp \left\{ \left( M - c \omega \rho (s, r, \tau) \right) \int_0^t \tau^{-c^*} \, d\tau \right\} + cA^s w_r(s, \tau, R) \]
\[ \leq cA^s w_r(s, t, R), \]
if we choose \( A' \) such that \( A' \geq 3c^* c_7 \) and note that \( s \) and \( M \) are sufficiently large.

Q.E.D.

Lemma 5.2. Let
\[ \Phi_r(t) = \sum_{i+j \leq n} t^{i+j} || A' \{ \sigma(x)^{i+j} A^i D_v \} ||, \]
then
\[ \Phi_r(t) \leq c g \left\{ \frac{1}{r^*} \right\} \Psi_r(t) + \hat{R}^r \Psi(t). \]

Proof. From Lemma 3.4 and Lemma 3.5, we get that
\[ t^{i+j} || A' \{ \sigma(x)^{i+j} A^i D_v \} || = || A' \{ \sum_{k=0}^{t^j} \sum_{a} A_k(x, t, D_a) \omega^a_v \} || \]
\[ \leq c g \sum_{k=0}^{t^j} \sum_{a} (|| A' \omega^a_v || + || A' A_k \omega^a_v ||). \]

Using Lemma A.3 in Appendix, we have
\[ || A' A_k \omega^a_v || \leq \sum_{j=1}^{r^*} \frac{1}{j} \hat{R}^j j^j \left( \frac{r^*}{j} \right)^{j-1} || A'^{-j} \omega^a_v || + \hat{R}^r \tau^j || \omega^a_v ||. \]

Thus we can obtain the desired inequality.

Q.E.D.

Corollary 5.1. For any \( r \geq 0 \), there exists a constant \( A' > 0 \) such that for sufficiently large \( R, M, s, \)
\[ \Phi_r(t) \leq c A^s w_r(s, t, R). \]
Proof. Applying Lemma 5.1 to Lemma 5.2, we find that
\[ \Phi_r(t) \leq c_6 \left\{ \sum_{j=0}^{r-1} \hat{R}^j r^j \left( \frac{r^*}{j} \right)^{cA' s^{-1} w_r(s, t, R) + \hat{R}^r r^r cA' s^{-1} w_0(s, t, R)} \right\} \]
\[ \leq c_6 \left\{ \sum_{j=0}^{r-1} (\hat{R}/R) r^j \left( \frac{r^*}{j} \right)^{cA' s^{-1} w_r(s, t, R) + (\hat{R}/R)^r cA' s^{-1} w_r(s, t, R)} \right\} \]
\[ \leq c \hat{A} s^{-1} w_r(s, t, R), \]
if we make \( R \geq 2 \hat{R} \) and choose \( \hat{A} \) such that \( \hat{A} \geq 3c_6 A' \). Q.E.D.

Lemma 5.3. For any \( r \geq 0 \) and \( i+j \leq m-1 \), there exists a constant \( A > 0 \) such that for sufficiently large \( R, M, s \),
\[ (5.4) \quad t^{i+j} \| A' \{ \sigma(x)^{i+j} A^l D^j \} \| \leq cA s^{- (m-i-j)} w_r(s, t, R). \]

Proof. It follows from Corollary 5.1 that
\[ \| A' \{ \sigma(x)^{i+j} A^l D^j \} \| \leq \int_0^t \cdots \int_0^t \| A' \{ \sigma(x)^{i+j} A^l D^j \} \| d\tau_1 \cdots d\tau_q \]
\[ \leq c \hat{A} s^{-1} R^r t^r \exp ( M r^* t^*) \int_0^t \cdots \int_0^t \tau_1^{-i} \cdots \tau_q^{-i} d\tau_1 \cdots d\tau_q \]
\[ \leq c(2^q \hat{A}) s^{-(q+1)} w_r(s-i-l-j, t, R). \]
Hence we get that if we put \( q = m-i-j-1 \),
\[ \| A' \{ \sigma(x)^{i+j} A^l D^j \} \| \leq \int_0^t \cdots \int_0^t \| A' \{ \sigma(x)^{i+j} A^l D^j \} \| d\tau_1 \cdots d\tau_q \]
\[ \leq c \hat{A} s^{-1} R^r t^r \exp ( M r^* t^*) \int_0^t \cdots \int_0^t \tau_1^{-i} \cdots \tau_q^{-i} d\tau_1 \cdots d\tau_q \]
\[ \leq c(2^q \hat{A}) s^{-(q+1)} w_r(s-i-l-j, t, R). \]
If we set \( A = 2^q \hat{A} \), we get (5.4). Q.E.D.

Lemma 5.4. For any \( r \geq 0 \) and \( i, j \) such that \( i+j = 0, \cdots, m-1 \),
\[ (5.5) \quad t^{i+j} \| \sigma(x)^{i+j} A^{r+i} D^j \| \leq c_0 cA w_{r+i}(s, t, R) \]
\[ \times \sum_{k=0}^j s^{- (m-i-j+k)} \{ (r+i) \cdots (r+k+1) \}^{-i} \{ (r+k) \cdots (r+1) \}^{-i}. \]

Proof. We carry out the proof by induction on \( i \). When \( i=0 \), (5.5) is trivial from (5.4). Using (5.4) and Lemma A.3 in Appendix and noting \( \mu \geq 1 \), we obtain that
\[ t^{i+j} \| \sigma(x)^{i+j} A^{r+i} D^j \| \]
\[ \leq t^{i+j} \| A' \{ \sigma(x)^{i+j} A^l D^j \} \| + t^{i+j} \| [A', \sigma(x)^{i+j}] A^l D^j \| \]
\[ \leq cA s^{- (m-i-j)} w_r(s, t, R) + \sum_{k=1}^j \hat{R}^k r^k t^k \left( \frac{r^*}{k} \right)^{t^{i+k} l_j \| \sigma(x)^{i+j} A^{r+i-k} D^j \|} \]
\[ + \sum_{k=1}^{i+j} \hat{R}^k r^k t^k \left( \frac{r^*}{k} \right)^t \| A^{r+i-k} D^j \| + \hat{R}^r r^r t^r \| A^l D^j \|. \]
\[
\sum_{k=0}^{i-1} s^{-(m-i-j+k)} \{(r+i-k) \cdots (r+k' +1)\}^{-\varepsilon} \{(r+k') \cdots (r+1)\}^{1-\varepsilon} \\
+ \sum_{k=0}^{i-1} \hat{R}^k k! \left( \frac{r+i}{k} \right) cA^{s-(m-i-j)} w_{r+i-k}(s, t, R) + \hat{R}^r r! cA^{s-(m-i-j)} w_i(s, t, R) \\
\leq cA^{s-(m-i-j)} \{(r+i) \cdots (r+1)\}^{-\varepsilon} w_{r+i}(s, t, R) + c_1 cA w_{r+i}(s, t, R) \\
\times \sum_{k=0}^{i-1} \sum_{k'=0}^{k} s^{-(m-i-j+k+k')} \{(r+i) \cdots (r+k+k'+1)\}^{-\varepsilon} \{(r+k+k') \cdots (r+1)\}^{1-\varepsilon} \\
+ cA^{s-(m-i-j)} w_{r+i}(s, t, R) \sum_{k=0}^{i-1} \hat{R}^k \left( \frac{r+i}{k} \right) \\
+ \hat{R}^r r! cA^{s-(m-i-j)} w_{r+i}(s, t, R) \\
\leq cA^{s-(m-i-j)} \{(r+i) \cdots (r+1)\}^{-\varepsilon} w_{r+i}(s, t, R) \\
+ c_1 cA w_{r+i}(s, t, R) \sum_{k=1}^{i} s^{-(m-i-j+k+k')} \{(r+i) \cdots (r+k+k'+1)\}^{-\varepsilon} \{(r+k+k') \cdots (r+1)\}^{1-\varepsilon} \\
+ cA^{s-(m-i-j)} \{(r+i) \cdots (r+1)\}^{-\varepsilon} w_{r+i}(s, t, R) \\
+ \hat{R}^r r! cA^{s-(m-i-j)} w_{r+i}(s, t, R) \\
\leq c_0 cA w_{r+i}(s, t, R) \sum_{k=0}^{i} s^{-(m-i-j+k+k')} \{(r+i) \cdots (r+k+k'+1)\}^{-\varepsilon} \{(r+k+k') \cdots (r+1)\}^{1-\varepsilon} \\
Q.E.D.
\]

\section{Estimate of $A^\varepsilon Q_v u(x, t)$}

**Lemma 6.1.** If $\sigma(x) \in \mathcal{B}(\mathbb{R}^n)$ and $0 \leq \nu < \mu$, then

\[
||\sigma(x)^\nu u|| \leq ||u||^{1-\nu/\mu} ||\sigma(x)^\mu u||^{\nu/\mu}. 
\]

**Proof.** By Holder’s inequality,

\[
||\sigma(x)^\nu u||^2 = \int |\sigma(x)^\nu u|^2 dx = \int |u|^{2(1-\nu/\mu)} |\sigma(x)^\mu u|^{2\nu/\mu} dx \\
\leq \left( \int |u|^{2\nu/\mu} dx \right)^{1-\nu/\mu} \left( \int |\sigma(x)^\nu u|^2 dx \right)^{\nu/\mu} \\
= ||u||^{2(1-\nu/\mu)} ||\sigma(x)^\mu u||^{2\nu/\mu}. \quad Q.E.D.
\]

**Lemma 6.2.** Let

\[
\rho_\theta(\alpha, j) = \begin{cases} 
\nu(\alpha, j)/|\alpha| \mu & \text{if } \nu(\alpha, j) < \mu s(\alpha, j) \\
\nu(\alpha, j) s(\alpha, j)/(|\alpha| \ell + \theta) & \text{if } \nu(\alpha, j) \geq \mu s(\alpha, j)
\end{cases}
\]

with respect to $0 < \theta \leq 1$, then for any $r \geq 0$,
(6.3) \[ t^{(\alpha,j) + j} || \sigma(x)^{\nu(\alpha,j)} A^{\nu(\alpha,j)} D_1^j || \]
\[ \leq c_1 \frac{c A}{\alpha + 1}(s + \varepsilon_1, t, R) \sum_{j=0}^{\infty} s^{-(m-j-(\alpha=1-k))\rho(\alpha,j)} \]
\[ \times \{(r + |\alpha|) \cdots (r + k + 1)\}^{-\varepsilon_0(\alpha,j)} \{(r + k) \cdots (r + 1)\}^{-(\varepsilon_1-1)\rho(\alpha,j)} , \]
where \( \varepsilon_1 = \min \{ s(\alpha, j) - \nu(\alpha, j)/\mu, s\theta/(|\alpha| l + \theta) \} > 0. \)

Proof. First we consider the case that \( \nu(\alpha, j) < s(\alpha, j) \). If we use Lemma 5.4 and Lemma 6.1, we get

Next in the case that \( \nu(\alpha, j) \geq s(\alpha, j) \), we have

\[ t^{(\alpha,j) + j} || \sigma(x)^{\nu(\alpha,j)} A^{\nu(\alpha,j)} D_1^j || \]
\[ \leq \frac{c_1}{\alpha + 1}(s + \varepsilon_1, t, R) \sum_{j=0}^{\infty} s^{-(m-j-(\alpha=1-k))\rho(\alpha,j)} \]
\[ \times \{(r + |\alpha|) \cdots (r + k + 1)\}^{-\varepsilon_0(\alpha,j)} \{(r + k) \cdots (r + 1)\}^{-(\varepsilon_1-1)\rho(\alpha,j)} \]

Next in the case that \( \nu(\alpha, j) \geq s(\alpha, j) \), we have

\[ \nu(\alpha, j) = 0 \] or there exists a non-negative integer \( p(\alpha, j) \) such that \( p(\alpha, j) \times \mu < j \alpha(j) \leq p(\alpha, j) + 1 \mu \). And there exists a non-negative integer \( q(\alpha, j) \) such that \( q(\alpha, j) l < s(\alpha, j) \leq q(\alpha, j) + 1 l \).
Lemma 6.3. For any $r \geq 0$ and $|\alpha| > 0$,

\begin{equation}
(6.5) \quad t^{r(|\alpha|^j + j)||[A^r, \sigma(x)^{(|\alpha|^j)}a_{\alpha,j}(x, t)D_x^{|\alpha|^j}D_t^j]}||
\leq c_{r_2}c_{A^r}w_{r+|\alpha|}(s + \varepsilon_2, t, R) \sum_{k=0}^{|\alpha|^j+1} s^{-(m-j-\delta(|\alpha|^j, k))} \times \{(r+|\alpha|-k)+1\}^{-(\varepsilon_2-1)},
\end{equation}

where $h(\alpha, j) = \begin{cases} p(\alpha, j) & \text{if } \nu(\alpha, j) < \mu s(\alpha, j) \\ q(\alpha, j) & \text{if } \nu(\alpha, j) \geq \mu s(\alpha, j) \end{cases}$ and $\varepsilon_2 = s(\alpha, j) - \theta h(\alpha, j) > 0$.

**Proof.** First we consider the case that $\nu(\alpha, j) < \mu s(\alpha, j)$. Since

\[
\sigma([A^r, \sigma(x)^{(|\alpha|^j)}a_{\alpha,j}(x, t)D_x^{|\alpha|^j}]) = \sum_{|\beta|=|\alpha|}^{r+|\alpha|+1} \frac{1}{\beta!} \partial_x^{\beta} \sigma(x)^{(|\alpha|^j)}a_{\alpha,j}(x, t) D_x^{\beta} D_t^j
\]

and if we note that

\[
\nu(\alpha, j) - k = (p(\alpha, j) + 1 - k)\mu + (\nu(\alpha, j) - p(\alpha, j)\mu - 1) + (k-1)(\mu-1),
\]

then we obtain that

\[
I(\alpha, j) = t^{r(|\alpha|^j + j)||[A^r, \sigma(x)^{(|\alpha|^j)}a_{\alpha,j}(x, t)D_x^{|\alpha|^j}D_t^j]}||
\leq c_{r_2}c_{A^r}w_{r+|\alpha|}(s + \varepsilon_2, t, R) \sum_{k=0}^{|\alpha|^j+1} s^{-(m-j-\delta(|\alpha|^j, k))} \times \{(r+|\alpha|-k)+1\}^{-(\varepsilon_2-1)}.
\]

Using Lemma 5.4 and noting Remark 1.2,
The calculation of the case that $\mathcal{L}(\alpha, j) \triangleright \mu \mathcal{L}(\alpha, j)$ is quite similar to the first case. Q.E.D.

From $A'Q_1 = [A', Q_1] + Q_1 A'$, Lemma 6.2 and Lemma 6.3, we obtain

**Lemma 6.4.**

\[
\| A'Q_1 v \| \leq \tilde{c} c A \sum_{|\alpha| + j \leq m - 1} K_f^a(s, r) w_{r + |\alpha|}(s + \varepsilon, t, R),
\]

where $\tilde{c} > 0$, $\varepsilon \equiv \min \{\varepsilon_1, \varepsilon_2\} > 0$ and

\[
K_f^a(s, r) = \sum_{k=0}^{r+j+1} s^{-m-j-(|\alpha|-k)g(\alpha, j)} \times \{(r+|\alpha|)\cdots(r+k+1)\}^{-\kappa} \{r+k\cdots(r+1)\}^{-\kappa} \times \{(r+|\alpha|)\cdots(r+|\alpha| - p(\alpha, j)+k)\}^{-\kappa} \times \{(r+|\alpha| - p(\alpha, j)+k-1)\cdots(r+|\alpha| - p(\alpha, j))\}^{-\kappa}.
\]
§7. Proof of Theorem 2.1

In order to prove Theorem 2.1, we prepare several lemmas.

**Lemma 7.1.** For any \( f(x, t) \in \mathcal{B}([0, T], \Gamma^{(e)}) \) and \( u(x) \in \Gamma^{(e)} \), \( 0 \leq i \leq m - k - 1 \), there exists a unique solution \( u(x, t) \in \mathcal{B}([0, T], \Gamma^{(e)}) \) of the equation:

\[
\begin{cases}
Q_d u(x, t) = t^{m-k} f(x, t) \\
D_i^t u(x, t) |_{t=0} = u^i(x), \quad 0 \leq i \leq m - k - 1.
\end{cases}
\]

And especially, if \( u^i(x) \equiv 0 \), \( 0 \leq i \leq m - k - 1 \) and \( D_i^t f(x, t) |_{t=0} = 0 \), \( 0 \leq i \leq s - 1 \), then we obtain that \( D_i^t u(x, t) |_{t=0} = 0 \), \( 0 \leq i \leq m - k - 1 + s \), where \( s \) is a positive integer.

**Proof.** It follows from Proposition 2.1 that there exists a unique solution \( u(x, t) \in \mathcal{B}([0, T], H^m) \) of (7.1). Therefore let us show that \( u(x, t) \in \mathcal{B}([0, T], \Gamma^{(e)}) \).

From (A-4), we note that we can calculate the derivatives of \( u(x, t) \) at \( t=0 \) and each derivatives belongs to \( \Gamma^{(e)} \).

For any fixed integer \( s \geq 1 \), let

\[
u_s(x, t) = u(x, t) - \sum_{j=0}^{s-1} \frac{t^j}{j!} \partial_j^i u(x, 0),
\]

then \( u_s(x, t) \) satisfies the equation

\[
Q_d u_s(x, t) = f(x, t) - Q_d \left( \sum_{j=0}^{s-1} \frac{t^j}{j!} \partial_j^i u(x, 0) \right) \equiv f_s(x, t).
\]

Thus we get that \( f_s(x, t) \in \mathcal{B}([0, T], \Gamma^{(e)}) \) such that \( D_i^t f_s(x, t) |_{t=0} = 0 \), \( 0 \leq i \leq s - 1 \). From the consequence of §5, it is easily seen that \( u_s(x, t) \in \mathcal{B}([0, T], \Gamma^{(e)}) \).

Hence \( u(x, t) \in \mathcal{B}([0, T], \Gamma^{(e)}) \).

The second assertion is clear from (A-4). Q.E.D.

**Lemma 7.2.** Let \( u_j(x, t) \) be the solution of (2.5) \( j \), then \( u_j(x, t) \in \mathcal{B}([0, T], \Gamma^{(e)}) \) for \( j \geq 0 \). Moreover there exists an integer \( s \geq 1 \) such that for \( j \geq 1 \), \( D_i^t u_j(x, t) |_{t=0} = 0 \), \( 0 \leq i \leq m - k - 1 + s(j-1) \).

**Proof.** It follows from the first assertion of Lemma 7.1 that \( u_0(x, t) \in \mathcal{B}([0, T], \Gamma^{(e)}) \). If we remember (2.2)~(2.4), then we find that

\[
-Q_i u_0(x, t) = t^{m-k} f_0(x, t)
\]

such that \( f_0(x, t) \in \mathcal{B}([0, T], \Gamma^{(e)}) \). Using Lemma 7.1 once more, we can get that \( u_0(x, t) \in \mathcal{B}([0, T], \Gamma^{(e)}) \). Therefore repeating these steps, we have \( u_j(x, t) \in \mathcal{B}([0, T], \Gamma^{(e)}) \).
\(B((0, T], \Gamma^{(\nu)})\) for \(j \geq 0\).

Let us consider the second assertion. From (2.5), \(D_t^j u_t(x, t)|_{t=0} = 0, 0 \leq i \leq m-k - 1\). Put
\[
\tilde{s} = \min_{|\alpha| + f \leq m-1, |\alpha| \neq 0} \{s(\alpha, j)\} \geq 1.
\]
Thus from (2.4) and the second assertion of Lemma 7.1, we obtain that
\(D_t^j u_t(x, t)|_{t=0} = 0, 0 \leq i \leq m-k - 1 + \tilde{s}\). Similarly we conclude the second assertion of Lemma 7.2.

Q.E.D.

From Lemma 7.2, for any fixed integer \(s \geq 1\), there exists \(N = N(s) \in \mathbb{N}\) such that for any \(j \geq N-1\), \(D_t^j u_t(x, t)|_{t=0} = 0, 0 \leq i \leq s-1\).

Therefore we may assume that for any \(r \geq 0\), there exist positive constants \(c\) and \(R\) such that
(7.2)
\[||A'Q_s u_{N-1}|| \leq c w_r(s, t, R).\]

Lemma 7.3. Under (7.2), if \(1 \leq \varepsilon < \rho/(\rho-1)\), there exist constants \(\tilde{A}, B, q > 0\) which are independent of \(r\) such that
(7.3)
\[||A' u_{N+r}|| \leq c \tilde{A} B^q n^{-q} w_r(s, t, 2^r R)\]
for \(n = 0, 1, 2, \ldots\).

Proof. From (7.2) and Lemma 5.3, we get that
\[||A' u_N|| \leq c A s^{-n} w_r(s, t, R).\]
It follows from Lemma 6.4 that
\[||A' Q_s u_N|| \leq c \tilde{A} \sum_{|\alpha| + f \leq m-1, |\alpha| \neq 0} K^{|\alpha|}(s, r) w_{r+|\alpha|}(s+\varepsilon, t, R).\]
If we use Lemma 5.3, we have that
\[||A' u_{N+1}|| \leq c A^2 \sum_{|\alpha| + f \leq m-1, |\alpha| \neq 0} (s+\varepsilon)^{-m} K^{|\alpha|}(s, r) w_{r+|\alpha|}(s+\varepsilon, t, R).\]
Applying Lemma 6.4 again, we obtain that
\[||A' Q_s u_{N+1}|| \leq c^2 \tilde{A} \sum_{|\alpha| + f \leq m-1, |\alpha| \neq 0} \sum_{|\alpha| \neq 0} K^{|\alpha|}(s, r) K^{|\alpha|}(s+\varepsilon, r+|\alpha|) \times w_{r+|\alpha|+1}(s+2\varepsilon, t, R).\]
Using Lemma 5.3 again, we get that
\[ ||A' u_{N+2}|| \leq e^2 c A^3 \sum \sum (s+2 \varepsilon)^{-m} K_i^m(s, r) K_{j_1}^{m_2} (s, r) \nu_{r+|\alpha_1|+|\alpha_2|}(s+2 \varepsilon, t, R) , \]

where \( K_i^m(s, r) = K_i^m(s+\varepsilon, r+|\alpha_1|) \).

Setting
\[ K_j^{m_2}(s, r) = K_j^m(s+(i-1)\varepsilon, r+|\alpha_1| + \cdots + |\alpha_{i-1}|) , \]

inductively we obtain that for any \( n \geq 0 \),
\[ ||A' u_{N+n}|| \leq c A(e A)^n \sum \sum K_i^m(s, r) K_{j_1}^{m_2}(s, r) \times \nu_{r+|\alpha_1|+\cdots+|\alpha_n|}(s+n \varepsilon, t, R) . \]

By the way,
\[ K_j^{m_2}(s, r) \cdots K_i^m(s, r) \]
\[ = \sum \cdots \sum s^{-\sigma_1(r+1)}^{k_1-\cdots-k_1 (r+|\alpha_1|)-b_1} \times (s+\varepsilon)^{-\sigma_2(r+|\alpha_1|+1)}^{k_2-\cdots-k_2 (r+|\alpha_1|+|\alpha_2|)-b_2} \times \cdots \times (s+(n-1)\varepsilon)^{-\sigma_n(r+1)+\cdots+|\alpha_n|}^{k_n-\cdots-k_n (r+|\alpha_1|+\cdots+|\alpha_n|)-b_n} , \]

where
\[ a_d \in \{ m-j_d-(|\alpha_d| - k_d)\rho_\theta(\alpha_d, j_d), m-j_d-h(\alpha_d, j_d)-1+k_d \} \]
and
\[ b_d^{k_d} \in \{ \kappa \rho_\theta(\alpha_d, j_d), (\kappa-1)\rho_\theta(\alpha_d, j_d), \kappa, \kappa-1, 0 \} . \]

We note the following.

(7.4) If \( a_d = m-j_d-(|\alpha_d|-k_d)\rho_\theta(\alpha_d, j_d) \), then \( b_d^k \), \( b_d^{k+1} \), \( b_d^{k+1} \) \( = \kappa \rho_\theta(\alpha_d, j_d) \).

(7.5) If \( a_d = m-j_d-h(\alpha_d, j_d)-1+k_d \), then \( b_d^k \), \( b_d^{k+1} \) \( = \kappa \rho_\theta(\alpha_d, j_d) \).

Let \( s \geq \varepsilon \) and \( a = \min \{ a_d \} \), and if we use Lemma A.4 in Appendix, then we have that
\[ s^{-\sigma_1 \cdots (s+(n-1)\varepsilon)^{-\sigma_n} \leq e^{-\sigma_1 \cdots (n \varepsilon)^{-\sigma_n}} \leq e^{-a_n} A_1 R_1^n (-e_1-\cdots-e_n) . \]

Let \( r = 0 \) and using Lemma A.4 again,
\[ (r+1)^{-k_1 \cdots (r+|\alpha_1|)^{-k_1} \cdots (r+|\alpha_1|+\cdots+|\alpha_n|+1)^{-k_1}} \]
\[ \times (r+|\alpha_1|+\cdots+|\alpha_n|)^{-b_1} \]
\[ \leq A_1 R_1^{|\alpha_1|+\cdots+|\alpha_n|} (b_1+b_2+\cdots+b_n) . \]
Further we estimate \( w_{r+\sum_{i=1}^{n} |\alpha_i|} (s+n \epsilon, t, R) \) as follows:

\[
R^{r+\sum_{i=1}^{n} |\alpha_i|} \lesssim R^R (m-1)^n,
\]

by Lemma A.5 in Appendix,

\[
(r+|\alpha_1|+\cdots+|\alpha_n|)! \lesssim 2^{(r+\sum_{i=1}^{n} |\alpha_i|)R} |(\alpha_1|+\cdots+|\alpha_n|)!^\epsilon
\]

\[
\lesssim 2^{2^{r+2(m-1)}\epsilon R} R^{2\eta (\sum_{i=1}^{n} |\alpha_i|)}
\]

and

\[
t^{1+\epsilon} \leq t^R \epsilon^n.
\]

Hence we find that

\[
\left| A^t u_{N+m} \right| \leq c A A_2^2 A_3 \left\{ c A R^2 R^m e^{-d R^m - T^m} \exp (M(m-1)T) \right\}^\epsilon w_{\epsilon}(s, t, 2^s R)
\]

\[
\times \sum \cdots \sum \eta^{(\sum_{i=1}^{n} |\alpha_i|+\sum_{j=1}^{n} |\alpha_j|)} \exp (-|\alpha_1|+\cdots+|\alpha_n|+|\epsilon^-1|\alpha_1^\epsilon+\cdots+|\epsilon^-1|\alpha_n^\epsilon)
\]

Let \( i \) be the number of \( \{m-j_d-\alpha_d|\alpha_d, j_d\} \) s in \{\alpha_d|1 \leq d \leq n\}. If we recall (7.4) and (7.5), then

\[
I = (a_1+\cdots+a_n)+(b_1+\cdots+b_n^\epsilon)
\]

\[
\equiv \{m-j_i-|\alpha_1| \rho_0(\alpha_1, j_1)\} + \cdots + \{m-j_i-|\alpha_i| \rho_0(\alpha_i, j_i)\}
\]

\[
+ \{m-j_{i+1}-h(\alpha_i, j_{i+1})-1+k_{i+1}\} + \cdots + \{m-j_n-h(\alpha_n, j_n)-1+k_n\}
\]

\[
+ (\epsilon-1) \rho_0(\alpha_i, j_i) k_i \exp \left\{ (|\alpha_1|+\cdots+|\alpha_n|+|\epsilon^-1|\alpha_1^\epsilon+\cdots+|\epsilon^-1|\alpha_n^\epsilon) \right\}
\]

Now recalling (6.2) and (6.4), then

\[
\{m-j-h(\alpha, j)-1+\epsilon h(\alpha, j)+\epsilon-|\alpha| \epsilon\} - \{m-j-|\alpha| \rho_0(\alpha, j)+|\alpha| \epsilon \rho_0(\alpha, j)-|\alpha| \epsilon\}
\]

\[
= (\epsilon-1) \left\{ h(\alpha, j)+1-|\alpha| \rho_0(\alpha, j) \right\}
\]

\[
= (\epsilon-1) \left\{ \rho(\alpha, j)+1-\nu(\alpha, j) \right\} \mu \quad \text{if} \quad \nu(\alpha, j) < \mu \epsilon \rho(\alpha, j)
\]

\[
\geq 0.
\]

Let us set
\[ \rho_0 = \max_{|\alpha| + j \leq m-1} \{ |m-j| - \rho_0(\alpha, j)|/(m-j| - |\alpha|) \}. \]

If \( 1 \leq \kappa < \rho_0/(\rho-1) \), then we find that
\[
I \geq \{ m-j-|\alpha_1| \rho_0(\alpha_1, j_1) + |\alpha_1| \kappa \rho_0(\alpha_1, j_1) - |\alpha_1| \} \]
\[+ \cdots + \{ m-l_n - |\alpha_n| \rho_0(\alpha_n, j_n) + |\alpha_n| \kappa \rho_0(\alpha_n, j_n) - |\alpha_n| \} \kappa \]
\[= (m-j_1-|\alpha_1|) [(m-j_1-|\alpha_1| \rho_0(\alpha_1, j_1))/(m-j_1-|\alpha_1|)] \]
\[- \{ (m-j_1-|\alpha_1| \rho_0(\alpha_1, j_1))/(m-j_1-|\alpha_1|) \} \kappa \]
\[\cdots + (m-j_n-|\alpha_n|) [(m-j_n-|\alpha_n| \rho_0(\alpha_n, j_n))/(m-j_n-|\alpha_n|)] \]
\[- \{ (m-j_n-|\alpha_n| \rho_0(\alpha_n, j_n))/(m-j_n-|\alpha_n|) \} \kappa \]
\[\geq n \{ \rho_0 - (\rho_0-1) \kappa \} > qn, \quad \text{where} \quad q > 0. \]

If we note that for fixed \( \kappa \) such that \( 1 \leq \kappa < \rho/(\rho-1) \), we can choose \( 0 < \theta \leq 1 \) such that \( 1 \leq \kappa < \rho_0/(\rho-1) \leq \rho/(\rho-1) \), then this completes the proof. Q.E.D.

**Corollary 7.1.** If \( 1 \leq \kappa < \rho/(\rho-1) \), the formal solution
\[ u(x, t) = \sum_{j=0}^{\infty} u_j(x, t) \]
converges in \( \mathcal{B}([0, T], \Gamma^{(\kappa)}) \).

**Proof.** If we devide \( u(x, t) \) as
\[ u(x, t) = \sum_{j=0}^{N-1} u_j(x, t) + \sum_{j=N}^{\infty} u_j(x, t), \]
then this corollary immeidiately follows from Lemma 7.2 and Lemma 7.3. Q.E.D.

Therefore we get the existence of solutions.

Next we shall show the uniqueness of solutions.

**Lemma 7.4.** If \( u(x, t) \in \mathcal{B}([0, T], \Gamma^{(\kappa)}) \) is a solution of the Cauchy problem:
\[
\begin{cases}
Lu(x, t) = 0 \\
D^i u(x, t)|_{t=0} = 0, \quad 0 \leq i \leq m-k-1,
\end{cases}
\]
where \( 1 \leq \kappa < \rho/(\rho-1) \), then \( u(x, t) \equiv 0 \).

**Proof.** We may assume that for sufficiently large \( s \), there exist constants \( c, R > 0 \) such that
\[ ||A'u|| \leq c w_c(s, t, R) \quad \text{for any} \quad r \geq 0. \]
therefore similar to the proof of Lemma 7.3, we can obtain that
Let $n \to \infty$, then we find that $u(x, t) \equiv 0$. Q.E.D.

Finally we shall prove assertion (2°).

**Lemma 7.5.** If supp $u'(x) \subset K$, $0 \leq i \leq m - k - 1$ and supp $f(x, t) \subset C_\delta(K)$ for compact set $K \subset \mathbb{R}^d$, then supp $u(x, t) \subset C_\delta(K)$, where $u(x, t) \in \mathcal{B}([0, T], F^{(\epsilon)})$ is a solution of (1.6).

**Proof.** From (2°) in Proposition 2.1 and (2.5), supp $u_0(x, t) \subset C_\delta(K)$. Next if we note how to make $Q_1$ and that $Q_1$ is a differential operator, then

$$-Q_1 u_0(x, t) = t^{m-k} f_1(x, t),$$

where $f_1(x, t) \in \mathcal{B}([0, T], F^{(\epsilon)})$ and supp $f_1(x, t) \subset C_\delta(K)$. Hence using (2°) in Proposition 2.1 again, supp $u_j(x, t) \subset C_\delta(K)$. Repeating these steps, we obtain that supp $u_j(x, t) \subset C_\delta(K)$ for any $j \geq 0$. Thus from the convergence of the formal solution, we find that supp $u(x, t) \subset C_\delta(K)$. Q.E.D.

This completes the proof of Theorem 2.1.

**Appendix**

Following Igari [3] and Uryu [17], we introduce a certain class of pseudo-differential operators.

**Definition A.1.** (1) For any $m \in \mathbb{R}$ and $x > 1$, we denote by $S^m(x)$ the set of functions $h(x, \xi) \in C^\infty(\mathbb{R}^d \times \mathbb{R}^n)$ satisfying the property that for any $\alpha, \beta$, there exist constants $c_\alpha$ and $R$ such that

$$|\partial_\xi^\alpha D_\xi^\beta h(x, \xi)| \leq c_\alpha R^{\beta |\xi|} |\xi|^{|\xi|^\alpha} |\xi|^{m-|\alpha|} \quad \text{for} \quad (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^n.$$

(2) For any $h(x, \xi) \in S^m(x)$, we shall define a semi-norm of $h(x, \xi)$ such that for any integer $\ell \geq 0$,

$$|h(x, \xi)|_\ell = \max_{|\alpha| + |\beta| \leq \ell} \sup_{|\xi| \leq R} |\partial_\xi^\alpha D_\xi^\beta h(x, \xi)| \langle \xi \rangle^{-m+|\xi|}.$$

Now we can define a pseudo-differential operator with a symbol $h(x, \xi) \in S^m(x)$ as follows:

$$H(x, D_x)u(x) = (2\pi)^{-\frac{d}{2}} \int \exp (ix \cdot \xi) h(x, \xi) u(\xi) d\xi.$$

**Lemma A.1.** (see Igari [3]). Let $h(x, \xi) \in S^m(x)$ and $r \geq 0$. Then
\[ 
\sigma(A' H) = \sum_{j=1}^{N} \frac{1}{|\alpha|} \partial_{x}^{*} \partial_{x}^{m} D_{x}^{*} h(x, \xi) + r_N(x, \xi), 
\]

where \( N = r^* + m \). And for any integer \( \ell \geq 0 \), there exist constants \( c_{\ell}, R > 0 \) such that

\[
|D_{x}^{*} h(x, \xi)\partial_{x}^{m} D_{x}^{*} h(x, \xi)| \leq c_{\ell} R |x|^{-m}(|x| - m)^{\ell}
\]

and

\[
|r_N(x, \xi)| \leq c_{\ell} R^\ell |x|^\ell.
\]

The following lemma is well-known.

Lemma A.2. For any \( h(x, \xi) \in S^0 \), there exist a constant \( c \) and non-negative integer \( \ell \) dependent only on dimension \( n \) such that

\[
\|H(x, D_x)u\| \leq c \|h(x, \xi)|_\ell\|u\|.
\]

Lemma A.3. (see Uryu [17] and Igari [3]). Under the assumptions of Lemma A.1, if we denote \( h_j(x, \xi) \) by

\[
h_j(x, \xi) = \sum_{|\alpha|=j} \frac{1}{|\alpha|} \partial_{x}^{*} \partial_{x}^{m} D_{x}^{*} h(x, \xi),
\]

then there exist \( \hat{c}, \hat{R} > 0 \) such that

\[
\|H_j(x, D_x)u\| \leq \hat{c} \hat{R}^j |x|^{-m}(|x| - m)^{\ell}\|A^{r^*+\ell} u\| \quad \text{for} \quad 1 \leq j \leq r^*,
\]

\[
\|H_j(x, D_x)u\| \leq \hat{c} \hat{R}^j |x|^{-m}(|x| - m)^{\ell}\|A^{r^*+\ell} u\| \quad \text{for} \quad r^* + 1 \leq j \leq N - 1,
\]

and

\[
\|R_N(x, D_x)u\| \leq \hat{c} \hat{R}^j |x|^{\ell}\|u\|.
\]

Lemma A.4. Let \( \{i_1, \ldots, i_n\} \) be a subset of non-negative numbers \( a_1, \ldots, a_m \), then there exist constants \( A_1, R_1 > 0 \) such that

\[
n^{i_1+\cdots+i_n} \leq A_1 R_1! i_1^{1}2^{i_2} \cdots n^{i_n}.
\]

Proof. Set \( S = n^{i_1+\cdots+i_n}/1^{i_1} \cdots n^{i_n} \). Then

\[
S = (n/1)^{i_1} \cdots (n/n)^{i_n}
\leq (n/1)^{a} \cdots (n/n)^{a}
= (n^a/n!)^a, \quad \text{where} \quad a = \max \{a_1, \ldots, a_n\}.
\]

Using Stirling's formula, we can get the desired inequality. Q.E.D.

Lemma A.5. Let \( \{i_1, \ldots, i_n\} \subset \{1, \ldots, m-1\} \), then there exist constants \( A_2, R_2 > 0 \) such that
Proof. By Stirling’s formula, there exists $R_3 > 0$ such that

$$ (i_1 + \cdots + i_d)! \leq R_3^{i_1 + \cdots + i_d}.$$

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