Domains of Holomorphy in Segre Cones

By

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Introduction

Let $X$ be a normal Stein space and $D$ a domain (open set) in $X$. If $D$ is Stein, then it is a domain of holomorphy. The converse is valid when $X$ is a manifold (Docquier-Grauert [2]). However, this is not the case in general, as was pointed out by Grauert-Remmert [6], [7]. They gave an example of a non-Stein domain of holomorphy in a Stein space (Segre cone).

This problem is naturally related with the Levi problem, which asks whether a domain in $X$ is Stein if it is locally Stein (at all boundary points). Concerning some results on the Levi problem for Stein spaces, see Andreotti-Narasimhan [1], Fornaess-Narasimhan [4], and Fornaess [3] particularly for Segre cones.

A domain of holomorphy is locally Stein at the boundary points which are non-singular points of $X$. So we pose the problem: Suppose that $D$ is locally Stein at the boundary points which are non-singular points of $X$. Under what additional condition is $D$ a domain of holomorphy, or a Stein domain?

In the present note we will give an answer to this problem for the case where $X$ is a Segre cone. The method used here is the same as the one in the previous note of the author [11], i.e., to go over to a domain in an affine space and to apply Oka’s theorem.

§ 1. Segre Cones

Let $r$, $s$ be integers $\geq 1$. We identify the complex affine space $\mathbb{C}^{(r+1)(s+1)}$ with the set of all $(r+1, s+1)$ matrices $z = (z_{ij})$, $i=0, 1, \ldots$, $j=0, 1, \ldots,$
The Segre cone $Z = \mathbb{C}^{(r+1)(s+1)}$ is the algebraic set

$$Z := \{ z \in \mathbb{C}^{(r+1)(s+1)} | \text{rank } z \leq 1 \},$$

which is naturally regarded as a normal and irreducible Stein space of dimension $r + s + 1$. The origin $0$ is the only singular point of $Z$.

(See Grauert-Remmert [8, Chap. 7, §5].)

We will describe three kinds of desingularizations of $Z$. Let $P^r$, $P^s$ be the projective spaces with homogeneous coordinate systems $[\xi] = [\xi_0 : \xi_1 : \cdots : \xi_s]$, $[\eta] = [\eta_0 : \eta_1 : \cdots : \eta_s]$ respectively. We set

$$Z_0 := \{ (z, [\xi]) \in Z \times P^r \times P^s | \text{there is a constant } \lambda \in \mathbb{C} \text{ such that } z_{ij} = \lambda \xi_i \eta_j \text{ for all } i, j \},$$

$$Z_1 := \{ (z, [\xi]) \in Z \times P^r | \text{there are constants } \nu_0, \nu_1, \cdots, \nu_s \in \mathbb{C} \text{ such that } z_{ij} = \xi_i \nu_j \text{ for all } i, j \},$$

$$Z_2 := \{ (z, [\eta]) \in Z \times P^s | \text{there are constants } \mu_0, \mu_1, \cdots, \mu_s \in \mathbb{C} \text{ such that } z_{ij} = \mu_i \eta_j \text{ for all } i, j \},$$

The projection $Z_0 \rightarrow P^r \times P^s$ defines a holomorphic line bundle over $P^r \times P^s$, whose zero section $O_0$ is canonically identified with $P^r \times P^s$. Let $\sigma_0$ denote the projection $Z_0 \rightarrow Z$. Then $\sigma_0^{-1}(0) = O_0$, and $\sigma_0 | Z_0 \setminus O_0$ is a biholomorphic map of $Z \setminus O_0$ onto $Z \setminus \{0\}$. Thus $Z$ is obtained from $Z_0$ by contracting $O_0$. The fibers of the line bundle $Z_0 \rightarrow P^r \times P^s$ correspond by $\sigma_0$ to lines in $\mathbb{C}^{(r+1)(s+1)}$ which pass through $0$.

The projection $Z_1 \rightarrow P^r$ defines a holomorphic vector bundle of rank $s + 1$ over $P^r$, whose zero section $O_1$ is canonically identified with $P^r$. Let $\sigma_1$ denote the projection $Z_1 \rightarrow Z$. Then $\sigma_1^{-1}(0) = O_1$, and $\sigma_1 | Z_1 \setminus O_1$ is a biholomorphic map of $Z_1 \setminus O_1$ onto $Z \setminus \{0\}$. Thus $Z$ is obtained from $Z_1$ by contracting $O_1$. The fibers of the vector bundle $Z_1 \rightarrow P^r$ correspond by $\sigma_1$ to vector subspaces of dimension $s + 1$ in $\mathbb{C}^{(r+1)(s+1)}$ which pass through $0$.

Now we define a holomorphic mapping $\tau_1 : Z_0 \rightarrow Z_1$ by $(z, [\xi], [\eta]) \mapsto (z, [\xi])$. $O_0$ is mapped by $\tau_1$ onto $O_1$ correspondingly to the projection $P^r \times P^s \rightarrow P^r$. The restriction $\tau_1 | Z_0 \setminus O_0$ is a biholomorphic map of $Z_0 \setminus O_0$ onto $Z_1 \setminus O_1$. We have obviously $\sigma_1 \circ \tau_1 = \sigma_0$.

The above observations for $Z_1$ are applied analogously to $Z_0$. We obtain the commutative diagram

$$
\begin{array}{ccc}
Z_0 & \xrightarrow{\tau_1} & Z_1 \\
\downarrow \sigma_2 & & \downarrow \sigma_1 \\
Z_2 & \xrightarrow{\sigma_2} & Z
\end{array}
$$
with \( \sigma_0 = \sigma_1 \circ \tau_1 = \sigma_2 \circ \tau_2 \).

Now we will represent the Segre cone as the quotient of an affine space. Let \( A := C^{(r+1)+(s+1)} \) be the affine space with the coordinates \((x, y) = (x_0, x_1, \ldots, x_r; y_0, y_1, \ldots, y_s)\). We set
\[
L_1 := \{ x = 0 \}, \quad L_2 := \{ y = 0 \}, \quad L_0 := L_1 \cup L_2, \\
A_1 := \overline{A} \setminus L_1, \quad A_2 := \overline{A} \setminus L_2, \quad A_0 := \overline{A} \setminus L_0.
\]
We define the holomorphic mapping \( \rho : A \to C^{(r+1)+(s+1)} \) by \((x, y) \to (x_i, y_j)\).

We have \( \rho(A) = Z \) and \( \rho^{-1}(0) = L_0 \). For the points \( z \in Z \setminus \{0\} \), the fibers are biholomorphic to \( C^* := C \setminus \{0\} \). Hence \( Z \setminus \{0\} \) is the quotient space of \( A_0 \) by the action \( A_0 \times C^* \to A_0 \), \((x, y), c) \to (cx, c^{-1}y)\). In other words, \( \rho|_{A_0} : A_0 \to Z \setminus \{0\} \) is regarded as a holomorphic principal \( C^* \)-bundle.

We define a holomorphic mapping \( \rho_1 : A_1 \to Z_1 \) by \((x, y) \to ((x_i, y_j), [x])\). \( \rho_1 \) is surjective and \( Z_1 \) is the quotient space of \( A_1 \) by the action \( A_1 \times C^* \to A_1 \), \((x, y), c) \to (cx, c^{-1}y)\). Thus \( \rho_1 \) defines a holomorphic principal \( C^* \)-bundle over \( Z_1 \). We have \( \sigma_1 \circ \rho_1 = \rho|_{A_1} \) and the pull-back of the bundle \( A_0 \to Z \setminus \{0\} \) by \( \sigma_1 \) coincides with the restriction of the bundle \( A_1 \to Z_1 \) to \( Z_1 \setminus O_1 \).

Analogously we define a holomorphic mapping \( \rho_2 : A_2 \to Z_2 \) by \((x, y) \to ((x_i, y_j), [y])\). The commutative diagram can be augmented in the following way:

\[
\begin{array}{ccc}
A_1 & \xrightarrow{\rho_1} & Z_1 \\
| & \downarrow{\rho} & | \\
A_2 & \xrightarrow{\rho_2} & Z_0 \\
\end{array}
\]

\[
\begin{array}{ccc}
A & \xrightarrow{\rho} & Z \\
| & \downarrow{\rho} & | \\
A & \xrightarrow{\rho} & Z_2 \\
\end{array}
\]

where \( A_1 \to A, A_2 \to A \) are inclusion mappings.

**Remark.** The map \( \rho \) is used also by Fornaess [3].

§ 2. Boundary Points

Let us recall some properties of boundary points of domains in complex spaces. Let \( E \) be a domain (open set) in a complex space
$X$, and let $\partial E$ denote the set of all boundary points of $E$ in $X$. The domain $E$ is said to be locally Stein at $q \in \partial E$ if there is a neighborhood $U$ of $q$ such that $U \cap E$ is a Stein space. When $E$ is locally Stein at a non-singular point $q$ of $X$, we say that $E$ is pseudoconvex at $q$.

Now let $S$ be an analytic set of positive codimension in $X$. A point $q \in \partial E \cap S$ is said to be removable along $S$ if there is a neighborhood $U$ of $q$ such that $U \setminus S \subset E$. We denote by $R$ the set of the boundary points of $E$ that are removable along $S$. We set $E^* := E \cup R$. Then $E^*$ is a domain in $X$, which will be called the extension of $E$ along $S$. The following lemma is essentially due to Grauert-Remmert [6].

**Lemma.** (Ueda [11]) Let $E$ be a domain in a complex manifold $X$ and $S$ be an analytic set of positive codimension in $X$. Suppose that $E$ is pseudoconvex at every point $q \in \partial E \setminus S$. (1) If there is no boundary point removable along $S$, then $E$ is pseudoconvex (at every boundary point.) (2) The extension $E^*$ of $E$ along $S$ is pseudoconvex.

**Remark.** When $X$ is a complex space, this lemma (with the word “pseudoconvex” replaced by “locally Stein”) is false in general, as is shown by the example in [6].

### § 3. Domains in a Segre Cone

Let $D$ be a domain in a Segre cone $Z$. For $k = 0, 1, 2$, we set $D_k := \sigma_k^{-1}(D)$. $D_k$ is a domain in $Z_k$. Since $\sigma_k|Z_k \setminus O_k$ is a biholomorphic map of $Z_k \setminus O_k$ onto $Z \setminus \{0\}$, the domain $D_k$ is biholomorphic to $D$ if $0 \not\in D$. Let $R_k$ be the set of all boundary points of $D_k$ that are removable along $O_k$, and $D_k^* := D_k \cup R_k$ be the extension of $D_k$ along $O_k$.

Assume that $D$ satisfies the condition

\[ (*) \quad D \text{ is pseudoconvex at every boundary point in } Z \setminus \{0\}. \]

Then $D_k$ is pseudoconvex at every boundary point in $Z_k \setminus O_k$. By the lemma, $D_k^*$ is pseudoconvex at every boundary point. Hence the set $R_k = D_k^* \cap O_k$ is empty or pseudoconvex, considered as a domain (not necessarily connected) in $O_k$.

The sets $O_0, O_1, O_2$ are naturally identified with $\mathbb{P}^r \times \mathbb{P}^s, \mathbb{P}^r, \mathbb{P}^s$. 

and the maps \( \tau_1|_{O_0}: O_0 \rightarrow O_1, \tau_2|_{O_0}: O_0 \rightarrow O_2 \) with the projections \( P^r \times P^s \rightarrow P^r, \ P^r \times P^s \rightarrow P^s \). The following proposition is an immediate consequence of a theorem of Fujita [5] on the Levi problem for the product of projective spaces.

**Proposition 1.** If \( D \) satisfies the condition (*), then one of the following four cases occurs:

(i) \( R^\circ_0 \) is empty or a Stein domain in \( O_0 \).

(ii) \( R^\circ_0 = \tau_1^{-1}(E_1) \cong E_1 \times P^s \), where \( E_1 \) is a Stein domain in \( O_1 \).

(iii) \( R^\circ_0 = \tau_2^{-1}(E_2) \cong P^r \times E_2 \), where \( E_2 \) is a Stein domain in \( O_2 \).

(iv) \( R^\circ_0 = O_0 \cong P^r \times P^s \).

Now we observe how \( R^\circ_0 \) is related with \( R^\circ_1, R^\circ_2 \).

Suppose that \( R_1 \neq \emptyset \), and take a point \( q \in R_1 \). We will show that \( \tau_1^{-1}(q) \subset R^\circ_0 \). Let \( U \) be a neighborhood of \( q \) such that \( U \setminus O_1 \subset D_1 \). Then

\[
\tau_1^{-1}(U) \setminus O_0 = \tau_1^{-1}(U \setminus O_1) \subset \tau_1^{-1}(D_1) = D_0.
\]

Here, \( \tau_1^{-1}(U) \) is an open set containing \( \tau_1^{-1}(q) \). Therefore all points in \( \tau_1^{-1}(q) \) are removable boundary points of \( D_0 \) along \( O_0 \), i.e., \( \tau_1^{-1}(q) \subset R^\circ_0 \).

Conversely, suppose that \( R_0 \) contains a set of the form \( \tau_1^{-1}(q) \), \( q \in O_1 \). We will show that \( q \in R_1 \). Since \( \tau_1^{-1}(q) \) is compact, we can take an open set \( V \) in \( Z_0 \) containing \( \tau_1^{-1}(q) \) such that \( V \setminus O_0 \subset D_0 \). Since \( \tau_1 \) is a proper map, we can take a neighborhood \( U \) of \( q \) in \( Z_1 \) such that \( \tau_1^{-1}(U) \subset V \). Then

\[
U \setminus O_1 \subset \tau_1(V \setminus O_0) \subset \tau_1(D_0) = D_1.
\]

Therefore \( q \in R_1 \).

These observations are valid also for \( R_2 \).

Combining these with Proposition 1, we obtain

**Proposition 2.** Suppose that \( D \) satisfies the condition (*). We have \( R_1 = \emptyset, R_2 = \emptyset \) in the case (i) of Proposition 1; \( R_1 = E_1, R_2 = \emptyset \) in the case (ii); \( R_1 = \emptyset, R_2 = E_2 \) in the case (iii); and \( R_1 = O_1, R_2 = O_2 \) in the case (iv).

Now we can state the main result.

**Theorem.** Let \( D \) be a domain in the Segre cone \( Z \) satisfying the
condition (*). In the case (i) of Proposition 1, $D$ is Stein. In the case (ii), $D^*_s$ is Stein, and $D$ is a non-Stein domain of holomorphy in $Z$. In the case (iii), $D^*_s$ is Stein and $D$ is a non-Stein domain of holomorphy. In the case (iv), 0 is an isolated boundary point of $D$, $D$ is not a domain of holomorphy and $D \cup \{0\}$ is a Stein domain.

The form of $R_0$ depends only on the form of $D$ in the vicinity of 0. Hence we have

**Corollary 1.** A domain in the Segre cone is Stein if it is locally Stein at every boundary point.

This is also an immediate consequence of a result of Andreotti-Narasimhan [1, Corollary 1 to Theorem 4].

In the case (ii), $D$ is biholomorphic to $D_1 = D^*_s \setminus R_1$, where $R_1$ is an analytic set of codimension $s+1$ in $D^*_s$. The situation is similar in the case (iii). Hence we have

**Corollary 2.** If $D$ is a domain in the Segre cone satisfying the condition (*), the set of all holomorphic functions on $D$ constitutes a Stein algebra.

### § 4. Proof of Theorem

Consider the domain $\bar{D} := \rho^{-1}(D)$ in $A$. We notice that $\bar{D} \subseteq A_0$ if $0 \notin D$, and $L_0 \subseteq \bar{D}$ if $0 \in D$. Further we have $\rho_k^{-1}(D_k) = \bar{D} \cap A_k$, $k = 1, 2$. Let $\bar{R}$ be the set of all boundary points of $\bar{D}$ that are removable along $L_0$, and let $\bar{D}^* = \bar{D} \cup \bar{R}$ be the extension of $\bar{D}$ along $L_0$. The set $\bar{R}$ is related with $R_1$, $R_2$ in the following way.

**Proposition 3.** We have $\rho_k^{-1}(R_k) = \bar{R} \cap A_k$ and $\rho_k^{-1}(D^*_k) = \bar{D}^* \cap A_k$, $k = 1, 2$.

**Proof.** First we remark that, if $0 \in D$, then the sets $\bar{R}$, $R_1$, $R_2$ are empty and the proposition is trivially true.

Suppose that $q \in R_1$. We choose a neighborhood $U$ of $q$ in $Z_1$ such that $U \setminus O_1 \subseteq D_1$. Then $\rho_1^{-1}(U)$ is an open set in $A_1$ such that $\rho_1^{-1}(U) \setminus L_2 = \rho_1^{-1}(U \setminus O_1) \subseteq \rho_1^{-1}(D_1) = \bar{D}$. 

This shows that all points in \( \rho_1^{-1}(q) \) are removable along \( L_2 \). Hence \( \rho_1^{-1}(R_i) \subset \bar{R} \cap A_i \).

Conversely suppose that \((x, 0) \in \bar{R} \cap A_i \). We choose a neighborhood \( W \) of \((x, 0)\) such that \( W \setminus L_2 \subset \bar{D} \). By shrinking \( W \), we can suppose that \( W \subset A_i \) and hence \( W \setminus L_2 \subset \bar{D} \). Then we have

\[
\rho_1(W) \setminus O_i = \rho_1(W \setminus L_2) \subset \rho_1(\bar{D}) = D_1.
\]

Here \( \rho_1(W) \) is an open set in \( Z_i \), since the projection \( \rho_1 \) of the vector bundle \( A_i \rightarrow \mathcal{Z}_i \) is an open map. Thus \( \rho_1(x, 0) = (0, [x]) \) is a removable boundary point of \( D_1 \) along \( O_i \). Hence \( \rho_1^{-1}(R_i) \supseteq \bar{R} \cap A_i \).

Thus we have shown \( \rho_1^{-1}(R_i) = \bar{R} \cap A_i \). From this follows immediately that \( \rho_1^{-1}(D_1^\ast) = \bar{D}^\ast \cap A_i \). The proof for \( k=2 \) is similar.

q. e. d.

**Proposition 4.** Suppose that \( D \) satisfies the condition 
\((*)\). Then \( \bar{D}^\ast \) is Stein. \( \bar{D} \) is Stein if and only if \( \bar{R} = \phi \).

**Proof.** The domain \( \bar{D} \) is pseudoconvex at every boundary point in \( A_0 = A \setminus L_0 \). Hence, by Lemma, \( \bar{D}^\ast \) is pseudoconvex at every boundary point. \( \bar{D}^\ast \) is Stein by Oka’s theorem. We have \( \bar{D} = \bar{D}^\ast \setminus \bar{R} \), where \( \bar{R} \) is an analytic set of codimension \( \geq 2 \) if it is not empty. The second assertion follows from this fact.

Let us consider the case (i) of Proposition 1, i. e., the case where \( R_0 \) is empty or Stein. We have \( R_1 = \phi \), \( R_2 = \phi \) by Proposition 2, and hence \( \bar{R} \cap A_1 = \phi \), \( \bar{R} \cap A_2 = \phi \) by Proposition 3. Since \( \bar{R} \) is an open subset of \( L_0 \), we have \( \bar{R} = \phi \). Therefore \( \bar{D} \) is Stein by Proposition 4. The theorem is proved for the case (i) if we show the following

**Proposition 5.** \( D \) is Stein if \( \bar{D} \) is Stein.

**Proof.** If \( 0 \notin D \), the map \( \rho \mid \bar{D} : \bar{D} \rightarrow D \) defines a holomorphic principal bundle with structure group \( C^\ast \). For this case, the proposition follows from a theorem of Matsushima–Morimoto [9, Théorème 5].

To cover the general case, we will go back to the construction of holomorphic functions on \( D \). The domain \( D \) is Stein if it has the following property: For any sequence \( \{z_\varepsilon\}_\varepsilon \) of points in \( D \) which has no accumulation point in \( D \) and for any sequence \( \{\varepsilon_\varepsilon\}_\varepsilon \) of complex numbers.
numbers, there exists a holomorphic function \( f \) no \( D \) such that \( f(z_\kappa) = \epsilon_\kappa \).

Consider the analytic set \( \rho^{-1}(\{z_\kappa\}) \) in \( \tilde{D} \). If \( \tilde{D} \) is Stein, there is a holomorphic function \( F \) on \( \tilde{D} \) such that \( F|\rho^{-1}(z_\kappa) = \epsilon_\kappa \) for all \( \kappa \). When \( z \in D \setminus \{0\} \), the fiber \( \rho^{-1}(z) \) is biholomorphic to \( \mathbb{C}^s \). Choose a fiber coordinate \( w \) on \( \rho^{-1}(z) \) so that \( \rho^{-1}(z) = \{w \in \mathbb{C} \mid w \neq 0\} \), and a smooth Jordan curve \( \gamma_z \) equipped with orientation such that

\[
\frac{1}{2\pi i} \int_{\gamma_z} \frac{dw}{w} = 1.
\]

We define

\[
f(z) := \frac{1}{2\pi i} \int_{\gamma_z} (F|\rho^{-1}(z))(w) \frac{dw}{w}.
\]

In other words, \( f(z) \) is the constant term of the Laurent expansion of \( (F|\rho^{-1}(z))(w) \) in \( w \). Clearly \( f(z) \) is defined independently of the choice of \( w \) and \( \gamma_z \). The function \( f \) is holomorphic on \( D \setminus \{0\} \), and \( f(z_\kappa) = \epsilon_\kappa \) if \( z_\kappa \neq 0 \).

When \( 0 \in D \), we define \( f(0) = F(0, 0) \), Then \( f \) is holomorphic at 0. To show this we specify \( \gamma_z \) as follows: Let

\[
\Gamma := \{(x, y) \in A \mid |x_0|^2 + \ldots + |x_r|^2 = |y_0|^2 + \ldots + |y_s|^2 \}
\]

and \( \gamma_z := \Gamma \cap \rho^{-1}(z) \), with an appropriate orientation. When \( z \) tends to \( 0 \in Z \), the set \( \gamma_z \) tends to \( (0, 0) \in A \); hence \( f(z) \) tends to \( F(0, 0) = f(0) \). Thus \( f \) is continuous and hence holomorphic at 0, because \( Z \) is a normal space. Clearly \( f(z_\kappa) = \epsilon_\kappa \) for \( z_\kappa = 0 \), too.

Thus \( f \) is a holomorphic function with the desired property.

q. e. d.

Let us next consider the case (ii) of Proposition 1, i.e., \( R_0 = \tau^{-1}(R_1), \ R_1 \neq \phi \) and \( R_2 = \phi \). We have \( \tilde{R} \cap A_2 = \phi \) by Proposition 3 and hence \( \tilde{D}^s \subseteq A_1 \). Therefore we have \( \rho^{-1}(D^*_i) = \tilde{D}^s \). The domain \( \tilde{D}^s \) is Stein by Proposition 4. The map \( \rho_i|\tilde{D}^s \) defines a holomorphic principal bundle with structure group \( \mathbb{C}^* \) over \( D^*_i \). By the theorem of Matsushima–Morimoto, mentioned above, \( D^*_i \) is Stein.

\( D \) is biholomorphic to \( D_1 = D^*_i \setminus R_1 \), where \( R_1 \) is a nonempty analytic set in \( D^*_i \) of codimension \( s+1 \). Therefore \( D \) is not Stein. Since \( D^*_i \) is Stein, it is a domain of holomorphy in \( Z_i \), i.e., there is a holomorphic function \( g_1 \) on \( D^*_i \) which is singular at all points in \( \partial D^*_i \). We set \( g = g_1 \circ (\sigma_1|D_i)^{-1} \). Then \( g \) is holomorphic on \( D \) and singular at all
points in $\partial D \setminus \{0\}$. Consequently it is singular also at 0. Thus $D$ is a domain of holomorphy.

We have thus proved the theorem for the case (ii). The case (iii) is treated in the same way.

Finally consider the case (iv). There is an open set $V$ containing $O_0$ such that $V \setminus O_0 \subset D_0$. Hence $\sigma_0(V) \setminus \{0\} = \sigma_0(V \setminus O_0) \subset \sigma_0(D_0) = D$. This implies that 0 is an isolated boundary point of $D$. $D \cup \{0\}$ is a domain in $Z$ containing 0, which falls upon the case (i).

This completes the proof of the theorem.

References
