The Radon-Nikodym theorem for $L^p$-spaces of $W^*$-algebras

By

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Abstract


§1. Main Results

Sakai's Radon-Nikodym theorem [15] describes the facial structure in the positive cone of the predual $L^1(\mathcal{M})$ of a $W^*$-algebra $\mathcal{M}$. Haagerup [9] succeeded to construct the spaces $L^p(\mathcal{M})$, $1 \leq p \leq \infty$. They are obtained as certain subspaces of measurable operators affiliated with the crossed product of $\mathcal{M}$ and modular automorphisms. In what follows we shall obtain an analogue of Sakai's result for $L^p(\mathcal{M})$, $1 \leq p \leq \infty$, by use of a Radon-Nikodym lemma for measurable operators affiliated with a semifinite $W^*$-algebra. This result is in fact a direct consequence of an underlying facial principle for measurable operators. Takesaki [20] obtained the uniqueness of Sakai's positive Radon-Nikodym derivative and gave a characterization of non-necessarily positive derivatives. Connes [3] obtained a Radon-Nikodym theorem for weights in terms of analytic extensions of the associated unitary cocycle. We shall prove the corresponding versions for $L^p(\mathcal{M})$.

1.1. Haagerup's definition of $L^p$-spaces of $W^*$-algebras [9], [22]

Consider a $W^*$-algebra $\mathcal{M}$. Let $\varphi$ be a normal, faithful, semifinite weight
on \( \mathcal{M} \) and let \( \sigma_t, t \in \mathbb{R} \), be the corresponding group of modular automorphisms of \( \mathcal{M} \). We denote the crossed product [5, Part 1, Def. 2.10] of \( \mathcal{M} \) and \( \sigma_t \) by \( \mathcal{N} = \mathcal{M} \overline{\otimes}_t R \). \( \mathcal{M} \) has a normal, faithful representation \( \pi \) in \( \mathcal{N} \) [5, Part 1, Prop. 2.5] and \( \pi(\mathcal{M}) := L^\infty(\mathcal{M}) \) is characterized as the set of fixed points under the dual action \( \tilde{\sigma} \) [5, Part 1, Prop. 4.12]. \( \mathcal{N} \) is semifinite and has a n.f.s. trace \( \tau \), which is canonically associated to \( \varphi \) — see [8, Lemma 5.2]. If \( \mathcal{H} \) denotes the space of \( \tau \)-measurable operators (compare §2), then

\[
L^p(\mathcal{H}) := \{ T \in \mathcal{N} | \delta_t(T) = e^{-it\tau} T \}.
\]

We shall write \( J \) for the natural involution \( T \mapsto T^* \) in \( L^p(\mathcal{H}) \). If \( \varphi \in \mathcal{M}_\tau^* \), then \( h_\varphi \) is defined as the Radon-Nikodym derivative [14, Thm. 5.12] of the dual weight \( \tilde{\varphi} \) [6, Def. 3.1] of \( \varphi \) with respect to \( \tau \), i.e. \( \tilde{\varphi} = \tau(h_\varphi) \). It can be shown [22, Chapter II, Cor. 6], that \( h_\varphi \in L^1(\mathcal{H}) \) and that the map \( \varphi \mapsto h_\varphi \) extends to an order isomorphism of \( \mathcal{M}_\tau^* \) onto \( L^1(\mathcal{H}) \) [22, Chapter II, Thm. 7]. This gives rise to a linear functional \( tr \) on \( L^p(\mathcal{H}) \) defined by

\[
tr(h_\varphi) := \varphi(1), \quad \varphi \in \mathcal{M}_\tau^*.
\]

The duality between \( L^p(\mathcal{H}) \) and \( L^q(\mathcal{H}) \), \( 1 \leq p < \infty, \frac{1}{p} + \frac{1}{q} = 1 \), is defined as [22, Chapter II, Thm. 32]:

\[
\langle S, T \rangle_{p,q} = tr(ST) = tr(TS), \quad S \in L^p(\mathcal{H}), \quad T \in L^q(\mathcal{H}).
\]

1.2. Radon-Nikodym theorem for \( L^p \)-spaces

Suppose \( T, S \in L^p(\mathcal{H}), 1 \leq p \leq \infty, \) and \( 0 \leq T \leq S \). Then there exists a unique \( h \in \mathcal{M}, 0 \leq \pi(h) \leq \text{supp}(S) \) such that

\[
T = \pi(h) J \pi(h) J S.
\]

1.3. Separating vectors in \( L^p \)-spaces:

Suppose \( 1 \leq p \leq \infty \) and \( S \in L^p(\mathcal{H})^+ \), i.e. \( S \in L^p(\mathcal{H}) \) and \( \text{supp}(S) \geq 0 \). We call \( S \) separating, if \( x \in \mathcal{M}, \pi(x) S = 0 \Rightarrow x = 0 \). We call \( S \) cyclic, if \( \pi(\mathcal{M}) S \) is dense in \( L^p(\mathcal{H}) \). As one might expect, we have \( \text{supp}(S) = 1 \Leftrightarrow S \) is separating \( \Leftrightarrow S \) is cyclic. Hence there exist separating vectors in \( L^p(\mathcal{H})^+ \) if and only if \( \mathcal{M} \) is \( \sigma \)-finite.

Let \( \mathcal{M} \) be a \( \sigma \)-finite \( W^* \)-algebra and consider two separating vectors \( S, T \in L^1(\mathcal{H})^+ \) as well as the corresponding linear functionals \( \varphi := tr(\pi(\cdot)) S \), \( \omega = tr(\pi(\cdot) T) \in \mathcal{M}_\tau^* \). Haagerup [10, §3] has shown that

1.3.1. \[
\pi(\sigma_t^\varphi(x)) = S^{it} \pi(x) S^{-it}, \quad x \in \mathcal{M},
\]
1.3.2. \((D\omega : D\varphi)_t = \pi^{-1}(T^{it}S^{-it})\).

In fact (1.3.2) holds also, if \((D\omega : D\varphi)_t\) is defined as in [4, p. 479] or [1, p. 394] and \(T\) is not necessarily a separating vector.

1.4. Analytic extensions of modular automorphism groups

Suppose \(\varphi\) is a n.f.s. weight on \(\mathcal{M}\). Then we shall write \(\mathcal{A}^\varphi(s), s \in \mathbb{R}\), for the set of \(x \in \mathcal{M}\) such that the map \(t \mapsto \sigma_t^\varphi(x), t \in \mathbb{R}\), has a \(\sigma(\mathcal{M}, \mathcal{M}_\tau)\)-continuous extension \(z \mapsto \sigma_z^\varphi(x) \in \mathcal{M}\) to the strip

\[\Gamma_\tau = \{z \mid |\text{Im } z| \leq |s|, s \cdot \text{Im } z \geq 0\},\]

which is \(\sigma(\mathcal{M}, \mathcal{M}_\tau)\)-analytic in \(\Gamma_\tau^0\).

The following result has been obtained by Takesaki [20, Thm. 15.3] in the case \(p=1\).

**Theorem.** Suppose \(\mathcal{M}\) is a \(\sigma\)-finite \(W^*\)-algebra and \(1 \leq p < \infty\). Let \(S \in L^p(\mathcal{M})^*\) be a separating vector, \(\varphi(\cdot) = \text{tr}(\pi(\cdot)S^p) \in \mathcal{M}_\tau^+\) be the corresponding linear functional and \(x \in \mathcal{M}\). Then the following conditions are equivalent:

(a) \(\pi(x)J\pi(x)JS \leq S\)

(b) \(x \in \mathcal{A}^\varphi(1)_{2p}^\tau\) and \(\|\sigma_z^\varphi(x)\| \leq 1\).

1.5. Analytic extensions of Radon-Nikodym cocycles

Our final result has been obtained by Connes [3] not only in the case \(p=1\) but for n.f.s. weights.

**Theorem.** Suppose \(\mathcal{M}\) is a \(\sigma\)-finite \(W^*\)-algebra and \(1 \leq p < \infty\). Let \(S, T \in L^p(\mathcal{M})^+\), \(\text{supp}(S) = 1\) and consider the corresponding linear functionals \(\varphi := \text{tr}(\pi(\cdot)S^p), \omega := \text{tr}(\pi(\cdot)T^p) \in \mathcal{M}_\tau^+\). Then the following two conditions are equivalent:

(a) \(T \leq S\)

(b) The function \(t \mapsto (D\omega : D\varphi)_t, t \in \mathbb{R}\), has a \(\sigma(\mathcal{M}, \mathcal{M}_\tau)\)-continuous extension to \(\Gamma_{-1/2p}\), which is analytic in \(\Gamma_{0-1/2p}\) and

\[\|(D\omega : D\varphi)_{-1/2p}\| \leq 1\].

Furthermore if one of the above conditions holds, then

\(y := (D\omega : D\varphi)_{-1/2p}\)

satisfies
\[ T = \pi(y) J \pi(y) J S. \]

**Remark.** The theorem shows that the (operator) order relation in \( L^p(\mathcal{M}) \) \( T \leq S \) is exactly the order relation \( \varphi \leq \omega \left( \frac{1}{2p} \right) \) for the corresponding functionals in the sense of Connes-Takesaki [4, Ch. II, Def. 4.1].

§2. **Measurable Operators with Respect to a Trace**


Suppose \( \mathcal{H} \) is a semifinite \( \mathcal{W}^* \)-algebra acting on a Hilbert space \( H \). Let \( \tau \) be a n.f.s. trace on \( \mathcal{H} \). A subspace \( \mathcal{X} \) of \( H \) is called \( \tau \)-dense, if for \( \varepsilon \in \mathbb{R}_+^* \), there exists a projection \( e \in \mathcal{H} \) such that

\[ eH \subseteq \mathcal{X} \text{ and } \tau(1-e) < \varepsilon. \]

A closed, densely defined operator \( T \) affiliated with \( \mathcal{H} \) is called \( \tau \)-measurable, if its domain of definition \( \mathcal{D}(T) \) is \( \tau \)-dense. We denote the set of \( \tau \)-measurable operators by \( \mathcal{H} \). If \( T \) is a selfadjoint operator affiliated with \( \mathcal{H} \) and \( T = \int_{-\infty}^{\infty} t \, dE(t) \) is its spectral decomposition, then \( T \in \mathcal{H} \) if and only if for every \( \varepsilon \in \mathbb{R}_+^* \), there exists \( t > 0 \) such that \( \tau(1 - E([-t, t])) < \varepsilon \) [22, Chapter I, Prop. 21]. If \( T \in \mathcal{H} \), then \( (T | \mathcal{X})^* = T \) for every \( \tau \)-dense subspace \( \mathcal{X} \subseteq \mathcal{D}(T) \) [22, Chapter I, Prop. 12]. If \( S, T \in \mathcal{H} \), then \( S^* \in \mathcal{H} \) and \( S + T, S \cdot T \) are densely defined, preclosed and \( S + T, S \cdot T \in \mathcal{H} \). In what follows, we shall always consider the strong sum and product in \( \mathcal{H} \) and omit the closure sign. \( \mathcal{H} \) is a \( * \)-algebra with respect to strong sum, strong product and adjoint operation [22, Chapter I, Prop. 24].

2.2. **Lemma.** Suppose \( S, T \in \mathcal{H} \).

(a) If \( T \cdot S = 0 \), then \( \text{rightsupp}(T) \cdot \text{leftsupp}(S) = 0 \).
(b) If \( \text{leftsupp}(T) \vee \text{rightsupp}(T) \leq \text{leftsupp}(S) \) and \( S^*TS = 0 \), then \( T = 0 \).
(c) Suppose \( 0 \leq T, S \) and \( \text{supp}(S) = 1 \). Then the following two statements are equivalent:

i) \( T \leq S^* \)

ii) The operator \( T^{1/2}S^{-1} \) has domain of definition \( \mathcal{D}(S^{-1}) \) and norm less than 1.

(d) **Facial principle:** If \( 0 \leq T \leq S^*S \), then there exists a unique \( x \in \mathcal{H} \),
0 ≤ x ≤ \text{leftsupp} (S) such that

\[ T = S^* x S. \]

(c) If leftsupp (S) = 1 and \( S^* T S ≥ 0 \), then \( T ≥ 0 \).

**Proof.** (a) Using polar decomposition and spectral calculus one reduces (a) to the bounded case, which is known.

In the remainder of the proof we shall assume w.l.o.g. that \( S ≥ 0 \) and \( \text{supp}(S) = 1 \).

(b) Let \( S = \int_0^\infty t \, dE(t) \) be the spectral decomposition of \( S \). Then we have

\[ E\left( \left( \frac{1}{n}, \infty \right) \right) T S = 0 \Rightarrow T = 0. \]

(c) and (d) \( 0 < T ≤ S^2 \) implies \( \mathcal{D}(S) \subseteq \mathcal{D}(T^{1/2}) \).

For \( \xi \in \mathcal{D}(S) = S^{-1} \mathcal{D}(S^{-1}) \) we set

\[ y(S \xi) = T^{1/2} \xi. \]

It follows from the hypothesis that \( y = T^{1/2} S^{-1} \) has norm less than 1.

Conversely, if \( T^{1/2} S^{-1} \) has domain of definitions \( \mathcal{D}(S^{-1}) \) and norm less than 1, then let \( y \in \mathcal{N} \) be its closure. We have

\[ y S = T^{1/2} \text{ on } \mathcal{D}(S) \]

and \( ||y|| ≤ 1 \).

Since \( \mathcal{D}(S) \) is \( r \)-dense we obtain

\[ y S = T^{1/2}. \]

Hence \( T = S y^* y S = S x S ≤ S^2 \) where \( 0 ≤ x := y^* y ≤ 1 \).

(e) We obtain from (b) that \( T = T^* \) since

\[ S T S = \frac{1}{2} S (T + T^*) S. \]

Now \( 0 ≤ S T S ≤ S |T| S \) and (d) imply the existence of \( x \in \mathcal{N}_+ \) such that

\[ S T S = S |T|^{1/2} x |T|^{1/2} S \]

So \( T = |T|^{1/2} x |T|^{1/2} ≥ 0 \) by (b).

The following lemma generalizes a well known result for bounded operators. The main idea in the proof is due to Pedersen [12].

2.3. **Lemma.** Suppose \( S, T \in \mathcal{N} \). If \( 0 ≤ T ≤ S \) and \( 0 ≤ \alpha ≤ 1 \), then \( T^\alpha ≤ S^\alpha \).
Proof. Assume first that $T$ is bounded and $S$ has a bounded inverse. Then the proof of [20, Chapter 1, Prop. 6.3] applies and yields the desired result. In general let

$$S = \int_0^\infty t \, dE(t) \quad \text{and} \quad T = \int_0^\infty t \, dF(t)$$

be the spectral decomposition of $S$ and $T$ respectively. If $e_n = E([0, n]) \land F([0, n])$, then $\mathcal{X} = \bigcup_{n \in \mathbb{N}} e_n H$ is a $\tau$-dense subspace of $H$. Hence it is a core for $S^a - T^a$. If $\xi \in e_n H$, then

$$\langle T^a \xi, \xi \rangle = \langle T^a F([0, n]) \xi, \xi \rangle$$
$$= \langle (T F([0, n]))^a \xi, \xi \rangle$$
$$\leq \inf_{\varepsilon > 0} \langle (S + \varepsilon I)^a \xi, \xi \rangle$$
$$= \inf_{\varepsilon > 0} \langle (S + \varepsilon I)^a E([0, n]) \xi, \xi \rangle$$
$$= \langle S^a \xi, \xi \rangle.$$

This completes the proof. \(\square\)

The following result has been obtained by Pedersen and Takesaki [13] in the bounded case

2.4. Lemma. Suppose $S, T \in \mathcal{B}$, $S, T \geq 0$. If

(a) $(S^{1/2} T S^{1/2})^{1/2} = |T^{1/2} S^{1/2}| \leq S$ and supp$(T) \leq$ supp$(S)$

then there exists a unique $h \in \mathcal{B}$ such that

(b) $0 \leq h \leq$ supp$(S)$

and $T = h S h$.

Conversely (b) implies (a). Furthermore, if $T \leq S$, then (a) is satisfied thanks to Lemma 2.3.

Proof. If (a) holds, then Lemma 2.2 (d) implies that

$$(S^{1/2} T S^{1/2})^{1/2} = S^{1/2} h S^{1/2}$$

for an $h \in \mathcal{B}$ with $0 \leq h \leq$ supp$(S)$. Hence

$$S^{1/2} T S^{1/2} = S^{1/2} h S^{1/2}.$$

Using Lemma 2.2 (b) this implies (b). If (b) is satisfied, then

$$S^{1/2} T S^{1/2} = S^{1/2} h S^{1/2}.$$

The uniqueness of the square root shows that
\[(S^{1/2} T S^{1/2})^{1/2} = S^{1/2} h S^{1/2} \leq S.\]

This and Lemma 2.2 (d) prove that \(h\) is uniquely determined.

§3. Proof of theorems

Proof of 1.2. This is an immediate consequence of Lemma 2.4 if one observes, that the uniqueness-statement forces the delivered \(h\) to be a fixed point under the dual action \(\delta\).

Proof of 1.3. Using Lemma 2.2 (a) we show that
\[\{\pi(\mathcal{H}) S\} = \text{supp}(S)^{1/2} L(\mathcal{H}).\]

With this in mind the first statement in (1.3) is a direct consequence of the bipolar theorem and Lemma 2.2 (a). (1.3.1) and (1.3.2) for separating \(T\) follow from [6, Thm. 3.2], [14, Thm. 4.6] and [2, Lemma 1.2.3]:
\[
\pi(\sigma^*(x)) = \sigma^*(\pi(x)) = h_0^* \pi(x) h_0^{*it} = S^{it} \pi(x) S^{-it},
\]
\[
\pi((D\omega : D\psi)_t) = (D\tilde{\omega} : D\tilde{\psi})_t = (D\tilde{\omega} : D\tau)(D\tau : D\tilde{\psi})_t = T^{it} S^{-it}.
\]

If \(T\) is not separating, fix \(R \subseteq L(\mathcal{H})^+\) with \(\text{supp}(R) = \text{supp}(T)^{1/2}\), let \(\omega_1 \in \mathcal{H}^\star\) denote the functional corresponding to \(T + R\) and \(p = \pi^{-1}(\text{supp } T) = \text{supp}(\omega)\). Now we obtain
\[
\pi(D\omega : D\psi)_t = \pi((D\omega : D\omega_1)_t, (D\omega_1 : D\psi)_t) = \pi(p(D\omega_1 : D\psi)_t) = \pi(p) (T + R)^{it} S^{-it} = T^{it} S^{-it}.
\]

Proof of 1.4 Theorem. (a) \(\Rightarrow\) (b). If (a) holds, then
\[\sigma^*(x) S \sigma^*(x^\star) \leq S, \ t \in \mathbb{R}.\]

Reducing to a \(\tau\)-dense subspace of \(\mathcal{D}(S^{1/2}) \cap \mathcal{D}(|S^{1/2} \sigma^*(x^\star)|)\) as in the proof of Lemma 2.3 one shows
\[(3.1) \quad |\langle \xi, S^{1/2} \sigma^*(x^\star) \eta \rangle| \leq ||\xi|| \cdot ||S^{1/2} \eta||, \ \xi, \eta \in \mathcal{D}(S^{1/2}).\]

Now for \(\xi, \eta \in \mathcal{D}(S^{1/2})\) we define
\[f_t, \eta = \langle \pi(x) S^{-it} \xi, S^{1/2} S^{1/2} \eta \rangle\]
\(f_t, \eta\) is continuous and bounded in \(\Gamma_{1/2}\) and is analytic in \(\Gamma_{1/2}^{0}\).

(3.1) and the three lines principle show that this defines a norm bounded, ope-
rator valued function \( z \mapsto \sigma_z(x) \), \( z \in \Gamma_{1/2p} \), which satisfies
\[
\langle \sigma_z(x) \xi, S^{1/2} \eta \rangle = f_{\xi, \eta}(z), \quad \xi, \eta \in \mathcal{D}(S^{1/2}).
\]

\[ ||\sigma_{1/2p}(x)|| \leq 1. \]

Since this construction is invariant under unitary operators in \( \mathcal{H} \) and under the dual action \( \delta \), we conclude \( \sigma_z(x) \in \pi(\mathcal{H}) \). If \( \xi, \eta \) are arbitrary vectors in \( \mathcal{H} \), then there exist sequences \( \xi_n, \eta_n \in \mathcal{D}(S^{1/2}) \) with \( \xi = \lim_{n \to \infty} \xi_n, \eta = \lim_{n \to \infty} S^{1/2} \eta_n \). Then
\[
\langle \sigma_z(x) \xi, \eta \rangle = \lim_{n \to \infty} f_{\xi_n, \eta_n}(z)
\]
uniformly on \( \Gamma_{1/2p} \). Hence \( z \mapsto \sigma_z(x) \) is continuous on \( \Gamma_{1/2p} \) and analytic in \( \Gamma_{1/2p}^0 \) for the weak operator topology. Using the lemma, in [19, Sec. 9.24] as well as the equivalence between \( \sigma(\mathcal{H}, \mathcal{H}) \) and the weak operator topology on bounded sets we obtain (b).

(b) \( \implies \) (a). First we claim
\[
(3.2) \quad \pi(x) S^{1/2} = S^{1/2}(\sigma_{1/2p}^0(x)) \quad \text{for} \quad x \in \mathcal{H}^0(\frac{1}{2p}).
\]

In order to obtain this we consider for \( \xi, \eta \in \mathcal{D}(S^{1/2}) \) the following two functions on \( \Gamma_{1/2p}^0 \):
\[
z \mapsto \langle \pi(x) S^{-i\xi} \xi, S^{1/2-i\eta} \eta \rangle
\]
\[
z \mapsto \langle \pi(\sigma_z^0(x)) \xi, S^{1/2} \eta \rangle.
\]

These functions are continuous \( \Gamma_{1/2p}^0 \) and analytic in \( \Gamma_{1/2p}^0 \). So they are equal, since they are equal on the real line by (1.3.1). This yields (3.2). (a) follows now from (b) and (3.2).

\[ \square \]

**Proof of 1.6 Theorem.** If (a) holds, then by Lemma 2.2 (c) we know that the operator
\[
(T \mp \text{supp}(T)^{-1/2}) \text{supp}(T) S^{-1/2}
\]
has domain of definition \( \mathcal{D}(S^{-1/2}) \) and norm less than 1. Hence by (i) \( \implies \) (ii) of the proposition in [19, Sec. 9.24] and 1.3.2 we obtain that the map
\[
f(z) = (T^{i} S^{-i})^{-} = ((T \pm \text{supp}(T)^{-1/2})^{-}) \text{supp}(T) S^{-i})^{-}
\]
is a well defined \( \pi(\mathcal{H}) \)-valued extension of
\[
t \mapsto \pi((D\omega; D\psi)_0)\]
which is continuous with respect to the strong operator topology on \( F_{-1/2p} \) and analytic in \( F_{0,1/2p} \). The three lines principle then shows that \( f \) is uniformly bounded by 1. So it is even \( s(\mathcal{M}, \mathcal{M}_\sigma) \)-continuous and we obtain (b).

If (b) holds, then by 1.3.2 and (iii) \( \Rightarrow \) (ii) of the proposition in [19, Sec. 9.24] we know that \( \mathcal{D}(T^{1/2S^{-1/2}}) = \mathcal{D}(S^{-1/2}) \). The uniqueness of analytic extensions shows that

\[
\pi((D\omega : D\varphi)_{-1/2}) = (T^{1/2S^{-1/2}}).
\]

Hence we can apply Lemma 2.2 (c) again to conclude (a). The remainder of Theorem 1.6 is clear by the proof of Lemma 2.2 (c).

3.3. Remark. By the way we have obtained the following:
Suppose \( \mathcal{M} \) is a \( \sigma \)-finite \( W^* \)-algebra, \( x \in \mathcal{M}_\sigma^+ \) and \( \varphi \in \mathcal{M}_\varphi^+ \) is faithful. Then for \( s \geq 0 \) there exists a unique \( y \in \mathcal{A}^\sigma(s) \) such that

\[
y \geq 0
\]

and

\[
x = \sigma^\phi_{-s}(y) \sigma^\phi_{-s}(y)^* = \sigma^\phi_{-s}(y) \sigma^\phi_{-s}(y).
\]

3.4. Remark. The facial principle has further applications:
For \( \sigma \)-finite \( W^* \)-algebras it allows to define completely positive, dense embeddings of \( \mathcal{M} \) into \( L^p(\mathcal{M}) \) and of \( L^p(\mathcal{M}) \) into \( L^q(\mathcal{M}) \). As in [17, Thm. 3.1], [18] this allows to transport semi-discreteness and injectivity from \( \mathcal{M} \) to \( L^p(\mathcal{M}) \) and vice versa.

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References


