Stationary Fourier Hyperprocesses

By

Yoshifumi ITO*

Introduction

In this paper, we will define stationary Fourier hyperprocesses as an extension of stationary random functions and stationary random distributions in a similar way to Ito [6] and study their properties.

In § 1, we will first introduce some fundamental notions and prepare some notations.

In § 2, we will define the covariance Fourier hyperfunctions of stationary Fourier hyperprocesses, which correspond to Khintchine's covariance functions and Ito's covariance distributions [15], [6].

In § 3, we will prove the spectral decomposition theorem of covariance Fourier hyperfunctions.

In § 4, we will prove the spectral decomposition theorem of stationary Fourier hyperprocesses.

In § 5, we will mention the derivatives of stationary Fourier hyperprocesses.

§ 1. Fundamental Notions and Notations

In this paper we will restrict ourselves to complex valued random variables with mean 0 and finite variance. Let $H$ be the Hilbert space constituted by all such variables. In $H$, we define the inner product by the following relation:

$$(X, Y) = E(X \cdot Y), \quad \text{for } X, Y \in H,$$

where $E$ denotes the expectation. We will here consider only the strong topology on $H$. A continuous random process $X(t), -\infty < t < \infty$, is an $H$-valued continuous function on $\mathbb{R} = (-\infty, \infty)$. The set of all continuous processes will be denoted by $C(H)$.

Now we will remember the notions of Fourier hyperfunctions and vector valued Fourier hyperfunctions following Sato [17], Kawai [13], [14], Ito and...
Let $D = (-\infty, \infty)$ be a directional compactification of $R = (-\infty, \infty)$. Let $\mathcal{A}$ be the sheaf of germs of rapidly decreasing real analytic functions over $D$. Let $A = \mathcal{A}(D)$ be the space of all sections of $\mathcal{A}$ on $D$. $\mathcal{A}$ is endowed with the usual DFS topology. A Fourier analytic functional defined on $\mathcal{A}$ is called a Fourier hyperfunction on $D$ and a Fourier analytic linear mapping from $\mathcal{A}$ to $\mathcal{H}$ is called an $\mathcal{H}$-valued Fourier hyperfunction on $D$. We will denote by $\mathcal{A}'$ the space of all Fourier hyperfunctions on $D$ and by $A'(\mathcal{H})$ the space of all $\mathcal{H}$-valued Fourier hyperfunctions on $D$.

**Definition 1.1.** A Fourier hyperprocess is defined to be an $\mathcal{H}$-valued Fourier hyperfunction.

**Remark.** Our concept of Fourier hyperprocesses is a generalization of Okabe's concept of hyperprocesses in [18], Definition 6.1.

Let $\mathcal{C}(\mathcal{H})$ be the set of all $\mathcal{H}$-valued continuous functions on $R$ which satisfy the following estimate:

$$\sup_{t \in R} \|X(t)e^{-\varepsilon|t|}\| < \infty,$$

where $\|\|$ denotes the norm on $\mathcal{H}$. An element of $\mathcal{C}(\mathcal{H})$ is called a slowly increasing continuous process. Then $\mathcal{C}(\mathcal{H})$ may be considered as a subsystem of $A'(\mathcal{H})$, since we can identify a slowly increasing continuous random process $X(t)$ with the following Fourier hyperprocess $X(\phi)$ determined by it:

$$X(\phi) = \int_{-\infty}^{\infty} X(t)\phi(t)dt,$$

for $\phi \in \mathcal{A}$.

The following notations will be often used in the theory of Fourier hyperfunctions. Let $F \in A'$ or $A'(\mathcal{H})$ and $\phi \in \mathcal{A}$.

- $\tau_h$ (shift transformation): $\tau_h\phi(t) = \phi(t+h)$, $\tau_hF(\phi) = F(\tau_{-h}\phi)$.
- $D$ (derivative): $D\phi(t) = \phi'(t)$, $DF(\phi) = -F(D\phi)$.
- $\gamma$ (inversion): $\overline{\phi}(t) = \overline{\phi(-t)}$, $\overline{F}(\phi) = F(\overline{\phi})$.
- $\overline{\gamma}$ (conjugate): $\overline{\phi}(t) = \overline{\phi(t)}$, $\overline{F}(\phi) = F(\overline{\phi})$.
- $\wedge$ (Fourier transformation): $\hat{\phi}(\lambda) = \int e^{-\lambda \tau \lambda} \phi(t)dt$, $\hat{F}(\phi) = F(\phi)$.

The following relation should be noted.

$$(F \ast \phi)(0) = F(\hat{\phi}) = \hat{F}(\phi), \quad (F \ast \phi) \gamma = \hat{\phi} \ast \hat{\phi}, \quad \hat{\phi} = \hat{\phi}.$$  

Generalizing Khintchine-Ito's notions of (weakly) stationary processes, we have the following
**Definition 1.2.** We will call $X \in \mathcal{A}'(H)$ weakly stationary or merely stationary for short if we have, for any $\phi, \psi \in \mathcal{A},$

$$\langle \tau_h X(\phi), \tau_h X(\psi) \rangle = \langle X(\phi), X(\psi) \rangle$$

and strictly stationary if the joint probability law of

$$\langle \tau_h X(\phi_1), \cdots, \tau_h X(\phi_n) \rangle$$

is independent of $h$ for any $n$ and $\phi_1, \cdots, \phi_n \in \mathcal{A}.$

We shall adopt here the following notations:

- $\mathcal{S}$: the totality of stationary Fourier hyperprocesses,
- $\mathcal{S}^0$: the totality of slowly increasing stationary processes,
- $\mathcal{S}^{\delta}$: the totality of strictly stationary Fourier hyperprocesses,
- $\mathcal{S}^{\delta 0}$: the totality of slowly increasing strictly stationary processes.

Clearly we have

$$\mathcal{S} \supseteq \mathcal{S} \cup \mathcal{S}^0, \quad \mathcal{S}^0 \supseteq \mathcal{S}^{\delta 0}.$$

**Definition 1.3.** A Fourier hyperprocess $X$ is called a complex normal Fourier hyperprocess if $X(\phi), \phi \in \mathcal{A},$ constitute a complex normal system and a real normal Fourier hyperprocess if $X$ is real viz. $X = \overline{X}$ and $X(\phi), \phi$ running over real functions in $\mathcal{A},$ constitute a (real) normal system (see Itô [4], [5] and Hida [3]).

This is an extension of normal processes or Gaussian processes (Doob [1], II, § 3) and complex (or real) normal random distributions (Itô [6]). A (complex as well as real) normal Fourier hyperprocess will be strictly stationary, if it is weakly stationary. The corresponding fact is well-known regarding stationary processes.

We shall here mention a typical example of real stationary Fourier hyperprocesses which are not stationary processes. Let $B(t)$ be a (real) Brownian motion process (Doob [1], p. 97). The derivative (in the sense of Fourier hyperfunctions) of this process $B' = DB$ is a Fourier hyperprocess defined by

$$B'(\phi) = -B(\phi') = \int \phi(t) dB(t)$$

(Wiener integral, see Itô [4]). This is evidently real normal and stationary, since

$$\langle \tau_h B'(\phi), \tau_h B'(\psi) \rangle = \langle B'(\tau_h \phi), B'(\tau_h \psi) \rangle$$

$$= \left( \int \phi(t-h) dB(t), \int \psi(t-h) dB(t) \right)$$

$$= \int \phi(t-h) \overline{\psi(t-h)} dt = \int \phi(t) \overline{\psi(t)} dt$$
which shows that $B' \in S$. The fact that $B' \in S^o$ will be proved in §2.

§2. Covariance Fourier Hyperfunctions

Similarly to Khintchine-Itô’s notion of covariance distributions, we will here define the notion of the covariance Fourier hyperfunctions.

**Theorem 2.1.** Let $X(\phi)$ be any stationary Fourier hyperprocess. Then there exists one and only one Fourier hyperfunction $\rho \in \Delta'$ satisfying the relation

$$(X(\phi), X(\phi)) = \rho(\phi \ast \phi), \quad \phi, \phi \in \Delta.$$ 

**Definition 2.2.** The Fourier hyperfunction $\rho$ in Theorem 2.1 is called the covariance Fourier hyperfunction of $X$.

**Proof of Theorem 2.1.** If we put

$$T_\phi(\phi) = (X(\phi), X(\phi)), \quad \phi, \phi \in \Delta,$$

then we get a Fourier hyperfunction $T_\phi \in \Delta'$ for each $\phi \in \Delta$. Taking into account the fact that $T_\phi(\phi)$ is continuous in $(\phi, \phi) \in \Delta \times \Delta$ and by virtue of Kernel Theorem, we will easily see that $\phi \rightarrow T_\phi$ is a continuous linear mapping from $\Delta$ into $\Delta'$ (see Grothendieck [2], Chap. II, Théorème 12 and Ito [7]). Furthermore this transformation commutes with the shift transformation:

$$(\tau_h T_\phi)(\phi) = T_{\phi}(\tau_h \phi)$$. 

Here we will use the following

**Lemma.** A continuous linear mapping $\phi \rightarrow T_\phi$ from $\Delta$ to $\Delta'$ commutes with the shift transformation if and only if there exists a Fourier hyperfunction $T$ such that $T_\phi = T \ast \phi$ holds.

Postponing the proof of this Lemma until the end of the proof of this Theorem, we will continue the proof of the Theorem. Thus by the above Lemma $T_\phi$ is expressible as a convolution of a Fourier hyperfunction $T$ and $\phi:

$$T_\phi = T \ast \phi.$$ 

Hence it follows that

$$(X(\phi), X(\phi)) = T_\phi(\phi) = (T \ast \phi)(\phi)$$

$$=(T \ast \phi \ast \phi)(0) = \rho(\phi \ast \phi),$$

where we put $\rho = \tilde{T}$.

The uniqueness of $\rho$ follows at once from the fact that the set of all
elements of the form $\phi * \phi$, $\phi, \phi \in \Delta$, is dense in $\Delta$. Q. E. D.

Proof of Lemma. It is evident that a continuous linear mapping $\phi \mapsto T_\phi = T * \phi$ commutes with the shift transformation. Thus we have only to prove that $T_\phi$ is expressible as a convolution of a Fourier hyperfunction $T$ and $\phi : T_\phi = T * \phi$ if it commutes with the shift transformation.

We will first show that, if $\alpha \in \Delta$, we have

$$T_{\alpha * \phi} = T_\alpha * \phi.$$ 

In fact, we have, by the definition of integration,

$$\langle \phi, \phi \rangle = \int \phi(t) \overline{\phi}(t) dt = \lim_{j} \sum_{n} a_n \phi(h_{nj}) = \langle \lim_{j} \sum_{n} a_n \tau_{h_{nj}} \delta, \phi \rangle$$

for any $\phi \in \Delta$ considering $\phi$ as a Fourier hyperfunction, where $\sum_n$'s are finite sums. Thus we have the relation

$$\phi = \lim_{j} \sum_{n} a_n \tau_{h_{nj}} \delta$$

as a Fourier hyperfunction. Therefore we have

$$\alpha * \phi(t) = \langle \lim_{j} \sum_{n} a_n \tau_{h_{nj}} \delta \rangle * \alpha(t) = \lim_{j} \sum_{n} a_n \tau_{h_{nj}} \alpha(t).$$

Hence we have, by the assumption,

$$T_{\alpha * \phi} = \lim_{j} \sum_{n} a_n \tau_{h_{nj}} T_\alpha = \langle \lim_{j} \sum_{n} a_n \tau_{h_{nj}} \delta \rangle * T_\alpha = \phi * T_\alpha = T_\alpha * \phi.$$ 

Then if we choose a sequence $\alpha_v \in \Delta$ which converges to $\delta$ in $\Delta'$, the sequence $\alpha_v * \phi$ converges to $\phi$ in $\Delta$. Hence $T_{\alpha_v * \phi}$ converges to $T_\phi$ in $\Delta'$. Hence, for the Fourier hyperfunctions $T_{\alpha_v}$, their regularizations $T_{\alpha_v} * \phi$ converges in $\Delta'$ for every $\phi \in \Delta$. Thus $T_{\alpha_v}$ itself converges in $\Delta'$. If $T$ is its limit, we have

$$T_\phi = T * \phi, \quad T \in \Delta'.$$ Q. E. D.

Theorem 2.3. If $X(\phi)$ is a real stationary Fourier hyperprocess, then its covariance Fourier hyperfunction is real, i.e. $\rho = \overline{\rho}$.

Proof. Let $\rho$ be the covariance Fourier hyperfunction. Then that of $\overline{X}$
will become \( \bar{p} \), since we have
\[
(X(\phi), X'(\phi)) = (X(\phi), X(\phi)) = (X(\phi), X(\phi)) = \rho(\phi * \phi) = p.
\]
Thus \( X = X' \) implies \( p = \bar{p} \).

**Example.** The covariance Fourier hyperfunction of \( B' \) is Dirac's \( \delta \)-function, since
\[
(B'(\phi), B'(\phi)) = \int \phi(t) \overline{\phi(t)} dt = \int \phi(t) \overline{\phi(-t)} dt = (\phi * \phi)(0) = \delta(\phi * \phi).
\]
Thus we see that \( B' \in S^0 \), because, if \( B' \in S^0 \), then the covariance Fourier hyperfunction would be induced by a slowly increasing continuous function as shown similarly to Itô [6], §2.

§ 3. Spectral Decomposition of Covariance Fourier Hyperfunctions

Let \( X(\phi) \) be any stationary Fourier hyperprocess with the covariance Fourier hyperfunction \( \rho \). Then we have
\[
\rho(\phi * \phi) = (X(\phi), X(\phi)) \geq 0,
\]
which implies that \( \rho \) is a positive semidefinite Fourier hyperfunction. Thus, by virtue of Bochner-Nagamachi-Mugibayashi-Junker's Theorem (see Nagamachi-Mugibayashi [16], Theorem 4.1 and Junker [12], Theorem 5.8), we have the following

**Theorem 3.1.** \( \rho \) is expressible in the form
\[
(\star) \quad \rho(\phi) = \int \phi(\lambda) d\mu(\lambda), \quad \phi \in A
\]
in one and only one way, where \( \mu \) is a nonnegative measure satisfying
\[
\int e^{-\epsilon|\lambda|} d\mu(\lambda) < \infty \quad \text{for every } \epsilon > 0.
\]

**Definition 3.2.** We will call the expression \( \star \) the spectral decomposition of \( \rho \) and \( \mu \) the spectral measure of \( \rho \).

Conversely we have

**Theorem 3.3.** Any Fourier hyperfunction of the above form \( \star \) is the
covariance Fourier hyperfunction of a stationary Fourier hyperprocess which is complex normal.

**Proof.** Let \( \rho \) be a Fourier hyperfunction of the above form. Put

\[
\Gamma(\phi, \phi') = \rho(\phi \ast \phi'), \quad \phi, \phi' \in \mathcal{A}.
\]

Then \( \Gamma(\phi, \phi') \) is positive semidefinite in \( (\phi, \phi') \), as we have

\[
\sum_{i,j=1}^n \Gamma(\phi_i, \phi_j) \xi_i \xi_j = \rho(\theta \ast \theta) \geq 0, \quad \theta = \sum_i \xi_i \phi_i.
\]

Therefore we can define a complex normal system \( X(\phi), \phi \in \mathcal{A} \), such that

\[
EX(\phi) = 0 \quad \text{and} \quad E(X(\phi) \cdot \overline{X(\phi)}) = \Gamma(\phi, \phi) = \rho(\phi \ast \phi^\ast) \quad \text{as in Ito [5] and Hida [3].}
\]

It remains only to show that \( X(\phi) \) is a Fourier hyperprocess. From the identity:

\[
\|X(\phi) - \epsilon X(\phi)\|^2 = (X(\phi), X(\phi)) - c(X(\phi), X(\phi))
\]

\[
-\epsilon(X(\phi), X(\phi)) + c\epsilon(X(\phi), X(\phi))
\]

\[
= \rho(\phi \ast \phi^\ast) - c\rho(\phi \ast \phi^\ast) - \epsilon \rho(\phi \ast \phi^\ast) + c\epsilon \rho(\phi \ast \phi^\ast)
\]

\[
= c\epsilon \rho(\phi \ast \phi^\ast) - \epsilon \rho(\phi \ast \phi^\ast) - \epsilon \rho(\phi \ast \phi^\ast) + c\epsilon \rho(\phi \ast \phi^\ast)
\]

\[
= 0,
\]

it follows that \( X(\phi) = -\epsilon X(\phi) \). By a similar way we can see that \( X(\phi + \phi') = X(\phi) + X(\phi) \). Therefore \( X(\phi) \) is linear in \( \phi \). By the identity \( \|X(\phi)\|^2 = \rho(\phi \ast \phi^\ast) \) we obtain the continuity of \( X \). Thus our theorem is completely proved.

Q. E. D.

Next we shall discuss the case of real stationary Fourier hyperprocesses. By Theorem 2.3 we see that \( \rho = \overline{\rho} \) in this case. But we have

\[
\rho(\phi) = \overline{\rho(\phi)} = \int_{-\mathbb{R}} \overline{\phi(\lambda)} d\mu(\lambda) = \int_{\mathbb{R}} \phi(\lambda) d\mu(\lambda) = \int_{-\mathbb{R}} \phi(\lambda) d\overline{\mu}(\lambda)
\]

\[
(\overline{\mu(E)} = \mu(-E), \quad -E = \{t; -t \in E\}).
\]

By the uniqueness of the spectral measure we will obtain the following

**Theorem 3.4.** In the case of a real stationary Fourier hyperprocess, the spectral measure \( \mu \) is symmetric with respect to 0, viz. \( \mu(E) = \mu(-E) \).

Conversely we have

**Theorem 3.5.** Any Fourier hyperfunction of the form (\( \ast \)) with a symmetric measure \( \mu \) is the covariance Fourier hyperfunction of a certain stationary Fourier hyperprocess which is real normal.

The proof is similar to that of Theorem 3.3; we use the existence theorem of real normal systems instead of complex normal ones.
Example. $B'$ is a real stationary Fourier hyperprocess whose spectral measure is the ordinary Lebesgue measure, because we have

$$
\delta(\phi) = \phi(0) = \int \phi(\lambda) d\lambda.
$$

§ 4. Spectral Decomposition of Stationary Fourier Hyperprocesses

We will first introduce a random hypomeasure. Let $\mu$ be a nonnegative measure defined for all Borel sets in $\mathbb{R}$ and $B^*$ denote the system of all Borel sets with finite $\mu$-measure.

Definition 4.1. An $H$-valued function $M(E)$ defined for $E \in B^*$ is called a random hypomeasure with respect to $\mu$ if

$$(M(E_1), M(E_2)) = \mu(E_1 \cap E_2), \quad E_1, E_2 \in B^*$$

holds.

We get, by the definition,

Theorem 4.2. Let $M(E)$ be a random hypomeasure with respect to $\mu$. Then we have

1. $\|M(E)\|^2 = \mu(E),$
2. $M(E_1) \perp M(E_2)$ if $E_1 \cap E_2 = \emptyset,$
3. If $E_1, E_2, \ldots$ are disjoint to each other and belong to $B^*$ with their sum $E = \sum_{n=1}^{\infty} E_n$, $M(E) = \sum_{n=1}^{\infty} M(E_n)$, (in $H$).

We can easily define the integral with respect to the random hypomeasure (Doob [1], IX, § 2):

$$M(f) = \int f(\lambda) dM(\lambda)$$

for $f \in L^2(\mathbb{R}, \mu)$.

Then we have the following

Theorem 4.3. Let $M(f)$ be as above. Then we have, for $f_1, f_2 \in L^2(\mathbb{R}, \mu)$ and $c_1, c_2 \in C,$

1. $\langle M(f_1), M(f_2) \rangle = \langle f_1, f_2 \rangle \left( = \int f_1(\lambda) f_2(\lambda) d\mu(\lambda) \right),$
2. $M(c_1 f_1 + c_2 f_2) = c_1 M(f_1) + c_2 M(f_2).$

Theorem 4.4. Let $X$ be any stationary Fourier hyperprocess with the spectral measure $\mu$. Then $X(\phi)$ will be expressible in the form

$$X(\phi) = \int \phi(\lambda) dM(\lambda) = M(\phi)$$

(***)}
in one and only one way, \( M \) being a random hypomeasure with respect to \( \mu \). Conversely, any Fourier hyperprocess of such form is a stationary Fourier hyperprocess.

**Definition 4.5.** We will call the expression (**) the spectral decomposition of \( X \) and \( M \) the spectral hypomeasure of \( X \).

**Proof of Theorem 4.4.** We will first remark that \( \mathcal{A} \) is dense in \( L^2 \equiv L^2(\mathbb{R}, \mu) \). Then the Fourier transformation is a topological isomorphism from \( \mathcal{A} \) onto itself. Thus the uniqueness of the expression is clear.

In order to prove the possibility of the expression, we will put

\[
T(\phi) = X(\phi) \quad \text{for} \quad \phi = \hat{\phi}.
\]

Then \( T \) will be a mapping from \( \mathcal{A} \) (\( \subset L^2 \)) into \( H \), which is clearly linear and isometric on account of the identity:

\[
\| T(\phi) \| = (X(\phi), X(\phi)) = \rho(\phi \ast \hat{\phi})
= \int |\phi(\lambda)|^2 d\mu(\lambda) = \| \phi \|^2,
\]

since \( (\phi \ast \hat{\phi})^\ast = |\phi|^2 \). \( \mathcal{A} \) being dense in \( L^2 \), we can extend \( T(\phi) \) to a linear isometric mapping from \( L^2 \) into \( H \). As the characteristic function \( 1_E(\lambda) \) of a set \( E \in \mathcal{B}^* \) belongs to \( L^2 \), we may define \( M(E) \) as follows:

\[
M(E) = T(1_E).
\]

Then we have

\[
(M(E_1), M(E_2)) = \int 1_{E_1}(\lambda) \overline{1_{E_2}(\lambda)} d\mu(\lambda) = \mu(E_1 \cap E_2),
\]

since \( T \) is isometric. In addition to this, we will have

\[
(***) \quad M(f) = T(f) \quad \text{for} \quad f \in L^2,
\]

for this is evidently true for any simple function \( f \) in \( L^2 \) by the definition and we will easily see that it is also true for any \( f \in L^2 \), by taking into account the fact that both sides of (***), are isometric in \( f \) and any \( f \in L^2 \) is expressed as the \( L^2 \)-limit of a sequence of simple functions. If we put \( f = \hat{\phi} \) in (***) , we obtain (**) at once. The last part of the theorem is clear by the definitions.

Q. E. D.

Making use of this theorem we can characterize the class of slowly increasing stationary processes.

**Theorem 4.6.** A slowly increasing stationary process \( X \) is a stationary Fourier hyperprocess with the spectral measure such that
Proof. A slowly increasing stationary process $X$ is a stationary continuous random process. By Khinchine's Theorem (see Khinchine [15]), its spectral measure $\mu$ satisfies the assumption of the theorem and, by virtue of Doob [1], XI, §4, we have the expression

$$X(\phi)=\int \hat{\phi}(\lambda) dM(\lambda)=M(\hat{\phi}).$$

Thus, by Theorem 4.4, $X$ induces a stationary Fourier hyperprocess with the spectral measure $\mu$.

Conversely, let $X$ be a stationary Fourier hyperprocess with the spectral measure $\mu$ which satisfies the assumption of the theorem. Then we have

$$X(\phi)=\int \hat{\phi}(\lambda) dM(\lambda), \quad (M(E_1), M(E_2)=\mu(E_1 \cap E_2),$$

where

$$\int d\mu(\lambda)<\infty.$$

Put

$$Y(t)=\int e^{-i2\pi \lambda t} dM(\lambda),$$

which may be defined, since the $\lambda$-function $e^{-i2\pi \lambda t}$ belongs to $L^2$ by virtue of the assumption on $\mu$, $Y(t)$ proves to be a stationary continuous random process and, what is more, it becomes a slowly increasing stationary continuous random process. Therefore, we have, for $\phi \in A$,

$$\int Y(t)\phi(t) dt=\int \hat{\phi}(t) \int e^{-i2\pi \lambda t} dM(\lambda) dt$$

$$=\int \hat{\phi}(\lambda) dM(\lambda)=X(\phi),$$

which implies that $X(\phi)$ is induced by a slowly increasing stationary process $Y$.

Q. E. D.

In the proof of the above theorem, we have the following

**Corollary.** A slowly increasing stationary continuous random process and a stationary continuous random process are identical.

By the same way as in Theorem 3.4, we will obtain

**Theorem 4.7.** In the case of a real stationary Fourier hyperprocess the spectral random hypomeasure $M$ is hermitian symmetric, i.e. $M(E)=\overline{M(-E)}$. 
§ 5. Derivatives of Stationary Fourier Hyperprocesses

Any Fourier hyperprocess has derivatives of any order, which are also Fourier hyperprocesses.

Theorem 5.1. Let \( X \) be a stationary Fourier hyperprocess with the spectral measure \( \mu \) and the spectral random hypomeasure \( M \). Then \( X^{(k)} (=D^kX) \) is also a stationary Fourier hyperprocess whose spectral measure \( \mu_k \) and spectral random hypomeasure \( M_k \) are given by

\[
d\mu_k(\lambda) = (2\pi \lambda)^k d\mu(\lambda), \quad dM_k(\lambda) = (i2\pi \lambda)^k dM(\lambda).
\]

Proof. We have, by definition,

\[
X^{(k)}(\phi) = (-1)^k X(\phi^{(k)}) = (-1)^k \int \phi^{(k)}(\lambda) dM(\lambda)
\]

since we have, for \( \phi \in \mathcal{A} \),

\[
\phi^{(k)}(\lambda) = (-1)^k (i2\pi \lambda)^k \phi(\lambda).
\]

Thus \( X^{(k)} \) proves to be a stationary Fourier hyperprocess satisfying the above conditions. Q. E. D.

By Theorem 4.6 we have the following

Theorem 5.2. In order that \( X^{(k)} \) is a stationary continuous process, it is necessary and sufficient that the spectral measure \( \mu \) of \( X \) satisfies

\[
\int \lambda^{2k} d\mu(\lambda) < \infty.
\]

References


