Squaring Operations in the 4-Connective Fibre Spaces over the Classifying Spaces of the Exceptional Lie Groups

Dedicated to Professor Nobuo Shimada on his 60th birthday

By

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§ 1. Introduction

In this paper we calculate the squaring operations in the 4-connective fibre spaces over the classifying spaces of the exceptional Lie groups.

Let $G$ be a compact, 1-connected and simple Lie group. As is well known, its classifying space $BG$ is 3-connected and $H^*(BG; \mathbb{Z}) \cong \mathbb{Z}$. Choose a generator $y_i \in H^i(BG; \mathbb{Z})$. Then the 4-connective fibre space $BG$ over $BG$ is, by definition, the homotopy fibre of $y_i : BG \to K(\mathbb{Z}, 4)$. Note that $BG$ is a classifying space of $G$, the 3-connective fibre space over $G$. Here we quote the results in [2] and [4].

Define the sets $J_l$ ($l = 2, 4, 6, 7, 8$) as follows:

\[
J_2 = \{9, 10, 12, 2^i+1 (i \geq 4)\}, \quad J_4 = J_2 \cup \{16, 24\},
\]
\[
J_6 = \{10, 12, 16, 18, 24, 33, 34, 2^i+1 (i \geq 6)\},
\]
\[
J_7 = \{12, 16, 20, 24, 28, 33, 34, 36, 2^i+1 (i \geq 6)\},
\]
\[
J_8 = \{16, 24, 28, 30, 31, 33, 34, 36, 40, 48, 2^i+1 (i \geq 6)\}.
\]

**Theorem 1.1.** Let $G$ be one of $G_2$, $F_4$ and $E_i$ ($l = 6, 7, 8$). Then

\[
H^*(BG; F_2) = F_2[y_j; j \in J_l] \quad (l = \text{rank } G, \text{ deg } y_j = j)
\]

where the generators can be taken so as to satisfy the following equalities whenever the suffixes in both sides appear in $J_l$:

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For the most part squaring operations on the \( y_j \) are determined from the data (1.1) by use of the Adem relations, but some remain undetermined. Our objective is to determine them completely.

In §2 we introduce a space \( B\hat{T} \) and a map \( \hat{\lambda}:B\hat{T}\to B\hat{G} \), where the induced homomorphism \( \hat{\lambda}^* \) is almost injective. In §§3 and 4 we investigate the action of the Weyl group \( W(G) \) on \( B\hat{T} \), and the \( \hat{\lambda}^*(y_j) \) are determined, and in the final section we give the complete list of \( Sq^\ast y_j \) and the correspondence of the generators between different groups.

Throughout this paper \( H^\ast(\_\_) \) denotes the mod 2 cohomology ring, and \( \rho:H^\ast(\_\_;A)\to H^\ast(\_\_;\hat{A}) \) denotes the mod 2 reduction for \( A=\mathbb{Z} \) or \( \mathbb{Z}_2 \). \( \sigma_i(x_1,\ldots,x_n) \) denotes the \( i \)-th elementary symmetric polynomial in the \( x_i \).

§ 2. Cohomology of \( B\hat{T} \) and \( B\hat{C} \)

In this and the following two sections \( G \) denotes the compact 1-connected exceptional Lie group of type \( E_l \) (\( l=6,7,8 \)), and \( T \) a maximal torus of \( G \). The Dynkin diagram of \( G \) is

\[
\begin{array}{cccccc}
\alpha_1 & \alpha_3 & \alpha_4 & \alpha_5 & \cdots & \alpha_l \\
\alpha_2 & & & & & \\
\end{array}
\]

where the \( \alpha_i \) are the simple roots. Define a 1-dimensional torus \( T^1 \) by the equations \( \alpha_i=0 \) \( (i\neq2) \), and let \( C\subset G \) be the centralizer of \( T^1 \). Note that (see [1], for example)

(2.1) \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad C=T^1\cdot SU(l).

The inclusions \( T\subset C\subset G \) induce maps \( \iota:BT\to BC \), \( \kappa:BC\to BG \) and \( \lambda=\kappa\circ\iota:BT\to BG \). Then the space \( B\hat{T} \) (resp. \( B\hat{C} \)) is, by definition, the homotopy fibre of \( y_1^1\lambda:BT\to K(\mathbb{Z},4) \) (resp. \( y_1^1\kappa:BC\to K(\mathbb{Z},4) \)). The maps \( \iota \), \( \kappa \) and \( \lambda \) induce maps \( \hat{\iota}:B\hat{T}\to B\hat{C} \), \( \hat{\kappa}:B\hat{C}\to B\hat{G} \) and \( \hat{\lambda}=\hat{\kappa}\circ\hat{\iota}:B\hat{T}\to B\hat{G} \), which make the following diagrams commutative:
where rows and columns are fiberings. From the first diagram we have

**Lemma 2.1.** \( \iota \) and \( \check{\iota} \) induce monomorphisms in \( H^* (\cdot; A) \) for any \( A \).

In fact, \( H^{\text{odd}} (C/T; A) = 0 \) by (2.1) and \( H^{\text{odd}}(BT; A) = 0 \), whence the Serre spectral sequence collapses for the lower row. Then \( \iota^* \) is onto, hence so is \( \check{\iota}^* \), and the spectral sequence collapses for the upper row.

From now on by use of \( \iota^* \) (resp. \( \check{\iota}^* \)) we regard \( H^*(BC°; A) \) (resp. \( H^*(BC'; A) \)) as a subalgebra of \( H^*(BT; A) \) (resp. \( H^*(BT'; A) \)).

Recall that the fundamental weights \( w_i \) (\( i = 1, 2, \ldots, l \)) form a basis of \( H^2(BT; \mathbb{Z}) \). For convenience of calculation we introduce \( t_i \) and \( c_j \in H^*(BT; \mathbb{Z}) \) in the following way. Let \( R_i \) be the reflection in the plane \( \alpha_i = 0 \). After [5] and [3] we define

\[
R_i = t_i = w_i, \quad t_1 = R_{i+1}(t_{i+1}), \quad t_1 = R_{i}(t_2)
\]

\[
c_j = c_j(t_1, \ldots, t_i) \quad \text{and} \quad t = \frac{1}{3} c_1 = w_2.
\]

Then each \( R_i \) (\( i \neq 2 \)) acts on \( \{t_j\} \) as a transposition, and

\[
(2.3) \quad R_2(t_j) = \begin{cases} 
   t - b_1 + t_j & (j \leq 3) \\
   t_j & (j \geq 4)
\end{cases}
\]

(\( b_1 = t_1 + t_2 + t_3 \)).

Since the Weyl group \( W(G) \) (resp. \( W(C) \)) is generated by \( \{R_i\} \) (resp. \( \{R_i; i \neq 2\} \)), we have from the data above that

\[
(2.4) \quad H^4(BT; \mathbb{Z})^{W(G)} = \mathbb{Z} \cdot (c_2 - 4t^2),
\]

\[
H^*(BT; \mathbb{Z})^{W(G)} = \mathbb{Z}[t, c_2, c_3, \ldots, c_l].
\]

Note that the Weyl group \( W(X) \) acts trivially on the image of \( H^*(BX; \mathbb{Z}) \rightarrow H^*(BT; \mathbb{Z}) \), and that \( H^*(X/T; \mathbb{Z}) \) is torsion free by the classical result of Bott. Consider the Poincaré polynomial of \( H^*(BC; \mathbb{Z}) \), which is obtained from (2.1). Then we have

**Theorem 2.2.** (i) \( \chi^* y_4 = \pm (c_2 - 4t^2) \).
Now consider the fiberings derived from the columns in (2.2):

\[
\begin{array}{ccc}
K(Z,3) & \longrightarrow & K(Z,3) \\
\downarrow & \downarrow & \downarrow \\
BT & \longrightarrow & BC \longrightarrow BG \\
\downarrow & \downarrow & \downarrow \\
BT & \longrightarrow & BC \longrightarrow BG \\
\end{array}
\quad (\lambda = \bar{e} \circ \tau)
\]

(2.5)

where the cohomology of the common fibre is given by

\[
H^*(K(Z,3)) = F_z[u_{i+1} ; i \geq 1] (\deg u_j = j), \quad Sq^i u_{i+1} = u_{i+1+j},
\]

By the definition of \(BT\) the fundamental class \(u_3\) transgresses to \(\rho(\lambda^* y) = \rho(c_3)\), by (i) of Theorem 2.2. To avoid complexity we will omit the symbol \(\rho\) except in the case of emphasis. Then

**Lemma 2.3.** (i) The transgression \(\tau\) is given by

\[
\tau(u_3) = c_2, \quad \tau(u_3) = c_3, \quad \tau(u_3) = c_4, \quad \tau(u_3) = c_5
\]

and

\[
\tau(u_{i+1}) = 0 \quad (i \geq 5) \text{ modulo the images in lower dimensions,}
\]

where \(c_5 = c_3 + c_4 c_5\) and \(c_i = c_2 c_1 + c_3 c_i + c_4 c_i^2 + c_5 c_i^3\).

(ii) The sequence \((c_2, c_3, c_5, c_6)\) is regular in both \(H^*(BC)\) and \(H^*(BT)\).

**Proof.** (i) This follows from the Wu formula and the commutativity of the transgression with \(Sq^i\).

(ii) Clearly the sequence is regular in \(H^*(BC)\). Then its regularity in \(H^*(BT)\) follows from the fact that \(H^*(BT)\) is a free \(H^*(BC)\)-module.

To simplify notation we will omit the symbol \(g^*\) in \(g^*x\) for \(x \in H^*(BT; \Lambda)\). Define \(J' = \{2^k + 1 ; k \geq 5\}\). Then the main theorem in this section is stated as follows:

**Theorem 2.4.** There exist

\[
\begin{array}{c}
\gamma_i \in H^{2i}(BC) \subset H^{2i}(BT) \quad (i = 3, 5, 9, 17) \\
v_j \in H^j(BC) \subset H^j(BT) \quad (j \in J')
\end{array}
\]

such that

\[
\begin{array}{c}
H^*(BC) = F_2[c_1, c_2, \ldots, c_i, \gamma_3, \gamma_9, \gamma_{17}, v_j ; j \in J']/(c_2, c_3, c_5, c_6), \\
H^*(BT) = F_2[t_1, t_2, \ldots, t_i, \gamma_3, \gamma_9, \gamma_{17}, v_j ; j \in J']/(c_2, c_3, c_5, c_6)
\end{array}
\]
where the generators are related by
\[ Sq^{2i-2}t_i = t_{3i-1} \quad (i = 3, 5, 9) \quad \text{and} \quad Sq^{i-1}v_j = v_{2j-1} \quad (j \in J'). \]

**Proof.** Consider the Serre spectral sequence for the middle column in (2.5). By (i) of Lemma 2.3 there exist \( \gamma_3 \in H^e(B\tilde{C}) \) and \( v_{33} \in H^{33}(B\tilde{C}) \) such that \( f^*\gamma_3 = u_3^2 \) and \( f^*v_{33} = u_{33}. \) Define \( \gamma_{2i+1} = Sq^{2i-1}t_i (i = 2, 4, 8) \) and \( v_{2j+1} = Sq^{2j-1}v_j (j \in J') \), so that \( f^*\gamma_i = u_i^2 \) (\( i = 3, 5, 9, 17 \)) and \( f^*v_j = u_j \) (\( j \in J' \)). Then by (ii) of the lemma we have
\[ H^*(B\tilde{C}) = H^*(B\tilde{C})/(c_2, c_3, c_9, c_9') \otimes F_2[\gamma_i, v_j; i = 3, 5, 9, 17, j \in J'] \]
and the same with \( C \) replaced by \( T \).

§ 3. The Action of the Weyl Group on \( H^*(BT) \)

Recall that \( y_4^3 = y_4 \) is \( W(G) \)-invariant. Thus the action of \( W(G) \) on \( BT \) lifts to \( BT \) in such a way that
\begin{align*}
(3.1) & \quad \text{the canonical map } g: BT \to BT \text{ is equivariant, and} \\
(3.2) & \quad W(X) \text{ acts trivially on the image of } H^*(BX; A) \to H^*(BT; A) \quad \text{where } X = C \text{ or } G.
\end{align*}

By (3.1) the action of \( W(G) \) on \( \{t_i\} \) in \( H^*(BT) \) is the same as that in \( H^*(BT) \). In order to determine the action on \( \{\gamma_i\} \) we consider the cohomology with coefficients \( Z_\omega \).

By Theorem 2.4 \( H^*(B\tilde{C}; Z_\omega) \) is torsion free for \( * \leq 32 \). Thus we can define \( g_3 \in H^{33}(B\tilde{C}; Z_\omega) \subset H^* (B\tilde{T}; Z_\omega) \) (\( i = 3, 5, 9 \)) by
\[ 2g_3 = c_3^2, \quad 2g_5 = c_5' = c_5 + c_4c_1 \quad \text{and} \quad 2g_9 = c_9' = c_9c_1 + c_8c_1^2 + c_7c_1^3 \]
since \( \rho(c_3) = \rho(c_5') = \rho(c_9') = 0. \) As a corollary to 2.2 in [3]

Lemma 3.1. \( g_3 \) is not divisible by 2.

Therefore \( f^* : H^*(BT) \to H^*(K(Z, 3)) \) sends \( \rho(g_3) \) to \( u_3^2 \), and so we may take \( \gamma_3 = \rho(g_3) \) in Theorem 2.4.

Now we shall determine the action of \( W(G) \) on \( \{\rho(c_i)\}, \{\rho(g_i)\} \) and \( \{\gamma_i\} \). Each \( R_i \) (\( i \neq 2 \)) acts trivially on them by (3.2) with \( X = C \). Our objective in this section is to determine the action of \( R_2 \).

From now on we will exclusively use the notations
\[ R' = R - 1 \quad \text{and} \quad R = R - 1. \]

Define \( b_i = \sigma_i(t_3, t_3) \) and \( a_i = \sigma_i(t_4, t_5, \ldots, t_i) \in H^*(BT; Z) \), so that
\[ c_n = \sum_{i+j=n} b_i a_j. \]

By (2.3) \( R(a_j) = a_j \) for any \( j \), and
\[
\sum R(b_i) = R(\sum b_i) = R(\prod_{i=1}^{n} (1 + t_i)) = \prod_{i=1}^{n} (1 + R(t_i)) \]
\[ = \prod_{i=1}^{n} (1 + t - b_i + t_i) = \sum (1 + t - b_i)^{3-i} b_i. \]

Substituting \( c_1 = 3t, c_2 = 4t^2, c_3 = 2g_3 \) and \( c_5 = 2g_5 - c_1 c_1 \) into (3.3), the \( b_i \) and \( a_5 \) are expressed in terms of \( g_3, g_5, a_j \) (\( 1 \leq j \leq 4 \)) and \( t \). Then so are the \( R(b_j) \): \[ R(b_1) \equiv -a_1, \quad R(b_2) \equiv a_1^2 \] and \( R(b_3) \equiv -a_2a_1 + a_3 \) mod \((4, 2t)\), and so are the \( R(c_n) = \sum R(b_j) a_{n-i} \). For instance, \( R(c_3) \equiv 2a_2a_1 + 2a_3 \) mod \((4, 2t)\), which implies \( R(g_3) \equiv a_3a_2 + a_3 \) mod \((2, t)\) since \( H^8(B^7; \mathbb{Z}_2) \) is torsion free. In other words, in \( H^* (B^7) \)
\[ R(\rho (g_j)) \equiv \rho (a_j) \rho (a_i) + \rho (a_3) \] mod \((t)\).

Similar calculations give \( R(\rho (g_j)) \) mod \((t)\) for \( j = 5 \) and \( 9 \). On the other hand, since \( \gamma_{2j+1} = S q \gamma_{2j+1} \) and \( R \) commutes with \( S q \), \( R(\gamma_j) \) \( (j = 5, 9, 17) \) are derived from (3.4) by use of the Wu formula.

The results are given in the following table, where for simplicity the symbol \( \rho \) is omitted again, and the \( a_i \) \( (i \geq 1) \) is abbreviated as \( i \); e.g., \( 321^3 \) is the abbreviation of \( a_3a_2a_1 \) \( (0 \) denotes not \( a_0 \) but the null):

<table>
<thead>
<tr>
<th>( x )</th>
<th>( x ) mod ((t))</th>
<th>( R(x) ) mod ((t))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c_1 )</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( c_4 )</td>
<td>( 4 + 2^2 + 21^2 + 1^4 )</td>
<td>( 31 + 1^4 )</td>
</tr>
<tr>
<td>( c_6 )</td>
<td>( 42 + 3^2 + 21^4 )</td>
<td>( 321 + 21^4 )</td>
</tr>
<tr>
<td>( c_7 )</td>
<td>( 43 + 421 + 321^2 + 31^4 + 2^21^3 + 21^5 )</td>
<td>( 421 + 3^4 - 21^5 )</td>
</tr>
<tr>
<td>( e_8 )</td>
<td>( 431 + 41^4 + 3^21^2 + 321^3 + 31^5 + 21^6 )</td>
<td>( 421^2 + 41^4 + 321^3 + 31^5 + 2^21^4 + 21^6 )</td>
</tr>
<tr>
<td>( \gamma_5 = g_3 )</td>
<td>( g_3 )</td>
<td>( 21 + 1^3 )</td>
</tr>
<tr>
<td>( g_5 )</td>
<td>( g_5 )</td>
<td>( g_3^2 + 41 + 21^3 )</td>
</tr>
<tr>
<td>( g_9 )</td>
<td>( g_3 )</td>
<td>( g_3^3 + 321 + 321^2 + 321^3 + 21^5 + 321^5 + 2^21^5 + 2^31^5 + 1^9 )</td>
</tr>
<tr>
<td>( \gamma_5 )</td>
<td>( \gamma_5 )</td>
<td>( 31^2 + 21^3 + 1^5 )</td>
</tr>
<tr>
<td>( \gamma_9 )</td>
<td>( \gamma_9 )</td>
<td>( 321^4 + 2^1 + 21^5 + 2^31^5 + 1^9 )</td>
</tr>
<tr>
<td>( \gamma_{17} )</td>
<td>( \gamma_{17} )</td>
<td>( 3^1^4 + 3^21^3 + 321^3 + 321^5 + 2^21^{10} + 2^41 + 2^41^9 + 2^41^{11} + 1^{17} )</td>
</tr>
</tbody>
</table>
Remark 3.2. The disappearance of $5 (=a_5)$ from the table results from the relation $5 \equiv 41 + 31^2 + 21^3 \mod(t)$, which implies $41 \equiv 31^2 + 21^3 \mod(t)$ for $l=6$, 7 and $31^2 \equiv 21^3 \mod(t)$ for $l=6$, since $i (=a_i=\sigma_i(t_4, \ldots , t_l))$ vanishes if $i > l - 3$.

§ 4. Invariants of the Weyl Group and the Image of $\mathfrak{I}^*$

By (3.2) with $X=G$ and (2.5), we see that

$$(4.1) \quad \text{Im} \mathfrak{I}^* \subset H^*(B\overline{C})^R = H^*(B\overline{C}) \cap \text{Ker} \mathcal{R}.$$  

The case of $l=8$. In this case it is easily seen that

Lemma 4.1. In $H^*(B\overline{T})/(t)$ the monomials in $\gamma_i$ and $a_j$ ($i=3, 5, 9, 17; j=1, 2, 3, 4$) are linearly independent over $\mathbb{F}_2$.

Consider the map $H^*(B\overline{C}) \to H^*(B\overline{T})/(t)$ induced by $\mathcal{R}$. Using the table in the previous section we have

Lemma 4.2. $H^*(B\overline{C})^R = 0$ for $0 < n < 16$.

Since $\mathcal{R}(Sq^2 \gamma_3) = Sq^2 \mathcal{R}(\gamma_3) = Sq^2(a_2a_1 + a_3^2) = a_2a_1 + a_3^4 = R(c_4)$ by the Wu formula, we see that $Sq^2 \gamma_3 = c_4$ by the previous lemma, which is sufficient to make the following table by use of the Adem relations:

$$
\begin{array}{|c|c|c|c|c|}
\hline
& \gamma_3 & \gamma_5 & \gamma_9 & \gamma_{17} \\
\hline
Sq^2 & c_4 & \gamma_3^2 & \gamma_5^2 & \gamma_9^2 \\
Sq^4 & \gamma_5 & c_7 & 0 & 0 \\
Sq^8 & 0 & \gamma_9 & c_7c_6 & 0 \\
Sq^{16} & 0 & 0 & \gamma_{17} & c_7^3c_4 + c_7c_6^3 \\
\hline
\end{array}
$$

(4.2) \quad (c_7 = c_7 + c_6c_4)

(For $Sq^2 \gamma_{17}$, see 5.6)

Remark 4.3. In the similar way we have the following relations:

$\gamma_5 = g_5 + g_3^2 + c_4^2$,

$\gamma_9 = g_9 + g_5(c_4 + t^4) + g_3(c_6 + c_4t^2 + t^6) + c_4^2t + c_4^2t^2$.

Now define polynomials $I_k \in H^{2k}(B\overline{C})$ ($k=8, 12, 14, 15, 17, 18, 20, 24$) as follows:

$I_8 = c_8 + c_6c_1^2 + c_4^2 + c_6c_1^4 + c_1^8$

$I_{12} = Sq^2 I_8 = c_8 + c_6c_1^2 + c_4^2 + c_6c_1^4 + c_1^8$

$I_{14} = Sq^2 I_{12} = c_8 + c_2^2 + c_4^2c_1^2 + c_6c_4c_1^4 + c_6c_1^8$

$I_{16} = Sq^2 I_{14} = c_8 + c_4^2 + c_4^2c_1^2 + c_6c_4c_1^4 + c_6c_1^8$

$I_{20} = Sq^2 I_{16} = c_8 + c_4^2 + c_4^2c_1^2 + c_6c_4c_1^4 + c_6c_1^8$

$I_{24} = Sq^2 I_{20} = c_8 + c_4^2 + c_4^2c_1^2 + c_6c_4c_1^4 + c_6c_1^8$
Then the main results in this section are stated as follows:

**Theorem 4.4.** For $E_8$ we have

(i) $H^*(B\tilde{C})^R = F_2[I_k, v_{23}; k = 8, 12, 14, 15, 17]$ for $* \leq 34$.

(ii) $\tilde{x}^*(y_j) = \begin{cases} v_i & (j = 2i+1 \text{ with } i \geq 5) \\ 0 & (j = 31) \end{cases}$

**Proof.** Denote by $T^*(m)$ (resp. $C^*(m)$) the subalgebra of $H^*(B\tilde{T})$ generated by $t_1, \ldots, t_8$ (resp. $c_1, \ldots, c_8$) and the $\gamma_i$ with $i \leq m$. From the table in the previous section follow

$(4.3) \quad R(C^*(m)) \subset T^*(m) \text{ and } R(\gamma_i) \subset T^*(0)$.

First we shall show $H^*(B\tilde{C})^R \subset F_2[I_k, v_{23}; k = 8, 12, 14, 15, 17]$ inductively on $n$. By Lemma 4.2 this holds for $n < 16$.

Let $x \in H^{16}(B\tilde{C})^R$ and write it in the form

$x = \gamma_8 p_3 + \gamma_3^2 q_2 + \gamma_8 q_8 + q_8 \quad (p_3 \in C^8(3), q_i \in C^{2i}(0)).$

Applying $R$ on both sides, we see, in view of the formula $R(XY) = XR(Y) + R(X)R(Y)$ and $(4.3)$, that $0 \equiv \gamma_8 R(p_3)$ mod $T^*(3)$. This implies $R(p_3) = 0$, whence $p_3 = 0$ since $H^*(B\tilde{C})^R = 0$. Then

$0 \equiv \gamma_3^2 R(q_2) + \gamma_3 R(q_3)$ mod $T^*(0)$

which implies $R(q_i) = 0$, whence $q_i = 0$ since $H^{2i}(B\tilde{C})^R = 0$ ($i = 2, 5$). Thus $x \in C^{16}(0)$, and after some calculations we see that $x = \alpha I_8 \quad (\alpha \in F_2)$ using Lemma 4.1.

Continuing this procedure yields the inclusion mentioned above for $n \leq 34$.

Next consider the Serre spectral sequence for the fibering $E_8/T \to B\tilde{T} \to B\tilde{E}_8$. According to Bott the odd dimensional part of $H^*(E_8/T)$ vanishes, and by Theorem 1.1 so dose that of $H^*(B\tilde{E}_8)$ for $* \leq 30$. Therefore for $p \leq 30$ we have $E_8^{2p,0} = E_8^{2p,0}$, which implies that $\tilde{x}^*: H^{2p}(B\tilde{E}_8) \to H^{2p}(B\tilde{T})$ is a monomorphism. In particular $\tilde{x}^*(y_{16})$ and
do not vanish. Then the theorem follows from (4.1) and (1.1).

Next we consider the case of \( l = 6 \) and 7. As is well known, there is a sequence of inclusions \( E_5 \subset E_7 \subset E_8 \). We may assume that the maximal tori \( T^l \) of \( E_l \ (l = 6, 7, 8) \) are chosen so that \( T^8 \subset T^7 \subset T^6 \). The inclusions induce maps \( \varphi_i : B T^{l-1} \to B T^l \) (\( BE_{l-1} \to BE_l \)) and \( \tilde{\varphi}_i : B T^{l-1} \to B T^l \) (\( B\tilde{E}_{l-1} \to B\tilde{E}_l \)) (\( l = 7, 8 \)) such that

\[
(4.4) \quad \varphi_i \circ f = f, \quad \varphi_i \circ g = g \circ \tilde{\varphi}_i \quad \text{and} \quad \varphi_i \circ \tilde{\lambda} = \tilde{\lambda} \circ \tilde{\varphi}_i.
\]

We may assume, in addition, that the systems of the simple roots \( \{\alpha_i ; i = 1, 2, \ldots, l\} \) are chosen so that \( \alpha_i T^{l-1} = \alpha_i \ (i < l), = 0 \ (i = l) \). Then the corresponding systems of the fundamental weights are in the similar relation, from which and the commutativity (4.4) it follows that

\[
\varphi_i^*(\gamma_i) = \gamma_i \ (i < l), = 0 \ (i = l); \quad \varphi_i^*(v_i) = v_i,
\]

for each \( l \). Moreover, we have the following:

**Lemma 4.5.** \( \varphi_6^*(y_{16}) = y_{16} \) and \( \varphi_7^*(y_{33}) = y_{33} \) for each \( l \).

**Proof.** Consider the Serre spectral sequence for the fibering \( E_{l-1} \to B\tilde{E}_{l-1} \to B\tilde{E}_l \) where the cohomology of the fibre is given by

\[
H^* (E_{l-1}) = \Lambda (x_{12}, x_{20}, x_{24}, x_{29}, x_{30}), \quad H^* (E_l) = \Lambda (x_{19}, x_{20}, x_{27}).
\]

It follows that \( E_{l-1}^p = E_l^p \ (p = 16, 33) \), which implies \( \varphi_i^*(y_p) \neq 0 \), and the lemma follows since dim \( H^p (B\tilde{E}_{l-1}) = 1 \).

The case of \( l = 7 \). Here the relation \( a_3a_1 = a_3a_1^2 + a_3a_1^3 \mod (t) \) (see 3.2) yields an invariant in dimension 12. To be precise, we have

**Lemma 4.6.** \( H^{12} (B\tilde{C}) \subset F_2^* \cdot (\gamma_3 + c \varphi_1^2 + c_1^6) \).

Define polynomials \( I_k \in H^{2k} (B\tilde{C}) \ (k = 6, 8, 10, 12, 14, 17, 18) \) by

\[
I_6 = \gamma_3^2 + c \varphi_1^2 + c_1^6, \quad I_8 = S q^3 I_6 = \gamma_3^3 + c \varphi_1^4 + c_1^8.
\]

and

\[
I_j = \varphi_6^* (I_j) \ (j = 8, 12, 14, 17, 18).
\]

Then Theorem 4.4 together with 4.5, 4.6 and (1.1) implies:

**Corollary 4.7.** For \( E_7 \) we have

\[
\tilde{\lambda}^* (y_j) = \begin{cases} 
I_j & (j = 12, 16, 20, 24, 28, 34, 36) \\
\nu_j & (j = 2^i + 1 \text{ with } i \geq 5)
\end{cases}
\]

The case of \( l = 6 \). Here the relation \( a_3a_1^2 = a_3a_1^3 \mod (t) \) yields \( H^{10} (B\tilde{C}) \subset F_2^* \cdot (\gamma_3 + c \varphi_1 + c_1^5) \). Define polynomials \( I_k \in H^{2k} (B\tilde{C}) \ (k = 5, 6, \ldots) \).
Corollary 4.8. For $E_8$ we have

$$\tilde{\lambda}^*(y_j) = \begin{cases} I_j^{12} & (j=10, 12, 16, 18, 24, 34) \\ v_j & (j=2^i+1 \text{ with } i \geq 5) \end{cases}$$

§ 5. Squaring Operations on the $y_j$

Now we are ready to compute $Sq^i y_j$. First we shall consider to what extent they are decidable by use of the Adem relations and the algebra structure of $H^*(B\tilde{E}_8)$. We use the Adem relations not only in the usual form but in the following forms:

(5.1) for $a > 2b$, $Sq^a Sq^b = Sq^{ab} Sq^{a-b} + \sum_{j=0}^{b-1} \left(\frac{a-b-1-j}{2}\right) Sq^{a+b-j} Sq^j$;

(5.2) for $r = 1$ and $2^{m-1}$, $Sq^{2^m+r} = Sq^{2^m-1} Sq^{2^m+r-2^m-1} + \sum_{j=0}^{m-2} Sq^{2^m+r-2^j} Sq^j$; etc.

Lemma 5.1. For $G = E_8$,

(i) $Sq^i y_{16} = y_{16+i}$ ($i=8, 12, 14, 15), = y_{16}^2$ ($i=16)$ and $=0$ otherwise.

(ii) $Sq^{16} y_{34} = y_{34} y_{16}$

Proof. (i) This follows from (1.1), (5.2) and the structure of $H^*(B\tilde{E}_8)$.

(ii) We may put $Sq^{16} y_{34} = \varepsilon y_{40} + \varepsilon' y_{24} y_{16}$ ($\varepsilon, \varepsilon' \in \mathbb{F}_2$). Applying $Sq^8$ we have $Sq^{20} y_{28} + y_{24}^2 = \varepsilon y_{48} + \varepsilon' (y_{24}^2 + (Sq^8 y_{28}) y_{16})$. But $Sq^{20} y_{28} = Sq^{12} Sq^{12} y_{16} = (Sq^{10} y_{24} + Sq^{13} y_{16}) y_{16} = 0$ and $Sq^8 y_{24} = Sq^8 y_{16} = (Sq^{10} y_{24} + Sq^{15} y_{16}) y_{16} = 0$ both by (i), which imply $\varepsilon = 0$ and $\varepsilon' = 1$.

Lemma 5.2. For $G = E_8$

(i) $Sq^i y_{33} = y_{33+i}$ ($i=1, 0$ ($i=2, 4, 8$), $y_{33} y_{16} = y_{33} y_{16}$ ($i=16$), $= y_{65}$ ($i=32$).

(ii) $Sq^{16} y_{48} = y_{48} y_{24} + y_{36} y_{28} + y_{34} y_{30} + y_{33} y_{31}$.

Proof. (i) From (1.1) and the structure of $H^*(B\tilde{E}_8)$ follow all but the case of $i=16$. We may put $Sq^{16} y_{33} = \varepsilon y_{33} y_{16}$ ($\varepsilon \in \mathbb{F}_2$). Apply $Sq^1$ and use $Sq^1 Sq^{16} = Sq^2 Sq^{15} + Sq^{16} Sq^1$ from (5.2). Then $Sq^{16} y_{34} = \varepsilon y_{34} y_{16} = \varepsilon y_{34} y_{16}$ ($\varepsilon \in \mathbb{F}_2$). Apply $Sq^1$ and use $Sq^1 Sq^{16} = Sq^2 Sq^{15} + Sq^{16} Sq^1$ from (5.2). Then $Sq^{16} y_{34} = \varepsilon y_{34} y_{16} = \varepsilon y_{34} y_{16}$ ($\varepsilon \in \mathbb{F}_2$).
Applying $\xi^*$, which is injective at dimension 50, we see that $\epsilon = 1$ since $Sq^{16}I_{17} = I_{17}I_8$ holds.

(ii) Since $Sq^{16}I_{24} = I_{20}I_{12} + I_{16}I_{14} + I_{17}I_{15}$ and the kernel of $\xi^*: H^*(B\tilde{E}_8) \to H^*(B\tilde{T}^q)$ is spanned by $y_{30}, y_{31}$, we may put $Sq^{16}y_{34} = y_{40}y_{24} + y_{36}y_{28} + y_{35}y_{30} + \delta y_{34}y_{31}$ ($\delta \in F_2$). Apply $Sq^1$ and use $Sq^1Sq^{16}y_{34} = Sq^1y_j = 0$ ($j = 34, 36, 40$). Then we see that $\delta = 1$.

These data together with (1.1) are sufficient to determine $Sq^iy_j$ for $G = E_8$ by use of the Adem relations.

For $G = E_6$ and $E_7$, we need

\begin{align*}
(5.3) & \quad Sq^1I_6 = I_6, \quad Sq^{16}I_{10} = I_{18} + I_{12}I_6 + I_{10}I_5; \\
& \quad Sq^1I_5 = I_5, \quad Sq^{16}I_9 = I_{17} + I_{12}I_5 + I_9I_4.
\end{align*}

\begin{align*}
(5.4) & \quad \phi^6_\ast I_{15} = 0, \quad \phi^6_\ast I_{20} = I_{14}I_6 + I_{12}I_8 + I_{10}I_5 + I_9I_4, \\
& \quad \phi^6_\ast I_{24} = I_{14}I_{10} + I_{12}I_6 + I_9I_8 + I_9I_5; \\
& \quad \phi^6_\ast I_{10} = I_6^2, \quad \phi^6_\ast I_{16} = 0, \quad \phi^6_\ast I_{18} = I_{12}I_6 + I_9I_4 + I_9I_5^2.
\end{align*}

Applying $(\xi^*)^{-1}$ to these equalities in view of (4.7), (4.8) and (4.4). The results are as follows:

**Theorem 5.3.** In $H^*(B\tilde{E}_l)$ $(l = 6, 7, 8)$ $Sq^iy_j$ are given by

<table>
<thead>
<tr>
<th>$j$</th>
<th>$i$</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>8</th>
<th>16</th>
<th>32</th>
</tr>
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<tbody>
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<td>16</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>$y_{24}$</td>
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</tr>
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<td>0</td>
<td>0</td>
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</tr>
<tr>
<td>28</td>
<td></td>
<td>$y_{30}$</td>
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<td>0</td>
<td>$y_{28}y_{16}$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>30</td>
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<td>$y_{31}$</td>
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<td>0</td>
<td>0</td>
<td>$y_{30}y_{16}$</td>
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</tr>
<tr>
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<td>0</td>
<td>0</td>
<td>0</td>
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<td>0</td>
</tr>
<tr>
<td>33</td>
<td></td>
<td>$y_{34}$</td>
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<td>0</td>
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<td>$y_{33}y_{16}$</td>
<td>$y_{65}$</td>
</tr>
<tr>
<td>34</td>
<td></td>
<td>0</td>
<td>$y_{36}$</td>
<td>0</td>
<td>0</td>
<td>$y_{34}y_{16}$</td>
<td>$y_{36}y_{30} + y_{33}^2$</td>
</tr>
<tr>
<td>36</td>
<td></td>
<td>0</td>
<td>0</td>
<td>$y_{40}$</td>
<td>0</td>
<td>$y_{36}y_{16}$</td>
<td>$y_{40}y_{28} + y_{24}^2$</td>
</tr>
<tr>
<td>40</td>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$y_{48}$</td>
<td>$y_{40}y_{16}$</td>
<td>$y_{40}y_{24} + y_{36}^2$</td>
</tr>
<tr>
<td>48</td>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$y_{48}y_{24} + y_{36}y_{28} + y_{34}y_{30} + y_{33}y_{31}$</td>
<td>$\ast$</td>
</tr>
</tbody>
</table>

($* = y_{48}y_{16}^2 + y_{46}^2 + y_{40}y_{24}y_{16} + y_{36}y_{28}y_{16} + y_{34}y_{30}y_{16} + y_{33}y_{31}y_{16}$)
where the \( y_j \) with \( j \in J \), must be read as
\[
y_{30} = 0, \quad y_{30} = y_{28}y_{12} + y_{24}y_{16} + y_{20} + y_{16}y_{12}^2 \quad \text{for } l = 7;
\]
\[
y_{30} = y_6^2, \quad y_{28} = 0, \quad y_{36} = y_{24}y_{12} + y_{18} + y_{16}y_{10}^2 \quad \text{for } l = 6.
\]

For \( G = F_4 \) and \( G_2 \), recall that the inclusions \( G_2 \subset F_4 \subset E_6 \) induce two fiberings \( E_6/F_4 \to B\tilde{F}_4 \to B\tilde{E}_6 \) and \( F_4/G_2 \to B\tilde{G}_2 \to B\tilde{F}_4 \). From
\[
H^*(E_6/F_4) = \Lambda(x_{9}, x_{17}) \quad \text{and} \quad H^*(F_4/G_2) = \Lambda(x_{15}, x_{23}) \quad (\deg x_i = i)
\]
it follows that \( \varphi_8^* y_{10} = y_{10}, \varphi_8^* y_{18} = y_9^2 \) and \( \varphi_8^* y_9 = y_9 \). Thus we have

**Corollary 5.4.** In \( H^*(B\tilde{F}_4) \) and \( H^*(B\tilde{G}_2) \) \( Sq^j y_j \) \( (j = 10, 12, 16, 24) \) are given by those in \( H^*(B\tilde{E}_6) \) with \( y_{18} \) replaced by \( y_9^2 \), and
\[
Sq^1 y_9 = y_{10}, \quad Sq^2 y_9 = Sq^4 y_9 = 0, \quad Sq^8 y_9 = y_{17}.
\]

**Remark 5.5.** The \( \varphi_i^* \) send the \( y_j \) in the following way:

(i) \( \varphi_8^* y_{30} = \varphi_8^* y_{33} = 0, \varphi_8^* y_{40} = y_{28}y_{12} + y_{24}y_{16} + y_{20} + y_{16}y_{12}^2 \),
\[
\varphi_8^* y_{36} = y_{24}y_{12} + y_{20} + y_{16}y_{12}^2 + y_9^2
\]
(ii) \( \varphi_7^* y_{30} = y_{18}^2, \varphi_7^* y_{33} = 0, \varphi_7^* y_{36} = y_{28}y_{12} + y_{24}y_{16} + y_{20} + y_{16}y_{12}^2 \),
(iii) \( \varphi_6^* y_{18} = y_9^2, \varphi_6^* y_{30} = y_{33} + y_{24}y_{12} + y_{18}y_{16} + \varphi_6^* y_{34} = y_{24}y_{10} + y_{17} + y_{16}y_{9}^2 \),
\[
\varphi_6^* y_{d + 1} = Sq^{i-1} \varphi_6^* y_{d-1} + (i \geq 6)
\]
(iv) \( \varphi_4^* y_{18} = \varphi_4^* y_{34} = 0 \)

and for the rest \( \varphi_i^* y_j = y_j \) holds.

Most of them follow from 4.7, 4.8 and (4.4). For \( \varphi_8^* y_{33} \) put \( \varphi_8^* y_{33} = \varepsilon_1 y_{33} + \varepsilon_2 y_{33} + \varepsilon_3 y_{33} + \varepsilon_4 y_{33} \) \( (\varepsilon_i \in F_4) \), and apply \( Sq^1, j^*: H^*(B\tilde{G}) \to H^*(K(\mathbb{Z}, 3)) \), etc.

**Remark 5.6.** From \( Sq^{32} y_{34} = y_{38} y_{30} + y_9^2 \) it follows that
\[
Sq^{32} y_{17} = v_3^2 + y_7 I_8^2 + y_5 I_4^2 + y_3 I_4^2 + y_3 I_3^2 + (I_3^2 c_0 + (c_1^2 + c_0 c_3^2)(c_0^2 + c_3^2)) c_i,
\]
which completes the table (4.2).

**References**


