Extension of Holomorphic Functions with Growth Conditions

By

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§1. Introduction and Summary

In his paper [4], Y. Nishimura discussed the problem of extending holomorphic functions on a submanifold $X$ of $\mathcal{O}^n$ to ones on the ambient space, with the order of growth in consideration. In this note we shall take up this problem in a slightly different setting. We consider a Stein manifold $X$ with a strictly plurisubharmonic $C^\infty$ exhaustion function $\psi$ such that $\sup \psi = +\infty$. We also set $t_0 = \inf \psi$. For a real valued continuous function $\lambda(t)$ on $[t_0, \infty)$, we set

$$M(X, \lambda) = \{h | h \text{ is holomorphic on } X \text{ and } \sup_{p \in X} |h(p)| \exp(-\lambda(p)) < \infty\}.$$  

$M(X, \lambda)$ is a Banach space with respect to the norm

$$||h||_\lambda = \sup_{p \in X} |h(p)| \cdot \exp\{-\lambda(p)\}.$$  

If $Y$ is a closed analytic submanifold of $X$, $Y$ is again a Stein manifold and $\psi|_Y$ is a strictly plurisubharmonic exhaustion function for $Y$. We shall use the similar notation as (1.1) and (1.2) for $Y$ and $\psi|_Y$.

For two $C^\infty$ functions $\mu(t)$ and $\lambda(t)$ on $[t_0, \infty)$, $\mu \ll \lambda$ shall mean $\mu(t) \leq \lambda(t)$, $\mu'(t) \leq \lambda'(t)$ and $\mu''(t) \leq \lambda''(t)$ for all $t \in [t_0, \infty)$.

Now our main assertion sounds as

**Theorem 1.** If $X$ and $\psi$ are as above and if $Y$ is a closed analytic submanifold of codimension 1, then there exists a non-decreasing convex $C^\infty$ function $\nu(t)$ of $t \in [t_0, \infty)$, such that for any $C^\infty$ function $\lambda(t)$ with $\nu \ll \lambda^{(1)}$, there exists a constant $C_\lambda$ such that any $h \in M(Y, \lambda)$ has an extension $H \in M(X, \lambda^{(2)} + \nu^{(1)})$.


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with \[ ||H||_{k(\omega + \nu(\Omega))} \leq C_\lambda ||h||_\lambda. \] Here \( \lambda^{(i)} \) is defined by \( \lambda^{(i)}(t) = \lambda(t+i) \) for \( i = 1, 2 \) and similarly for \( \nu^{(i)} \).

There are many works in this direction. To mention some of them, we have [0] and [6]. They have their own general scope or concrete viewpoints. The author hopes that the present work is not completely covered by them.

Many thanks are due to the referee for valuable suggestions.

§2. Construction of \( \nu \)

Let \( X, \psi \) and \( Y \) be as above. Let us take locally finite open covers \( \mathcal{U} = \{ U_i \}_i \) and \( \mathcal{U}' = \{ U'_i \}_i \) with the following properties:

1. For each \( i \in I \), there exists a system of local coordinates \( (z_1', \ldots, z_n') \) which is valid on an neighborhood \( U''_i \) containing \( U_i \), and in this coordinate system \( U_i \) and \( U'_i \) have forms

\[
\begin{cases}
U_i = \{(z_\alpha) \mid |z_\alpha| < 1 \text{ for } \alpha = 1, \ldots, n\}, \\
U'_i = \{(z_\alpha) \mid |z_\alpha| < 1 - \varepsilon_i \text{ for } \alpha = 1, \ldots, n\},
\end{cases}
\]

\( 0 < \varepsilon_i < 1 \).

2. \( I \) is the disjoint union of subsets \( I_1 \) and \( I_2 \). For \( i \in I_1 \), \( U_i \cap Y \neq \emptyset \) and \( z_i^n = 0 \) is a local equation for \( Y \) in \( U'_i \). For \( i \in I_2 \), \( U_i \cap Y = \emptyset \).

3. If \( p \) and \( q \) belong to \( U_i \), then \( |\psi(p) - \psi(q)| < 1 \).

We set \( R_i = z_i^n \) for \( i \in I_1 \) and \( R_i = 1 \) for \( i \in I_2 \). Then \( \{ R_i \} \) form a system of local equations for \( Y \) and

\[
b_{ij} = R_i/R_j
\]

defines a system of transition functions for \( [Y] \). It is clear that such coverings exist provided \( \varepsilon_i \) are suitably chosen. For \( i \in I_1 \), we define the projection \( \pi_i: U''_i \to U'_i \cap Y \) by

\[
\pi_i(z_i, z_i') = z_i, \quad \text{where } z_i = (z_i, \ldots, z_i^{n-1}).
\]

Now there exists, for each \( p \in U_j \cap U'_i \) (\( j \in I_i \)), a positive number \( s_{ij}(p) \) such that

\[
|z_i^n(p)| \geq s_{ij}(p) \quad \text{or} \quad (z_j(p), \zeta) \in U_j \cap U_i \text{ for } |\zeta| < s_{ij}(p).
\]

If we set

\[
\pi_i^{-1}(\pi_j(p)) \cap U_j \cap U_i = \pi_j(p) \times W_j, \\
\pi_i^{-1}(\pi_j(p)) \cap U_j \cap U'_i = \pi_j(p) \times W'_j,
\]

then there exists a positive number \( s_{ij}(p) \) such that
$W_p$ and $W'_p$ being open sets on the $z_j$-plane, we have

\begin{align}
(2.4) \quad \begin{cases} 
    r_{ji}(p) = \max \{z_j^*(p), \sup \{s \mid \{z_j^* \mid <s\} \subset W_p\}\} & \text{if } W_p \not\equiv 0, \\
    r_{ji}(p) = |z_j^*(p)| & \text{if } W_p \equiv 0.
\end{cases}
\end{align}

(Note that $r_{ji}(p) \geq |z_j^*(p)|$ in any cases.) Then we have the following:

**Lemma 1.** \[ \liminf r_{ji}(q) \geq r_{ji}(p) \text{ when } q \to p. \] $r_{ji}$ is bounded from below by a positive number.

**Proof.** Suppose $W_p \not\equiv 0$ and $r_{ji}(p) = \sup \{s \mid \{z_j^* \mid <s\} \subset W_p\}$. For any $s$ with $0 < s < r_{ji}(p)$, $\pi_j(p) \times \{z_j^* \leq s\}$ is a compact set in $U_j \times U_i$. Hence there exists an open neighborhood $V$ of $\pi_j(p)$ such that $V \times \{z_j^* \leq s\} \subset U_j \cap U_i$. Then for $q$ with $q \in U_j \cap U_i$ and $\pi_j(q) \in V$, $W_q \supset \{z_j^* \geq s\}$. Hence (2.4) holds for $q$ and we have $r_{ji}(q) \geq s$. Hence $\liminf r_{ji}(q) \geq s$.

In other cases we have $\liminf r_{ji}(q) \geq \liminf |z_j^*(q)| = |z_j^*(p)| = r_{ji}(p)$.

Next, set

$$
e_{ji} = \frac{1}{2} \inf \{|x-y| \mid x, y \in C, x \in \overline{W_p}, y \in \overline{W_p}\}. $$

It is clear that $\epsilon_{ji} > 0$. We have $r_{ji}(p) \geq \epsilon_{ji}$ for $p \in U_j \cap U_i$. In fact, if $W_p \not\equiv 0$, then

$$r_{ji}(p) = |z_j^*(p)| \geq |z_j^*(p)| - 0 \geq 2\epsilon_{ji}. $$

($z_j^*(p)$ is in $W_p$ always.) Suppose $W_p \equiv 0$. If $\epsilon_{ji} < \sup \{s \mid \{z_j^* \mid <s\} \subset W_p\}$, then clearly $r_{ji}(p) > \epsilon_{ji}$. Otherwise there is a $y \in C - W_p$ with $|y| = \epsilon_{ji}$. Hence $r_{ji}(p) \geq |z_j^*(p)| \geq 2\epsilon_{ji} - |y| = \epsilon_{ji}$.

For each $p$, there are only a finite number of pairs $(j, l)$ such that $p \in U_j \cap U_l$. Hence

$$r(p) = \min_{(j, l)} r_{ji}(p)$$

is a well defined positive valued function on a neighborhood of $Y$. This can be extended to a function on $X$ with the same properties, by setting $r(p) = 1$ outside.

We choose a partition of unity $\{\rho_i\}$ subordinate to $U'$. We also take a metric along the fibres of the bundle $[Y]^{-1}$. This is expressed by a system $\{a_i\}$ of positive valued $C^\infty$ functions $a_i: U_i \to \mathbb{R}$ such that

$$a_j/a_k = |b_{jk}|^2 \text{ on } U_j \cap U_k.$$
Because the coordinates \((z_j)\) exist on \(U'_j\) which contains \(\bar{U}_j\) and because we can assume \(R_j\) is also defined on \(U'_j\), we can assume that \(a_j\) is defined as a differentiable function on \(U'_j\). This we shall assume to be the case. Set, for each \(p \in U_j\), \(b_j(p) = \max_i |\log |b_{j_l}(p)|^2|, \ l\) ranging over such indices that \(p \in U_l\). Then we have
\[-b_j(p) \leq \log a_j(p) - \log a_j(p) \leq b_j(p).

We put \(b(p) = \sum \rho_j(p) b_j(p)\), then we have
\[-b(p) \leq \log a_j(p) - \sum \rho_j(p) \log a_j(p) \leq b(p).

We replace \(a_j(p)\) by \(a_j(p) \exp (- \sum \rho_j(p) \log a_j(p))\), then we have the estimate
\[
\exp (-b(p)) \leq a_j(p) \leq \exp (b(p)).
\]

Finally we choose a complete Kähler metric \(d\sigma^2\) on \(X\). In terms of local coordinate system \((z_j)\), \(d\sigma^2\) will be expressed as
\[
ds^2 = 2 \sum g_{j_k \bar{k}} dz_j^2 d\bar{z}_k^2.
\]

We set
\[
\tau_j = \det (g_{j_k \bar{k}}),
\]
then \(\{\tau_j\}\) defines a metric along the fibres of \(K_X^{-1}\), where \(K_X\) denotes the canonical bundle of \(X\).

All these being settled, we take a \(C^\infty\) non-decreasing convex function \(v(t)\) on \([t_0, \infty)\), such that
\[
\begin{align*}
\text{(a)} & \quad \int_X \exp \{ - v(\psi(p)) + 3 b(p) \} \frac{1}{r(p)^2} \max \{1, (\sum \rho_j(p))_j\} dv < \infty , \\
\text{(b)} & \quad 3 \left( \frac{\partial^2 v(\psi)}{\partial z_j^2 \partial \bar{z}_k^2} - \left( \frac{\partial^2 \log a_j}{\partial z_j^2 \partial \bar{z}_k^2} \right) \right) \geq (g_{j_k \bar{k}}), \\
\text{(c)} & \quad \exp \{ - v(\psi(p)) + b(p) \} \cdot \tau_j(p)^{-1} \cdot e^{-2n} \leq C ,
\end{align*}
\]
where \(\epsilon_j\) appeared in the definition of \(U'_j, j\) is any index such that \(p \in U'_j\), and \(C\) is a positive constant.

§3. The Proof of the Theorem

For a function \(h \in \mathcal{A}(Y, \lambda)\), we extend it to the holomorphic function \(h_j\ in \ U_j (j \in I_1)\), by
For \( j \in I_n \), we simply set \( \tilde{h}_j = 0 \). In \( U_j \cap U_k \), \( \tilde{h}_j - \tilde{h}_k \) and \( R_j \) have the zero set \( U_j \cap U_k \cap Y \) in common. Hence

\[
\tilde{h}_{jk} = (\tilde{h}_j - \tilde{h}_k)/R_j
\]

is holomorphic in \( U_j \cap U_k \) and \( \{\tilde{h}_{jk}\} \) form a 1-cocycle in \( Z^1(U, \mathcal{O}(Y)^{-1}) \). As usual we consider the 1-cocycle \( \{H_{jk}\} \) in \( Z^1(U, \mathcal{O}^n([Y]^{-1} \otimes K_X)) \) defined by

\[
H_{jk} = \tilde{h}_{jk} \, dz_j^1 \wedge \cdots \wedge dz_j^n
\]

and set

\[
\eta_j = -\sum_{m} \rho_{nm} e_{jm} H_{mj} ,
\]

where \( e_{jk} = (R_k/R_j) \det (\partial z_{ij} / \partial z_{jk}) \), then we have \( \eta_j - e_{jk} \eta_k = H_{jk} \). We set

\[
f_j = \tilde{\delta} \eta_j ,
\]

then \( f = \{f_j\} \) is a \([Y]^{-1} \otimes K_X^{-1}\)-valued \((n, 1)\)-form on \( X \) with \( \partial f = 0 \).

With the aid of a function \( \mu : \mathbb{R} \to \mathbb{R} \), we introduce the inner product \((\varphi, \chi)\) of \([Y]^{-1} \otimes K_X^{-1}\)-valued \((n, q)\)-forms \( \varphi = \{\varphi_j\} \) and \( \chi = \{\chi_j\} \) by

\[
(\varphi, \chi) = \int_X \exp \left( -\mu(\varphi) \cdot a_j \varphi_j \wedge * \chi_j \right)
\]

and denote by \( \mathcal{L}^n(X, \mu) \) the Hilbert space of \([Y]^{-1} \otimes K_X^{-1}\)-valued measurable \((n, q)\)-forms \( \varphi \) with \((\varphi, \varphi) < \infty \).

**Lemma 2.** For any \( h \in \mathcal{H}(Y, \lambda) \), \( f \) constructed as \((3.1) \sim (3.4)\) belongs to \( \mathcal{L}^n(X, 2\lambda^{(1)} + \nu) \) and the correspondence \( h \mapsto f \) is a continuous linear map. For \( \eta_j \) we also have

\[
||\eta_j||_{U_j}^2 = \int_{U_j} \exp \{ -2(\lambda^{(1)}(\varphi) + \nu(\varphi)) \} a_j \eta_j \wedge * \eta_j \leq C' \cdot ||h||_{\lambda}^2 .
\]

**Proof.** For \( h \in \mathcal{H}(Y, \lambda) \), \( \tilde{h}_j \) satisfies \( |\tilde{h}_j|(p) \leq ||h||_{\lambda} \exp \{\lambda^{(1)}(\varphi(p))\} \), because \( p \) and \( \pi_j(p) \) belong to the same \( U_j \). Consider \( \tilde{h}_{jk}(p) \) for \( p \in U_j \cap U_k \), and in particular as a function of \( z_j^p \), other \( z_j^p \)'s being considered as parameters. If \( |z_j^p(p)| \geq r(p) \), then \((3.7)\) below holds trivially. If \( (z_j^p(p), \zeta) \in U_j \cap U_k \) for \( |\zeta| \leq r(p) \), then \( \tilde{h}_{jk}(z_j^p, \zeta) \) has the maximum of its absolute values on the circle \(|\zeta| = r(p)\). Hence

\[
|\tilde{h}_{jk}(p)| \leq \frac{2}{r(p)} ||h||_{\lambda} \cdot \exp \{\lambda^{(1)}(\varphi(p))\}
\]
holds. Since supp $\rho_m$ is contained in $U'_m$ for any $m$, we can estimate $\eta_j$ and $f_j$ by virtue of (3.7), and see

\[
(f, f) = \int_X \exp \left\{ - (2\lambda^{(3)}(\psi) + \nu(\psi)) \right\} a_j r_j \left( \sum \delta \rho_m e_j \bar{h}_{ij} dz_j \wedge \cdots \wedge dz_l \right)
\]

\[
\left. \left. \wedge \ast \left( \sum \delta \rho_m e_j \bar{h}_{mj} dz_m \wedge \cdots \wedge dz_m^m \right) \right) \right\}
\]

\[
= \int_X \exp \left\{ - (2\lambda^{(3)}(\psi) + \nu(\psi)) \right\} a_j r_j \left( \sum \delta \rho_m b_{ij} \bar{h}_{ij} dz_j \wedge \cdots \wedge dz_l \right)
\]

\[
\left. \left. \wedge \ast \left( \sum \delta \rho_m b_{mj} \bar{h}_{mj} dz_m \wedge \cdots \wedge dz_m^m \right) \right) \right\}
\]

\[
\leq C_0 \int_X \exp \left\{ - (2\lambda^{(3)}(\psi) + \nu(\psi)) \right\} a_j r_j \left( \sum |b_{ij} \bar{h}_{ij} \bar{h}_{mj} u_{im} | w_j dv ,
\right.
\]

where $u_{im}$ and $w_j$ denote the pointwise inner product of $\delta \rho_1$ with $\delta \rho_m$ and $dz_j \wedge \cdots \wedge dz_l$ with itself respectively, that is: $\delta \rho_1 \wedge \ast (\delta \rho_m) = u_{im} dv$ and $(dz_j \wedge \cdots \wedge dz_l) \wedge \ast (dz_j \wedge \cdots \wedge dz_l) = w_j dv$. $C_0$ is the constant which appears in [5], Lemma A (p. 18). (The same lemma is referred to as Lemma 11.1 in [3].) Hence we have

\[
(f, f) \leq 4C_0 \|h\|^2 \int_X \exp \left\{ - \nu(\psi) + 3b \right\} r(p)^{-2} \sum |u_{im}| dv
\]

\[
\leq 4C_0 \|h\|^2 \int_X \exp \left\{ - \nu(\psi) + 3b \right\} r(p)^{-2} \sum |\delta \rho_1|^2 dv
\]

\[
\leq C_1 \|h\|_\lambda^2,
\]

where $C_1$ is a constant independent of $h$. Thus we have

\[
(3.8) \quad \|f\|_\lambda \leq \sqrt{C_1} \|h\|_{\lambda}.
\]

The proof of (3.6) is similar.

Next we apply Hörmander's $\bar{\partial}$-theory. Since components of the curvature form $\Theta$ of the metric along the fibres are

\[
2 \left[ \frac{\partial^2 \lambda^{(1)}(\psi)}{\partial z_j^a \partial z_l^a} \right] + \left[ \frac{\partial^2 \nu(\psi)}{\partial z_j^a \partial z_l^a} \right] - \left[ \frac{\partial^2 \log a_j}{\partial z_j^a \partial z_l^a} \right] - \left[ \frac{\partial^2 \log \tau_j}{\partial z_j^a \partial z_l^a} \right] \geq (g_{ja}),
\]

we have

\[
((e(\Theta) A - Ae(\Theta)) \varphi, \varphi) \geq (\varphi, \varphi)
\]

for $(n, q)$-form $\varphi$ with compact support ($q \geq 1$). Hence

\[
(3.9) \quad (\varphi, \varphi) \leq (\delta \varphi, \delta \varphi) + (\delta^* \varphi, \delta^* \varphi)
\]

for such a $\varphi$, where $\delta^*$ denotes the adjoint operator to $\delta$. Because $ds^2$ is complete, (3.9) holds for any $\varphi$ of $\mathcal{L}^q(X, 2\lambda^{(3)} + \nu)$, which belongs to the inter-
section of the domains of $\bar{\partial}$ and $\bar{\partial}^*$. ([5], Theorem 1.1, p. 18. See also [3], theorem 14.1.) Then Hörmander's theorem ([2], Theorem 1.1(170,122),(892,958)(3,3),(994,992).4) tells that

$(*)$ For $f$ defined by (3.4), there exists a $\xi \in L^0(\mathcal{X}, 2\lambda^{(\nu)} + \nu)$ such that $f = \bar{\partial}\xi$, $\|\xi\|_{L^0} \leq \|f\|_{L^0}$ and $\xi$ is $C^\omega$.

Set

\[ \eta_j = g_j \, dz_j^1 \wedge \cdots \wedge dz_j^n, \]

then $g_j$ is a holomorphic function on $U_j$ and we have

\[ h_{j_0} = g_j - (R_k/R_j)g_k \quad \text{on} \quad U_j \cap U_k. \]

Hence

\[ H = h_j - R_j g_j = h_k - R_k g_k \]

is a global holomorphic function on $\mathcal{X}$ which extends $h$ on $Y$.

We have only to estimate $H$. Take any point $p$ of $\mathcal{X}$. $p$ is contained in $U_j$ for some $j$. Set $\zeta^a = z_j^a - z_j^a(p)$, then the multi-disk $D = \{(\zeta)| \zeta^a| < \epsilon_j\}$ is contained in $U_j$ and $g_j$ is holomorphic there. Hence we have, for $|\tau_a| < \epsilon_j$

\[ g_j(p) = \frac{1}{(2\pi)^n} \int_{|\zeta^a| = \epsilon_j} \cdots \int_{|\zeta^a| = \epsilon_j} \frac{g_j(z_j^a(p) + \zeta)}{\zeta \cdots \zeta} \, d\zeta^1 \cdots d\zeta^n \]

\[ = \frac{1}{(2\pi)^n} \int_{\zeta^a} \cdots \int_{\zeta^a} g_j(z_j^a(p) + \tau_a e^{-i\theta_a}) d\theta_1 \cdots d\theta_n. \]

Multiplying both hand sides by $\tau_1 \cdots \tau_n$ and integrating from $\tau_a = 0$ to $\tau_a = \epsilon_j$, we obtain

\[ \frac{\epsilon_j^{2n}}{2^n} g_j(p) = \frac{1}{(2\pi)^n} \int_D g_j(z) dz^1 dz^2 \cdots dz^n. \]

Hence

\[ \frac{\epsilon_j^{2n}}{2^n} |g_j(p)| \leq \frac{1}{(2\pi)^n} \int_D (\eta_j(z) - \xi_j(z)) dz^1 dz^2 \cdots dz^n, \]

where $\eta_j$ and $\xi_j$ denote the coefficients of $dz^1 \wedge \cdots \wedge dz^n$ in $\eta_j$ and $\xi_j$ respectively. We have, by Schwarz inequality,
\[
\cdot \left\{ \int_D a_j \exp \left\{ -2 \lambda^{(1)}(\psi) - \nu(\psi) \right\} |\eta_j - \xi_j|^2 d\nu \right\}^{1/2}
\]
\[
= 2^{-n/2} \int_D a_j^{-1} \gamma_j^{-1} \exp \left\{ 2 \lambda^{(3)}(\psi) + \nu(\psi) \right\} dz^1 dz^2 \cdots dz^n dz^m |\eta_j - \xi_j|_{D}^{1/2}
\]
where \( || \cdot ||_D \) denotes the latter factor in the preceding expression. Because of (2.7) and (2.9c), we have
\[
\left| \int_D a_j^{-1} \gamma_j^{-1} \exp \left\{ 2 \lambda^{(2)}(\psi(p)) + 2 \nu^{(1)}(\psi(p)) \right\} \right| \leq \exp \left\{ 2 \lambda^{(2)}(\psi(p)) + 2 \nu^{(1)}(\psi(p)) \right\} ||\gamma_j^{-1}||_{D}^{1/2}
\]
while
\[
||\eta - \xi||_{D}^{1/2} \leq ||\eta - \xi||_{U_j} = ||\eta - \xi||_{U_j} \leq 2||\eta||_{U_j} + 2||\xi||_{U_j} \leq 2C_1||h||_{\lambda}.
\]
Thus we have, from (3.12),
\[
\frac{e^{2n}}{2^n} |g_j(p)| \leq \frac{1}{(2\pi)^n \sqrt{2^n \pi^n}} \exp \left\{ 2 \lambda^{(2)}(\psi(p)) + 2 \nu^{(1)}(\psi(p)) \right\} ||\gamma_j^{-1}||_{D}^{1/2} \leq 2C_1||h||_{\lambda}.
\]
Putting \( C_2 = \frac{2\sqrt{C \cdot C_1}}{(2\pi)^n} \), we finally have
\[
\left(3.13\right) \quad |g_j(p)| \cdot \exp \left\{ -\lambda^{(2)}(\psi(p)) - \nu^{(1)}(\psi(p)) \right\} \leq C_2 ||h||_{\lambda}.
\]
Because \( |R_j(p)| \leq 1 \) and \( |H_j(p)| \leq ||h||_{\lambda} \exp \left\{ \lambda^{(1)}(\psi(p)) \right\} \), we have proved that
\[
\left(3.14\right) \quad |H(p)| \cdot \exp \left\{ -\lambda^{(2)}(\psi(p)) - \nu^{(1)}(\psi(p)) \right\} \leq C_3 ||h||_{\lambda}.
\]

§4. Nishimura’s Case

In this section we shall sketch that we can deal with Nishimura’s case by our method partly. In this case \( \mathcal{X} = \mathbb{C}^n \) with the standard cartesian coordinates \( (\zeta^1, \cdots, \zeta^n) \) and \( \psi(\zeta) = \left\{ 1 + \sum a \zeta^a \cdot \zeta \right\}^{1/2} \). \( Y \) is defined by an entire function \( F \) as \( Y = \{ \zeta | F(\zeta) = 0 \} \) and \( F \) satisfies the conditions
\[
\left\{ \begin{array}{ll}
|F(\zeta)| \exp \left\{ -B \psi(\zeta)^a \right\} \leq M & \zeta \in \mathcal{X}, \\
|dF(\zeta)| \exp \left\{ B' \psi(\zeta)^a \right\} \geq M' > 0 & \zeta \in \mathcal{Y},
\end{array} \right.
\]
where \( |dF(\zeta)|^2 = \sum a |\partial F(\zeta) / \partial \zeta^a|^2 \) and \( \eta, M, M', B \) and \( B' \) are positive constants.
Nishimura's main result is as follows:

**Theorem N.** Suppose that above conditions are satisfied, then for any $B_i > 0$, there exist positive constants $B_2$ and $C_2$ such that for any $h \in \mathcal{J}(Y, B,t^n)$, there exists an $H \in \mathcal{J}(\mathbb{C}^n, B_2 t^n)$ satisfying $H|_Y = h$ and $\|H\|_{B_2 t^n} \leq C_2 \|h\|_{B_1 t^n}$.

We shall assume that $\eta \geq 1$, and shall show in the following that

**Theorem 2.** If $X = \mathbb{C}^n$ and $Y$ is defined by an $F$ satisfying (4.1) with $\eta \geq 1$, then the function $\varphi(t)$ mentioned in Theorem 1 can be taken as $\varphi(t) = B_0 t^n$, where $B_0$ is a suitable positive constant.

Since $(t+i)^a \ll A t^n$ for a constant $A$ and for $i = 1, 2$, it is clear that Theorem N follows from Theorems 1 and 2.

We shall sketch direct construction of the covers $\mathcal{U}$ and $\mathcal{U}'$. First we recall a classical lemma in the theory of functions of a complex variable.

**Lemma 3.** Suppose $w = f(z)$ is a function of a complex variable $z$, holomorphic in the disk $|z| < R$ and $|f(z)| \leq M$ there. Moreover assume that $f(0) = 0$ and $f'(0) = c_1 \neq 0$. Then the inverse function $z = g(w)$ (with $g(0) = 0$) is defined at least in the domain $|w| < |c_1|^2 R^2 / 6M$ and we have $|g(w)| \leq |c_1| \cdot R^2 / 4M$ there.

For the proof see, for example, [1] Lemma 17. 7.1 pp. 385–386. We shall also make use of the following\(^*)

**Lemma 4.** ([4], Lemma 2) If $H$ is a function in $\mathcal{J}(\mathbb{C}^n, Bt^n)$ for some $\eta > 0$, $B > 0$, then $|1/m! \partial^m H / \partial \zeta^m| \leq \|H\|_{B^n(\sqrt{m})} \exp \{-K B t^n\}$, where $m$ denotes the multi-index $(m_1, \cdots, m_n)$, $|m| = \sum m_n$ and $K = 2^n + 1$.

These being settled, we first verify without difficulty, that if $p$ and $q \in \mathbb{C}^n$ are such that $\psi(p) \geq 2$ and $\sum| \zeta^*(p) - \zeta^*(q) |^2 < \frac{1}{9} \{ \sum| \zeta^*(p) |^2 \}^{-1}$, then we have $|\psi(q) - \psi(p)| < 1$. In the construction of $\mathcal{U}$ and $\mathcal{U}'$ we pay attentions to that part for which $\psi(p) \geq 2$. Remaining part can be dealt with separately.

Take $p \in Y$, we change the coordinate into $(\xi^*_p)$ by a unitary transformation;

\[(4.2) \quad \xi^*_p = \sum\limits_p c^*_p(p) (\zeta^p - \zeta^*(p)),\]

\(^*)\) Note that our $\{1 + \sum| \zeta^* |^{2\gamma / 2}\}$ and Nishimura's $(\sum| \zeta^* |^{\gamma / 2} + 1$ define the equivalent norms in the sense of (1.1).
so that \( \partial F/\partial \xi^\alpha_p(p) = 0 \) for \( \alpha < n \). Then \( \partial F/\partial \xi^\alpha_p(p) \), which we denote by \( c_1 \), has the absolute value

\[
|c_1| = \left\{ \sum_{\alpha} \left| \frac{\partial F}{\partial \xi^\alpha_p(p)} \right|^2 \right\}^{1/2} \geq \exp \left\{ -(A' + B'\psi(p)\gamma) \right\},
\]

where \( B' \) appears in (4.1) and \( A' \) is a constant determined by \( M' \).

For the moment we fix \( \xi^\alpha_p, \ldots, \xi^{n-1}_p \) to be 0 and consider \( F \) as a function \( f(\xi) = w \) of a single variable \( \xi = \xi^p_p \). If we set

\[
R = \frac{1}{3} \psi(p)^{-1/2},
\]

\[
M = \exp \{ A + B\psi(p)\gamma \},
\]

then we can apply Lemma 3 to \( f \) with \( R, M \) in (4.4). (\( A \) is determined in connection with \( M \) in (4.1).) Thus we have the inverse function \( \xi = g(w) \), defined in a domain containing \( \{ w | |w| < \exp \{ -(A_3 + B_3\psi(p)\gamma) \} \} \). (Here and in the following, \( A_i, B_i \) are suitably chosen constants independent of \( p \).)

In order to take the change of \( \xi^\alpha_p, \ldots, \xi^{n-1}_p \) into account, we consider the relation

\[
w' = H(\tilde{\xi}_p, \xi) = G_p(\tilde{\xi}_p, \xi) - G_p(\tilde{\xi}_p, 0),
\]

where \( \tilde{\xi}_p \) stands for \( (\xi^1_p, \ldots, \xi^{n-1}_p) \) and \( G_p \) is the function \( F \) expressed in terms of \( (\xi^\alpha_p) \). Because of Lemma 4 we have

\[
|H(\tilde{\xi}_p, \xi)| = \left| G_p(\tilde{\xi}_p, \xi) - G_p(\tilde{\xi}_p, 0) \right| = \left| \int_0^\xi \frac{\partial G_p}{\partial \xi}(\tilde{\xi}_p, \tau) d\tau \right| \leq \sqrt{n} M \exp \{ KB(\psi(p) + 1)\gamma \} |\xi|.
\]

\[
\left| G_p(\tilde{\xi}_p, 0) \right| = \left| \int_0^1 \frac{d}{d\tau} \{ G_p(\tau \tilde{\xi}_p, 0) \} d\tau \right| = \left| \int_0^1 \sum_{\alpha=1}^{n-1} \xi^\alpha_p \frac{\partial G_p}{\partial \xi^\alpha_p}(\tau \tilde{\xi}_p, 0) d\tau \right| \leq \sum_{\alpha=1}^{n-1} |\xi^\alpha_p| \cdot \sqrt{n} M \exp \{ KB(\psi(p) + 1)\gamma \},
\]

\[
\left| \frac{\partial H}{\partial \xi}(\tilde{\xi}_p, 0) \right| = \left| \frac{\partial G_p}{\partial \xi}(\tilde{\xi}_p, 0) \right| = \left| \frac{\partial G_p}{\partial \xi}(0, 0) + \int_0^1 \frac{d}{d\tau} \left\{ \frac{\partial G_p}{\partial \xi}(\tau \tilde{\xi}_p, 0) \right\} d\tau \right| \geq \left| c_1 - \sum_{\alpha=1}^{n-1} |\xi^\alpha_p| \cdot 2n M \exp \{ KB(\psi(p) + 1)\gamma \}.\]

Hence if we choose \( A_4 \) and \( B_4 \) suitably and set

\[
V_p = \{ (\xi^\alpha_p) | |\xi^\alpha_p| < \exp \{ -(A_4 + B_4\psi(p)\gamma) \} \},
\]
we have, in $V_p$,

$$
|H| \leq \exp\{- (A_\alpha + B_\alpha \psi(p)^\alpha)\},
$$

$$
\frac{\partial H}{\partial \xi}(\xi_p, 0) \geq \frac{c_1}{2}, \quad |c_1| \geq \exp\{- (A'+B'\psi(p)^\alpha)\}.
$$

We apply Lemma 3 to $H$. When $|\xi_p| < \exp\{- (A_\alpha + B_\alpha \psi(p)^\alpha)\}$ for $\alpha=1, \ldots, n-1$, the relation $\xi = H(\xi_p, \xi)$ can be solved as $\xi = \psi(\xi_p, w')$ for $|w'| < \exp\{- (A_\alpha + B_\alpha \psi(p)^\alpha)\}$. We change $A_\alpha$ and $B_\alpha$ if necessary and can achieve that $|G_p(\xi_p, 0)| < \frac{1}{2} \exp\{- (A_\alpha + B_\alpha \psi(p)^\alpha)\}$, then

$$
\xi = \psi(\xi_p, w - G_p(\xi_p, 0))
$$

is holomorphic for $|w| < \frac{1}{2} \exp\{- (A_\alpha + B_\alpha \psi(p)^\alpha)\}$. In other words, we have constructed a neighborhood $V_p$ of $p$ and local coordinates $(\xi_p, w)$ such that

$$
V_p = \{(\xi_p, w) | |\xi_p| < \exp\{- (A_\alpha + B_\alpha \psi(p)^\alpha)\},
$$

$$
|w| < \frac{1}{2} \exp\{- (A_\alpha + B_\alpha \psi(p)^\alpha)\}\},
$$

where $w(q) = F(q)$ for $q \in V_p$.

Take $U_\rho = \{(\xi_p, w) | |\xi_p| < \rho, \quad |w| < \rho\}$ and $U'_\rho = \{(\xi_p, w) | |\xi_p| < (1-\epsilon)\rho, \quad |w| < (1-\epsilon)\rho\}$, where $\rho = \exp\{- (A_\alpha + B_\alpha \psi(p)^\alpha)\}$, $\epsilon$ is small positive and independent of $p$, in such a way that $\overline{U_\rho} \subset V_p$. Choose a locally finite system $\{U'_j\}_{j \in I_1}$ from among $\{U'_\rho\}_{\rho \in \mathbb{R}}$ so that $\bigcup U'_j \supset Y$. As a matter of fact we choose $\{p_j\}$ and denote $U'_j$ by $U'_j$. (Similarly for $U_j$.) We add $\{U'_j\}_{j \in I_2}$ and $\{U_j\}_{j \in I_2}$ to obtain covers of $X$, as described in §2. Those $U'_j$ and $U_j$ $(j \in I)$ which intersect $\bigcup U_j$ shall be taken in similar forms as $U'_\rho, U_\rho$ above with the origin $p_j$ of $(\xi_j)$-coordinates as the centre of $U'_j$, and $w_j = F - F(p_j)$ instead of $w_j = F$. Coordinates for $U_j$ which does not intersect any $U_k (k \in I_2)$ can be taken as linear coordinates unitary equivalent to $(\zeta)$. $\epsilon$ will be taken to be common all through.

These coordinate systems are not normalized as described in §2, but dependence of $\rho_j$ on $j$ is known and its effect on the choice of the function $\nu$ is clear.

We have only to estimate $r_{jk}$ appeared in §2. Take a pair $(j, k)$ of indices in $I_1$ such that $U_j \cap U_k \neq \emptyset$. $\xi_k^\alpha (\alpha = 1, \ldots, n-1)$ are linear functions of $(\xi_j, \ldots, \xi_k)$:
(4.6) \[
\xi_k^\alpha = \sum_{p_j} c_{p_j}^\alpha (p_k) \cdot \xi_j^\alpha \cdot \sum_{p_j} c_{p_j}^\alpha (p_k) (\xi_j^\alpha (p_j) - c_{p_j}^\alpha (p_k)) .
\]

$c_{p_j}^\alpha (p_k)$ and $c_{p_j}^\alpha (p_k)$ are entries of unitary matrices and hence coefficients of the linear terms in (4.6) are bounded when $j$ and $k$ vary. In $U_j$ we take $\xi_j^\alpha (\alpha = 1, \cdots, n-1)$ and $w_j = F$ as coordinates. $\xi_k^\alpha (1 \leq \alpha \leq n-1)$ are holomorphic functions of these coordinates. We have

\[
\frac{\partial \xi_k^\alpha}{\partial w_j} = \sum_{p_j} c_{p_j}^\alpha (p_k) \cdot \frac{\partial F}{\partial \xi_j^\alpha} ^\alpha .
\]

Hence

\[
\left| \frac{\partial \xi_k^\alpha}{\partial w_j} (q) \right| < \exp \{ A_\alpha + B_\alpha \psi (q) \} \quad \text{for } q \in U_j .
\]

Consider a point $q \in U_j \cap U_k$, for which

\[
|w_j (q)| < \exp \{ -(A_\alpha + B_\alpha \psi (q)) \}
\]

where $A_\alpha$ and $B_\alpha$ are suitably chosen. We have $|\xi_k^\alpha (q)| < (1-\epsilon) \rho_\alpha$ for $1 \leq \alpha \leq n-1$, hence for $q' \in \pi_j^{-1} (\pi_j (q))$ for which $|w_j (q')| \leq \exp \{ -(A_\alpha + B_\alpha \psi (q')) \}$, we have

\[
|\xi_k^\alpha (q')| < \rho_\alpha \quad (\alpha = 1, \cdots, n-1),
\]

and we see that $q' \in U_j \cap U_k$.

This shows that $r_j$ is not less than $\exp \{ -(A_\alpha + B_\alpha \psi (q)) \}$, and combined with above considerations, shows that we can take $v(t)$ to be $B_\alpha \exp (\eta t)$.

Note added in Proof. The condition $\eta \geq 1$ in Theorem 2 can be replaced by $\eta > 0$, if we take $\psi (\zeta)$ to be $\psi (\zeta) = \frac{1}{2} \log \{ 1 + \sum \| \xi_k^\alpha \|^2 \}$ and $v(t) = B_\alpha \exp (\eta t)$.

References