A Gauge Theory for the Kadomtsev-Petviashvili System

By

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Abstract

A Lagrangian formalism of scalar fields is considered and a new concept of "connection" is introduced. By this a gauge-theoretic understanding of the Sato theory on the K.-P. system is obtained. Our gauge group \( \mathcal{G} \) is the group consisting of pseudo-differential operators of non-positive orders with certain growth conditions. Then it can be concluded that the space \( \mathbb{R}^n \) of elements of \( \mathcal{G} \) giving solutions of the K.-P. system defines a flat \( \mathbb{R}^n \)-connection which we call the K.-P. connection. This connection can be regarded as a special gauge field.

Introduction

It is well known that various soliton equations can be obtained by using the theory of isospectral deformations of linear differential operators. A remarkable unification of soliton equations has been established by M. and Y. Sato [5] in terms of isospectral deformations of \( D=d/dx \) in the category of pseudo-differential operators. This unified system of equations is called the Kadomtsev-Petviashvili system (=K.-P. system). They discovered the surprising fact: The space of solutions of the K.-P. system makes the Grassmann manifold of infinite dimension and moreover, any solution of the K.-P. system can be reduced to that of a system of certain linear equations. Several attempts of understandings on the Sato theory and its generalizations have been presented. Some of them are the method of Riemann-Hilbert transforms [10], the method of group-decompositions [4], [7] and the field-theoretic method [1]. The co-adjoint orbit method for the K.-P. system is given by

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using groups of pseudo-differential operators [11]. Our attempt which we
present here is a new one, which we call a gauge-theoretic understanding.
Although the method in [11] is based on the notion of Hamiltonians rather
than connections, the result obtained there is in close relation to our discussion.

In this paper, we see that the K.-P. system can be understood in the view
point of Uchiyama's gauge theory [9]. We note that our gauge group is an
infinite dimensional Lie group. Hence our gauge theory for soliton equations
is contrasted with that of Yang-Mills equations and nonlinear Heisenberg
equation in dimensions of their gauge groups [3]. First, we consider the
Lagrangian action:

\[ \mathcal{L} = \int_{\mathbb{R}} \bar{\psi} D\psi dx \quad (D = d/dx) \]

for scalar fields \( \psi, \bar{\psi} \), i.e., wave functions on the real line \( \mathbb{R} \). We analyse the
symmetry of \( \mathcal{L} \) and obtain as the gauge group of the first kind a group consisting
of invertible pseudo-differential operators with constant coefficients of the form:

\[ \cdots + c_n D^n + \cdots + c_1 D + c_0 + c_{-1} D^{-1} + \cdots + c_{-n} D^{-n} + \cdots. \]

Secondly, we apply the Uchiyama's gauge theory to our Lagrangian formalism.
In this case, the gauge group of the second kind becomes a group consisting
of invertible pseudo-differential operators with function coefficients of the form:

\[ \cdots + u_n(x) D^n + \cdots + u_1(x) D + u_0(x) + u_{-1}(x) D^{-1} + \cdots + u_{-n}(x) D^{-n} + \cdots. \]

Then in order to obtain a new Lagrangian action which is invariant under
this group, a connection, i.e., gauge field, necessarily arises in our considera-
tion. It has a worth mentioning that pseudo-differential operators with negative
orders, extended from usual differential operators, may be introduced
as elements of the gauge group of the first or the second kind.

In Section 1, from a gauge group of pseudo-differential operators we
introduce a new concept of "connection". Here we have to pay attention
to the fact that our connection has been defined not only for a subgroup but
also for a special subset \( R \) of the gauge group, although \( R \) does not admit
a structure of subgroup. We prove that the decomposition law of pseudo-
differential operators into the parts of non-negative and negative orders gives
rise to the flat connection (Theorem 1). This is our first step to a gauge-
theoretic understanding on the K.-P. system. In Section 2, we shall treat the
Lagrangian action of scalar fields \( \psi, \bar{\psi} \) with infinitely many parameters
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For this Lagrangian action we consider the gauge groups \( \tilde{G}_0, \tilde{G} \) of the first and the second kind, and then \( \tilde{G} \)-connections. Then we can conclude that the space \( R^* \) of elements of \( \tilde{G} \) giving solutions of the K.-P. system defines the flat \( R^* \)-connection which we call the K.-P. connection (Theorem 2).

Our discussions show that the space of solutions of soliton equations determines a special gauge field. Hence, we may expect to extend our discussions to the Yang-Mills equation and nonlinear Heisenberg equation by a gauge-theoretic version of the Sato theory on the Minkowski space-time [3].

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§ 1. A Lagrangian Formalism and \( R \)-connections

We consider complex valued functions defined on the real line and a collection of pseudo-differential operators. A pseudo-differential operator is called an operator simply. Let \( \psi \) and \( \tilde{\psi} \) denote two functions. Here \( \tilde{\psi} \) may not be the complex conjugate of \( \psi \).

First, we deal with a Lagrangian action for \( \psi \) and \( \tilde{\psi} \) given by

\[
\mathcal{L}_t = \int_R \tilde{\psi} D\psi \, dx, \quad D = d/dx.
\]

For a function \( \psi \) and an operator \( \tilde{\psi} = \tilde{\psi} \cdot 1 \), identified with the function \( \tilde{\psi} \), we act an operator \( W \) on the pair as

\[
\psi \to \psi' = W\psi, \quad \tilde{\psi} \to \tilde{\psi}' = \tilde{\psi} W^{-1}.
\]

Under this action the function \( \tilde{\psi} \psi \) is invariant. We are interested in a set of invertible operators \( W \) which makes a group \( G_0 \) and preserves \( \tilde{\psi} D\psi \) invariant, equivalently satisfies \( WD = DW \). Choices of such groups are not unique. One of possible groups can be obtained by

\[
G_0 = \{ W \mid W = \sum_{n=0}^{\infty} c_n D^n \text{ with constant coefficients} \}.
\]

For an invertible operator \( W \) we put

\[
\psi_W = W\psi, \quad \tilde{\psi}_W = \tilde{\psi} W^{-1}.
\]
Proposition (1.5). The Lagrangian action

\[ \mathcal{L}_0 = \int_R \overbar{\psi}^W D\psi^W dx , \quad W \in G_0 \]

is invariant under the action of the group \( G_0 \).

Proof. We choose arbitrary elements \( W \) and \( W' \) of \( G_0 \) and set \( \phi \) by \( W = \phi W' \), namely, \( \phi = WW'^{-1} \). Since

\[ \psi^W = \phi \psi^{W'} , \quad \overbar{\psi}^W = \overbar{\psi}^{W'} \phi^{-1} , \]

we obtain

\[ \overbar{\psi}^W D\psi^W = \overbar{\psi}^{W'} \phi^{-1} D\psi^{W'} = \overbar{\psi}^{W'} D\psi^{W'} . \]

The group \( G_0 \) is called the gauge group of the first kind. Next we proceed to a group

\[ G = \{ W | W = \sum_{n=0}^{\infty} u_n(x) D^n \text{ with function coefficients} \} . \]

We call an element of \( G \) a formal pseudo-differential operator [5]. \( G \) is called the gauge group of the second kind. In order to obtain exact mathematical meanings, we have to restrict our considerations to special groups. For example, we may choose a group \( G \) consisting of elements \( W \) with the following condition: Every \( u_n(x) \) is analytic function and there exists an integer \( n_0 \) such that \( \text{ord } u_n(x) \geq n - n_0 \) for any sufficiently large \( n \) ([4], [7], [8]). For a complex valued analytic function \( u \) with the Taylor expansion

\[ u = c_n x^n + c_{n+1} x^{n+1} + \cdots \quad (c_n \neq 0) , \]

the order of \( u \) is defined by \( \text{ord } u = n \). We have to pay attention to the fact that the Lagrangian action \( \mathcal{L}_0 \) is not invariant under \( G \), because the commutator \( [D, W] = DW - WD \) does not vanish identically. Hence we note that the following equalities hold:

\[ [D, W] = \sum (Du_n(x)) D^n \quad \text{for } W = \sum u_n(x) D^n \]

and

\[ WDW^{-1} = -[D, W]W^{-1} + D \quad \text{for } W \in G . \]

The Uchiyama gauge theory [9] says that in order to get a new Lagrangian action which is invariant under the group of the second kind, a connection, i.e., a gauge field, has to be introduced. Then we can make the following definition:

Definition (1.11). Let \( G \) be a group of operators described in (1.8) and
let $R$ be a subset of $G$. A collection $\{\mathcal{Q}(W) \mid W \in R\}$ of operators is called an $R$-connection if

(1) there exists a pair $(G_1, \rho)$ constituted with an injective set-map $\rho : G_1 \rightarrow G$ of a group $G_1$ to $G$ such that $R = \rho(G_1)$ and

(2) $L_\rho(W) \equiv D - \mathcal{Q}(W)$ satisfies

\[ L_\rho(W) = \phi L_\rho(W') \phi^{-1} \quad \text{for} \quad W, W' \in R \quad \text{where} \quad W = \phi W'. \]

In particular, we call it a $G$-connection if in addition $\rho$ is a group-isomorphism.

The following are examples of $G$-connections:

Examples

(1) $\mathcal{Q}(W) = D$.

(2) $\mathcal{Q}(W) = [D, W]W^{-1}$, in this case

\[ L(W) \equiv L_\rho(W) = WDW^{-1}. \]

(3) Let $G'$ be a subgroup of $G$ and $\iota : G' \rightarrow G$ be the natural inclusion mapping. If $\mathcal{Q}(W)$ $(W \in G)$ is a $G$-connection, then $\mathcal{Q}(W)$ $(W \in G')$ becomes a $G'$-connection.

Immediately from (1.12) we see that if $\mathcal{Q}_1(W)$ and $\mathcal{Q}_2(W)$ are $R$-connections, then the relation

\[ (1.13) \quad \mathcal{Q}_1(W) - \mathcal{Q}_2(W) = \phi(\mathcal{Q}_1(W') - \mathcal{Q}_2(W'))\phi^{-1} \]

holds for $W, W' \in R$ where $W = \phi W'$. This fact and Example (2) show that operators $\hat{\mathcal{Q}}(W)$ given by

\[ (1.14) \quad \hat{\mathcal{Q}}(W) = W^{-1}([D, W]W^{-1} - \mathcal{Q}(W))W \quad \text{for} \quad W \in R \]

satisfy the condition $\hat{\mathcal{Q}}(W) = \hat{\mathcal{Q}}(W')$ for any pair of $W$ and $W'$ of $R$, namely $\hat{\mathcal{Q}}(W)$ does not depend on a choice of $W \in R$. Therefore, we may write as $\hat{\mathcal{Q}} = \hat{\mathcal{Q}}(W)$. We call $\hat{\mathcal{Q}}$ the connection form determined by $\mathcal{Q}(W')$. An $R$-connection is called to be flat if its connection form vanishes identically, namely $\mathcal{Q}(W) = [D, W]W^{-1}$.

By an application of Uchiyama theory to the Lagrangian action (1.6), we obtain

**Proposition (1.16).** Let $\mathcal{Q}(W)$ be a $G$-connection. The Lagrangian action

\[ (1.17) \quad \mathcal{L} = \int_R \bar{\psi}^W(D - \mathcal{Q}(W))\psi_W d\chi \quad W \in G \]

is invariant under the group $G$.

**Proof.** For arbitrary elements $W$ and $W'$ where $W = \phi W'$ in $G$ we have
which implies the invariance of $\mathcal{L}$ under $G$.

The following group is important for a study on the K.-P. system. We put

\begin{equation}
G_+ = \{ \sum_{n=0}^{\infty} v_n(x)D^{-n} \in G \mid v_0(x) = 1 \} .
\end{equation}

Further we make the following definition:

**Definition (1.19).**

\begin{equation}
\mathfrak{g} = \{ \sum_{n=-\infty}^{\infty} u_n(x)D^n \} ,
\end{equation}

\begin{align*}
\mathfrak{g}_+ &= \{ \sum_{n=0}^{\infty} u_n(x)D^n \} \quad \text{and} \quad \mathfrak{g}_- = \{ \sum_{n=1}^{\infty} u_n(x)D^{-n} \} .
\end{align*}

Then the following decomposition holds:

\begin{equation}
\mathfrak{g} = \mathfrak{g}_+ + \mathfrak{g}_- ,
\end{equation}

which implies that any element $S$ of $\mathfrak{g}$ has the decomposition: $S = (S)_+ + (S)_-$ for $(S)_+ \in \mathfrak{g}_+$ and $(S)_- \in \mathfrak{g}_-$. Then we can prove

**Theorem 1.** $\omega(W)$ ($W \in G_-$) is the flat $G_-$-connection if and only if

\begin{equation}
\omega(W) = -(L(W))_- \quad \text{for} \quad W \in G_- .
\end{equation}

**Proof.** For $W, W' \in G_-$, where $W = \phi W'$, it holds that

\begin{align*}
(L(W))_- &= (\phi L(W')\phi^{-1})_- = (\phi(L(W'))_+\phi^{-1})_- + (\phi(L(W'))_-\phi^{-1})_- \\
&= (\phi D\phi^{-1})_- + \phi(L(W'))_\phi^{-1} \\
&= (-[D, \phi]\phi^{-1} + D)_- + \phi(L(W'))_\phi^{-1} \quad \text{(by (1.10))} \\
&= -D + \phi D\phi^{-1} + \phi(L(W'))_\phi^{-1} ,
\end{align*}

which implies $D - \omega(W) = \phi(D - \omega(W'))\phi^{-1}$. Hence $\omega(W)$ is a $G_-$-connection. Comparing the non-positive orders of the both sides of (1.10), we obtain $\omega(W) = -(L(W))_- = [D, W]W^{-1}$, i.e., $\omega(W)$ is flat. Conversely, if $\omega(W)$ ($W \in G_-$) is the flat $G_-$-connection, then $\omega(W)$ reduces to $\omega(W) = [D, W]W^{-1} = -(L(W))_-$ by (1.10).

**§ 2. A Gauge Theory for the K.-P. System**

We consider a Lagrangian formalism for scalar fields, $\psi = \psi(x, t)$ and
\( \bar{\psi} = \overline{\psi}(x, t) \) defined on the real line \((x \in \mathbb{R})\) with infinitely many parameters

\[ t = (t_1, t_2, \cdots) , \]

and for some collections of operators including \( D = d/dx \) and \( D_n = \partial / \partial t_n \). The total differential operator with respect to the parameters is denoted by

\[
(2.1) \quad d = \sum_{n=1}^{\infty} D_n dt_n .
\]

The Lagrangian action which we treat here is given by

\[
(2.2) \quad \mathcal{L}(t) = \int_{\mathbb{R}} \overline{\psi}(x, t) d\psi(x, t) dx
\]

for functions \( \psi \) and \( \bar{\psi} \). We proceed to our discussions analogous to the one done in the previous section. We are interested in invertible operators \( W = W(x, t) \), considering together with the action law for \( \psi \) and \( \bar{\psi} \):

\[
(2.3) \quad \psi \rightarrow \psi^\prime = W \psi \quad (= \psi_w) , \quad \bar{\psi} \rightarrow \bar{\psi}^\prime = \bar{\psi} W^{-1} \quad (= \bar{\psi}^w) .
\]

Hence, the function \( \bar{\psi} \psi \) is invariant under this action.

First, we consider a group

\[
(2.4) \quad \tilde{G}_0 = \{ W \mid W = \sum_{n=-\infty}^{\infty} c_n(x) D^n \} .
\]

In this case, we observe that coefficients \( c_n(x) \) are constant with respect to \( t \). Immediately, from \( Wd = dW \) we have

**Proposition (2.5).** *The Lagrangian*

\[
(2.6) \quad \mathcal{L}_0 = \int_{\mathbb{R}} \bar{\psi}^w d\psi_w dx , \quad W \in \tilde{G}_0 ,
\]

possesses the symmetry of the group \( \tilde{G}_0 \).

Following the Uchiyama theory, next we deal with a group

\[
(2.7) \quad \tilde{G} = \{ W \mid W = \sum_{n=-\infty}^{\infty} u_n(x, t) D^n \text{ with the property (*)} \}
\]

\[ (*) \quad u_n(x, t) \quad (n = 0, \pm 1, \pm 2, \cdots) \] are analytic functions of \( x \) and \( t \) satisfying the following growth condition: There exists an integer \( n_0 \) such that \( \text{ord} u_n(x, t) n \geq n - n_0 \) for any sufficiently large \( n \)

(see [4], [7], [8]). The Lagrangian action (2.6) gives rise to a gauge group \( \tilde{G}_0 \) of the first kind and a gauge group \( \tilde{G} \) of the second kind respectively. \( \mathcal{L}_0 \) is not invariant under \( \tilde{G} \), since commutators
\[ [D_m, W] = \sum (D_m u_m(x, t)) D^n \quad (m = 1, 2, \cdots) \]

for \( W = \sum u_n(x, t) D^n \), do not vanish identically, i.e., \([d, W] \neq 0\). Hence we have to make

\[ D_n - \mathcal{Q}_n(W) = \phi(D_n - \mathcal{Q}_n(W'))\phi^{-1} \]

for \( W, W' \in \tilde{R} \), where \( W = \phi W' (\phi \in \tilde{G}) \). \( \mathcal{Q}_n(W) \) is called the partial connection of \( \mathcal{Q}(W) \).

We note that an \( \tilde{R} \)-multiconnection \( \mathcal{Q}(W) \) implies

\[ d - \mathcal{Q}(W) = \sum \phi(D_n - \mathcal{Q}_n(W'))\phi^{-1} = \phi(d - \mathcal{Q}(W'))\phi^{-1} \]

for \( W, W' \in \tilde{R} \) with \( W = \phi W' \).

By use of Uchiyama’s theory, we obtain

**Proposition (2.9).** Let \( \mathcal{Q}(W) \) be a \( \tilde{G} \)-connection. The Lagrangian

\[ \mathcal{L} = \int_{\tilde{R}} \tilde{g}^W (d - \mathcal{Q}(W))\psi_\psi \, dx \quad \text{for} \quad W \in \tilde{G} \]

is invariant under the group \( \tilde{G} \).

We set

\[ (2.10) \quad \tilde{G}_+ = \left\{ \sum_{n=0}^{\infty} u_n(x) D^n \in G \mid u_0 \equiv 0 \right\}, \quad \tilde{G}_- = \left\{ \sum_{n=0}^{\infty} u_n(x) D^{-n} \in G \mid u_0 \equiv 1 \right\}. \]

Corresponding to \( \tilde{G}, \tilde{G}_+ \) and \( \tilde{G}_- \), we consider the spaces of operators \( \tilde{g} = \left\{ \sum_{n=0}^{\infty} u_n(x) D^n \right\} \), and its complementary subspaces

\[ (2.11) \quad \tilde{g}_+ = \left\{ \sum_{n=0}^{\infty} u_n(x, t) D^n \right\}, \quad \tilde{g}_- = \left\{ \sum_{n=1}^{\infty} u_n(x, t) D^{-n} \right\}, \]

that is the direct sum \( \tilde{g} = \tilde{g}_+ \oplus \tilde{g}_- \). Hence, any element \( X \in \tilde{g} \) is written as \( X = (X)_+ + (X)_- \) for \( (X)_+ \in \tilde{g}_+ \) and \( (X)_- \in \tilde{g}_- \).

Here we recall the K.-P. system. The operator \( L = WD W^{-1} \) for \( W \in G_- \) derived from the flat connection implies that \( L^n = WD^n W^{-1} \) and its decomposition \( L^n = (L^n)_+ + (L^n)_- \). In this case, \( (L^n)_+ \) is the \( n \)-th order differential operator. The K.-P. system is a system of equations defined by

\[ (2.12) \quad \partial L / \partial t_n = [(L^n)_+, L] \quad (n=1, 2, \cdots). \]
When $W (\in \tilde{G}_+)$ is an element described in the solution $L = WDW^{-1}$ of the K.-P. system, we shall say that $W$ gives a solution of the K.-P. system. It is known ([1], [5], [6]) that an element $W$ of $\tilde{G}_-$ gives a solution of the K.-P. system if and only if $W$ satisfies

$$\frac{\partial W}{\partial t_n} + (L^n(W))_\cdot W = 0 \quad (n = 1, 2, \ldots).$$

The following theorem is our main result:

**Theorem 2.** Let $R^*$ be the space of all elements of $\tilde{G}_-$ each of which gives a solution of the K.-P. system. Then the set $\{Q_{K,P}(W) \mid W \in R^*\}$ defined by

$$Q_{K,P}(W) = \sum_n Q_n(W)dt_n, \quad Q_n(W) = -(L^n(W)).$$

becomes the flat $R^*$-connection (say, the K.-P. connection).

**Remark.** (1) The K.-P. connection is a direct generalization of the connection given in Theorem 1, when we identify $t_1$ with $x$ and set $t_n=0$ $(n=2, 3, \ldots)$. (2) The flatness of the K.-P. connection is well known as the Zakharov-Shabat equation.

For the proof of this theorem we need the following two lemmas:

**Lemma 1 (Mulase's decomposition theorem [4]).** The group $\tilde{G}$ described in (2.7) can be decomposed into

$$\tilde{G} = \tilde{G}_- \cdot \tilde{G}_+,$$

in a sense that any element $g \in \tilde{G}$ determines the unique pair of elements $g_1 \in \tilde{G}_-$ and $g_2 \in \tilde{G}_+$ such that $g = g_1 \cdot g_2$.

**Lemma 2 ([4], [6]).** There exists a one-to-one correspondence between the space $R^*$ and the space $Q$ of solutions $U$ of the initial value problem:

$$\frac{\partial U}{\partial t_n} = [D^n, U], \quad U_{\mid t=0} = U_0 \in G_-,$$

where $G_-$ is given in (1.18). The exact correspondence is described in the following manner: A solution $U$ of (2.15) determines an element $W$ of $\tilde{G}_-$ by the decomposition $U = W^{-1}V$ in Lemma 1. Then $L(W) = WDW^{-1}$ gives a solution of (2.12). Conversely, for a solution $W$ of (2.12), we can find a unique element $V$ of $\tilde{G}_+$ such that $V_{\mid t=0}$=identity and $U = W^{-1}V$ gives a solution of (2.15).

**The proof of Theorem 2.** Let $U_0$ be any element of $G_-$. $U_0$ determines a unique solution $U (\in \tilde{G}_-)$ of (2.15) by Lemma 2. $U$ can be decomposed uniquely as $U = W^{-1}V$ with $W \in \tilde{G}_-$ and $V \in \tilde{G}_+$ by Lemma 1. This gives rise
to a mapping $\rho: G_\rightarrow \tilde{\mathcal{G}}_-$ which maps $U_0$ to $W$. This mapping $\rho$ is injective ([4], [6]). Then we see that $R^\ast = \rho(G_-).$ Next we show that $\mathcal{Q}_{K,\rho}(W)$ becomes an $R^\ast$-connection. Let $W$ and $W'$ be elements of $R^\ast$ and set $\phi (\phi \in \tilde{\mathcal{G}}_-)$ by $W = \phi W'$. It follows from

$$\partial W/\partial t_n = (\partial \phi/\partial t_n)W' + \phi(\partial W'/\partial t_n)$$

and from (2.13) that

$$-(L^\ast(W))_- W = (\partial \phi/\partial t_n)W' - \phi(L^\ast(W'))_- W'.$$

Hence

$$\omega_n(W) = (\partial \phi/\partial t_n)\phi^{-1} + \phi \omega_n(W')\phi^{-1}$$

holds, which implies that $\omega_n(W) (W \in R^\ast)$ is a partial $R^\ast$-connection. Therefore, $\mathcal{Q}_{K,\rho}(W) (W \in R^\ast)$ is an $R^\ast$-connection. The flatness of the connection follows from (2.13):

$$0 = \sum_n (\partial W/\partial t_n + (L^\ast(W))_- W)dt_n = \sum_n (\partial W/\partial t_n - \omega_n(W)W)dt_n$$

$$= [d, W] - \mathcal{Q}_{K,\rho}(W) W.$$

References


