On $Z_q$-Equivariant Immersions for $q=2^r$

Dedicated to Professor Nobuo Shimada on his 60th birthday

By

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§ 1. Introduction

Let $Z_q$ be the cyclic group of order $q$, where $q$ is an integer $>1$. A $C^\omega$-differentiable map $f$ of a $Z_q$-manifold in another $Z_q$-manifold is called a $Z_q$-equivariant immersion (or simply a $Z_q$-immersion) if $f$ is an immersion and a $Z_q$-equivariant map.

Let $m$ and $k$ be non-negative integers. Euclidean $(m+2k)$-space $R^{m+2k}$ has a structure of a $Z_q$-manifold $(R^{m+2k}, Z_q)$ defined by the action:

$Z_q \times R^{m+2k} \to R^{m+2k}$;

$(T, (t_1, \ldots, t_m, z_{m+1}, \ldots, z_{m+k})) \mapsto (t_1, \ldots, t_m, Tz_{m+1}, \ldots, Tz_{m+k})$,

where $T=\exp(2\pi i/q)$ is the generator of $Z_q$, $t_1, \ldots, t_m$ are real numbers ($\in R$) and $z_{m+1}, \ldots, z_{m+k}$ are complex numbers ($\in C=\mathbb{R}^2$). This $Z_q$-manifold is also written by $R^{m,2k}$.

The unit $(2n+1)$-sphere $S^{2n+1}$ in complex $(n+1)$-space $C^{n+1}$ has a structure of a $Z_q$-manifold $(S^{2n+1}, Z_q)$ defined by the action:

$Z_q \times S^{2n+1} \to S^{2n+1}$;

$(T, (z_0, \ldots, z_n)) \mapsto (Tz_0, \ldots, Tz_n)$,

where $z_0, \ldots, z_n$ are complex numbers with $\sum_{j=0}^{n} |z_j|^2 = 1$. This action is free and differentiable of class $C^\omega$. The orbit differentiable manifold $S^{2n+1}/Z_q$ is the mod $q$ standard lens space $L^n(q)$. As is easily seen, there is a $Z_q$-immersion of $(S^{2n+1}, Z_q)$ in $R^{m,0}$ if and only if there is an immersion of $L^n(q)$ in $R^m$.

A. Jankowski obtained in [1] some non-existence theorems for $Z_q$-immersions. In [2] we considered $Z_{2^r}$-immersions, where $p$ is an odd prime. In this note we prove some non-existence theorems for $Z_{2^r}$-immersions.

§ 2. Statements of Results

Theorem 1. Let $r$ be an integer $>1$, and $n$ and $k$ be integers with $0 \leq k \leq n$. Assume that there is an integer $m$ satisfying the following conditions:
Corollary 2. Let $r$ be an integer $>1$. Assume that there is an integer $m$ satisfying the following conditions:

(i) $0 < m \leq n/2$,

(ii) $\binom{n+m}{k+m} \equiv (-1)^k \cdot 2^{(2s+1)^2} \mod 2^r$ for some integer $s$,

(iii) $n + m + 1 \equiv 0 \mod 2^{n-m-1}$.

Then there does not exist a $Z_r$-immersion of $(S^{2n+1}, Z_r)$ in $(R^{2n+z2+k+1}, Z_r) = R^{2n+z2+k+1}$. 

If $k = 0$, we have a new result on the non-existence of an immersion of $L^n(2^r)$ in $R^{2n+z2+k+1}$.

Corollary 4. There does not exist an immersion of $L^n(2^r)$ in $R^{2n+2L}$ where $L = L(r, n, 0)$.

This corollary is known (cf. [3, Corollary 1.5] or [4, Chapter 6, Proposition 4.16]).

Corollary 2 is very restricted. But, in some cases, this gives better results than Corollary 4. For example, $L^{21}(4)$ (resp. $L^{36}(4)$) is not immersible in $R^{25}$ (resp. $R^{109}$) by Corollary 4, but $L^{21}(4)$ (resp. $L^{36}(4)$) is not immersible in $R^{25}$ (resp. $R^{109}$) by Corollary 2.

§ 3. Preliminaries

In this section we recall some known results according to [2, Lemmas 2.1–2.3 and Proposition 2.4].

For a $Z_q$-space $(X, Z_q)$, let $\theta(X, Z_q)$ denote a $Z_q$-vector bundle $(X \times R^3, X, \pi, R^3)$ defined as follows:
(1) \( \pi : X \times R^2 \to X \) is the projection onto the first factor.

(2) \( Z_q \) acts on \( X \times R^2 \) diagonally; \( T(x, z) = (Tx, Tz) \), where \( x \in X, z \in R^2 \) and \( T = \exp(2\pi i/q) \).

**Lemma 3.1.** If \( X \) and \( Y \) are \( Z_q \)-spaces and \( f : X \to Y \) is a \( Z_q \)-map, then \( f^*\theta(Y, Z_q) = \theta(X, Z_q) \).

A \( G \)-vector bundle \( E \to X \) determines a vector bundle \( E/G \to X/G \) and this correspondence induces a homomorphism \( \rho : KO(G) \to KO(X/G) \).

Let \( r \gamma \) be the real restriction of the canonical complex line bundle \( \gamma \) over \( L^n(q) \). Then we see

**Lemma 3.2.** \( \rho(\theta(S^{2n+1}, Z_q)) = r \gamma \).

Define the action of \( Z_q \) on the total space of the Whitney sum \( m \oplus k \theta(R^{m+2k}, Z_q) \) of the \( m \)-dimensional trivial bundle \( m \) over \( R^{m+2k} \) and \( k \theta(R^{m+2k}, Z_q) \) by

\[
T((u, t_1), \ldots, (u, t_m), (u, z_{m+1}), \ldots, (u, z_{m+k})) = ((Tu, t_1), \ldots, (Tu, t_m), (Tu, Tz_{m+1}), \ldots, (Tu, Tz_{m+k})),
\]

where \( u \in R^{m, 2k}, t_i \in R (i = 1, \ldots, m), z_{m+j} \in R^2 (j = 1, \ldots, k) \) and \( T \) is the generator of \( Z_q \). Then we have

**Lemma 3.3.** There is a \( Z_q \)-bundle isomorphism of the tangent \( Z_q \)-bundle \( \pi(R^{m, 2k}) \) onto the \( Z_q \)-bundle \( m \oplus k \theta(R^{m+2k}, Z_q) \).

Using \( \gamma \)-operations, we obtain

**Proposition 3.4.** Let \( n \) and \( k \) be integers with \( 0 \leq k \leq n \), and put

\[
L = \max \left\{ j \left| \binom{n-k+j}{j} (r \gamma - 2)^j \neq 0 \right. \right\}.
\]

Then there does not exist a \( Z_q \)-immersion of \( (S^{2n+1}, Z_q) \) in \( (R^{2n+2L}, Z_q) = R^{2n+2L-2k, 2k} \).

§ 4. Proofs of Theorems 1 and 3

Two spaces \( X \) and \( Y \) are said to be mod \( q \) \( S \)-related, if there are non-negative integers \( m \) and \( n \) and a map \( f : S^m X \to S^n Y \) which induces isomorphisms of all homology groups with \( Z_q \)-coefficients, where \( S^k Z \) denotes the \( k \)-fold suspension of a space \( Z \). The following is proved in the line of the proof of Proposition 3.1 of [2].

**Proposition 4.1.** Let \( r \) be a positive integer, and \( l \) and \( n \) be integers with \( 0 < l \leq n/2 \). Assume that there is a positive integer \( t \) satisfying the following conditions:

1. \( \binom{n-t}{i} (r \gamma - 2)^i \neq 0 \) for all \( 0 \leq i < t \).
2. \( \binom{n-t}{t} (r \gamma - 2)^t = 0 \).

Then there does not exist an \( S^r Z \)-immersion of \( (S^{2n+1}, Z_q) \) in \( (R^{2n+2L}, Z_q) = R^{2n+2L-2k, 2k} \).
(i) \((l+t)r\eta\) has linearly independent \(2t\) cross-sections, where \(r\eta\) is the real restriction of the canonical complex line bundle \(\eta\) over \(L^n(2^r)\).

(ii) 
\[
\left(\frac{l+t}{l}\right) \equiv (2s+1)^s \mod 2^r \text{ for some integer } s.
\]

Then the stunted lens spaces \(L^n(2^r)/L^{l-1}(2^r)\) and \(L^{n+t}(2^r)/L^{l-1+t}(2^r)\) are mod \(2^r\) \(S\)-related.

Combining this proposition with Proposition 3.2 in [2], we have

**Proposition 4.2.** Let \(r\) be an integer \(>1\). Then, under the assumption of Proposition 4.1, \(s \equiv 0 \mod 2^{n-1}\).

**Proof of Theorem 1.** Put \(q = 2^r\), \(r > 1\). Suppose that there exists a \(Z_q\)-immersion \(f : (S^{2n+1}, Z_q) \to R^{2n+2m+1}\). Let \(\nu\) be the normal \(Z_q\)-bundle of \(f\). Then we have

\[
\tau(S^{2n+1}, Z_q) \oplus \nu = f*(R^{2n+2m+1}, \varepsilon).
\]

Since \(\rho(\tau(S^{2n+1}, Z_q) = \tau(L^n(q)) (= \text{the usual tangent bundle of } L^n(q))\), we have, by Lemmas 3.1-3.3,

\[
\tau(L^n(q)) \oplus \rho \nu = \rho f*(R^{2n+2m+1}, \varepsilon, Z_q)
\]

\[
= (2n+2m+1) \oplus k \rho \theta(S^{2n+1}, Z_q) = (2n+2m+1) \oplus kr\eta.
\]

It is well-known that \(\tau(L^n(q)) \oplus 1 = (n+1)\kappa\eta\). Thus

\[
(n+1-k)\kappa\eta + \rho \nu = 2n+2m+2.
\]

Let \(A = u \cdot 2^{n-1}\), where \(u\) is some positive integer. Then \(A(r\eta - 2) = 0\), because \(r\eta - 2 = \varepsilon K^\wedge(\langle L^n(q)\rangle)\) is of order \(2^{n-1} - \varepsilon\), where \(\varepsilon = 0\) or 1 according to \(n\) being even or odd respectively (cf. [3, Theorem 1.4]). Hence, if we take \(u\) such that \(2A - 2n - 2 + 2k > 2n + 1\), we have

\[
(A - n - 1 + k)\kappa\eta = (2A - 2n - 2m - 2) \oplus \rho \nu.
\]

Put \(l = k + m\) and \(t = A - n - m - 1\). Then the above equality implies that \((l+t)r\eta\) has linearly independent \(2t\) cross-sections. Since we may choose \(u\) so that \(\left(\frac{A-n-1+k}{k+m}\right) = \left(-\frac{n-1+k}{k+m}\right) \mod 2^r\), we have, by (ii),

\[
\left(\frac{l+t}{l}\right) = \left(\frac{A-n-1+k}{k+m}\right) \equiv \left(-\frac{n-1+k}{k+m}\right) = (-1)^{k+m} \left(\frac{n+m}{k+m}\right) \equiv (2s+1)^s \mod 2^r.
\]

We therefore see, by Proposition 4.2, that \(s \equiv 0 \mod 2^{n-m-k-1}\), and hence \(n+m+1 \equiv 0 \mod 2^{n-m-k-1}\). But this contradicts (iii). q.e.d.

There are errors in [2]. As is seen in the proof of Theorem 1, we must correct them as follows:
Line 14 in p. 344 should be replaced by

(ii) \( \left( \frac{n+m}{n-k} \right) \equiv (-1)^{k+m}(ap+b)^n \mod r \) for some integers \( a \) and \( b \) with \( (b, p) = 1 \)

Line 14 in p. 347 should be replaced by

(ii) \( \left( \frac{l+1}{l} \right) \equiv (ap+b)^n \mod r \) for some integers \( a \) and \( b \) with \( (b, p) = 1 \).

Proof of Theorem 3. For \( 1 \leq j \leq n/2 \), the order of \( \langle \tau \eta-2 \rangle \) \((\in K\tilde{O}(L^*(\mathbb{Z}^r)))\) is equal to \( 2^{r+n-2j+1-\varepsilon} \), where \( \varepsilon = 0 \) or 1 according to \( n \) being even or odd respectively (cf. [3, Theorem 1.4]). Thus the result follows from Proposition 3.4.

q. e. d.

References


