Extended Affine Root Systems I
(Coxeter transformations)

By
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Introduction

1. The present paper is the first part of the study on invariants for extended affine root systems with markings, which may be regarded as a development of the work of E. Looijenga [12] on Root systems and Elliptic curves, extending and strengthening the results by introducing the flat structure on the invariants.

We introduce in this paper the concept of an extended affine root system with a marking ((2.1) Definition).

The objective of this first part is the study of the Coxeter transformations for a
marked extended affine root system. It is achieved in the paragraphs § 9, § 10 and § 11. The results are summarized in Lemma A (9.7), Lemma B (10.1) and Lemma C (11.3).

These Lemmas are used essentially in the second part of the study [20] for the construction of flat θ-invariants for marked extended affine root systems.

2. An extended affine root system $R$ is, by definition, a root system belonging to a positive semi-definite quadratic form $I$, whose radical has rank two. A marking $G$ for the root system $R$ is a rank 1 subspace of the radical. (cf. (1.2) Definition, (2.1) Definition)

The main difficulty for the study of such extended affine root systems arise from the fact that there does not exist an analogous of a Weyl chamber, compared with the cases of finite or affine root systems, since the group $W_R$ generated by the reflexions with respect to all roots of $R$: i.e. the extended affine Weyl group, does not act anywhere properly on the ambient real vector space*. Nevertheless we define a Dynkin diagram $\Gamma_{R,G}$ for $(R, G)$ by the help of the exponents introduced in § 7. (See (8.2) for a definition of the diagram, and (8.6) for discussions on the diagram.) The diagrams are listed in the following Table 1.

Then a Coxeter transformation for $(R, G)$ is defined as a product of reflexions of roots corresponding to the nodes of the diagram ((9.7) Definition).

*) $W_R$ acts properly on a domain in the complexification of the real ambient space.

3. Let us explain briefly some geometric backgrounds, which helps but is not necessary for the understanding of this paper.

A rational double point of a complex analytic surface and its universal deformation are described by a Dynkin diagram and a simple Lie group (or algebra) of type $A_{i}, D_{i}$ or $E_{i}$ by E. Brieskorn [3,4]. (See also P. Slodowy [21].) Then there exists canonically a vector space $\mathcal{Q}$ and a non-degenerate symmetric bilinear form $J$ on $\mathcal{Q}$ such that the base space $S$ of the deformation (whose coordinate ring is the ring of invariant polynomials on the Cartan algebra by the action of the Weyl group) is canonically isomorph to $\mathcal{Q}$ (see [16]). We call such structure, the flat structure of $S$ or the flat structure on the invariant ring.**

A simple elliptic singularity of a complex analytic surface is, by definition [28], obtained by blowing down a smooth elliptic curve in a smooth surface, whose deformation is studied by E. Looijenga [11][12], P. Slodowy [22] and others, where
an affine root system and affine Lie algebra (i.e. Kac-Moody Lie algebra (or group) of Euclidean type) are used for the construction of the family.

The present paper gives a construction of a flat structure for the base space $S$ of the universal deformation of a simple elliptic singularity or equivalently a flat structure on the invariant ring of $\theta$-functions, which will be actually done in the second half of this paper [20].

For the purpose, it was necessary to introduce a new root system, which is an extension of an affine root system by one dimensional radical: the extended affine root system with a marking, as introduced in this paper.

Even the extended affine root system does not correspond to a Kac-Moody Lie algebra (cf. [33], [34]), one may naturally ask an existence of a Lie algebra corresponding to them which would describe the universal deformation of the simple elliptic singularity (cf. (8.5), P. Slodowy [23]).

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system \( R/ \text{rad} I \) and \( t_i := t_i(R, G) \) \((i = 1, 2)\) are tier numbers. We call \( P^{(t_1, t_2)} \) as the type for \((R, G)\). For the exceptional cases, we define types by a slight modification such as \( A^{(1,1)*}_i, B^{(2,2)*}_i, C^{(1,1)*}_i, BC^{(2,2)}(1) \) and \( BC^{(2,2)}(2) \), which are called exceptional types (cf. (5.1), (5.2)).

The result of classification is exposed in the table in (5.2). The proof is reduced in § 6 to classify weighted diagrams \((\Gamma', (k(a))_{a \in |\Gamma'|})\) where \( \Gamma' \) is a Dynkin diagram for an affine root system and \( k(a) \) \((a \in |\Gamma'|)\) are positive integers called the counting (6.1).

If one admits the result of the classification in § 5, one may skip these paragraphs till § 7.

iii) After these preparations, we arrive basically important concepts for a marked extended affine root system: the exponents \( m_i \) \((i = 0, \ldots, l)\) in § 7 and the Dynkin diagram \( \Gamma_{R,G} \) in § 8.

The following Table 1 is the complete list of the types and the Dynkin diagrams for marked extended affine root systems with an assumption that the quotient affine root system \( R/G \) is reduced.

The author is grateful to W. Ebeling, who noticed him that the figures \( \tilde{E}_6, \tilde{E}_7, \tilde{E}_8 \) appear already in the representation theory of algebras [2], and the importance of the branching points of the diagrams.

iv) The Dynkin diagram \( \Gamma_{R,G} \) gives a most intrinsic description of the marked extended affine root system \((R, G)\) in the sense that \((R, G)\) can be reconstructed only from the data of the diagram (9.6) Theorem). In the reconstruction of \((R, G)\), a (pre-) Coxeter transformation, which is a product of reflexions w.r.t. the nodes of the diagram (cf. (9.3)(9.7)), plays an essential role.

v) The Coxeter transformation \( c \) is unique up to (auto) conjugacy in \( W_R \). Then we shall show in § 9, § 10 and § 11 that;

i) \( c \) is a semi-simple of finite order \( l_{\max} + 1 \), whose eigenvalues are described by the exponents (Lemma A (9.7)).

ii) No element of \( R \) can be expressed as an image of the transformation \( c - 1 \) (Lemma B (10.1)).

iii) The hyperbolic extension \( \tilde{W}_{R,G} \) is an extension of \( W_R \) by an infinite cyclic group, which is generated by the power \( \tilde{c}^{l_{\max} +1} \) of a hyperbolic Coxeter transformation \( \tilde{c} \).
Table 1. Dynkin Diagrams for Extended Affine Root Systems

<table>
<thead>
<tr>
<th>System</th>
<th>Diagram</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1^{(1,1)}$</td>
<td><img src="image1.png" alt="Diagram" /></td>
<td>$(I \geq 2)$</td>
</tr>
<tr>
<td>$A_1^{(1,1)_s}$</td>
<td><img src="image2.png" alt="Diagram" /></td>
<td>$I = 1$</td>
</tr>
<tr>
<td>$B_1^{(1,1)}$</td>
<td><img src="image3.png" alt="Diagram" /></td>
<td>$(I \geq 3)$</td>
</tr>
<tr>
<td>$B_1^{(1,2)}$</td>
<td><img src="image4.png" alt="Diagram" /></td>
<td>$(I \geq 3)$</td>
</tr>
<tr>
<td>$B_2^{(1,1)}$</td>
<td><img src="image5.png" alt="Diagram" /></td>
<td>$(I \geq 2)$</td>
</tr>
<tr>
<td>$B_2^{(1,2)}$</td>
<td><img src="image6.png" alt="Diagram" /></td>
<td>$(I \geq 2)$</td>
</tr>
<tr>
<td>$C_1^{(1,1)}$</td>
<td><img src="image7.png" alt="Diagram" /></td>
<td>$(I \geq 2)$</td>
</tr>
<tr>
<td>$C_1^{(1,2)}$</td>
<td><img src="image8.png" alt="Diagram" /></td>
<td>$(I \geq 2)$</td>
</tr>
<tr>
<td>$C_2^{(1,1)}$</td>
<td><img src="image9.png" alt="Diagram" /></td>
<td>$(I \geq 3)$</td>
</tr>
</tbody>
</table>
$C^{[2,2]}_i$ \hspace{1cm} $(i \geq 3)$

$B^{[2,2]}_i$ \hspace{1cm} $(i \geq 2)$

$C^{[1,1]}_i$ \hspace{1cm} $(i \geq 2)$

$BC^{[2,1]}_i$ \hspace{1cm} $(i \geq 2)$, \hspace{1cm} $(i = 1)$

$BC^{[2,4]}_i$ \hspace{1cm} $(i \geq 2)$, \hspace{1cm} $(i = 1)$

$BC^{[2,2]}_i(1)$ \hspace{1cm} $(i \geq 2)$

$BC^{[2,2]}_i(2)$ \hspace{1cm} $(i \geq 2)$, \hspace{1cm} $(i = 1)$

$D^{[1,1]}_i$ \hspace{1cm} $(i \geq 4)$

$E^{[1,1]}_6$ \hspace{1cm}

$E^{[1,1]}_7$
$E_{8}^{(1,1)}$

$F_{4}^{(1,1)}$

$F_{4}^{(1,2)}$

$F_{4}^{(2,1)}$

$F_{4}^{(2,2)}$

$G_{2}^{(1,1)}$

$G_{2}^{(1,3)}$

$G_{2}^{(3,1)}$

$G_{2}^{(3,3)}$
vi) The last paragraph § 12 treats the folding relations among Dynkin diagrams for marked extended affine root systems. The folding relation induces a hierarchy relations among the diagrams so that all diagrams are divided into 4 groups, illustrating the classification in § 5. Particularly the exceptional types form one group.

The study of foldings also illustrate the importance of rank two-ness of the radical of the quadratic form for an extended affine root system, since the two extensions correspond to the two types of foldings defined in (12.3).

5. Some part of the result, including the classification of marked extended affine root systems, is published in [19].

Part of this work was carried out in Sept. '82 – Dec. '82, when the author was a visitor of the University of Nijmegen under the support of Z.W.O, in Jan. '83 – Feb. '83, when he was a visitor of the Universiroy Bonn and the Max Planck Institut in Bonn and March '83, when he was a visitor of the Ecole Polytechniques in Paris. He expresses his gratitude for the organizations and the mathematicians for the hospitality.


§ 1. Root Systems Belonging to a Quadratic Form

This paragraph is devoted for generalities on root systems belonging to quadratic forms. We prepare terminologies and concepts for the uses in later paragraphs.

A rough view of the paragraph is the following.

i ) The axioms for a root system \( R \) belonging to a quadratic form \( I \) and its examples are given in (1.2), (1.3). The isomorphism class of the root systems determines the quadratic form \( I \) up to a constant factor (1.4).

ii ) In (1.5)-(1.13) one is concerned with the rational structure (1.7) and the finiteness of the set of length of roots (1.9) and their direct consequences.
One obtains the concepts of dual root system $R^\vee$ (1.5), quotient root system $R/G$ (1.8), marking $G$, tier numbers $t(R)$, $t(R, G)$ (1.10) (1.12), and even lattice structure $I_R$ (1.11).

iii) In (1.14)-(1.20) the group $W_R$ generated by reflections of roots is investigated. The main tool is the use of Eichler–Siegel map and its inverse (1.14.1)(1.14.5).

For the name of the transformation, the author is indebted to W. Ebeling, who pointed out him that such transformations (1.14.1)(1.14.5) are used implicitly in the works of M. Eichler and C. L. Siegel.

A criterion for $W_R$ to split into a semi-direct product of another $W_R'$ and a free abelian group is given in (1.15).

We introduce a hyperbolic extension $\tilde{F}_G$ (1.17) of the space $F$ w.r.t. a marking $G$, which induces a central extension $\tilde{W}_{R,G}$ of $W_R$ in (1.18). Some structural study of $\tilde{W}_{R,G}$ is done in (1.19) (1.20) using a map $r: \tilde{W}_{R,G} \to M_G$.

(1.1) Reflection $w_\alpha$ of $\alpha$. Let $F$ be a real vector space equipped with a quadratic form $q$, which induces a symmetric bilinear form,

\[(1.1.1) \quad I: F \times F \to \mathbb{R}, \quad I(x, y) := q(x + y) - q(x) - q(y).\]

If an element $\alpha \in F$ is non-isotropic (i.e. $q(\alpha) \neq 0$), then we define the dual $\alpha^\vee \in F$ and the reflection $w_\alpha \in GL(F)$ as follows.

\[(1.1.2) \quad \alpha^\vee := \frac{1}{q(\alpha)} \alpha = \frac{2}{I(\alpha, \alpha)} \alpha,\]

\[(1.1.3) \quad w_\alpha(u) := u - I(u, \alpha^\vee) \alpha \quad \text{for} \quad u \in F.\]

By the definition we have,

\[(1.1.4) \quad \alpha^{\vee \vee} = \alpha, \quad \frac{I(\alpha, \alpha)}{2}, \frac{I(\alpha^\vee, \alpha^\vee)}{2} = 1,\]

\[(1.1.5) \quad w_\alpha = w_{\alpha^\vee}, \quad w_\alpha = id_F,\]

\[(1.1.6) \quad w_\alpha \in O(F, I),\]

where

\[O(F, I) := \{g \in GL(F) : I \circ g = I\}.\]
is the orthogonal group of the metric $I$.

For a non isotropic subset $B$ of $F$ (i.e. $q(a)\neq 0$ for any $a \in B$), we define a reflexion group by,

$$(1.1.7) \quad W_B := \text{the subgroup of } O(F, I) \text{ generated by } w_a \text{ for } a \in B.$$ 

**Note.** To avoid a confusion on the word “dual”, we shall denote by $F^*$ the dual vector space $\text{Hom}_{\mathbb{R}}(F, \mathbb{R})$ of $F$. Note that the element $\alpha^*$ belongs to $F$ but not to $F^*$.

**1.2 The axioms for a root system $R$ belonging to $I$.**

As in (1.1) let $F$ be a real vector space of finite rank with a metric $I$, whose signature is $(\mu_+, \mu_0, \mu_-)$. (i.e. $\mu_+$, $\mu_0$, or $\mu_-$ are number of positive, zero or negative eigenvalues of $I$ respectively.)

**Definition 1.** A subset $R$ of $F$ is called a root system belonging to $I$ or a root system of sign $(\mu_+, \mu_0, \mu_-)$, if it satisfies the following Axioms 1), ..., 5).

1) Let $Q(R)$ be the additive subgroup of $F$ generated by $R$. Then $Q(R)$ is a full lattice of $F$. (i.e. the natural map induces an isomorphism $\mathbb{R} \otimes Q(R) \cong F$.)

2) For any $a \in R$, $I(a, a) \neq 0$.

3) For any $a \in R$, $w_a R = R$.

4) For any $a, \beta \in R$, $I(a, \beta^*) \in \mathbb{Z}$.

5) Irreducibility. If $R = R_1 \perp R_2$ and $R_1 \perp R_2$ with respect to $I$ for subsets $R_i$ of $R$, then either $R_1 = \emptyset$ or $R_2 = \emptyset$.

2. Two root systems $R$ in $F$ and $R'$ in $F'$ are isomorphic if there exists a linear isomorphism $\varphi : F \cong F'$ which induces a bijection $\varphi : R \cong R'$ (cf. (1.4) Lemma).

**Note 1.**

i) If $a \in R$, then $-a \in R$. (∴ $w_a a = -a$)

ii) If $a, ca \in R$ for a constant $c \in \mathbb{R}$, then

$$c \in \{\pm 2, \pm 1, \pm 1/2\}.$$ 

(∴ Axiom 4) implies $I(ca, a^*) = 2c$, $I(a, (ca)^*) = 2/c \in \mathbb{Z}$)

iii) A root system $R$ is called reduced if $a, ca \in R$ implies $c \in \{\pm 1\}$. 
Note 2. Let us call a subset \( R \) of \( F \) to be a root system belonging to \( I \), which may not be irreducible, if \( R \) satisfies the axioms 1), ..., 4) of Def. 1. Then we have the following.

Assertion. Let \( R \) be a root system belonging to \( I \), which may not be irreducible. Then there exists a unique disjoint decomposition \( R = \bigoplus_{j=1}^{N} R_j, R_j \neq \phi \) \((j=1, ..., N)\) such that

i) \( R_j \) is an irreducible root system belonging to \( I \mid \mathbb{R}R_j \) for \( j=1, ..., N \). (Here \( \mathbb{R}R_j \) is the linear span of \( R_j \)).

ii) \( R_i \perp R_j \) for \( i \neq j \). (i.e. \( I(a_i, a_j) = 0 \) for \( \forall a_i \in R_i, \forall a_j \in R_j \).

Proof. If \( R = \bigoplus_{j=1}^{N} R_j, R_j \neq \phi \) for \( j=1, ..., N(\geq 2) \) is a decomposition of \( R \) satisfying only ii) of the assertion, then one check easily that automatically each \( R_j \) is a root system which may not be irreducible. Furthermore \( \mathbb{R}R_j \subset F \) for all \( i \), since otherwise ii) implies \( R_j \subset \text{rad} \ I \) for \( j \neq i \), which contradicts to the fact that \( R_j \subset R \) is non-isotropic. Thus the existence of the decomposition with i) and ii) can be shown by induction on rank \( F \).

If \( \bigoplus_j R_j \) and \( \bigoplus_i S_i \) are two irreducible decompositions. Then \( R_j = \bigoplus_i (R_j \cap S_i) \) is a decomposition of \( R_j \). Since \( R_j \) is irreducible \( R_j \cap S_i = \phi \) or \( R_j \). The same argument for \( S_i \) shows that up to a permutation of the indexes \( i, j \), the two decompositions coincide. This is the uniqueness of the decomposition. q.e.d.

Note 3. In the above situation we have \( F = \bigoplus_j \mathbb{R}R_j \) if \( I \) is non-degenerate. \((\because \mathbb{R}R_j \cap \bigoplus_{i \neq j} \mathbb{R}R_i \subset \text{rad} \ I \) for all \( j \))

Note 4. By definition \( W_R \) preserve the metric \( I \). Conversely up to a constant factor, \( I \) is the unique symmetric bilinear form on \( F \) which is invariant by \( W_R \).

Proof. Let \( J \) be another symmetric invariant form on \( F \). The equality \( J(w_a u, w_a a) = J(u, a) \) for \( a \in R, u \in F \) implies \( J(u, a) = c_a I(u, a) \) where \( c_a = J(a, a)/I(a, a) \). If \( I(a, \beta) \neq 0 \) for \( a, \beta \in R \), the symmetry of \( J \) implies \( c_a = c_\beta \). Thus the irreducibility of \( R \) implies \( J = c I \) for \( c = c_a \) for any \( a \in R \).

(1.3) Example 1. Let \( R \) be a root system belonging to \( I \), where \( I \) is positive semi definite of sign \((l_+, l_0, 0)\).

Then
For any $a, \beta \in R$.

(Semi-definiteness of $q | R\alpha + R\beta$ implies that $I(\alpha^\vee, \beta)I(\alpha, \beta^\vee) \leq 4$. Then apply Axiom 4.)

i) If $l_0 = 0$ (i.e. $I$ is positive definite), then $R$ is finite and is a root system in the classical sense (see [1] Ch. VI).

Proof. We have only to show that $#R < \infty$. Let $a_1, \ldots, a_t \in R$ be a $R$ basis for $F$. Since $I$ is non-degenerate, the correspondence $a \in R \rightarrow (I(a, \alpha_i^\vee))_{i=1, \ldots, t} \in (Z \cap [-4, 4])^t$ is injective.

Note 1. We shall generalize this finiteness property for general $I$ in (1.21) Lemma.

Note 2. If a root system $R$ which belongs to $I$ is finite, then $I$ is definite, due to the uniqueness of $I$ in (1.2) Note 3.

ii) If $l_0 = 1$, then $R$ is an affine root system. (For a definition, see [14, 2], [7]).

Proof. All axioms for an affine root system stated in [14] are direct consequences of the axioms in (1.2), except an axiom (AR4), which asks that $W_R$ acts properly on an affine hyperplane $E := \{b = 1\} \subset F^*$, where $b$ is a generator of $\text{rad} \ I \simeq Rb$. This is a direct consequence of an exact sequence $0 \rightarrow T_{\text{rad}} \rightarrow W_R \rightarrow W_{R/\text{rad}} \rightarrow 0$ in (1.15) Assumption, where $T_{\text{rad}}$ is a lattice of the translation group of $E$ and $W_{R/\text{rad}}$ is a finite Weyl group. (cf. (1.3) Ex 1. i )}, (1.7) Corollary, (1.8) Assumption).

iii) In general if $l_0 = k \geq 2$, let us call $R$ a $k$-extended affine root system. Particularly we shall investigate the case $k = 2$ in this paper, when we call $R$ an extended affine root system simply.

Example 2. Let $L$ be a free abelian group of finite rank and $I$ be a symmetric bilinear form on it. Put,

$$R(L, I) := \{a \in L : I(a, a) \neq 0, I(a^\vee, \beta) \in Z \text{ for all } \beta \in L\}.$$ 

If $R(L, I)$ spans $R \otimes L$ over $R$, $R(L, I)$ is a root system belonging to $I$, which may not be irreducible. We shall call $R(L, I)$ the maximal root system belonging
to \((L, I)\).

In [29, § 7 (7.4)], all maximal root systems belonging to an indefinite form of sign \((1, 0, 1)\) are classified into seven types.

**Example 3.** All root systems of sign \((1, 0, 1)\) are classified into 72 types in [29, § 1 (1.6)].

**Example 4.** Let \(B\) be a (finite) subset of a vector space \(F\) with a form \(I\). If \(B\) satisfies the axioms for a root system except axiom 3, then \(R := W_B B\) is a root system belonging to \(I\). Sometimes we shall call \(B\) a basis for \(R\). We have relations, \(Q(R) = Q(B), W_R = W_B\).

(1.4) **Isomorphisms of Root Systems.**

**Lemma.** Let \(\varphi\) be an isomorphism between two root systems \(R\) and \(R'\) which are belonging to \(I\) and \(I'\) respectively. Then there exists a non-zero constant \(c \in \mathbb{R}\) such that \(I = cI' \circ \varphi\).

**Proof.** Let \(\alpha, \beta \in R\) be roots s.t. \(I(\alpha, \beta) \neq 0\) and \(R\alpha \neq R\beta\). Put \(F_2 := R\alpha + R\beta\) and \(F'_2 := \varphi(F_2)\). Then \(R_2 := F_2 \cap R\) and \(R'_2 := F'_2 \cap R'\) are root systems belonging to \(I|F_2\) and \(I'|F'_2\) respectively which are isomorphic by \(\varphi|F_2\). Therefore due to the axiom 5) of irreducibility, one can reduce the proof of the lemma to the followings.

i ) **Lemma is true for the case when rank \(F = 2\).**

ii ) Let \(\varphi\) be an isomorphism of root systems \(R, R'\) which may not be irreducible in rank 2 vector spaces. If \(R\) is irreducible, then \(R'\) is irreducible.

**Proof of i).** Let \(A_R\) and \(A_{R'}\) be the accumulating sets in \(\mathcal{P}(F)\) and \(\mathcal{P}(F')\) of the image set of \(R\) and \(R'\) respectively. According as \(I\) is definite, semi-definite or indefinite, \(A_R\) is either void, one point or two points, which correspond to the isotropic vectors in \(F\) of \(I\). (cf. (1.3) Ex 1., i ), ii ) and Ex 3. ([29] (1.6) **Note 6)). Since \(\varphi\) induces a bijection among \(A_R\) and \(A_{R'}\), \(\varphi\) maps the isotropic vectors of \(I\) to that of \(I'\). This implies already the statement i ) for the cases when \(I\) is semi-definite or indefinite.

In the case when \(I\) is definite, \(R\) is either of the types \(A_2, B_2, BC_2\) or \(G_2\). Then the coincidences of some geometric invariants (e.g. \(\#R, \#(R/R^\perp)\), etc.) show that \(R\)
and \( R' \) are of the same type.

**Proof of ii).** Reducible root systems for rank \( F = 2 \) is either \( A_1 \times A_1, A_1 \times BC_1 \) or \( BC_1 \times BC_1 \). Thus they are characterized among all root systems by an equation \( \#(R/R^\ast) = 2 \), which is invariant by \( \varphi \). q.e.d.

(1.5) **The dual root system** \( R^\vee \).

For a given subset \( R \subseteq F \), put

\[
R^\vee := \{ a^\vee \in F : a \in R, I(a, a) \neq 0 \}.
\]

**Lemma.** If \( R \) is a root system belonging to \( I \), then \( R^\vee \) is a root system belonging to \( I \).

**Proof.** The axioms 2)-5) for \( R^\vee \) are directly shown by definition. The axiom 1) for \( R^\vee \) will be shown in (1.10) after a preparation of a finiteness lemma (1.9).

(1.6) **Irreducibility.**

**Assertion.** For any \( \alpha, \beta \in R \), there exists a sequence \( \alpha_0 := \alpha, \alpha_1, ..., \alpha_k := \beta \) of elements of \( R \) s.t.

\[
\begin{align*}
\text{i ) } & I(\alpha_i, \alpha_j) = \begin{cases} 0 & \text{for } i = j, j \neq 1, \ldots, k \\ 0 & \text{otherwise} \end{cases}, \\
\text{ii ) } & k \leq \mu_+ + \mu_-.
\end{align*}
\]

**Proof.** An existence of a sequence with a property i ) is an immediate consequence of the axiom of irreducibility. The property i ) implies that \( \text{rank}(I(\alpha_i, \alpha_j))_{i,j=0, \ldots, k} \geq k \). This implies ii ). q.e.d.

**Corollary 1.** Let \( R \) be a root system which may not be irreducible. Then \( R \) is irreducible iff it has the following property :

\( \ast \) If a linear subspace \( G \) of \( F \) is invariant under \( W_R \) and \( G \cap R \neq \emptyset \), then \( G = F \).

**Proof.** If \( R \) is irreducible and \( G \) is a \( W_R \)-invariant subspace with \( \alpha \in G \cap R \). For any \( \beta \in R \), take the sequence \( \alpha_0, ..., \alpha_k \) of the assertion. Then \( (w_{\alpha_k} - 1)w_{\alpha_{k-1}} \ldots w_{\alpha_1} \alpha_0 \in G \) is a non-zero multiple of \( \beta \), and therefore \( \beta \in G \). Then the axiom 1) implies \( G = F \).
Conversely if $R$ satisfies *) and $R = R_1 \sqcup R_2$ with $R_1 \perp R_2$. Then $\mathbb{R}Q(R_1)$, $\mathbb{R}Q(R_2)$ are $W_R$-invariant subspaces which are orthogonal to each other. If $R_1 \neq \phi$, then by *), $\mathbb{R}Q(R_1) = F$ and hence $\mathbb{R}Q(R_2)$ is contained in the radical $I := \{x \in F : I(x, y) = 0 \text{ for } \forall y \in F\}$. Hence $I(a, a) = 0$ for $a \in R_2$. This implies $R_2 = \phi$.

**Corollary 2.** Let $R$ be an irreducible root system belonging to $I$. For a constant $r \in \mathbb{R}$, let us put,

$$R_r := \{ a \in R : I(a, a) = 2r \}.$$

If $R_r \neq \phi$, then $R_r$ is a root system belonging to $I$, which may not be irreducible.

**Proof.** All the axioms of (1.2) except the axiom 1) are trivially verified. For a proof of the axiom 1), we have only to show that $\mathbb{R}Q(R_r) = F$, since $Q(R_r) \subseteq Q(R)$. Since $I$ is invariant by $W_R$, the axiom 3) implies $W_R R_r = R_r$. Hence $\mathbb{R}Q(R_r)$ is a $W_R$-invariant subspace of $F$ containing $R_r \neq \phi$. Therefore $\mathbb{R}Q(R_r) = F$ due to (1.6) Corollary 1.

(1.7) **Rational structure of** $F$.

**Assertion.** There exists a non-zero constant $c \in \mathbb{R}$ such that $cl$ is an integral bilinear form on $Q(R) \times Q(R)$.

**Proof.** Let us fix an element $a \in R$ and put $c := I(a, a)^{-1}$. For any $\beta \in R$, take a sequence $a_0, \ldots, a_k$ of the assertion (1.6). Then

$$cl(\beta, \beta) = (\prod_{j=1}^{k} I(a_j^{-1}, a_j))/ (\prod_{j=1}^{k} I(a_j, a_j)) \in \mathbb{Q}.$$ 

Thus for any $\beta, \gamma \in R$,

$$cl(\beta, \gamma) = \frac{1}{2} cl(\beta, \beta) I(\beta^{-1}, \gamma) \in \mathbb{Q}.$$ 

Thus for a suitable integral $n \in \mathbb{N}$, $ncI$ is an integral bilinear form on $Q(R)$.

q.e.d.

**Definition.** A linear subspace $G$ of $F$ is said to be defined over $\mathbb{Q}$ if $G \cap Q(R)$ is a full lattice of $G$. 

Corollary. Put
\[ \text{rad} \ I = \{ x \in F : I(x, y) = 0 \text{ for any } y \in F \} . \]
Then \ rad \ I \ is \ defined \ over \ Q .

Proof. Since \ cI \ is \ rational \ valued \ on \ Q(R) \ for \ a \ c \neq 0 , \ the \ system \ of \ equations \ cI(x, y) = 0 \ (y \in Q(R)) \ are \ rational \ coefficients. \q.e.d.

(1.8) Quotient Root Systems. Let \ G be a linear subspace of \ rad \ I \ defined \ over \ Q \ and let \ \rho : F \to F/G \ be \ the \ linear \ projection \ map. \ The \ bilinear \ form \ on \ F/G \ induced \ from \ I \ on \ F \ is \ denoted \ by \ I_G \ so \ that \ I_G(\rho(x), \rho(y)) = I(x, y) \ for \ x, y \in F . \]

Assertion. Let us denote by \ R/G the image of \ R in \ F/G . \ Then \ R/G \ is \ a \ root \ system \ belonging \ to \ I_G , \ such \ that \ Q(R/G) \approx Q(R)/(Q(R) \cap G) . \]

Proof. The axioms 2), ..., 5) are trivially verified for \ R/G . The axiom 1) follows from \ Q(R/G) = \rho(Q(R)) = Q(R)/(Q(R) \cap G) . \q.e.d.

Definition. We shall call \ R/G the quotient root system of \ R by \ G .

Note. For the study of \ R, sometimes it is convenient to fix a flag \ G_0 = 0 \subset G_1 \subset \cdots \subset G_r = \text{rad} \ I \ defined over \ Q , \ which we shall call a marking of \ R. (cf. (2.1) Def. 2.)

This concept of a marking a root system comes from a study of primitive forms for the period mapping of simple elliptic singularities. (cf. [17] [20]).

(1.9) Lemma of finiteness of root length.

The following simple lemma plays an important role.

Lemma. Let \ R be a root system belonging to \ I . \ Then there exists a non zero constant \ c \ such that the set
\[ \{ cI(a, a) : a \in R \} \]
is a finite set of integers.

Proof. Due to (1.7) Cor. and (1.8) Assertion, \ R/\text{rad} \ I \ is \ a \ root \ system
belonging to $I_{\text{rad}}$ such that

$$\{I(a, a) : a \in R\} = \{I_{\text{rad}}(\beta, \beta) : \beta \in R/\text{rad } I\}.$$ 

Therefore without loss of generality, one assumes that $I$ is non-degenerate.

Define, $P^\vee := \{f \in F : I(f, Q(R)) \subset \mathbb{Z}\}$. Since $I$ is non-degenerate, $P^\vee$ is a full lattice of $F$. Due to (1.7) Assertion, there exists a non-zero constant $c$ such that $cI(Q(R)), Q(R)) \subset \mathbb{Z}$. Therefore $cQ(R) \subset P^\vee$. Since both $cQ(R)$ and $P^\vee$ are full lattices of $F$, there exists an integer $N \neq 0$ such that $NP^\vee \subset cQ(R)$.

Hence $I(P^\vee, P^\vee) \subset I(P^\vee, \frac{C}{N}Q(R)) = \frac{C}{N}I(P^\vee, Q(R)) \subset \frac{C}{N}\mathbb{Z}$.

Due to the axiom 4), $R^\vee \subset P^\vee$. In particular, $4/I(a, a) = I(a^\vee, a^\vee) \subset \frac{C}{N}\mathbb{Z}$ for $a \in R$. This implies $cI(a, a)$ is an integer which divides $4N$. This completes the proof of the finiteness. q.e.d.

Note. A modification of the proof shows that, if $Q(R)$ is an even lattice w.r.t. $cI$, then $cI(a, a) | 2N^2$.

(1.10) The total tier number $t(R)$.

Let $R$ be a root system belonging to $I$, where $cI$ is an integral bilinear form on $Q(R)$ for a positive constant $c$.

Due to (1.9) Lemma, we define now,

$$t(R) := \frac{\text{l.c.m.} \{cI(a, a) : a \in R\}}{\text{g.c.d.} \{cI(a, a) : a \in R\}}.$$ 

Let us call $t(R)$ the total tier number of $R$ which takes the value in $\mathbb{N}$.

Proof of (1.5) Lemma.

Let us fix a root $a \in R$. Then for any $\beta$, one computes,

$$\beta^\vee = \frac{2}{I(\beta, \beta)^2} = \frac{2}{t(R)I(a, a)^2}t(R)cI(a, a)\beta \subset \frac{2}{t(R)I(a, a)^2}$$

This implies, $Q(R^\vee) \subset \frac{2}{t(R)I(a, a)}Q(R)$, so that $Q(R^\vee)$ is a lattice in $F$. q.e.d.

Note. The above proof shows also the following:
A linear subspace $G$ of $F$ is defined over $Q$ with resp. to $R$, iff it is defined over $Q$ w.r.t. $R^\vee$.

(1.11) **Even lattice structure.**

Let $R$ be a root system belonging to $I$ and $c$ is a constant as in (1.9) Lemma. Put

$$I_R := \frac{2c}{\gcd\{cI(a, a) : a \in R\}} I,$$

$$I_{R^\vee} := \frac{1}{2c} \text{l.c.m.}\{cI(a, a) : a \in R\} I.$$

Then we have the following properties.

i) $Q(R)$ (resp. $Q(R^\vee)$) is an even lattice w.r.t. $I_R$ (resp. $I_{R^\vee}$).

ii) If $Q(R)$ (resp. $Q(R^\vee)$) is an even lattice w.r.t. $cI$ for a suitable constant $c' \neq 0$, then $c'I$ is an integral multiple of $I_R$ (resp. $I_{R^\vee}$).

iii) Formula

$$I_R \otimes I_{R^\vee} = t(R) I \otimes I.$$

In particular,

$$\frac{I_R(a, a)}{2}, \frac{I_{R^\vee}(a^\vee, a^\vee)}{2} = t(R) \text{ for } a \in R.$$

(1.12) **Relative tier number.** Let the notations be as in (1.11). Let $G, H$ are linear subspace of $F$ defined over $Q$ s.t. $G \supseteq H$.

We define tier numbers relative to $G$ and $H$ as follows.

$$t_R(G, H) := \# \frac{Q(R) \cap G / Q(R) \cap H}{Q(R)^\vee \cap G / Q(R^\vee) \cap H} \times (I_{R^\vee} : I)^{\text{rank } G / H},$$

$$t_{R^\vee}(G, H) := \# \frac{Q(R^\vee) \cap G / Q(R^\vee) \cap H}{Q(R) \cap G / Q(R) \cap H} \times (I_R : I)^{\text{rank } G / H}.$$

Here in the above expression, $\# \frac{L}{M}$ for two lattices $L$ and $M$ means $|\det A|$ where $A$ is a transformation matrix from a $Z$-basis of $L$ to a $Z$-basis of $M$, and $A : B$ means a constant $c$ s.t. $A = cB$.

Note that if one replace $I$ by $dI$ for a constant $d \neq 0$, then $R$ and $I_R$ are
unchanged, but $R^{\vee}$ and $I_R^{\vee}$ are replaced by $d^{-1}R^{\vee}$ and $d^2I_R^{\vee}$ so that the tier numbers in (1.12.1) and (1.12.2) are independent of $d$.

Assertion. i) The relative tier numbers are positive integers.
ii) $t_R(G, H) \cdot t_R^{\vee}(G, H) = t(R)^{\text{rank} G/H}$.

Proof. i) Since the tier number does not depend on a constant factor of $I$, we take $I$ to be $(l.c.m.(I_R^\vee(a, a) : a \in R)^{-2}I_R$. Then we have $I_R^{\vee}=I$ and $a^{\vee} = I(a^{\vee}, a^{\vee})a \in Z \cdot a \subset Q(R)$. Therefore $Q(R^{\vee}) \subset Q(R)$. These show that $t_R(G, H)$ is an integer. In a similar manner, one sees that $t_R^{\vee}(G, H)$ is an integer.
ii) Apply the formula (1.11.3). q.e.d.

(1.13) The following is a simple consequence of the finiteness lemma.

Assertion. Let $\varphi$ be an isomorphism of a root system $R$ belonging to $I$. Then

$I \circ \varphi = \pm I$.

Proof. Due to (1.4) Lemma, there exists a constant $c \neq 0$ s.t. $I = cI \circ \varphi$. If $c \neq \pm 1$, then the set $\{I(a, a) : a \in R\}$ which is invariant by the multiplication by $c$ is infinite. This contradicts to (1.9) Lemma. q.e.d.

(1.14) Eichler Siegel presentation $E$. For a study of the Weyl group $W_R$, we introduce a map $E : F \otimes (F/\text{rad} I) \to \text{End}(F)$ (Def. (1.14.1)) and its inverse $E^{-1} : W_R \to F \otimes (F/\text{rad} I)$ (Def. (1.14.5)) in this paragraph.

Using $E$, we give a sufficient condition for $R, G$ so that $W_R$ splits into semi direct product in (1.15) and we study central extensions $\tilde{W}_{R,G}$ of $W_R$ associated to hyperbolic extensions in (1.19).

Definition 1. Eichler Siegel map $E$ for $(F, I)$ is the following,

\begin{equation}
E : F \otimes (F/\text{rad} I) \to \text{End}(F),
\end{equation}

where

\begin{equation}
E(\sum_i f_i \otimes g_i)(u) := u - \sum_i f_i I(g_i, u).
\end{equation}

2. We define a semi group structure $\circ$ on $F \otimes (F/\text{rad} I)$ by,

\begin{equation}
\varphi \circ \psi := \varphi + \psi - I(\varphi, \psi),
\end{equation}
where we use the following convention,

\[ I : F \otimes (F/\text{rad} I) \times \cdots \times F \otimes (F/\text{rad} I) \to F \otimes (F/\text{rad} I) \]

\[ \varphi_1 \times \cdots \times \varphi_k \mapsto I(\varphi_1, \ldots, \varphi_k), \]

\[ I(\varphi_1, \ldots, \varphi_k) := \sum_{i_1, \ldots, i_h} f_{i_1}^1 \otimes I(g_{i_1}^1, f_{i_2}^2) \cdots I(g_{i_{k-1}}^{k-1}, f_{i_k}^k) g_{i_k}, \]

for \( \varphi_j := \sum f_{i_j}^j \otimes g_{i_j}^j \in F \otimes (F/\text{rad} I) \) \( (j = 1, \ldots, k). \)

From the definition directly we obtain following assertions.

\[ \text{i) The map } E \text{ is injective. It is bijective iff } \text{rad} I = 0. \]

\[ \text{ii) } E \text{ is a homomorphism of semi groups.} \]

\[ (1.14.3) \quad E(\xi \circ \eta) = E(\xi)E(\eta). \]

\[ \text{iii) For an non isotropic } a \in F, \text{ the reflection } w_a \text{ (1.1.2) is given by} \]

\[ (1.14.4) \quad w_a = E(a \otimes a^\vee). \]

\[ \text{iv) The inverse of the Eichler Siegel map on } W_R \text{ is well defined.} \]

\[ (1.14.5) \quad E^{-1} : W_R \to F \otimes (F/\text{rad} I). \]

The image of (1.14.5) is contained in \( Q(R) \otimes (Q(R^\vee)/Q(R^\vee) \cap \text{rad} I). \)

\[ (\because \quad \text{The lattice } Q(R) \otimes (Q(R^\vee)/Q(R^\vee) \cap \text{rad} I) \text{ of } F \otimes (F/\text{rad} I) \text{ is closed under the product } \circ \text{ and contains } a \otimes a^\vee \text{ for all } a \in R. \text{ Taking on account of (1.14.4), } W_R \]

\[ \text{is contained in the image of the lattice by } E. \]

\[ \text{v) The subspace } \text{rad} I \otimes (F/\text{rad} I) \text{ is closed under the product } \circ, \text{ where } \circ \text{ coincides with the additive structure of the vector space on the subspace. We have} \]

\[ (1.14.5) \quad (E(\phi) - id_F)^2 = 0 \quad \text{for } \phi \in \text{rad} I \otimes (F/\text{rad} I). \]

For a later use, let us show an assertion.

Assertion. Suppose \( w := E(\xi) \in O(F, I) \) for a \( \xi \in F \otimes (F/\text{rad} I). \)

Then,

\[ \xi + \xi - I(\xi, \xi) \equiv 0 \mod \text{rad} I \otimes F + F \otimes \text{rad} I. \]

Proof. The orthogonality of \( w \) implies,
Hence

\[ I(f, g) = I(wf, wg) = I(wf, g) - I(wf, \xi, g) \]

for all \( f, g \in F \).

\[ f - wf + I(wf, \xi) \equiv 0 \mod \text{rad } I \]

\[ = I(\xi, f) + I(wf, \xi) = I(\xi w + \xi, wf) = I(\xi + '\xi - I(\xi, '\xi), wf) \]

for all \( f \in F \). This implies the assertion. q.e.d.

(1.15) Splitting of \( W_R \). As in (1.8) let \( G \) be a linear subspace of \( \text{rad } I \) defined over \( \mathbb{Q} \).

Define a lattice of \( G \otimes (F/\text{rad } I) \) by,

(1.15.1) \[ T_g := E^{-1}(W_R) \cap (G \otimes (F/\text{rad } I)). \]

(\( T_g \) is a lattice, due to (1.14) Assertion iv), v).)

Assertion. i) Following is an exact sequence.

(1.15.2) \[
1 \longrightarrow T_G \overset{E}{\longrightarrow} W_R \overset{p_*}{\longrightarrow} W_{R/G} \longrightarrow 1.
\]

ii) The adjoint action of \( W_R \) on \( T_G \) is given by

(1.15.3) \[
wE(\sum f_i \otimes g_i) w^{-1} = E(\sum f_i \otimes wg_i),
\]

for \( w \in W_R, \sum f_i \otimes g_i \in G \otimes (F/\text{rad } I) \).

Proof. i) The following diagram is naturally commutative.

\[
1 \longrightarrow T_G \overset{E}{\longrightarrow} W_R \overset{p_*}{\longrightarrow} W_{R/G} \longrightarrow 1
\]

\[
\cap \overset{\text{rad } I}{\longrightarrow} \cap \overset{E^{-1}}{\longrightarrow} \cap \overset{E^{-1}}{\longrightarrow} \cap \overset{E^{-1}}{\longrightarrow} \cap
\]

\[
1 \longrightarrow G \otimes (F/\text{rad } I) \longrightarrow F \otimes (F/\text{rad } I) \longrightarrow (F/G) \otimes (F/\text{rad } I) \longrightarrow 1
\]

where the second line is exact.

ii) Let us show (1.15.3) for \( w = E(\alpha \otimes a^\vee), \alpha \in R \).

\[
E(\sum f_i \otimes wg_i) w = E((\sum f_i \otimes (g_i - I(\alpha \otimes a^\vee, g_i))^o a \otimes a^\vee) \]

\[
= E(\sum f_i \otimes g_i + \alpha \otimes a^\vee)
\]

\[
= E(\alpha \otimes a^\vee o (\sum f_i \otimes g_i)) = wE(\sum f_i \otimes g_i).
\]

q.e.d.
We state now a lemma giving a condition for (1.15.2) to split.

**Lemma.** Let $R$ be a root system belonging to $I$ and let $G \subset \text{rad} I$ be defined over $Q$. Let $L$ be a linear subspace of $F$ such that

(1.15.4) \[ F = L \oplus G \quad \text{(i.e. } L \text{ is complementary to } G), \]

(1.15.5) \[ \phi^* | W_{R \cap L} : W_{R \cap L} \to W_{R/G} \text{ is surjective.} \]

Then

i ) The homomorphism (1.15.5) is an isomorphism. Hence (1.15.2) splits into a semi direct product,

(1.15.6) \[ W_R = W_{R \cap L} \rtimes T_G. \]

ii) $T_G$ is a full lattice of $G \otimes (F/\text{rad} I)$, which is generated by

(1.15.7) \[ a_G \otimes a_G^\vee \quad \text{for } a \in R. \]

Here $a = a_L + a_G$ is the splitting (1.15.4) for an $a \in F$ and $a_G^\vee$ means the element $(a_L)^\vee = (a^\vee)_L$ of $L$.

**Proof.** i ) The subgroup $E^{-1} W_{R \cap L}$ is contained in $L \otimes (F/\text{rad} I)$, since it is generated (as a semi-group) by $a \otimes a_G^\vee \in L \otimes (F/\text{rad} I)$ for $a \in R \cap L$ and $L \otimes (F/\text{rad} I)$ is closed under the product $\circ$.

Therefore,

\[ E^{-1} W_{R \cap L} \cap T_G \subset (L \otimes (F/\text{rad} I)) \cap (G \otimes (F/\text{rad} I)) = 0. \]

Hence $W_{R \cap L} \cap E(T_G) = \{1\}$, which implies injectivity of (1.15.5) due to (1.15.2).

ii) Let $a = a_L + a_G$ be the decomposition of (1.15.4) of $a \in R$. Then in $F \otimes (F/\text{rad} I)$, one computes as follows.

\[ a \otimes a^\vee = (a_L + a_G) \otimes (a_L)^\vee = a_L \otimes a_G^\vee + a_G \otimes a_G^\vee = (a_L \otimes a_G^\vee) \circ (a_G \otimes a_G^\vee). \]

Therefore we obtain a decomposition (1.15.6) of the reflection,

(1.15.8) \[ w_a = E(a_L \otimes a_G^\vee) E(a_G \otimes a_G^\vee) \quad \text{for } E(a_L \otimes a_G^\vee) \in W_{R \cap L}, \]

\[ a_G \otimes a_G^\vee \in T_G. \]

Thus $T_G$ is generated by $a_G \otimes a_G^\vee$ for $a \in R$.

To show that $T_G$ is a full lattice of $G \otimes (F/\text{rad} I)$, it is enough to show that $T_G$
spans the whole vector space since $T_G$ is already contained in a lattice $(Q(R) \cap G) \otimes (Q(R)/(Q(R) \cap \text{rad } I))$.

Due to (1.15.3) the space $RT_G$ is invariant under the action of $W_R/T_G \cong W_{R/G}$. Therefore due to (1.6) Cor. 1 and (1.8) Assertion, $RT_G$ contains $a_G \otimes (F/\text{rad } I)$ for any $a \in R$. Since $G$ is defined over $Q$, $G$ is spanned by $a_G$ for $a \in R$. This implies $RT_G = G \otimes (F/\text{rad } I)$.

q.e.d.

Note. Under the same assumption of the lemma, we have,

i) $R \cap L$ is a root system belonging to $I \mid L$.

ii) The linear isomorphism $p|_L : L \cong F/G$ induces an injection map $R \cap L \rightarrow R/G$ among two root systems, which induces an isomorphism $W_{R \cap L} \cong W_{R/G}$.

Proof. i) We have only to show Axiom 1 and 5 for $R \cap L$. Since $R \cap L$ is invariant under the action of $W_{R \cap L}$, the image of $RQ(R \cap L))$ in $F/G$ by $p$ is invariant under the action of $W_{R/G} \cong p_* W_{R \cap L}$ which contains $p(R \cap L) \subset R/G$. Thus due to (1.6) Cor. 1 and (1.8) Assertion, $p|_L(RQ(R \cap L)) = F/G$. Hence $Q(R \cap L)$ spans $L$.

If $R \cap L$ were reducible, there exists a linear subspace $H \subseteq L$, which is invariant under the action of $W_{R \cap L}$ containing an element of $R \cap L$. Then $pH \subseteq F/G$ is invariant under $W_{R/G}$ containing an element of $p(R \cap L) \subset R/G$. This is a contradiction to the fact that $R/G$ is an irreducible root system.

q.e.d.

(1.16) Counting $K_G(a)$. Let $R$ be a root system belonging to $I$, and $G$ be a subspace of $\text{rad } I$ defined over $Q$. To give a tentative description of the “extension” of the root system $R/G$ to the root system $R$, we introduce $K_G$, which associates a subset $K_G(a)$ of the lattice $Q(R) \cap G$ for each root $a \in R$ as follows,

$$K_G(a) := \{ x \in G : a + x \in R \}.$$ 

We shall call the $K_G(a)$, the counting set of $a \in R$.

Assertion 1. i) $0 \in K_G(a)$ and $K_G(a) = -K_G(a)$ for $a \in R$.

ii) $K_G(\varphi a) = K_G(a)$ for an automorphism $\varphi$ of $R$ and $a \in R$.

iii) $K_G(a)$ is closed under the reflection centered at each point of $K_G(a)$.

(i.e. If $x, y \in K_G(a)$, then $2x - y \in K_G(a)$.)

iv) $K_G(a)$ is closed under the translation by $I(\alpha, \beta \vee)K_G(\beta)$ for $a, \beta \in R$.

(i.e. $K_G(a) \supseteq K_G(a) + I(\alpha, \beta \vee)K_G(\beta)$.)
\[ \frac{2}{I(a, a)} K_G(a) = K_G(a^\vee), \text{ where } K_G(a^\vee) := G \cap \{ R^\vee - a^\vee \}. \]

**Proof.** i) is a consequence of (1.2) Note 1, i).

ii) and v) are trivial by definition.

iii) If \( a, a+x, a+y \in R \), then \( w_{a+x}(a+y) = -a+y-2x \in R \).

iv) If \( a, \beta, a+x, \beta+y \in R \), then \( w_{\beta+y}(a+x) = w_{\beta}(a)+x-I(\alpha, \beta^\vee)y \in R \).

q.e.d.

**Assertion 2.** Suppose that there exists a linear subspace \( L \) of \( F \) satisfying the conditions (1.15.4), (1.15.5) of (1.15) Lemma such that \( R \cap L \to R/G \) is surjective. Then

i) \( K_G(a) \) contains a full lattice of \( G \) for \( a \in R \).

ii) \( R = \bigoplus_{a \in R \cap L} \{ a + K_G(a) \} \).

iii) \( \bigcup_{a \in R \cap L} K_G(a) \) generates the lattice \( Q(R) \cap G \).

**Proof.** By assumptions \( R \subseteq (R \cap L) \oplus G \), so that we get the disjoint union presentation of \( R \) as ii). iii) is an immediate consequence of ii). i) is a consequence iii) taking in account the facts Assertion 1. iv) and the irreducibility of \( R \).

q.e.d.

(1.17) Hyperbolic extension. For the description of the central extension \( \tilde{W}_{R,G} \) in (1.17.1), we prepare some linear algebraic assertions. Proofs are elementary and omitted.

**Assertion 1.** Let \( F \) be a vector space over \( \mathbb{R} \) of finite rank equipped with a symmetric bilinear form \( I \). For a given linear subspace \( G \subseteq \text{rad} I \), there exists a triple \( (\tilde{F}_G, \tilde{I}_G, \iota_G) \) of a vector space \( \tilde{F}_G \) of rank \( \text{rank } F + \text{rank} (\text{rad } I/G) \), a symmetric bilinear form \( \tilde{I}_G \) on \( \tilde{F}_G \) and an injective linear map \( \iota_G : F \to \tilde{F}_G \), s.t.

i) \( I = \tilde{I}_G \circ \iota_G \).

ii) \( \text{rad } \tilde{I}_G = \iota_G(G) \).

**Definition.** We call \( (\tilde{F}_G, \tilde{I}_G, \iota_G) \) a hyperbolic extension of \( (F, I) \) w.r.t. \( G \). So far there is no confusion, we identify \( F \) with the subspace \( \iota_G(F) \) of \( \tilde{F}_G \) and \( G \) with the radical of \( \tilde{I}_G \).

**Assertion 2.** Hyperbolic extension is unique up to an isomorphism in the
following sense. Let $G, H$ be subspaces of $\text{rad } I$ s.t. $H \subset G$ and let $(\tilde{F}_G, \tilde{I}_G, \iota_G), (\tilde{F}_H, \tilde{I}_H, \iota_H)$ be hyperbolic extensions w.r.t. them. Then there exists an injective linear map $\varphi : \tilde{F}_G \to \tilde{F}_H$ such that

1) $\tilde{I}_G = \tilde{I}_H \circ \varphi,$
2) $\varphi \circ \iota_G = \iota_H.$

Assertion 3. Let $(\tilde{F}_G, \tilde{I}_G, \iota_G)$ be a hyperbolic extension w.r.t. $G$. Then the automorphism group is given by,

$$(1.17.2) \quad \text{Aut}(\tilde{F}_G, \tilde{I}_G, \iota_G) = E_G(M_G),$$

where

$$E_G : \tilde{F}_G \otimes (\tilde{F}_G/G) \to \text{End}(\tilde{F}_G),$$

is the Eichler Siegel map for $\tilde{F}_G$ and $\tilde{I}_G$ (1.14.1), and

$$M_G := \text{ker}(\text{rad } I \otimes (\text{rad } I/G) \to S^2(\text{rad } I/G))$$

$$\xi \quad \psi \quad \longrightarrow \frac{1}{2}(\xi + ^t\psi) \mod G.$$

(Here $S^2(V)$ is a symmetric tensor product of $V$.)

The rank of $M_G$ is given by,

$$(1.17.4) \quad \text{rank}(M_G) = \frac{1}{2}\text{rank}(\text{rad } I/G)(\text{rank}(\text{rad } I) + \text{rank } G - 1).$$

The product $\circ$ structure on $M_G$ coincides with the addition structure on $M_G$ as a linear space.

(1.18) The extension $\tilde{W}_{R,G}$. Let $R$ be a root system belonging to $I$ and $G$ be a subspace of $\text{rad } I$. We fix a hyperbolic extension $(\tilde{F}_G, \tilde{I}_G, \iota_G)$ and regard $\iota_G : F \to \tilde{F}_G$ as an inclusion map. Therefore as an element in $\tilde{F}_G$ each $\alpha \in R$ defines a reflexion of $\tilde{F}_G$ denoted by $\tilde{\omega}_\alpha = E_G(\alpha \otimes ^\alpha \gamma)$.

Put,

$$(1.18.1) \quad \tilde{W}_{R,G} := \text{the subgroup of } O(\tilde{F}_G, \tilde{I}_G) \text{ generated by } \tilde{\omega}_\alpha (\alpha \in R).$$

Lemma. The group $\tilde{W}_{R,G}$ is a central extension of $W_R$.

$$1 \to \tilde{K}_G \xrightarrow{E_G} \tilde{W}_{R,G} \xrightarrow{\rho_*} W_R \to 1,$$
where

i) $p_*$ is a surjective homomorphism induced from the restriction of the action of $\tilde{W}_{R,G}$ on $\tilde{F}_G$ to the subspace $F$.

ii) $\tilde{K}_G$ is a lattice of $M_G$ ((1.17.3)) defined by

$$(1.18.3) \quad \tilde{K}_G : = E_G^{-1}(\tilde{W}_{R,G}) \cap M_G.$$ 

where $E_G$ is the Eichler Siegel map for $\tilde{F}_G$ and $\tilde{I}_G$ in (1.17.2).

Proof. Since $\tilde{W}_{R,G}$ is generated by $E_G(a \otimes a')$ for $a \otimes a' \in F \otimes (F/G)$ ($a \in R$) and $F \otimes (F/G)$ is closed under the product $\circ$, the group $\tilde{W}_{R,G}$ is contained in $E_G(F \otimes (F/G))$. Hence the inverse Eichler Siegel map is well defined as an injective map

$$(1.18.4) \quad E_G^{-1} : \tilde{W}_{R,G} \rightarrow F \otimes (F/G).$$

The action of $E_G(F \otimes (F/G))$ on $\tilde{F}_G$ leaves the subspace $F$ invariant so that the restriction $p_*$ is well defined to make the diagram commutative:

$$\begin{array}{ccc}
\tilde{W}_{R,G} & \xrightarrow{p_*} & W_R \\
\cap E_G^{-1} & \xrightarrow{E^{-1}} & F \otimes (F/G) \rightarrow F \otimes (F/\text{rad } I).
\end{array}$$

Therefore by putting $\tilde{K}_G : = E_G^{-1}(\tilde{W}_{R,G}) \cap (F \otimes (\text{rad } I/G))$, one obtains the exact sequence (1.18.2).

Since $M_G \subset F \otimes (\text{rad } I/G)$, it is enough to show the inclusion relation $\tilde{K}_G \subset M_G$ to show (1.17.2).

Let us apply (1.14) Assertion for $\tilde{F}_G$, $\tilde{I}_G$ and $\xi \in \tilde{K}_G$, since $\tilde{w} : = E_G(\xi) \in \tilde{W}_{R,G}$ $\subset O(\tilde{F}_G, \tilde{I}_G)$. Hence

$$\xi + 't \xi \equiv I(\xi, t'\xi) \mod \tilde{F}_G \otimes G + G \otimes \tilde{F}_G.$$ 

Since $\xi \in F \otimes (\text{rad } I/G)$ we have $I(\xi, t'\xi) = 0$. Therefore

* ) $\xi + 't \xi \equiv 0 \mod \tilde{F}_G \otimes G + G \otimes \tilde{F}_G.$

Again noting $\xi \in F \otimes (\text{rad } I/G)$, * ) implies

** ) $\xi \in \text{rad } I \otimes (\text{rad } I/G).$

Then * ) and ** ) implies $\xi \in M_G$.

$\tilde{K}_G$ is discrete in $M_G$, since it is contained in a lattice $Q(R) \otimes (Q(R^\vee) / Q(R^\vee) \cap G)$. q.e.d.
The definition of the $r$ depends on a decomposition,

$$(1.19.1) \quad F = L \oplus H \oplus G$$

where $L$ is a subspace of $F$ complementary to $\text{rad} I$ and $H$ is a subspace of $\text{rad} I$ complementary to $G$.

Fixing one such decomposition, we introduce mappings,

$$(1.19.2) \quad \xi : \tilde{W}_{R,G} \rightarrow L \otimes L,$$

$$(1.19.3) \quad \rho : \tilde{W}_{R,G} \rightarrow H \otimes L,$$

$$(1.19.4) \quad \eta : \tilde{W}_{R,G} \rightarrow G \otimes L,$$

as components of the inverse Eichler Siegel map (1.14.5),

$$(1.19.5) \quad E^{-1}(\rho_*(g)) = \xi(g) + \rho(g) + \eta(g) \quad \text{for} \quad g \in \tilde{W}_{R,G},$$

where we use the following identification,

$$F \otimes (F/ \text{rad} I) \simeq (L \oplus H \oplus G) \otimes L = (L \otimes L) \oplus (H \otimes L) \oplus (G \otimes L).$$

The orthogonality of $g$ implies the condition (cf. (1.18) Assertion),

$$(1.19.6) \quad \xi(g) + \iota \xi(g) = I(\xi(g), \iota \xi(g)) \quad \text{for} \quad g \in \tilde{W}_{R,G}.$$

Let us reformulate the relation (1.14.3) into some formulae for components of $E$.

$$(1.19.7) \quad \xi(g_1 g_2) = \xi(g_1) + \xi(g_2) - I(\xi(g_1), \xi(g_2)),$$

$$(1.19.8) \quad \rho(g_1 g_2) = \rho(g_1)E_0(\iota \xi(g_2)) + \rho(g_2),$$

$$(1.19.9) \quad \eta(g_1 g_2) = \eta(g_1)E_0(\iota \xi(g_2)) + \eta(g_2),$$

where

$$E_0 : L \otimes L \rightarrow \text{End}(L),$$

$$E_0(\sum_i f_i \otimes g_i)(u) = u - \sum_i f_i I(g_i, u),$$

the Eichler Siegel map for $L$ and $I|L$.

As a corollary to (1.19.7), we have,
(1.19.10) \[ t\xi(g) = \xi(g^{-1}) \quad \text{for} \quad g \in \widetilde{W}_{R,G}. \]

\( \therefore \) If (1.19.10) is true for \( g_1, g_2, \) (1.19.7) implies that it is also true for \( g_1 g_2. \) On the other hand (1.19.10) is obviously true for reflexions \( g := \tilde{w}_a. \)

(1.20) The map \( r. \) We introduce the map \( r \) (1.20.1) and summarize its properties in the following Assertions 1—3.

**Assertion 1.** There exists a map,

(1.20.1) \[ r: \widetilde{W}_{R,G} \longrightarrow M_G \]

such that the inverse Eichler Siegel map (1.18.4) is given by,

(1.20.2) \[ E_{g^{-1}}(g) = \xi(g) + p(g) + q(g) - E_0(\xi(g)) t\xi(g) \]

\[ + \frac{1}{2} I(p(g), t\xi(g)) + r(g) \quad \text{for} \quad g \in \widetilde{W}_{R,G}. \]

Here the terms in the right hand corresponds to the following decomposition.

(1.20.3) \[ F \otimes (F/G) \approx (L \otimes L) \oplus (H \otimes L) \oplus (G \otimes L) \oplus (L \otimes H) \oplus \text{Sym}^2(H) \oplus M_G, \]

\[ M_G \approx A^2(H) \oplus \text{Sym}^2(H), \]

where \( H \otimes H = \text{Sym}^2(H) \oplus A^2(H) \) is the direct sum decomposition of \( H \otimes H \) into the symmetric and the anti-symmetric tensors.

**Assertion 2.** The multiplicative law for \( r \) is given by,

(1.20.4) \[ r(g_1 g_2) = r(g_1) + r(g_2) - A(I(p(g_1), E_0(\xi(g_2)) t\xi(g_2))) \]

\[ - I(q(g_1), E_0(\xi(g_2)) t\xi(g_2)) \quad \text{for} \quad g_1, g_2 \in \widetilde{W}_{R,G}. \]

Here \( A(\eta) = \frac{1}{2}(\eta - t\eta) \) is the anti-symmetric part of \( \eta \in H \otimes H. \)

**Assertion 3.** The ranges of \( \xi, p, q, r \) are followings

(1.20.5) \[ \xi(\widetilde{W}_{R,G}) \subset \sum_{\alpha, \beta \in \mathbb{R}} \mathbb{Z} \text{int}(\alpha, \beta) aL \otimes \beta L, \]

(1.20.6) \[ p(\widetilde{W}_{R,G}) \subset \sum_{\alpha, \beta \in \mathbb{R}} \mathbb{Z} \text{int}(\alpha, \beta) aH \otimes \beta L, \]

(1.20.7) \[ q(\widetilde{W}_{R,G}) \subset \sum_{\alpha, \beta \in \mathbb{R}} \mathbb{Z} \text{int}(\alpha, \beta) aL \otimes \beta L, \]

(1.20.8) \[ r(\widetilde{W}_{R,G}) \subset \sum_{\alpha, \beta \in \mathbb{R}} \mathbb{Z} \text{int}(\alpha, \beta) A(aH \otimes \beta H) \]


Here we denote by,

\[ (1.20.9) \quad a = a_L + a_H + a_G, \]

the decomposition (1.19.1) for a root \( a \in R \), and by \( \text{int}(a, \beta) \) for \( a, \beta \in R \) a positive real number defined by

\[
\text{int}(a, \beta) := \left\{ \begin{array}{ll}
2 \langle \alpha, \alpha \rangle & \text{for } \alpha = \beta \\
g.\text{c.\,d.} \left\{ \frac{2^k \prod_{j=1}^{k-1} I(a_j, a_{j+1})}{\prod_{j=1}^{k} I(a_j, a_j)} : a_1, \ldots, a_k \in R \right. \\
\left. \text{s.t. } a_1 = a \text{ and } a_k = \beta \right\}
\end{array} \right.
\]

(1.20.10)

(Note that \( \text{int}(a, \beta) = \text{int}(\beta, a) \) and \( \frac{I(a, a)}{2} \) \( \text{int}(a, \beta) \in \mathbb{Z} \) for \( a, \beta \in R \).)

**Proof for Assertions 1—3.**

1. Let

\[ E_G^{-1}(g) = \xi(g) + p(g) + q(g) + u(g) + v(g) + r(g), \]

be the decomposition of \( E_G^{-1}(g) \) according to the splitting (1.20.3). Apply the criterion for the orthogonality (1.14) Assertion for this. Namely

\[ \Xi + t^* \Xi - I(\Xi, t^* \Xi) \equiv 0 \mod G \otimes F + F \otimes G \quad \text{for} \quad \Xi = E_G^{-1}(g). \]

An explicit calculation of this relation gives

\[
u(g) = -t^* p(g) + I(\xi(g), t^* p(g)) \quad \text{and} \quad 2v(g) = I(p(g), t^* p(g)).\]

2. Apply (1.20.2) to the relation \( E_G^{-1}(g_1, g_2) = E_G^{-1}(g_1) \circ E_G^{-1}(g_2) \). An explicit calculation of this gives (1.20.4).

3. If \( a \in R \) is decomposed as (1.20.9), then

\[ a \otimes a^* = a_L \otimes a_L^* + a_H \otimes a_H^* + a_G \otimes a_G^* + a_L \otimes a_H + \frac{2}{I(a, a)} a_H \otimes a_H + \frac{2}{I(a, a)} a_G \otimes a_G \]

in \( F \otimes (F/G) \), so that we have
\[ \xi(\bar{a}) = a_L \otimes a_L = \frac{2}{I(a, a)} a_L \otimes a_L, \]
\[ p(\bar{a}) = a_H \otimes a_L = \frac{2}{I(a, a)} a_H \otimes a_L, \]
\[ q(\bar{a}) = a_C \otimes a_L = \frac{2}{I(a, a)} a_C \otimes a_L, \]
\[ r(\bar{a}) = \frac{2}{I(a, a)} a_C \otimes a_H. \]

Therefore for the reflexions \( \bar{a} (a \in R) \), \( \xi \), \( p \), \( q \), \( r \) takes value in the right hand of the formula of the assertion. Let us denote by \( M_\xi, M_p, M_q \) and by \( M_r \) the modules in the right hand of the formulae (1.20.5) – (1.20.8) respectively.

What is enough to show is that if for \( g_1, g_2 \in \bar{W}_{R,G} \) the mappings \( \xi, p, q, r \) take values in \( M_\xi, M_p, M_q \), and in \( M_r \) respectively then so is also for \( g_1 g_2 \).

In view of formulae (1.19.7) (1.19.8) (1.19.9) and (1.20.4), it is sufficient to show,

\[ I(M_\xi, M_\xi) \subset M_\xi, \quad M_pE_0(M_\xi) \subset M_\xi, \quad M_qE_0(M_\xi) \subset M_\xi, \quad \text{and} \quad AI(M_p, 'M_p) \subset M_r, \quad I(M_q, 'M_p) \subset M_r. \]

All these relations are reduced to a relation,

(1.20.11) \quad \int(a, \beta)I(\beta, \gamma) \int(\gamma, \delta) \subset \int(\alpha, \delta), \quad \text{for} \quad \alpha, \beta, \gamma, \delta \in R.

This is a direct consequence of the definition of \( \int(a, \delta) \) by noting the fact 
\[ I(\beta_L, \gamma_L) = I(\beta, \gamma) \quad \text{for} \quad \beta, \gamma \in R. \]

q.e.d.

One can sharpen the Assertion 3 by a slight modification of the proof as follows.

**Assertion 3'.** The ranges of \( p + q, r \) are followings

\[ (p + q)(\bar{W}_{R,G}) \subset \sum_{\alpha, \beta \in R} \int(\alpha, \beta) (a_H + a_C) \otimes a_L, \]
\[ r(\bar{W}_{R,G}) \subset \sum_{\alpha, \beta \in R} \int(\alpha, \beta) (a_H \otimes a_H) + a_C \otimes \beta_H. \]

**§ 2. Marked Extended Affine Root System \((R, G)\)**

(2.1) Let us recall the definition of an extended affine root system ((1.3) Ex. 1 iii)).

**Definition 1.** A root system \( R \) belonging to a symmetric bilinear form \( I \), whose sign is \((1, 2, 0)\), is called an extended affine root system of rank 1.

Namely; \( I \) is a positive semi-definite bilinear form on a real vector space \( F \).
of rank $l+2$, such that rank of $\text{rad} I := \{ x \in F : I(x, y) = 0 \forall y \in F \}$ is equal two. $R$ is a subset of $F$ satisfying the axioms 1), ..., 5) for a root system belonging to $I$ in (1.2) Def.

For short we shall use an abbreviation e.a.r.s. for an extended affine root system.

2. A marking $G$ for an e.a.r.s. $R$ is a linear subspace of $\text{rad} I$ of rank 1 defined over $\mathbb{Q}$ (cf. (1.8)). The pair $(R, G)$ is called a marked extended affine root system belonging to $I$ or a m.e.a.r.s. for short.

3. Two e.a.r.s.'s $R \subset F$ and $R' \subset F'$ are said to be isomorphic, if there exists a linear isomorphism $\varphi : F \cong F'$ s.t. $\varphi R = R'$ ((1.2) Def. 2.). Furthermore let $G$ and $G'$ be markings for $R$ and $R'$ respectively, and the map $\varphi$ induces $\varphi G = G'$. Then we say that the two m.e.a.r.s.'s are isomorphic.

$$\varphi : (R, G) \cong (R', G').$$

Note 1. If $\varphi$ is an isomorphism between e.a.r.s.'s belonging to $I$ and $I'$, then there exists a positive constant $c \in \mathbb{R}^+$ s.t. $I = d' \circ \varphi$ (cf. (1.4) Lemma).

Note 2. The same e.a.r.s. can split into non isomorphic m.e.a.r.s.'s, by different choice of markings. In fact there are at most two isomorph classes of m.e.a.r.s.'s for the same e.a.r.s. (cf. (5.4) Appendix).

(2.2) The dual $(R^\vee, G)$. Let $(R, G)$ be a m.e.a.r.s. belonging to $I$. Then $R^\vee := \{ \alpha^\vee \in F : \alpha \in R \}$ is also an e.a.r.s. belonging to $I$ (cf. (1.5) Lemma) and the same space $G$ defines a marking for $R^\vee$ (cf. (1.10) Note). We call the pair $(R^\vee, G)$ the dual m.e.a.r.s. of $(R, G)$.

(2.3) $\mathbb{Z}$-basis $a, b$ of $\text{rad} I$. Let $(R, G)$ be a m.e.a.r.s.. Recall that we denote by $Q(R)$ the lattice in $F$ generated by $R$. ((1.2) Def. 1.1))

Then $Q(R) \cap \text{rad} I$ is a full lattice of $\text{rad} I$, which has rank 2 (cf. (1.7) Cor.). We choose a $\mathbb{Z}$-basis $a, b$ of the module in the following way,

(2.3.1) $Q(R) \cap G^1 = \mathbb{Z}a,$

(2.3.2) $Q(R) \cap G^2 = \mathbb{Z}a + \mathbb{Z}b,$

where
The ambiguity of such basis \((a, b)\) is described by a group \(
\begin{pmatrix}
\pm 1 & * \\
0 & \pm 1
\end{pmatrix} : * \in \mathbb{Z}
\).}

\section{Quotient Root Systems of \((R, G)\)}

(3.1) Let \((R, G)\) be a m.e.a.r.s. Put \(G^1 := G, G^2 := \text{rad} I\) as in (2.3). Denote by \(R/G^i\) the image set of \(R\) in \(F/G^i\) by the projection \(p_i : F \to F/G^i\) and by \(I_G\), the metric on \(F/G^i\) induced from \(I, i = 1, 2\).

Recall that we denote by \(W_a\) the group generated by reflexions \(w_a\) for \(a \in B\) for a subset \(B\) in \(F\) (cf. (1.1.7)).

\textbf{Lemma 1.} i) \(R/\text{rad} I := R/G^2\) is a finite root system belonging to \(I_{\text{rad} I}\).

ii) \(R/G := R/G^1\) is an affine root system belonging to \(I_G\).

2. The following is an exact sequence,

\[(3.1.1) \quad 1 \to T_{G^i} \to E \to W_{R^i} \to W_{R/G^i} \to 1, \quad \text{for } i = 1, 2,\]

where

i) \(p_{i*}\) is a homomorphism induced naturally from the projection \(p_i\).

ii) \(E : F \otimes (F/\text{rad} I) \to \text{End}(F), E(\sum_i f_i \otimes g_i)(u) := u - \sum_i f_i I(g_i, u)\),

is the Eichler Siegel presentation for \(F\) and \(I\), defined in (1.14).

iii) \(T_{G^i} := E^{-1}(W_R) \cap (G^i \otimes F/\text{rad} I)\).

3. i) The sequence (3.1.1) splits into a semi-direct product.

ii) \(T_{G^i}\) is a full lattice of \(G^i \otimes (F/\text{rad} I)\) of rank \(iI\).

\textbf{Definition.} \(R/\text{rad} I\) and \(R/G\) are called the quotient finite root system and the quotient affine root system of \((R, G)\) respectively.

\textbf{Proof} 1. Due to (1.8) Assertion, \(R/G^i, i = 1, 2\) are root systems. The examples (1.3) 1. i) and ii) show that they are either a finite root system or an affine root system respectively.

2. The exactness of (3.1.1) is shown in (1.15) Assertion.
3. Due to the Lemma of (1.15), it is enough to show the existences of linear subspaces \( L^i \) and \( L^{i+1} \) of \( F \) such that

i) \( L^{i+2-i} \) is a complementary subspace of \( G^i \) in \( F \).

(3.1.2) \[ F = L^{i+2-i} \oplus G^i \quad \text{for} \quad i = 1, 2. \]

ii) The following homomorphisms are surjective.

(3.1.3) \[ \rho_{i}\mid_{W_{R \cap L^{i+2-i}}} : W_{R \cap L^{i+2-i}} \rightarrow W_{R/G^i} \quad \text{for} \quad i = 1, 2. \]

Since the decomposition (3.1.2) and the splitting of (3.1.1) play an important role in the later study, we explain details of the construction of \( L^i \) and \( L^{i+1} \) in the next (3.2) and (3.3).

(3.2) Subspace \( L^i \).

Let \( \beta_1, ..., \beta_l \in R/rad I \) be a basis of the finite root system.

For each \( 1 \leq i \leq l \), choose an element \( \alpha_i \in R \cap p_2^{-1}(\beta_i) \) where \( p_2 : F \rightarrow F/rad I \) is the linear projection. Define,

(3.2.1) \[ L^i = \bigoplus_{i=1}^{l} \mathbb{R} \alpha_i. \]

The fact that \( L^i \) satisfies (3.1.2), (3.1.3) for \( i = 2 \) follows directly from the facts,

i) any root \( \beta \in R/rad I \) is a linear combination of \( \beta_1, ..., \beta_l \) with integral coefficients which are either all \( \geq 0 \) or all \( \leq 0 \),

ii) the reflections \( \rho_{\beta_i}, i = 1, ..., l \) generates \( W_{R/rad I} \).

Let us summarize direct consequences of this splitting. (cf. (1.15) Lemma)

Assertion. i) \( Q(R) \) splits over \( \mathbb{Z} \) as follows,

(3.2.2) \[ Q(R) = (Q(R) \cap L^i) \oplus (Q(R) \cap rad I) \]

and

(3.2.3) \[ Q(R) \cap L^i = \bigoplus_{i=1}^{l} \mathbb{Z} \alpha_i. \]

ii) The subgroup \( W_{R \cap L^i} \) of \( W_R \) is generated by \( w_{\alpha_1}, ..., w_{\alpha_l} \) and is isomorphic to \( W_{R/rad I} \) so that

(3.2.4) \[ W_R = W_{R \cap L^i} \times T_{rad I} \]

iii) For any \( \alpha \in R \), the reflexion \( w_\alpha \) is decomposed as,

\[ w_\alpha = w_{\alpha L^i} \cdot E(\alpha_{rad I} \otimes \alpha_{rad I}^\vee), \quad \text{for} \quad w_{\alpha L^i} \in W_{R \cap L^i}, \alpha_{rad I} \otimes \alpha_{rad I}^\vee \in T_{rad I}. \]
where $a = a_L + a_{\text{rad}}$ is the splitting (3.1.2) and $a^{\vee} := (a_L)^{\vee} = (a^{\vee})_L = 2I(a, a)^{-1}a_L$.

iv) The lattice $T_{\text{rad}}$ is generated by $a_{\text{rad}} \otimes a^{\vee}_L$ for $a \in R$.

Note 1. If $R/\text{rad} I$ is reduced, then $p_2 : F \to F/\text{rad} I$ induces an isomorphism,

$$p_2 : R \cap L^i \approx R/\text{rad} I.$$  

(∴ $R/\text{rad} I = \bigcup_{i=1}^t W_{R/\text{rad} I} \beta_i \approx \bigcup_{i=1}^t W_{R \cap L^i} \alpha_i \subset R \cap L^i \subset R/\text{rad} I$.)

Note 2. The group $W_R$ acts nowhere properly on $F$ and $F^*$. (otomy: The subgroup $E(T_{\text{rad}})$ is a free abelian group of rank $= 2l > \text{rank } F = l + 2$ for $l > 2$, which cannot act properly anywhere on $F$ and $F^*$. Remember that $\text{rad} I$ is pointwisely fixed by $W_R$.)

(3.3) Subspace $L^{i+1}$. Let us recall the concept of basis $\beta_0, ..., \beta_l$ for the affine root system $R/G$ from [14] 4.

First, fix a sign of a generator $\overline{b}$ of $\text{rad} I_G = \text{rad} I/G$ and put,

$$(3.3.1) \quad E^i = \{x \in (F/G)^*: \overline{b}(x) = 1\}.$$  

The set $E^i$ is an affine space of rank $l$, whose translation group is $(F/\text{rad} I)^*$. $F/G$ is identified with the vector space of affine linear functions on $E^i$. The contragradient action of $W_{R/G}$ on $E^i$ is proper and the set of regular points of the action $E^i - \bigcup_{\beta \in R/G} H_{\beta}$. (Here $H_{\beta} := \{x \in E^i: \beta(x) = 0\}$) decomposes into open connected components, called chambers. $W_{R/G}$ acts faithfully and transitively on the set of chambers.

A basis $\{\beta_0, ..., \beta_l\}$ of $R/G$ is by definition a set of indivisible roots $\beta \in R/G$, such that $H_{\beta_0}, ..., H_{\beta_l}$ form the set of walls of a chamber $C$ and $\beta_i(x) > 0$ for all $x \in C, i = 0, ..., l$.

Let $\beta_0, ..., \beta_l$ be a basis for $R/G$ and $p_1 : F \to F/G$ be the projection. For each $0 \leq i \leq l$ choose an element $\alpha_i \in R \cap p_1^{-1}(\beta_i)$ and we define,

$$(3.3.2) \quad L^{i+1} := \bigoplus_{i=0}^t R\alpha_i.$$  

The fact that $L^{i+1}$ satisfies (3.1.2), (3.1.3) for $i = 1$, follows from the following
properties i), ii) of the basis $\beta_0, ..., \beta_t$.

i) Any root $\beta \in R/G$ is a linear combination of $\beta_0, ..., \beta_t$ with integral coefficients which are either all $\geq 0$ or all $\leq 0$.

ii) The group $W_{R/G}$ is generated by $w_{\beta_i}, i = 0, ..., l$.

iii) There exists positive integers $n_0, ..., n_t$ with $\text{gcd} (n_0, ..., n_t) = 1$, such that

\[ \sum_{i=0}^{t} n_i \beta_i \]

is a positive generator, say $\delta$, of the integral constant functions $Q(R/G) \cap \text{rad} I/G$ on $E$ (cf. [14](6.7)).

Let us summarize direct consequences of this splitting (cf. (1.15)).

**Assertion.** i) $Q(R)$ splits over $\mathbb{Z}$ as follows.

\[ Q(R) = (Q(R) \cap L^{t+1}) \bigoplus (Q(R) \cap G), \]

\[ = (\bigoplus_{i=0}^{t} \mathbb{Z} \alpha_i) \bigoplus \mathbb{Z} a. \]

Put

\[ b := \sum_{i=0}^{t} n_i \alpha_i. \]

Then

\[ Q(R) \cap \text{rad} I = Zb \bigoplus \mathbb{Z} a, \]

\[ Q(R) \cap L^{t+1} \cap \text{rad} I = Zb. \]

ii) The subgroup $W_{R \cap L^{t+1}}$ of $W_R$ is generated by $w_{\alpha_0}, ..., w_{\alpha_t}$ and is isomorphic to $W_{R/G}$, so that

\[ W_R = W_{R \cap L^{t+1}} \ltimes T_G. \]

**Note 1.** In the above choice of $\alpha_0, ..., \alpha_t$, if necessary after a change of the ordering of them, one can always assume that $p_2(\alpha_1), ..., p_2(\alpha_t)$ is a basis for $R/\text{rad} I$, and hence $L^i := \bigoplus_{i=1}^{t} \mathbb{R} \alpha_i$ is a splitting factor (3.2.1). Then $n_0$ of (3.3.3) is equal to 1 (cf. [14] (6.6)). Therefore,

\[ \bigoplus_{i=0}^{t} \mathbb{Z} \alpha_i = \left( \bigoplus_{i=1}^{t} \mathbb{Z} \alpha_i \right) \bigoplus Zb, \]

\[ Q(R) = \left( \bigoplus_{i=1}^{t} \mathbb{Z} \alpha_i \right) \bigoplus Zb \bigoplus \mathbb{Z} a. \]
Note 2. If the affine root system \( R/G \) is reduced, then the projection \( p_1 : F \to F/G \) induces an isomorphism,

\[
p_1 : R \cap L^{t+1} \simeq R/G.
\]

(\because \quad R/G = \bigcup_{i=0}^t \mathbb{W}_{R/G} \beta_i \simeq \bigcup_{i=0}^t \mathbb{W}_{R \cap L^{t+1}} \alpha_i \subseteq R \cap L^{t+1} \subseteq R/G.)

(3.4) Definition. A set \{\alpha_0, \ldots, \alpha_t\} of roots of \( R \) is called a basis for \( (R, G) \), if their image in \( R/G \) form basis for the affine root system \( R/G \).

We shall show the following assertion in (6.2).

Assertion. Let \( (R, G) \) be a m.e.a.r.s. such that \( R/G \) is reduced. Let \{\alpha_0, \ldots, \alpha_t\} and \{\beta_0, \ldots, \beta_t\} be basis for \( (R, G) \). Then there exists an automorphism \( \varphi \) of \( (R, G) \) such that

\[
\{\varphi(\alpha_0), \ldots, \varphi(\alpha_t)\} = \{\beta_0, \ldots, \beta_t\}.
\]

§ 4. Tier Numbers \( t(R), t_1(R, G), t_2(R, G) \)

For a m.e.a.r.s. \( (R, G) \), we introduce some numerical invariants \( t(R), t_1(R, G), t_2(R, G) \) which we shall call tier numbers.

(4.1) The total tier number.

Definition. Let \( R \) be an e.a.r.s. belonging to \( I \). The total tier number of \( R \) is,

\[
t(R) := \max\{I(\alpha, \alpha)/I(\beta, \beta) : \alpha, \beta \in R\}.
\]

Obviously this number depends only on the finite root system \( R/\text{rad} I \) as follows.

<table>
<thead>
<tr>
<th>type of ( R/\text{rad} I )</th>
<th>( A_t )</th>
<th>( B_t )</th>
<th>( C_t )</th>
<th>( BC_t )</th>
<th>( D_t )</th>
<th>( E_t )</th>
<th>( F_4 )</th>
<th>( G_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t(R) )</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>4</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

(4.2) Even lattice structure on \( Q(R) \). Since \( R/\text{rad} I \) is a finite root system, by suitable choice of positive constants \( c_1, c_2 \), the metrics \( I_R := c_1 I, I_{R^\vee} := c_2 I \) have the following properties (cf. (1.11.1) (1.11.2) and (1.11.4)).

i) \( Q(R) \) (resp. \( Q(R^\vee) \)) is an even lattice w.r.t. \( I_R \) (resp. \( I_{R^\vee} \)).
ii) \( \inf \{I_R(a, a) : a \in R\} = \inf \{I_{R^\vee}(a^\vee, a^\vee) : a^\vee \in R^\vee\} = 2. \)

iii) \( t(R) = \frac{I_R(a, a)}{2} \cdot \frac{I_{R^\vee}(a^\vee, a^\vee)}{2} \) for \( a \in R. \)

(4.3) Relative tier numbers. Let \( (R, G) \) be a m.e.a.r.s. Let \( a, b \) (resp. \( a^\vee, b^\vee \)) be a basis for \( Q(R) \cap \text{rad} I \) (resp. \( Q(R^\vee) \cap \text{rad} I \)) as in (2.3).

**Definition.** The first and the second tier numbers for \( (R, G) \) and \( (R^\vee, G) \) are,

\[

t_1(R, G) := \left| (b \mod a^\vee) : (b \mod a) \right| \times (I_{R^\vee} : I),
\]

\[

t_1(R^\vee, G) := \left| (b \mod a) : (b^\vee \mod a^\vee) \right| \times (I_R : I),
\]

\[

t_2(R, G) := \left| (a^\vee : a) \right| \times (I_{R^\vee} : I),
\]

\[

t_2(R^\vee, G) := \left| (a : a^\vee) \right| \times (I_R : I).
\]

*Here the notion \( A : B \) means the constant number \( c \) s.t. \( A = cB. \)*

The definition of the tier numbers above is invariant by a change of \( I \) by a constant multiple of \( I. \) A more general combinatorial definition of them is given in (1.12.1), (1.12.2).

(4.4) **Assertion.** i) Tier numbers are positive integers.

ii) \( t(R) = t_1(R, G) \cdot t_1(R^\vee, G), \) \( t(R) = t_2(R, G) \cdot t_2(R^\vee, G). \)

**Proof** (cf. **Assertion (1.12)).** i) Take \( I \) to be \( I_R. \) Since \( a = \frac{I_R(a, a)}{2} a^\vee \in Z a^\vee \subset Q(R^\vee), \) we have \( Q(R) \subset Q(R^\vee) \) and therefore \( Q(R) \cap G^i \subset Q(R^\vee) \cap G^i \) for \( i = 1, 2. \) Thus \( a : a^\vee, b \mod a : b^\vee \mod a^\vee \) are integers. This implies \( t_1(R^\vee, G), \) \( t_2(R^\vee, G) \) are integers. Other cases are shown similarly by taking \( I = I_R. \)

ii) By definition, the right hand is equal to \( (I_R : I) \cdot (I_{R^\vee} : I). \) Taking \( I \) to be equal to \( I_R \) and using (1.1.4) this number is equal to \( (I_{R^\vee}(a^\vee, a^\vee)/2) \times (I_R(a, a)/2) \) for an \( a \in R, \) which is equal to \( t(R) \) by (4.2) iii). q.e.d.

(4.5) The first tier number. By definition, the first tier number \( t_1(R, G) \) depends only on the affine root system \( R/G. \) If \( R/G \) is reduced, it is calculated as follows.

Let \( \beta_0, ..., \beta_l \) (resp. \( \beta_0^\vee , ..., \beta_l^\vee ) \) be a basis of the affine root system \( R/G \) (resp. \( R^\vee/G), \) so that \( \bar{b} = \sum_{i=0}^l n_i \beta_i \) (resp. \( \bar{b}^\vee = \sum_{i=0}^l n_i^\vee \beta_i^\vee \)) is the generator (3.3.3) of \( Q(R/G) \cap (\text{rad} I/G). \) Then \( \bar{b} : \bar{b}^\vee = 2 n_i / n_i^\vee I(a_i^\vee, a_i^\vee) \) for \( i = 0, ..., l \) and therefore,
\[ t_i(R, G) = \frac{n_i}{n^{\gamma}} \frac{I_{R, G}(a_i^{\gamma}, \alpha_i^{\gamma})}{2}, \quad (i = 0, \ldots, l). \]

The following is a table of the first tier number.

1. Reduced quotient affine root system case.

<table>
<thead>
<tr>
<th>Type of ( R/G ) (according to [14])</th>
<th>( A_1 )</th>
<th>( B_1 )</th>
<th>( B_1^{\gamma} )</th>
<th>( C_1 )</th>
<th>( C_1^{\gamma} )</th>
<th>( BC_1 )</th>
<th>( D_1 )</th>
<th>( E_1 )</th>
<th>( F_4 )</th>
<th>( F_4^{\gamma} )</th>
<th>( G_2 )</th>
<th>( G_2^{\gamma} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t_1(R, G) )</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>

2. Non reduced quotient affine root system case.

<table>
<thead>
<tr>
<th>Type of ( R/G ) (according to [14])</th>
<th>( BCC_1 )</th>
<th>( C^{\gamma}BC_1 )</th>
<th>( BB_1^{\gamma} )</th>
<th>( C^{\gamma}C_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t_1(R, G) )</td>
<td>1</td>
<td>4</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

(4.6) \textit{Note.} If \( R/G \) is reduced, then \( R/G \) is uniquely determined by the type of the quotient finite root system \( R/\text{rad } I \) and the tier number \( t_1(R, G) \), as we see in the first table above.

§ 5. Classification of Marked Extended Affine Root Systems

In this paragraph (5.2), we present a complete list of isomorphism classes of marked extended affine root systems \((R, G)\), which satisfy an assumption,

A) the quotient affine root system \( R/G \) is reduced.

For each isomorphism class of a m.e.a.r.s. with this assumption, we associate the type of the isomorphism class in (5.1). For each type of a m. e. a. r. s., we shall exhibit in the table of (5.2):

1) The first and the second tier numbers \( t_1(R, G) \) and \( t_2(R, G) \) (cf. (4.3)).

2) The type of the isomorphism class of dual \((R^{\gamma}, G)\) (cf. (2.2)).

3) The set \( R \) of roots in a vector space \( F \) with a metric \( I \) and a marking \( G \) (cf. Notation below) such that \((R, G)\) is a m.e.a.r.s. of the type of the table.

4) A basis, \( a_0, \ldots, a_l \) for \((R, G)\) (cf. (3.4)).

5) The counting, \( k(a_0), \ldots, k(a_l) \) (cf. (6.1), (6.4)).

6) The exponents \( m_{a_0}, \ldots, m_{a_l} \) (cf. (7.1.1)) w. r. t. the basis.

7) The Dynkin diagram \( \Delta_{R,G} \) (cf. (8.2)) of the type.
EXTENDED AFFINE ROOT SYSTEMS

\[ \# |\Gamma| = l(R) + \text{cod}(R, G) + 1 \text{ and } \text{cod}(R, G) = \text{codimension of } (R, G) \text{ (cf. (8.1)).} \]

**Notation.** Let \( \varepsilon_0, \varepsilon_1, \varepsilon_2, \cdots \) be a sequence of the orthogonal vectors in a real Hilbert space \( H \). The ambient space \( F \) and the marking \( G \) for a root system in the following table are defined as linear subspaces of \( \mathbb{R} a \oplus \mathbb{R} b \oplus H \) by,

\[
F : = \mathbb{R} a \oplus \mathop{\sum}_{i=0}^{l} \mathbb{R} a_i = \mathbb{R} a \oplus \mathbb{R} b \oplus \mathop{\sum}_{i=1}^{l} \mathbb{R} a_i
\]

\[
G : = \mathbb{R} a,
\]

where \( a_0, \cdots, a_l \) are the basis given in 3). The metric \( I \) on \( F \) is induced from that of \( H \) by regarding \( \mathbb{R} a \oplus \mathbb{R} b \) as the radical.

(5.1) **Definition of the type for a marked extended affine root system.**

Using a classification of m.e.a.r.s.'s in § 6, we define the types for a m.e.a.r.s. in the following manner.

Let \( (P_t, t_1, t_2) \) be a triple of a type \( P_t \) of a finite root system of rank \( l \) and two positive integers \( t_1 \) and \( t_2 \). A m.e.a.r.s. \( (R, G) \) is said to belong to \( (P_t, t_1, t_2) \) if \( P_t \) is the type of \( R/\text{rad } I \) and \( t_i = t_i(R, G), i = 1, 2 \).

i) **Type** \( P^{(t_1, t_2)} \)

Let \( t_1, t_2 \) be divisors of the total tier number \( t(P_t) \) of a finite root system \( P_t \) (cf. (4.1)), and \( (P_t, t_1, t_2) \) corresponds to a reduced affine root system (cf. Note 1.).

Then there exists a unique isomorphism class of m.e.a.r.s.'s which belong to \( (P_t, t_1, t_2) \) except the following 4 cases; \( (A_1, 1, 1), (B_t, 2, 2), l \geq 2, (C_t, 1, 1), l \geq 2, (BC_t, 2, 2), l \geq 1 \). Except for these cases we shall call \( P^{(t_1, t_2)} \) to be the type of the isomorphism class.

ii) **Type** \( P^{(t_1, t_2)*} \)

Let \( (P_t, t_1, t_2) \) be one of \( (A_1, 1, 1), (B_t, 2, 2), l \geq 2 \) or \( (C_t, 1, 1), l \geq 2 \). Then there exist two isomorphism classes of m.e.a.r.s.'s which belong to \( (P_t, t_1, t_2) \). In this case, one isomorphism class is called to be of type \( P^{(t_1, t_2)} \) and the other to be of type \( P^{(t_1, t_2)*} \).

iii) **Types** \( BC^{(2,2)}(1) \) and \( BC^{(2,2)}(2) \)

If \( (P_t, t_1, t_2) \) is \( (BC_t, 2, 2) \) for \( l \geq 1 \), then there are two isomorphism classes of
m.e.a.r.s.'s which belong to \((BC_t, 2, 2)\). Then one isomorphism class is called to be of type \(BC(2,2)(1)\) and the other to be of type \(BC(2,2)(2)\).

iv) Exceptional types.

Let us call the above \(A^{(1,1)*}, B^{(2,2)*}(l \geq 2), C^{(1,1)*}(l \geq 2), BC(2,2)(1)(l \geq 2)\) and \(BC(2,2)(2)(l \geq 1)\), exceptional types of marked extended affine root systems.

Note 1. The condition A discloses the cases \((BC_t, t_1, t_2)\) such that \(t_1=1\) or 4.

Note 2. The treatments and the studies of exceptional types in this and the following paragraphs are rather case by case study using the explicit description of the set of roots in (5.3). After introducing a concept of (mean) foldings of Dynkin diagrams in paragraph 12, we shall see that the exceptional types form naturally a group (cf. (12.5) Hierarchy).

Note 3. If \(P\) is a type for a m.e.a.r.s, let us denote also by \(P^\vee\) the type for the dual m.e.a.r.s. Then as a consequence of the classification, we see easily:

\[
(P_t^{(t_1,t_2)})^\vee = P_t^{(t/t_1,t/t_2)} \quad \text{for} \quad t = t(P),
\]

\[
(P_t^{(t_1,t_2)*})^\vee = P_t^{(t/t_1,t/t_2)*} \quad \text{for} \quad t = t(P).
\]

Hereafter we shall use a convention that if a statement on a m.e.a.r.s. \((R, G)\) depends only on the isomorphism class of \((R, G)\), then we use the type \(P\) instead of \((R, G)\) in the statement.

(5.2) Extended Affine Root Systems with Markings.

Type \(A_l^{(1,1)}\) \((l \geq 1)\)

1) \(t_1(A_l^{(1,1)})=1, \quad t_2(A_l^{(1,1)})=1, \quad (A_l^{(1,1)})^\vee = A_l^{(1,1)}\).

2) \(R: \pm (\varepsilon_i - \varepsilon_j) + nb + ma \quad (0 \leq i < j \leq l) \quad (n, m \in \mathbb{Z})\).

3) \(a_0 = -\varepsilon_0 + \varepsilon_1 + b, \quad a_i = \varepsilon_{i-1} - \varepsilon_i \quad (1 \leq i \leq l)\).

4) \(k_i=1 \quad (0 \leq i \leq l), \quad m_i=1 \quad (0 \leq i \leq l)\).

5) \(\text{cod}(A_l^{(1,1)}) = l+1, \quad \# \Gamma(A_l^{(1,1)}) = 2l+2\).
1) \( t_1(A^{(1,1)^0}_1) = 1, \quad t_2(A^{(1,1)^0}_1) = 1, \quad (A^{(1,1)^0}_1)^\vee = A^{(1,1)^0}_1 \)

2) \( R : \pm \varepsilon + nb + ma \quad (n, m \in \mathbb{Z}) \quad \text{s.t.} \quad n \cdot m \equiv 0 \mod 2) \).

3) \( a_0 = -\varepsilon + b, \quad a_1 = \varepsilon. \)

4) \( k_0 = 2, \quad k_1 = 1, \quad m_0 = \frac{1}{2}, \quad m_1 = 1. \)

5) \( \text{cod}(A^{(1,1)^0}_1) = 1, \quad \#\Gamma(A^{(1,1)^0}_1) = 3. \)

**Type \( B^{(1,1)}_l \) (\( l \geq 3 \))**

1) \( t_1(B^{(1,1)}_l) = 1, \quad t_2(B^{(1,1)}_l) = 1, \quad (B^{(1,1)}_l)^\vee = C^{(2,2)}_l. \)

2) \( R : \pm \varepsilon_i + nb + ma \quad (1 \leq i \leq l) \quad (n, m \in \mathbb{Z}), \)
   \[ \pm \varepsilon_i \pm \varepsilon_j + nb + ma \quad (1 \leq i < j \leq l) \quad (n, m \in \mathbb{Z}). \]

3) \( a_0 = -\varepsilon_1 - \varepsilon_2 + b, \quad a_i = \varepsilon_i - \varepsilon_{i+1} \quad (1 \leq i \leq l-1), \quad a_l = \varepsilon_1. \)

4) \( k_1 = 1 \quad (0 \leq i \leq l), \quad m_0 = 2, \quad m_1 = 2, \quad m_i = 4 \quad (2 \leq i \leq l-1), \quad m_l = 2. \)

5) \( \text{cod}(B^{(1,1)}_l) = l-2, \quad \#\Gamma(B^{(1,1)}_l) = 2l-1. \)
Type $B^{[1,2]} \ (l \geq 2)$

1) $t_1(B^{[1,2]}_l) = 1, \ t_2(B^{[1,2]}_l) = 2, \ (B^{[1,2]}_l) = C^{[2,1]}_l$.

2) $R : \pm \epsilon_i + nb + ma \ (1 \leq i \leq l) \ (n, m \in \mathbb{Z})$,
   $\pm \epsilon_i \pm \epsilon_j + nb + 2ma \ (1 \leq i < j \leq l) \ (n, m \in \mathbb{Z})$.

3) $a_0 = -\epsilon_1 - \epsilon_2 + b, \ a_i = \epsilon_i - \epsilon_{i+1} \ (1 \leq i \leq l-1), \ a_l = \epsilon_l$.

4) $k_i = 2 \ (0 \leq i \leq l-1), \ k_l = 1$,
   $m_0 = 1, \ m_1 = 1, \ m_i = 2 \ (2 \leq i \leq l)$.

5) $\text{cod}(B^{[1,2]}_l) = l - 1, \ \# \Gamma(B^{[1,2]}_l) = 2l$.

Type $B^{[2,1]} \ (l \geq 2)$

1) $t_1(B^{[2,1]}_l) = 2, \ t_2(B^{[2,1]}_l) = 1, \ (B^{[2,1]}_l) = C^{[1,2]}_l$.

2) $R : \pm \epsilon_i + nb + ma \ (1 \leq i \leq l) \ (n, m \in \mathbb{Z})$,
   $\pm \epsilon_i \pm \epsilon_j + 2nb + ma \ (1 \leq i < j \leq l) \ (n, m \in \mathbb{Z})$.

3) $a_0 = -\epsilon_1 + b, \ a_i = \epsilon_i - \epsilon_{i+1} \ (1 \leq i \leq l-1), \ a_l = \epsilon_l$.

4) $k_i = 1 \ (0 \leq i \leq l)$,
   $m_0 = 1, \ m_i = 2 \ (1 \leq i \leq l-1), \ m_l = 1$.

5) $\text{cod}(B^{[2,1]}_l) = l - 1, \ \# \Gamma(B^{[2,1]}_l) = 2l$. 
EXTENDED AFFINE ROOT SYSTEMS

Type $B_l^{(2,2)}$ ($l \geq 2$)

1) $t_1(B_l^{(2,2)}) = 2$, $t_2(B_l^{(2,2)}) = 2$, $(B_l^{(2,2)})^\vee = C_l^{(1,1)}$.

2) $R : \pm \epsilon_i + nb + ma$ ($1 \leq i \leq l$) ($n, m \in \mathbb{Z}$),
   \hspace{1cm} \pm \epsilon_i \pm \epsilon_j + nb + 2ma$ ($1 \leq i < j \leq l$) ($n, m \in \mathbb{Z}$).

3) $\alpha_0 = -\epsilon_1 + b$, $\alpha_i = \epsilon_i - \epsilon_{i+1}$ ($1 \leq i \leq l - 1$), $\alpha_l = \epsilon_l$.

4) $k_0 = 1$, $k_1 = 2$ ($1 \leq i \leq l - 1$), $k_l = 1$,
   \hspace{1cm} m_i = 1$ ($0 \leq i \leq l$).

5) $\text{cod}(B_l^{(2,2)}) = l + 1$, $\# \Gamma(B_l^{(2,2)}) = 2l + 2$.

\begin{center}
\begin{figure}
\end{figure}
\end{center}

Type $C_l^{(1,1)}$ ($l \geq 2$)

1) $t_1(C_l^{(1,1)}) = 1$, $t_2(C_l^{(1,1)}) = 1$, $(C_l^{(1,1)})^\vee = B_l^{(2,2)}$.

2) $R : \pm 2\epsilon_i + nb + ma$ ($1 \leq i \leq l$) ($n, m \in \mathbb{Z}$),
   \hspace{1cm} \pm \epsilon_i \pm \epsilon_j + nb + ma$ ($1 \leq i < j \leq l$) ($n, m \in \mathbb{Z}$).

3) $\alpha_0 = -2\epsilon_1 + b$, $\alpha_i = \epsilon_i - \epsilon_{i+1}$ ($1 \leq i \leq l - 1$), $\alpha_l = 2\epsilon_l$.

4) $k_1 = 1$ ($0 \leq i \leq l$),
   \hspace{1cm} m_i = 2$ ($0 \leq i \leq l$).

5) $\text{cod}(C_l^{(1,1)}) = l + 1$, $\# \Gamma(C_l^{(1,1)}) = 2l + 2$.

\begin{center}
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Type $C_l^{(1,2)}$ ($l \geq 2$)

1) $t_1(C_l^{(1,2)}) = 1$, $t_2(C_l^{(1,2)}) = 2$, $(C_l^{(1,2)})^\vee = B_l^{(2,1)}$.

2) $R : \pm 2\epsilon_i + nb + 2ma$ ($1 \leq i \leq l$) ($n, m \in \mathbb{Z}$),
   \hspace{1cm} \pm \epsilon_i \pm \epsilon_j + nb + ma$ ($1 \leq i < j \leq l$) ($n, m \in \mathbb{Z}$).
3) $a_0 = -2\varepsilon_1 + b$, $a_i = -2\varepsilon_i - \varepsilon_{i+1}$, $1 \leq i \leq l - 1$, $a_l = 2\varepsilon_1$.

4) $k_0 = 2$, $k_i = 1$, $1 \leq i \leq l - 1$, $k_l = 2$, $m_0 = 1$, $m_i = 2$, $1 \leq i \leq l - 1$, $m_l = 1$.

5) $\text{cod}(C^{[1,2]}) = l - 1$, $\# \Gamma(C^{[1,2]}) = 2l$.

**Type $C^{[2,1]}$ ($l \geq 2$)**

1) $t_1(C^{[2,1]}) = 2$, $t_2(C^{[2,1]}) = 1$, $(C^{[2,1]})^\gamma = B^{[1,2]}$.

2) $R: \pm 2\varepsilon_i + nb + ma$, $1 \leq i \leq l$ ($n, m \in \mathbb{Z}$),

$$\pm \varepsilon_i \pm \varepsilon_j + nb + ma$$

$1 \leq i < j \leq l$ ($n, m \in \mathbb{Z}$).

3) $a_0 = -\varepsilon_1 - \varepsilon_2 + b$, $a_i = -\varepsilon_i - \varepsilon_{i+1}$, $1 \leq i \leq l - 1$, $a_l = 2\varepsilon_i$.

4) $k_i = 1$, $0 \leq i \leq l$,

$m_0 = 1$, $m_i = 1$, $m_l = 2$, $2 \leq i \leq l$.

5) $\text{cod}(C^{[2,1]}) = l - 1$, $\# \Gamma(C^{[2,1]}) = 2l$.

**Type $C^{[2,2]}$ ($l \geq 3$)**

1) $t_1(C^{[2,2]}) = 2$, $t_2(C^{[2,2]}) = 2$, $(C^{[2,2]})^\gamma = B^{[1,1]}$.

2) $R: \pm 2\varepsilon_i + nb + ma$, $1 \leq i \leq l$ ($n, m \in \mathbb{Z}$),

$$\pm \varepsilon_i \pm \varepsilon_j + nb + ma$$

$1 \leq i < j \leq l$ ($n, m \in \mathbb{Z}$).

3) $a_0 = -\varepsilon_1 - \varepsilon_2 + b$, $a_i = -\varepsilon_i - \varepsilon_{i+1}$, $1 \leq i \leq l - 1$, $a_l = 2\varepsilon_i$.

4) $k_i = 1$, $0 \leq i \leq l - 1$, $k_l = 2$,

$m_0 = 1$, $m_i = 1$, $m_l = 2$, $2 \leq i \leq l - 1$, $m_i = 1$.

5) $\text{cod}(C^{[2,2]}) = l - 2$, $\# \Gamma(C^{[2,2]}) = 2l - 1$. 


Type $B^{(2,2)^o}_{l\geq2}$

1) $t_1(B^{(2,2)^o}_l) = 2$, $t_2(B^{(2,2)^o}_l) = 2$, $(B^{(2,2)^o}_l)^\vee = C^{(1,1)^o}_l$.

2) $R: \pm \varepsilon_i + nb + ma \ (1 \leq i \leq l) \ (n, m \in \mathbb{Z} \ s.t. \ nm \equiv 0 \ mod \ 2),$
\[ \pm \varepsilon_i \pm \varepsilon_j + 2nb + 2ma \ (1 \leq i < j \leq l) \ (n, m \in \mathbb{Z}). \]

3) $\alpha_0 = -\varepsilon_1 + b$, $\alpha_i = \varepsilon_i - \varepsilon_{i+1} \ (1 \leq i \leq l-1)$, $\alpha_i = \varepsilon_i$,

4) $k_l = 2 \ (0 \leq i \leq l-1)$, $k_l = 1$,

\[ m_0 = \frac{1}{2}, \ m_i = 1 \ (1 \leq i \leq l). \]

5) $\text{cod}(B^{(2,2)^o}_l) = l$, $\# \Gamma(B^{(2,2)^o}_l) = 2l + 1$.

Type $C^{(1,1)^o}_{l\geq2}$

1) $t_1(C^{(1,1)^o}_l) = 1$, $t_2(C^{(1,1)^o}_l) = 1$, $(C^{(1,1)^o}_l)^\vee = B^{(2,2)^o}_l$.

2) $R: \pm 2\varepsilon_i + nb + ma \ (1 \leq i \leq l) \ (n, m \in \mathbb{Z} \ s.t. \ nm \equiv 0 \ mod \ 2),$
\[ \pm \varepsilon_i \pm \varepsilon_j + nb + ma \ (1 \leq i < j \leq l) \ (n, m \in \mathbb{Z}). \]

3) $\alpha_0 = -2\varepsilon_1 + b$, $\alpha_i = \varepsilon_i - \varepsilon_{i+1} \ (1 \leq i \leq l-1)$, $\alpha_i = 2\varepsilon_i$.

4) $k_0 = 2$, $k_i = 1 \ (1 \leq i \leq l)$,

\[ m_0 = 1, \ m_i = 2 \ (1 \leq i \leq l). \]

5) $\text{cod}(C^{(1,1)^o}_l) = l$, $\# \Gamma(C^{(1,1)^o}_l) = 2l + 1$. 
Type $BC^{(2,1)} \ (l \geq 1)$

1) $t_1(BC^{(2,1)}) = 2, \ t_2(BC^{(2,1)}) = 1, \ (BC^{(2,1)})^\vee = BC^{(2,4)}$.

2) $R : \pm \epsilon_i + nb + ma \ (1 \leq i \leq l) \ (n, m \in \mathbb{Z}),$
   $\pm 2\epsilon_i + (2n+1)b + ma \ (1 \leq i \leq l) \ (n, m \in \mathbb{Z}),$
   $\pm \epsilon_i \pm \epsilon_j + nb + ma \ (1 \leq i < j \leq l) \ (n, m \in \mathbb{Z})$.

3) $a_0 = -2\epsilon_1 + b, \ a_i = \epsilon_i - \epsilon_{i+1} \ (1 \leq i \leq l-1), \ a_l = \epsilon_l.$

4) $k_i = 1 \ (0 \leq i \leq l),$
   $m_i = 4 \ (0 \leq i \leq l-1), \ m_l = 2.$

5) $\text{cod}(BC^{(2,1)}) = l, \ \# \Gamma(BC^{(2,1)}) = 2l + 1.$

Type $BC^{(2,4)} \ (l \geq 1)$

1) $t_1(BC^{(2,4)}) = 2, \ t_2(BC^{(2,4)}) = 4, \ (BC^{(2,4)})^\vee = BC^{(2,1)}$.

2) $R : \pm \epsilon_i + nb + ma \ (1 \leq i \leq l) \ (n, m \in \mathbb{Z}),$
   $\pm 2\epsilon_i + (2n+1)b + 4ma \ (1 \leq i \leq l) \ (n, m \in \mathbb{Z}),$
   $\pm \epsilon_i \pm \epsilon_j + nb + 2ma \ (1 \leq i < j \leq l) \ (n, m \in \mathbb{Z})$.

3) $a_0 = -2\epsilon_1 + b, \ a_i = \epsilon_i - \epsilon_{i+1} \ (1 \leq i \leq l-1), \ a_l = \epsilon_l.$

4) $k_0 = 4, \ k_i = 2 \ (1 \leq i \leq l-1), \ k_l = 1,$
   $m_0 = 1, \ m_i = 2 \ (1 \leq i \leq l).$

5) $\text{cod}(BC^{(2,4)}) = l, \ \# \Gamma(BC^{(2,4)}) = 2l + 1.$
Type $BC^{[2,2]}(1)$ \((l \geq 2)\)

1) \(t_1(BC^{[2,2]}(1)) = 2, \ t_2(BC^{[2,2]}(1)) = 2, \ (BC^{[2,2]}(1))^\vee = BC^{[2,2]}(1)\).

2) \(R: ±\varepsilon_i + nb + ma \ (1 \leq i \leq l) \ (n, m \in \mathbb{Z}),\)
   \[±2\varepsilon_i + (2n+1)b + 2ma \ (1 \leq i \leq l) \ (n, m \in \mathbb{Z}),\]
   \[±\varepsilon_i ± \varepsilon_j + nb + ma \ (1 \leq i < j \leq l) \ (n, m \in \mathbb{Z}).\]

3) \(a_0 = -2\varepsilon_1 + b, \ a_i = \varepsilon_i - \varepsilon_{i+1} \ (1 \leq i \leq l-1), \ a_l = \varepsilon_l.\)

4) \(k_0 = 2, \ k_i = 1 \ (1 \leq i \leq l),\)
   \(m_0 = 2, \ m_i = 4 \ (1 \leq i \leq l-1), \ m_l = 2.\)

5) \(\text{cod}(BC^{[2,2]}(1)) = l - 1, \ \# \Gamma(BC^{[2,2]}(1)) = 2l.\)

Type $BC^{[2,2]}(2)$ \((l \geq 1)\)

1) \(t_1(BC^{[2,2]}(2)) = 2, \ t_2(BC^{[2,2]}(2)) = 2, \ (BC^{[2,2]}(2))^\vee = BC^{[2,2]}(2).\)

2) \(R: ±\varepsilon_i + nb + ma \ (1 \leq i \leq l) \ (n, m \in \mathbb{Z}),\)
   \[±2\varepsilon_i + (2n+1)b + 2ma \ (1 \leq i \leq l) \ (n, m \in \mathbb{Z}),\]
   \[±\varepsilon_i ± \varepsilon_j + nb + ma \ (1 \leq i < j \leq l) \ (n, m \in \mathbb{Z}).\]

3) \(a_0 = -2\varepsilon_1 + b, \ a_i = \varepsilon_i - \varepsilon_{i+1} \ (1 \leq i \leq l-1), \ a_l = \varepsilon_l.\)

4) \(k_i = 2 \ (0 \leq i \leq l-1), \ k_l = 1,\)
   \(m_i = 2 \ (0 \leq i \leq l).\)

5) \(\text{cod}(BC^{[2,2]}(2)) = l + 1, \ \# \Gamma(BC^{[2,2]}(2)) = 2l + 2.\)
Type $D^{(1,1)}(l \geq 4)$

1) $t_1(D^{(1,1)}_l) = 1$, $t_2(D^{(1,1)}_l) = 1$, $(D^{(1,1)}_l)^\vee = D^{(1,1)}_l$.

2) $R : \pm \varepsilon_i \pm \varepsilon_j + nb + ma$ $(1 \leq i < j \leq l)$ $(n, m \in \mathbb{Z})$.

3) $a_0 = -\varepsilon_1 - \varepsilon_2 + b$, $a_i = \varepsilon_i - \varepsilon_{i+1}$ $(1 \leq i \leq l-1)$, $a_i = \varepsilon_{i-1} + \varepsilon_i$.

4) $k_l = 1$ $(0 \leq i \leq l)$,
   $m_0 = 1$, $m_1 = 1$, $m_2 = 2$ $(2 \leq i \leq l-2)$, $m_{l-1} = 1$, $m_l = 1$.

5) $\text{cod}(D^{(1,1)}_l) = l - 3$, $\# \Gamma(D^{(1,1)}_l) = 2l - 2$.

In the next three types $(E_8^{(1,1)}$, $E_7^{(1,1)}$ and $E_6^{(1,1)}$), let $\omega_i : = \varepsilon_i - \frac{1}{9} \sum_{j=0}^{8} \varepsilon_j$ $(0 \leq i \leq 8)$, so that $\sum_{i=0}^{8} \omega_i = 0$ and $I(\omega_i, \omega_j) = \frac{1}{9} + \delta_{ij}$ for $0 \leq i, j \leq 8$.

Type $E_8^{(1,1)}$

1) $t_1(E_8^{(1,1)}) = 1$, $t_2(E_8^{(1,1)}) = 1$, $(E_8^{(1,1)})^\vee = E_8^{(1,1)}$.

2) $R : \pm (\omega_i - \omega_j) + nb + ma$ $(1 \leq i < j \leq 6)$ $(n, m \in \mathbb{Z})$,
   $\pm (\omega_i + \omega_j + \omega_k) + nb + ma$ $(1 \leq i < j < k \leq 6)$ $(n, m \in \mathbb{Z})$,
   $\pm (\omega_1 + \omega_2 + \cdots + \omega_6) + nb + ma(n, m \in \mathbb{Z})$.

3) $a_0 = -(\omega_1 + \cdots + \omega_6) + b$, $a_i = \omega_i - \omega_{i+1}$ $(1 \leq i \leq 5)$, $a_6 = \omega_4 + \omega_5 + \omega_6$.

4) $k_l = 1$ $(0 \leq i \leq 6)$,
   $m_0 = 1$, $m_1 = 1$, $m_2 = 2$, $m_3 = 3$, $m_4 = 2$, $m_5 = 1$, $m_6 = 2$.

5) $\text{cod}(E_8^{(1,1)}) = 1$, $\# \Gamma(E_8^{(1,1)}) = 8$. 
Type $E^{(1,1)}_7$

1) $t_1(E^{(1,1)}_7) = 1$, $t_2(E^{(1,1)}_7) = 1$, $(E^{(1,1)}_7)^\vee = E^{(1,1)}_7$.

2) $R : \pm (\omega_i - \omega_j) + nb + ma \ (1 \leq i < j \leq 7) \ (n, m \in \mathbb{Z})$,
   $\pm (\omega_i + \omega_j + \omega_k) + nb + ma \ (1 \leq i < j < k \leq 7) \ (n, m \in \mathbb{Z})$,
   $\pm (\omega_1 + \cdots + \omega_i + \cdots + \omega_7) + nb + ma \ (1 \leq i \leq 7) \ (n, m \in \mathbb{Z})$.

3) $\alpha_0 = -(\omega_1 + \cdots + \omega_6) + b$, $\alpha_1 = \omega_i - \omega_{i+1} \ (1 \leq i \leq 6)$, $\alpha_7 = \omega_8 + \omega_6 + \omega_7$.

4) $k_i = 1 \ (0 \leq i \leq 7)$,
   $m_0 = 1$, $m_1 = 1$, $m_2 = 2$, $m_3 = 3$, $m_4 = 4$, $m_5 = 3$, $m_6 = 2$, $m_7 = 2$.

5) cod$(E^{(1,1)}_7) = 1$, $# \Gamma(E^{(1,1)}_7) = 9$.

Type $E^{(1,1)}_8$

1) $t_1(E^{(1,1)}_8) = 1$, $t_2(E^{(1,1)}_8) = 1$, $(E^{(1,1)}_8)^\vee = E^{(1,1)}_8$.

2) $R : \pm (\omega_i - \omega_j) + nb + ma \ (0 \leq i < j \leq 8) \ (n, m \in \mathbb{Z})$,
   $\pm (\omega_i + \omega_j + \omega_k) + nb + ma \ (0 \leq i < j < k \leq 8) \ (n, m \in \mathbb{Z})$.

3) $\alpha_0 = \omega_8 - \omega_1 + b$, $\alpha_1 = \omega_i - \omega_{i+1} \ (1 \leq i \leq 7)$, $\alpha_8 = \omega_6 + \omega_7 + \omega_8$.

4) $k_i = 1 \ (0 \leq i \leq 8)$,
   $m_0 = 1$, $m_1 = 2$, $m_2 = 3$, $m_3 = 4$, $m_4 = 5$, $m_5 = 6$, $m_6 = 4$, $m_7 = 2$, $m_8 = 3$.

5) cod$(E^{(1,1)}_8) = 1$, $# \Gamma(E^{(1,1)}_8) = 10$. 
Type $F_4^{(1,1)}$

1) $t_1(F_4^{(1,1)})=1$, $t_2(F_4^{(1,1)})=1$, $(F_4^{(1,1)})^\vee = F_4^{(2,2)}$.

2) $R: \pm \varepsilon_i + nb + ma \ (1 \leq i \leq 4) \ (n, m \in \mathbb{Z})$, 
   \[ \pm \varepsilon_i \pm \varepsilon_j + nb + ma \ (1 \leq i < j \leq 4) \ (n, m \in \mathbb{Z}) \], 
   \[ \frac{1}{2}(\pm \varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4) + nb + ma \ (n, m \in \mathbb{Z}). \]

3) $\alpha_0 = \varepsilon_1 - \varepsilon_2 + b$, $\alpha_1 = \varepsilon_2 - \varepsilon_3$, $\alpha_2 = \varepsilon_3 - \varepsilon_4$, $\alpha_3 = \varepsilon_4 - \sigma$, $\alpha_4 = \sigma$,
   where $\sigma = \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)$.

4) $k_i = 1 \ (0 \leq i \leq 4)$,
   $m_0 = 2$, $m_1 = 4$, $m_2 = 6$, $m_3 = 4$, $m_4 = 2$.

5) $\text{cod}(F_4^{(1,1)}) = 1$, $\# \Gamma(F_4^{(1,1)}) = 6$.

Type $F_4^{(1,2)}$

1) $t_1(F_4^{(1,2)}) = 1$, $t_2(F_4^{(1,2)}) = 2$, $(F_4^{(1,2)})^\vee = F_4^{(2,1)}$.

2) $R: \pm \varepsilon_i + nb + ma \ (1 \leq i \leq 4) \ (n, m \in \mathbb{Z})$, 
   \[ \pm \varepsilon_i \pm \varepsilon_j + nb + 2ma \ (1 \leq i < j \leq 4) \ (n, m \in \mathbb{Z}) \], 
   \[ \frac{1}{2}(\pm \varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4) + nb + ma \ (n, m \in \mathbb{Z}). \]

3) $\alpha_0 = \varepsilon_1 - \varepsilon_2 + b$, $\alpha_1 = \varepsilon_2 - \varepsilon_3$, $\alpha_2 = \varepsilon_3 - \varepsilon_4$, $\alpha_3 = \varepsilon_4 - \sigma$, $\alpha_4 = \sigma$.

4) $k_0 = 2$, $k_1 = 2$, $k_2 = 2$, $k_3 = 1$, $k_4 = 1$,
   $m_0 = 1$, $m_1 = 2$, $m_2 = 3$, $m_3 = 4$, $m_4 = 2$. 
5) \( \text{cod}(F_4^{(2,1)}) = 1, \ # \Gamma'(F_4^{(2,1)}) = 6. \)

**Type \( F_4^{(2,1)} \)**

1) \( t_1(F_4^{(2,1)}) = 2, \ t_2(F_4^{(2,1)}) = 1, \ (F_4^{(2,1)})^\vee = F_4^{(1,2)}. \)

2) \( R : \pm 2\varepsilon_i + 2nb + ma \ (1 \leq i \leq 4) \ (n, m \in \mathbb{Z}), \)
\( \pm \varepsilon_i \pm \varepsilon_j + nb + ma \ (1 \leq i < j \leq 4) \ (n, m \in \mathbb{Z}), \)
\( \pm \varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4 + 2nb + ma \ (n, m \in \mathbb{Z}). \)

3) \( a_0 = \varepsilon_1 - \varepsilon_2 + b, \ a_1 = \varepsilon_2 - \varepsilon_3, \ a_2 = \varepsilon_3 - \varepsilon_4, \ a_3 = 2\varepsilon_4, \ a_4 = -\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4. \)

4) \( k_1 = 1 \ (0 \leq i \leq 4), \)
\( m_0 = 1, \ m_1 = 2, \ m_2 = 3, \ m_3 = 4, \ m_4 = 2. \)

5) \( \text{cod}(F_4^{(2,1)}) = 1, \ # \Gamma'(F_4^{(2,1)}) = 6. \)

**Type \( F_4^{(2,2)} \)**

1) \( t_1(F_4^{(2,2)}) = 2, \ t_2(F_4^{(2,2)}) = 2, \ (F_4^{(2,2)})^\vee = F_4^{(1,1)}. \)

2) \( R : \pm 2\varepsilon_i + 2nb + 2ma \ (1 \leq i \leq 4) \ (n, m \in \mathbb{Z}), \)
\( \pm \varepsilon_i \pm \varepsilon_j + nb + ma \ (1 \leq i < j \leq 4) \ (n, m \in \mathbb{Z}), \)
\( \pm \varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4 + 2nb + 2ma \ (n, m \in \mathbb{Z}). \)

3) \( a_0 = \varepsilon_1 - \varepsilon_2 + b, \ a_1 = \varepsilon_2 - \varepsilon_3, \ a_2 = \varepsilon_3 - \varepsilon_4, \ a_3 = 2\varepsilon_4, \ a_4 = -\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4. \)
4) $k_0=1$, $k_1=1$, $k_2=1$, $k_3=2$, $k_4=2$, 
$m_0=1$, $m_1=2$, $m_2=3$, $m_3=2$, $m_4=1$.

5) $\text{cod}(F_4^{(2,2)})=1$, $\# \Gamma(F_4^{(2,2)})=6$.

In the last four types ($G_2^{(1,1)}$, $G_2^{(1,3)}$, $G_3^{3,1}$ and $G_3^{(3,3)}$), let $\phi_i = \varepsilon_i - \frac{1}{3}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$, so that $\sum_{i=1}^{3} \phi_i = 0$ and $I(\phi_i, \phi_j) = -\frac{1}{3} + \delta_{ij}$ ($1 \leq i, j \leq 3$).

**Type $G_2^{(1,1)}$**

1) $t_1(G_2^{(1,1)})=1$, $t_2(G_2^{(1,1)})=1$, $(G_2^{(1,1)})^\vee = G_2^{(3,3)}$.

2) $R: \pm \phi_i + nb + ma$ ($1 \leq i \leq 3$) $(n, m \in \mathbb{Z})$, 
$\pm (\phi_i - \phi_j) + nb + ma$ ($1 \leq i < j \leq 3$) $(n, m \in \mathbb{Z})$.

3) $a_0 = -\phi_1 + \phi_3 + b$, $a_1 = \phi_1 - \phi_2$, $a_2 = \phi_2$.

4) $k_1 = 1$ ($0 \leq i \leq 2$), 
$m_0 = 3$, $m_1 = 6$, $m_2 = 3$.

5) $\text{cod}(G_2^{(1,1)})=1$, $\# \Gamma(G_2^{(1,1)})=4$.

**Type $G_2^{(1,3)}$**

1) $t_1(G_2^{(1,3)})=1$, $t_2(G_2^{(1,3)})=3$, $(G_2^{(1,3)})^\vee = G_2^{(3,1)}$.

2) $R: \pm \phi_i + nb + ma$ ($1 \leq i \leq 3$) $(n, m \in \mathbb{Z})$, 
$\pm (\phi_i - \phi_j) + nb + 3ma$ ($1 \leq i < j \leq 3$) $(n, m \in \mathbb{Z})$. 
3) \( \alpha_0 = -\phi_1 + \phi_3 + b, \quad \alpha_1 = \phi_1 - \phi_2, \quad \alpha_2 = \phi_2. \)

4) \( k_0 = 3, \quad k_1 = 3, \quad k_2 = 1, \)
\( m_0 = 1, \quad m_1 = 2, \quad m_2 = 3. \)

5) \( \text{cod}(G^{(3,3)}_2) = 1, \quad \# \Gamma(G^{(3,3)}_2) = 4. \)

\[ \begin{array}{c}
\text{Type } G^{(3,1)}_2 \\
1) \ t_1(G^{(3,1)}_2) = 3, \quad t_2(G^{(3,1)}_2) = 1, \quad (G^{(3,1)}_2) = G^{(1,3)}_2.
2) \ R: \pm 3\phi_i + 3nb + ma \ (1 \leq i \leq 3) \ (n, m \in \mathbb{Z}), \]
\[ \quad \pm (\phi_i - \phi_j) + nb + ma \ (1 \leq i < j \leq 3) \ (n, m \in \mathbb{Z}). \]
3) \( \alpha_0 = -\phi_1 + \phi_3 + b, \quad \alpha_1 = \phi_1 - \phi_2, \quad \alpha_2 = 3\phi_2. \)

4) \( k_i = 1 \ (1 \leq i \leq 3) \)
\( m_0 = 1, \quad m_1 = 2, \quad m_2 = 3. \)

5) \( \text{cod}(G^{(3,1)}_2) = 1, \quad \# \Gamma(G^{(3,1)}_2) = 4. \)

\[ \begin{array}{c}
\text{Type } G^{(3,3)}_2 \\
1) \ t_1(G^{(3,3)}_2) = 3, \quad t_2(G^{(3,3)}_2) = 3, \quad (G^{(3,3)}_2) = G^{(1,1)}_2.
2) \ R: \pm 3\phi_i + 3nb + 3ma \ (1 \leq i \leq 3) \ (n, m \in \mathbb{Z}), \]
\[ \quad \pm (\phi_i - \phi_j) + nb + ma \ (1 \leq i < j \leq 3) \ (n, m \in \mathbb{Z}). \]
3) \( \alpha_0 = -\phi_1 + \phi_3 + b, \quad \alpha_1 = \phi_1 - \phi_2, \quad \alpha_2 = 3\phi_2. \)
4) $k_0 = 1, \ k_1 = 1, \ k_2 = 3, \ m_0 = 1, \ m_1 = 2, \ m_2 = 1.$

5) $\text{cod}(G_2^{(3,3)}) = 1, \ # \Gamma(G_2^{(3,3)}) = 4.$

\[ a_1 \]
\[ a_2 \]
\[ a_3 \]
\[ a_4 \]

(5.3) **A description of the set of roots.**

Here we give a systematic description of the set $R$ from the knowledge of the type of an e.a.r.s. $(R, G)$.

In the following we use the notation $R_P$ as for the set of finite roots of type $P$. The Weyl group invariant metric is denoted by $I_P$, which is normalized s.t. $\inf \{I_P(a, a) : a \in R_P\} = 2$.

i) **Type** $P^{(1,1)}$

\[ R = \bigcup_{a \in R_P} \{a + Zb + Za\}. \]

ii) **Type** $P^{(1,t)}$ for $t = t(P)$

\[ R = \bigcup_{a \in R_P} \{a + Zb + Z I_P(a, a)\}. \]

iii) **Type** $P^{(t,1)}$ for $t = t(P)$

\[ R = \bigcup_{a \in R_P} \{a + Z I_P(a, a) b + Za\}. \]

iv) **Type** $P^{(t,t)}$ for $t = t(P)$

\[ R = \bigcup_{a \in R_P} \{a + Z I_P(a, a) b + Z I_P(a, a) a\}. \]

v) **Type** $P^{(1,1)*}$ where $1 = t(P)$ (Case $A^{(1,1)*}_1$)

\[ R = \bigcup_{a \in R_P} \{a + nb + ma : m, n \in Z, mn \equiv 0 \mod 2\}. \]
vi) Type \( P^{(1,1)*} \) where \( 1 < t(P) \) (Case \( C^{(1,1)*}, l \geq 2 \))

\[
R = \bigcup_{a \in R_P} \left\{ a + nb + ma : m, n \in \mathbb{Z}, mn \equiv 0 \pmod{2} \text{ if } a \text{ is a long root} \right\}.
\]

vii) Type \( P^{(t,t)*} \) where \( t = t(P) + 1 \) (Case \( B^{(2,2)*}, l \geq 2 \))

\[
R = \bigcup_{a \in R_P} \left\{ a + nb + ma : m, n \in \mathbb{Z}, mn \equiv 0 \pmod{I_P(a, a)} \right\}. \\
mn \equiv 0 \pmod{2} \text{ if } a \text{ is a short root}.
\]

viii) Type \( BC^{(2,1)} \) (\( l \geq 1 \))

\[
R = \bigcup_{a \in R_{BC}} \left\{ a + nb + ma : m, n \in \mathbb{Z}, n \equiv 1 \pmod{2} \text{ for } a \text{ longest root} \right\}.
\]

ix) Type \( BC^{(2,4)} \) (\( l \geq 1 \))

\[
R = \bigcup_{a \in R_{BC}} \left\{ a + nb + ma : m, n \in \mathbb{Z}, n \equiv 1 \pmod{2} \right\}. \\
\text{for } a \text{ longest root}
\]

x) Type \( BC^{(2,2)(1)} \) (\( l \geq 2 \))

\[
R = \bigcup_{a \in R_{BC}} \left\{ a + nb + ma : m, n \in \mathbb{Z}, n \equiv 1 \pmod{2} \text{ for } a \text{ longest root} \right\}. \\
mn \equiv 0 \pmod{2} \text{ for } a \text{ longest root}
\]

xi) Type \( BC^{(2,2)(2)} \) (\( l \geq 1 \))

\[
R = \bigcup_{a \in R_{BC}} \left\{ a + nb + ma : m, n \in \mathbb{Z}, n \equiv 1 \pmod{2} \text{ for } a \text{ longest root} \right\}. \\
mn \equiv 0 \pmod{2} \text{ for } a \text{ not shortest root}
\]

(5.4) Appendix. Isomorphism as Root Systems. 

In the following, we list up all pair of non isomorphic marked extended affine root systems, which are isomorphic as root systems forgetting about their markings. For each such pair, we give an isomorphism \( \Phi \) explicitly by using the explicit description of the root system in (5.2).

1) \( B^{(1,2)} \cong B^{(2,1)} \) (\( l \geq 3 \))

\[
\Phi : F_{B^{(1,2)}} \cong F_{B^{(2,1)}} \\
\varepsilon_i \mapsto \varepsilon_i \quad i = 1, \ldots, l
\]
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2) \( C_{1,2}^{(1,3)} \cong C_{1}^{(2,1)} \) \((l \geq 3)\)

\[ \Phi : F_{C_{1,2}^{(1,3)}} \cong F_{C_{1}^{(2,1)}} \]

\[ \varepsilon_{1} \rightarrow \varepsilon_{1} + \varepsilon_{2} \]
\[ \varepsilon_{2} \rightarrow \varepsilon_{1} - \varepsilon_{2} \]
\[ \varepsilon_{3} \rightarrow \varepsilon_{3} + \varepsilon_{4} \]
\[ \varepsilon_{4} \rightarrow \varepsilon_{3} - \varepsilon_{4} \]
\[ a \rightarrow b \]
\[ b \rightarrow a \]

3) \( F_{4}^{(1,3)} \cong F_{4}^{(2,1)} \)

\[ \Phi : F_{F_{4}^{(1,3)}} \cong F_{F_{4}^{(2,1)}} \]

\[ \varepsilon_{1} \rightarrow \varepsilon_{1} + \varepsilon_{2} \]
\[ \varepsilon_{2} \rightarrow \varepsilon_{1} - \varepsilon_{2} \]
\[ \varepsilon_{3} \rightarrow \varepsilon_{3} + \varepsilon_{4} \]
\[ \varepsilon_{4} \rightarrow \varepsilon_{3} - \varepsilon_{4} \]
\[ a \rightarrow b \]
\[ b \rightarrow a \]

4) \( G_{2}^{(1,3)} \cong G_{2}^{(3,1)} \)

\[ \Phi : F_{G_{2}^{(1,3)}} \cong F_{G_{2}^{(3,1)}} \]

\[ \phi_{1} \rightarrow \phi_{2} - \phi_{3} \]
\[ \phi_{2} \rightarrow \phi_{3} - \phi_{1} \]
\[ \phi_{3} \rightarrow \phi_{1} - \phi_{2} \]
\[ a \rightarrow b \]
\[ b \rightarrow a \]

(To see that these 4 give a complete list of isomorphism among two m.e.a.r.s.'s \((R, G)\) and \((R', G')\), we proceed the following. If \( R \cong R' \) then i) \( R/\text{rad}I \cong R'/\text{rad}I' \) and ii) the counting set \( K_{\text{rad}}(a) \) for \( a \in R \) and \( K_{\text{rad}}'(a') \) for \( a' \in R' \) (cf. (1.16)) should behave similarly. For a m.e.a.r.s. \((R, G)\), such data are easily calculated from the description of \((R, G)\) in (5.3). Thus in this way we can pick up all possible pair of marked e.a.r.s.'s from the table of (5.3), which might be isomorphic as root systems. Then as we see in the above list, one constructs explicitly
§ 6. The Second Tier Number $\iota_2(R,G)$ and the Counting $k(\alpha)$ ($\alpha \in R$)

In this paragraph we give a proof of the classification of m.e.a.r.s.'s stated in the last paragraph. The idea is to calculate $\iota_2(R,G)$ by introducing counting number $k(\alpha)$ for $\alpha \in R$ which is a positive integer s.t. $\{\alpha + Zk(\alpha)\alpha\} = R \cap \{\alpha + Z\alpha\}$ (cf. (6.1)). The counting number $k(\alpha)$ plays a role in the definition of exponents in § 7 and in the study of the Coxeter transformation in § 9.

(6.1) Let $(R,G)$ be a m.e.a.r.s. such that $R/G$ is reduced.

Assertion i) For any root $\alpha \in R$, there exists a positive integer $k(\alpha)$, which we shall call the counting of $\alpha$, such that

\[(6.1.1) \quad \{\alpha + Zk(\alpha)\alpha\} = R \cap \{\alpha + Z\alpha\}.\]

ii) For any two roots $\alpha, \beta \in R$,

\[(6.1.2) \quad k(\beta) | I(\beta, \alpha)k(\alpha).\]

Particularly if $I(\beta, \alpha) = \pm 1$, then

\[(6.1.3) \quad 1 | k(\alpha)/k(\beta) | I(\alpha, \beta).\]

iii) If $\varphi$ is an automorphism of $(R,G)$, then

\[(6.1.4) \quad k(\alpha) = k(\varphi(\alpha)) \quad \text{for} \quad \alpha \in R.\]

iv)

\[(6.1.5) \quad R = \bigcup_{x \in \mathbb{N}} \{\alpha + Zk(\alpha)\alpha\},\]

for a linear subspace $L^{t+1}$ of $F$ spanned by a basis (cf. (3.4)).

v) $g.c.d.\{k(\alpha) : \alpha \in R\} = 1,$

$g.c.d.\{k(\alpha) : \alpha_0, ..., \alpha_t \text{ a basis for } (R,G)\} = 1.$

Proof. i) Put $K(\alpha) = \{x \in \mathbb{Z} : \alpha + x\alpha \in R\}$ for a root $\alpha \in R$ (cf. (1.16)). Since $0 \in K(\alpha)$ and $x, y \in K(\alpha)$ implies $2x - y \in K(\alpha)$ (cf. (1.16) Assertion i)
iii)), \(K(a)\) is an ideal of \(\mathbb{Z}\), whose non negative generator is denoted by \(k(a)\). Then \(k(a)\neq 0\) due to (1.16) Assertion 2 and (3.3) Note 2.

Then ii) iii) iv) and v) are direct consequences of (1.16) Assertion 1, Assertion 2.

(6.2) Corollary. Let \((R, G)\) and \((R', G')\) be m.e.a.r.s.'s and let \(\{\alpha_0, \ldots, \alpha_i\}\) and \(\{\beta_0, \ldots, \beta_i\}\) be basis of \((R, G)\) and \((R', G')\) respectively (cf. (3.4) Def.). Suppose \((R, G)\) and \((R', G')\) are isomorphic. Then there exists an isomorphism \(\varphi : (R, G) \simeq (R', G')\) such that \(\{\varphi(\alpha_0), \ldots, \varphi(\alpha_i)\} = \{\beta_0, \ldots, \beta_i\}\).

Proof. It is enough to show the case when \((R, G) = (R', G')\). Recall the projection \(p_1 : F \rightarrow F/G\). Since \(p_1(\alpha_0), \ldots, p_1(\alpha_i)\) and \(p_1(\beta_0), \ldots, p_1(\beta_i)\) are basis for the affine root system \(R/G\), there exists an element \(\tilde{w} \in \mathbb{W}_{RIG}\) and a sign \(\varepsilon \in \{\pm 1\}\) such that \(\{\varepsilon \tilde{w}(p_1(\alpha_0)), \ldots, \varepsilon \tilde{w}(p_1(\alpha_i))\} = \{p_1(\beta_0), \ldots, p_1(\beta_i)\}\). (\(\mathbb{W}_{RIG}\) acts transitively on the set of chambers.) Therefore if \(w \in \mathbb{W}_R\) is a lifting of \(\tilde{w}\), using the Assertion i) we see that there exists a permutation \(\sigma \in \mathbb{S}_{i+1}\) and integers \(m_i \in \mathbb{Z}\) \((i = 0, \ldots, l)\) such that

\[
\varepsilon w(\alpha_{\sigma(i)}) = \beta_i + m_ik(\beta_i)a \quad i = 0, \ldots, l.
\]

Let us define two elements \(\phi\) and \(\varphi\) of \(GL(F)\) by

\[
\phi : F = \bigoplus_{i=0}^{l+1} \mathbb{R} \alpha_i \oplus \mathbb{R}a \ni \sum c_i \alpha_i + pa \mapsto \varepsilon(\sum c_i \alpha_i) + pa \in F,
\]

\[
\varphi : F = \bigoplus_{i=0}^{l+1} \mathbb{R} \beta_i \oplus \mathbb{R}a \ni \sum c_i \beta_i + pa \mapsto \sum c_i \beta_i + (\varepsilon \sum c_i m_i k(\beta_i))a \in F.
\]

The map \(\phi\) is shown to be an automorphism of \((R, G)\) using the expression (6.1.5) for \(L^{i+1} = \bigoplus_{i=0}^{l+1} \mathbb{R} \alpha_i\) and the fact \(k(a) = k(-a)\).

Let us show that \(\varphi\) is an automorphism of \((R, G)\). First let us show that for any \(\beta \in R \cap \tilde{L}^{i+1}\) with \(\tilde{L}^{i+1} = \bigoplus_{i=0}^{l+1} \mathbb{R} \beta_i\), \(\varphi(\beta)\) belongs to \(R\) such that \(k(\beta) = k(\varphi(\beta))\).

By definition \(\varphi \beta_i = \beta_i + m_ik(\beta_i)a \in R\) and \(\varphi \omega_{\beta_i}^{-1} = w_{\varphi \beta_i} \in \mathbb{W}_R\) for \(i = 0, \ldots, l\). Therefore

\[
\varphi(R \cap \tilde{L}^{i+1}) = \varphi(\bigcup_{i=0}^{l+1} \mathbb{W}_R \cap \tilde{L}^{i+1} \beta_i) = \varphi(\bigcup_{i=0}^{l+1} \langle \beta_i, \ldots, \omega_{\beta_i} \rangle \beta_i) = \bigcup_{i=0}^{l+1} \langle \omega_{\varphi \beta_i}, \ldots, w_{\varphi \beta_i} \rangle \beta_i \subset \tilde{L}^{i+1} \cap R = R.
\]

If \(\beta = w\beta_i\) for \(\beta \in \tilde{L}^{i+1}\), \(w \in \mathbb{W}_R \cap \tilde{L}^{i+1}\) and \(0 \leq i \leq l\), then using Assertion i) and iii) one computes

\[
k(\varphi(\beta)) = k(\varphi \omega_{\beta_i}^{-1} \beta_i) = k(\varphi \beta_i) = k(\beta_i) = k(w \beta_i) = k(\beta).
\]
Using the expression (6.1.5) for $L^{t+1} = \bigoplus_{t=0}^{t} R\beta_{t}$ again, one computes $\varphi R = \bigcup_{\beta \in R \cap L^{t+1}} (\varphi(\beta) + Z\varphi(\beta))a = \bigcup_{\beta \in R \cap L^{t+1}} (\varphi(\beta) + Z\varphi(\beta))a \subset R$. Samely one computes that $\varphi^{-1}R \subset R$. This shows that $\varphi$ is an automorphism of $(R, G)$.

Therefore the composition $\varphi^{-1}w\psi$ is an automorphism of $(R, G)$ such that $(\varphi^{-1}w\psi)\alpha_{\sigma(t)} = \beta_{t}, t = 0, \ldots, l$. This completes the proof of the corollary.

(6.3) Let us give a formula for the second tier number $t_2(R, G)$ using the counting $k(a), a \in R$.

$$t_2(R, G) = \text{g.c.d.} \{k(a)\frac{I_{R^*}(a, a^*)}{2} : a \in R\}$$

(6.3.1)

$$= \text{g.c.d.} \{k(a)\frac{I_{R^*}(a, a^*)}{2} : a_0, ..., a_t \text{ a basis} \}$$

**Proof.** For the dual root system $(R^*, G)$, parallel to (6.1.1) let us define,

(6.3.2) $k^*: R^* \to N$ by $\{a^* + Zk^*(a^*)a^*\} = R^* \cap \{a^* + Za^*\}$.

By definition, we have a relation,

(6.3.3) $k^*(a^*)a^* = \frac{2}{I(a, a)} k(a)a = \frac{I(a, a^*)}{2} k(a)a$ for $a \in R$.

Taking the proportion of (6.3.3), for $a, \beta \in R$, we have

$$\frac{k^*(a^*)}{k^*(\beta^*)} = \frac{I(a^*, a^*)}{I(\beta^*, \beta^*)} \frac{k(a)}{k(\beta)}.$$ 

Therefore

(6.3.4) $k^*(a^*) = \frac{1}{c} \frac{I(a^*, a^*)}{2} k(a)$ for $a \in R$,

where $c = \frac{I(\beta^*, \beta^*)}{2} \frac{k(\beta)}{k^*(\beta)}$.

Applying (6.3.4) to $\text{g.c.d.} \{k^*(a^*) : a \in R^*\} = 1$, one gets,

(6.3.5) $c = \text{g.c.d.} \{\frac{I(a^*, a^*)}{2} k(a) : a \in R\}$.

By definition of $t_2(R, G)$ in (4.3), applying (6.3.3), (6.3.4), (6.3.5),

$t_2(R, G) = |(a^* : a)|(I_{R^*} : I) = \frac{k(a)}{k^*(a^*)} \frac{I(a^*, a^*)}{2} (I_{R^*} : I)$. 


\[ (6.3.6) \quad k^\vee(\alpha^\vee) = \frac{1}{I^2(R, G)} I_{R}^\vee(\alpha^\vee, \alpha^\vee)k(\alpha) \quad \text{for} \quad \alpha \in R. \]

**Corollary.**

(6.4) Let \{\alpha_0, \ldots, \alpha_t\} be a basis for a m.e.a.r.s. \((R, G), (3.4)\). Since it can be regarded as a basis for the affine root system \(R \cap \bigoplus_{i=0}^{t} R\alpha_i\), one can associate a Dynkin diagram \(\Gamma\) for this basis. (See for instance \([14,5]\)). By \(|\Gamma|\) let us denote the set of nodes of \(\Gamma\), which is identified with the set \{\alpha_0, \ldots, \alpha_t\} of the basis. At each node of the diagram corresponding to a base \(\alpha_i\), let us associate the counting integer \(k(\alpha_i)\), so that we obtain weighted Dynkin diagram \((\Gamma, (k(\alpha))_{\alpha \in |\Gamma|})\).

**Lemma**

i ) The weighted diagram \((\Gamma, (k(\alpha))_{\alpha \in |\Gamma|})\) is uniquely determined by the isomorphism class of \((R, G)\).

ii ) The weighted diagram \((\Gamma, (k(\alpha))_{\alpha \in |\Gamma|})\) for \((R, G)\) determines uniquely the isomorphism class of \((R, G)\).

**Proof.**

i ) (6.1) Assertion iii) and (6.2) Corollary.

ii ) Let \{\alpha_0, \ldots, \alpha_t\} be a basis for \((R, G)\). Let us take the expression (6.1.5) for \(L^{i+1} = \bigoplus_{i=0}^{t} R\alpha_i\),

\[
R = \bigcup_{\alpha \in R \cap \bigoplus_{i=0}^{t} L^{i+1}} \{a + Zk(\alpha)a\} = \bigcup_{i=0}^{t} \bigcup_{\omega \in S_{\alpha_0,\ldots,\alpha_t}} \{wa_1 + Zk(\alpha_1)a\},
\]

where the right hand is uniquely determined by the weighted diagram \((\Gamma, (k(\alpha))_{\alpha \in |\Gamma|})\). The marking \(G\) is given by \(Ra\). q.e.d.

**Note.** Let \((\Gamma, (k(\alpha))_{\alpha \in |\Gamma|})\) be the weighted diagram for a m.e.a.r.s. \((R, G)\). Then the weighted diagram for the dual \((R^\vee, G)\) is given by \((\Gamma^\vee, (k^\vee(\alpha^\vee))_{\alpha \in |\Gamma^\vee|})\), where \(\Gamma^\vee\) is the dual of \(\Gamma\) (i.e. the set of nodes of \(\Gamma^\vee\) is bijective to \(\Gamma\) and the arrows on the bonds are reversed) and \(k^\vee(\alpha^\vee)\) is given by (6.3.6).

(6.5) Due to the Lemma (6.4), the classification of m.e.a.r.s.'s is reduced to the
classification of weighted diagram \((\Gamma,(k(\alpha))_{\alpha \in |\Gamma|})\) where

i) \(\Gamma\) is a Dynkin diagram for a reduced affine root system,

ii) the counting weights \(k(\alpha) \in |\Gamma|\) are positive integers s.t.

\[ k(\beta)|I(\beta, \alpha^\vee)k(\alpha) \text{ for } \alpha, \beta \in |\Gamma| \text{ and } \gcd\{k(\alpha) : \alpha \in |\Gamma|\} = 1. \]

In the following we list all such weighted diagrams. For each weighted diagram, we calculate the first and the second tier numbers, using (4.5.1) and (6.3.1). For each weighted diagram, let us define the type according to a similar rule as in (5.1).

(6.6) Existence of m.e.a.r.s.'s. To finish the classification of m.e.a.r.s.'s in § 5, we want to show the existence of a m.e.a.r.s. \((R, G)\) for each weighted diagram \((\Gamma, (k(\alpha))_{\alpha \in |\Gamma|})\) in the list of (6.5) s.t. a basis for \((R, G)\) gives the weighted diagram.

This can be achieved by showing the following steps.

i) An explicit construction of \((R, G)\) is given in (5.2) (or (5.3)). (The author owes to the appendix of [14] for a description of affine root systems).

Therefore we need to show the following ii) – v).

ii) Each \((R, G)\) in (5.2) satisfies the axioms for an e.a.r.s. given in (2.1).

iii) The set \(a_0, ..., a_i\) given in 3) of the table (5.2) is a basis for \((R, G)\) in the sense of (3.4).

iv) The weighted Dynkin diagram (defined in (6.4)) associated to the basis belongs to the list in (6.5).

v) By this correspondence \((R, G) \Rightarrow (\Gamma, (k(\alpha))_{\alpha \in |\Gamma|})\), the type for \((R, G)\) in (5.2) coincides with the type for \((\Gamma, (k(\alpha))_{\alpha \in |\Gamma|})\) in (6.5).

To check all these steps for all types of m.e.a.r.s.'s is a rather long cumbersome routin work, which we do not proceed in this note.

(6.7) As a result of the classification, we have the following.

Assertion. Let \((R, G)\) be a m.e.a.r.s. of type \(P(t_1, t_2)\).

If \(t_2 = 1\), then \(k(\alpha) = 1\) for any \(\alpha \in R\).

If \(t_2 = t(P)\), then \(k(\alpha) = \frac{1}{2}I_R(\alpha, \alpha)\) for any \(\alpha \in R\).

(6.8) Note. It is curious to observe that the coefficients \(n_i\) \((i = 0, ..., l)\) of \(b\)
The first tier number | The second tier number | The type | The dual type
--- | --- | --- | ---
1 | 1 | $A_1^{(1,1)}$ | $A_1^{(1,1)}$
1 | 1 | $A_1^{(1,1)}$ | $A_1^{(2,1)}$
1 | 1 | $A_1^{(1,1)}$ | $A_1^{(1,1)^*}$
1 | 1 | $B_1^{(2,1)}$ | $C_1^{(2,3)}$
1 | 1 | $B_1^{(1,1)}$ | $C_1^{(1,3)}$
1 | 1 | $B_1^{(2,2)}$ | $C_1^{(1,1)}$
1 | 1 | $B_1^{(2,2)^*}$ | $C_1^{(1,1)^*}$
1 | 1 | $C_1^{(1,1)}$ | $B_1^{(2,2)}$
1 | 1 | $C_1^{(1,2)}$ | $B_1^{(2,1)}$
1 | 1 | $C_1^{(2,1)}$ | $B_1^{(1,2)}$
| 1 | 1 | 1 | 1 | 2 | 2 | 2 | \(C^{(2,2)}\) | \(B^{(1,1)}\) |
| 2 | 1 | 1 | 1 | 1 | 2 | 1 | \(C^{(1,1)*}\) | \(B^{(2,2)*}\) |
| 2 | 1 | 1 | 1 | 1 | 2 | 1 | \(BC^{(2,1)}\) | \(BC^{(2,4)}\) |
| 2 | 2 | 1 | 4 | \(BC^{(2,1)}\) | \(BC^{(2,4)}\) |
| 2 | 2 | 1 | 4 | \(BC^{(2,4)}\) | \(BC^{(2,1)}\) |
| 2 | 2 | 1 | 4 | \(BC^{(2,4)}\) | \(BC^{(2,1)}\) |
| 2 | 2 | 1 | 4 | \(BC^{(2,1)}\) | \(BC^{(2,3)}(1)\) | \(BC^{(2,3)}(1)\) |
| 2 | 2 | 1 | 4 | \(BC^{(2,1)}\) | \(BC^{(2,3)}(2)\) | \(BC^{(2,3)}(2)\) |
| 2 | 1 | 4 | \(BC^{(2,3)}\) | \(BC^{(2,3)}\) |
| 1 | 1 | 1 | 1 | 1 | 1 | \(D^{(1,1)}\) | \(D^{(1,1)}\) |
| 1 | 1 | 1 | 1 | 1 | 1 | \(E^{(1,1)}\) | \(E^{(1,1)}\) |
(3.3.6) and the counting coefficients \( k(\alpha_i) \) \((i = 0, \ldots, l)\) behave somehow parallely in connection with the first and the second tier numbers (cf. (4.5.1) and (6.3.6)).

This fact leads us to a definition and the studies of exponents in the following paragraphs. Particularly it becomes clear again in the study of foldings in § 12.

$$\S 7. \text{ Exponents } m_i \((i = 0, \ldots, l)\) \text{ for a m.e.a.r.s.}$$

We introduce in this paragraph exponents for a m.e.a.r.s. \((R, G)\), which we shall use for a definition of the Dynkin diagram for \((R, G)\). The exponents appear as eigen values for Coxeter transformation of \((R, G)\) in § 9 and as the degrees of \(\widetilde{W}_{R,G}\) invariant \(\theta\)-functions in [20].

(7.1) Let \((R, G)\) be a m.e.a.r.s. and \(\{a_0, \ldots, a_l\}\) be a basis for \((R, G)\) (cf. (3.4) Def.).

Definition. \textit{The exponents for } \((R, G)\) \textit{are}

\[
m_i := \frac{I_R(\alpha_i, \alpha_i)}{2k(\alpha_i)} n_i, \quad i = 0, \ldots, l.
\]

where \(n_i\) are the coefficients of (3.3.3) and, \(k(\alpha)\) is the counting in (6.1) and \(I_R\) is the normalized metric on \(F\) defining the even lattice structure on \(Q(R)\) (cf. (4.2)).

(7.2) For each type of isomorphism class of m.e.a.r.s.'s, the exponents are calculated and given in 4) of the table (5.2).

Example. Let \((R, G)\) be of type \(P^{(t_1, t_2)}\).

If \(l_2 = l(P)\), then \(m_i = n_i, \quad i = 0, \ldots, l.\)

If \(l_2 = 1\), then \(m_i = \frac{I_R(\alpha_i, \alpha_i)}{2} n_i, \quad i = 0, \ldots, l.\)

(\(\therefore\) (6.7) Assertion.)

(7.3) Assertion. i) The exponents are half integers.

ii) The set of exponents does not depend on the choice of the bases \(\{a_0, \ldots, a_l\}\).

(\(\therefore\) (6.2) Corollary and (6.1) Assertion iii).)

iii) Let
be the exponents for the dual \((R^\vee, G)\). Then \(m_i\)'s and \(m_{i}^\vee\)'s are proportional. Precisely,

\[
m_i^\vee = \frac{t_1(R, G)t_2(R, G)}{t(R)} m_i \quad i=0, \ldots, l.
\]

Proof.

\[
m_i^\vee := \frac{I_{R^\vee}(a_i^\vee, a_i^\gamma)}{2k^\vee(a_i^\gamma)} n_i
\]

\[= t_1(R, G)\frac{n_i}{k^\vee(a_i^\gamma)} \quad \text{(by (4.5.1))}
\]

\[= t_1(R, G)\frac{I(a_i, a_i^\gamma)}{2k(a_i)} (a^\vee: a)n_i \quad \text{(by (6.3.3))}
\]

\[= \frac{t_1(R, G)}{t_2(R^\vee, G)} m_i = \frac{t_1(R, G)t_2(R, G)}{t(R)} m_i \quad \text{(by (4.3), (4.4)). q.e.d.}
\]

(7.4) We prepare an assertion which is rather of technical nature, but will be used in crucial steps of the proofs of (8.6) Assertion, (9.6) Theorem and (10.1) Assertion 5 for the proof of Lemma B.

**Assertion.** Let \(\{a_0, \ldots, a_l\}\) be a basis for \((R, G)\) and \(m_{a_0}, \ldots, m_{a_l}\) be the exponents. For \(\alpha, \beta \in \{a_0, \ldots, a_l\}\) suppose \(m_{a_0} < m_{a_l}\) and \(I(\alpha, \beta) = 0\).

Then

\[k(\beta)/k(\alpha) = -I(\beta, a^\vee).
\]

Proof. From the formula (6.3.6), we have,

\[\frac{k(\alpha)}{k(\beta)} = \frac{k(\beta)}{k(\alpha)} = \frac{I_R(\alpha, \alpha)}{I_R(\beta, \beta)} \left( = \frac{I_{R^\vee}(\beta^\vee, \beta^\gamma)}{I_{R^\vee}(\alpha^\vee, \alpha^\gamma)} \right) = 1 \text{ or } 2^{\pm 1}.
\]

Except for the types \(A_1^{(1,1)*}\) or \(BC_2^{(2,2)}\), we have either \(I(\alpha, \beta^\vee) = -1\) or \(I(\beta, \alpha^\vee) = -1\). Thus taking in account of (6.1.3), \(\ast\) implies,

\[
either \frac{k(\alpha)}{k(\beta)} = \frac{I_R(\alpha, \alpha)}{I_R(\beta, \beta)} \quad or \quad \frac{k(\beta)}{k(\alpha)} = \frac{I_R(\beta^\vee, \beta^\gamma)}{I_R(\alpha^\vee, \alpha^\gamma)}.
\]

On the other hand due to the proportionality of exponents (7.3) iii) and (6.3.6), the statement of the assertion is equivalent for \((R, G)\) and \((R^\vee, G)\) so that one may
prove only one of them. Therefore, except the types $A_i^{(1,1)*}$ and $BC_i^{(2,2)}$, we may assume $I_R(\alpha, \alpha)/2k(\alpha) = I_R(\beta, \beta)/2k(\beta)$ by choosing one of $(R, G)$, $(R^\vee, G)$.

Thus the problem is reduced to show,

$$\bullet$$  If $n_\beta < n_\alpha$ and $I(\alpha, \beta) \neq 0$ for $\alpha, \beta \in |\Gamma|$, then $I(\alpha, \beta^\vee) = -1$, where $\Gamma$ is a Dynkin diagram for a reduced affine root system.

If $I(\alpha, \beta^\vee) = -t \neq 0$, $-1$ for $\alpha, \beta \in |\Gamma|$, then $\alpha \rightarrow \beta \rightarrow \alpha$ and therefore $n_\beta \geq n_\alpha$

(\*\* Folding of Dynkin diagrams). This proves $\bullet$).

The cases $A_i^{(1,1)*}$ and $BC_i^{(2,2)}$ can be directly verified. q.e.d.

§ 8. Dynkin Diagram for a m.e.a.r.s.

In this paragraph, we introduce a Dynkin diagram $\Gamma_{\mathfrak{R}, G}$ for an isomorphism class of $(R, G)$. The Dynkin diagram gives a most intrinsic way of describing the marked extended affine root system $(R, G)$ as we see in § 9.

(8.1) Codimension of $(R, G)$. Let $\{a_0, ..., a_l\}$ be a basis for a m.e.a.r.s. $(R, G)$ (cf. (3.4)) and $\Gamma$ be the Dynkin diagram for the affine root system $R \cap \bigoplus_{i=0}^{l} \mathbb{R} a_i$. Recall the exponent $m_i := \frac{I_R(a_i, a_i)}{2k(a_i)} n_i$ at each node $a_i$ of $\Gamma$.

**Definition 1.** Let us denote by $\Gamma_m$ the subdiagram of $\Gamma$ consisting of nodes,

$$|\Gamma_m| := \{a_i \in |\Gamma| : m_i = m_{\text{max}} \}.$$  

Here $m_{\text{max}} := \max\{m_0, ..., m_l\}$.

2. Let us denote by $\text{cod}(R, G)$ the number $\#|\Gamma_m|$ and call it the codimension of $(R, G)$.

**Note 1.** Due to the proportionality of exponents ((7.3 iii)), the set $|\Gamma_m|$ is naturally bijective to

$$|\Gamma_{\vee, \vee}| := \{a_i^\vee \in |\Gamma^\vee| : m_i^\vee = \max\{m_0^\vee, ..., m_l^\vee \} \}.$$ 

Therefore $\text{cod}(R, G) = \text{cod}(R^\vee, G)$.

**Note 2.** The name codimension for $\text{cod}(R, G)$ is introduced here, since it is identified with the codimension of a Hamiltonian system introduced in [30, (1.12)]
(i.e. \( \text{cod}(R, G) \) is the number of (energy) functions which is necessary to describe the system constructed from \((R, G)\). For the precise identification, see [20].)

(8.2) Hereafter, we use the following notations

i) \( a^\circ := a + k(a)a \) for \( a \in R \). (\( k(a) \) is the counting defined in (6.1)).

ii) \( |\Gamma^w| := \{a^\circ : a \in |\Gamma^w|\} \).

**Definition.** The Dynkin diagram \( \Gamma_{R,G} \) for a m.e.a.r.s. \((R, G)\) is defined as the intersection diagram for the set \( |\Gamma| \cup |\Gamma^w| \). i.e.

i) The set of nodes \( |\Gamma_{R,G}| \) of the diagram is identified with the set \( |\Gamma| \cup |\Gamma^w| \) consisting of \( l(R) + \text{cod}(R, G) + 1 \) points.

ii) Bonds and arrows among nodes are inserted according to the same rules for a finite root system with an additional case. Namely as follows.

\[
\begin{array}{c|c}
\text{Diagram} & \text{Condition} \\
\hline
\begin{array}{c}
\circ \quad \circ \\
\end{array} & I(\alpha, \beta^\vee) = 0 \iff I(\beta, \alpha^\vee) = 0, \\
\begin{array}{c}
\circ \quad \circ \\
\end{array} & I(\alpha, \beta^\vee) = I(\beta, \alpha^\vee) = -1, \\
\begin{array}{c}
\circ \quad \circ \\
\end{array} & I(\alpha, \beta^\vee) = -1, I(\beta, \alpha^\vee) = -t, \text{ for } t = 2, 3, 4, \\
\begin{array}{c}
\circ \quad \circ \\
\end{array} & I(\alpha, \beta^\vee) = I(\beta, \alpha^\vee) = -2, \\
\begin{array}{c}
\circ \quad \circ \\
\end{array} & I(\alpha, \beta^\vee) = I(\beta, \alpha^\vee) = 2.
\end{array}
\]

(8.3) i) The diagram \( \Gamma_{R,G} \) depends only on the isomorphism class of \((R, G)\). (\( \because \) (6.2) Corollary, (7.3) ii)).

ii) The diagram \( \Gamma_{R^\vee,G} \) for the dual \((R^\vee, G)\) is the dual diagram of \( \Gamma_{R,G} \). (i.e. arrows are reversed.). (\( \because \) (8.1) Note 1.)

iii) For each type of m.e.a.r.s., the Dynkin diagram is explicitly given in 5) of Table (5.2).

iv) The Dynkin diagrams distinguish the isomorphism classes of m.e.a.r.s.'s. (i.e. if two m.e.a.r.s.'s have the same diagram, they are isomorphic.)

(\( \because \) cf. (9.6) Theorem)

**Note 1.** The diagrams for \( E_6^{(1,1)}, E_7^{(1,1)}, E_8^{(1,1)} \) appeared already in a study of the lattice for simple elliptic singularities. (See W. Ebeling [5], where he gave more generalized Dynkin diagrams for the lattice of singularities.) Also W. Ebeling has noticed to the author that the diagram appear also in a study of the presentation of algebras (cf. [2]).
Note 2. B. Verdier has told to the author that he got also some of the diagrams in his note [6].

(8.4) We collect here some elementary facts on the Dynkin diagram and exponents for a \((R, G)\), which we shall use in later paragraphs freely.

i) The diagram is isomorphic by a transposition of nodes \(a \in \Gamma_m\) and \(a^* \in \Gamma_m^*\) which are connected by the bond \(==\).

ii) The complement \(\Gamma_{R,G} - \Gamma_m \cup \Gamma_m^*\) decomposes into a finite disjoint union of diagrams of type \(A_i\), \(i = 1, \ldots, r\).

iii) Each component \(\Gamma_{A_i}\) is connected to \(\Gamma_m\) only at one node, say \(\beta_i \in |\Gamma_m|\), at a terminal node of \(\Gamma_{A_i}^*\).

iv) On the branch \(\Gamma_{A_i} \cup \{\beta_i\}\) of \(\Gamma_{R,G}\), the exponents \(m_j\)'s are in arithmetic progression. Namely

\[
m_j/m_{\text{max}} = j/(l_i + 1) \quad \text{for} \quad j = 1, \ldots, l_i + 1,
\]

where \(\alpha_1, \ldots, \alpha_{l_i+1}\) are renumbered nodes of \(\Gamma_{A_i} \cup \{\beta_i\}\).

v) Put

\[
l_{\text{max}} := \max\{l_i : 1 \leq i \leq r\}.
\]

Then

\[
l_i + 1 | l_{\text{max}} + 1 \quad \text{for} \quad 1 \leq i \leq r.
\]

This implies that \(l_{\text{max}} + 1\) is the smallest common denominator of \(m_{\text{max}}/m_i\), \(i = 0, \ldots, l\).

(8.5) As an application of the above facts (8.4), we give a reduction of a calculation of some numerical invariants, which practically helps much.

Assertion. Let the notations be as before. Then,

\[
g.c.d.\{k(\alpha) : \alpha \in |\Gamma_m|\} = 1,
\]

(8.5.1)

\[
t_2(R, G) = g.c.d.\{k(\alpha)\Gamma_{R,G}(\alpha^*, \alpha^*)/2 : \alpha \in |\Gamma_m|\}.
\]

(8.5.2)

Proof. The formulae were shown readily if we replace the running index set \(\alpha \in |\Gamma_m|\) by \(\alpha \in |\Gamma|\) ((6.1) Assertion v), (6.3.1)).

Therefore in account of (8.5) iv), it is enough to show the following.
Let $a, \beta \in |\Gamma|$ s.t. $I(a, \beta) \neq 0$ and $m_a < m_\beta$.

Then $k(a)/k(\beta)$ and $k(\alpha)\frac{I_R(\alpha^\vee, \alpha^\vee)}{2}/k(\beta)\frac{I_R(\beta^\vee, \beta^\vee)}{2}$ are integers.

(Proof of *) Due to the (7.4) Assertion, in the above situation, $k(\alpha)/k(\beta) = -I(\beta, a^\vee)$ is an integer and hence $(k(\alpha)\frac{I_R(\alpha^\vee, \alpha^\vee)}{2})/(k(\beta)\frac{I_R(\beta^\vee, \beta^\vee)}{2}) = -I(\beta^\vee, a)$ is also an integer.

Corollary. Let $(R, G)$ be a m.e.a.r.s. such that $\text{cod}(R, G) = 1$. Then

$$t_\alpha(R, G) = \frac{I_R(\alpha^\vee, \alpha^\vee)}{2}$$
where $\alpha \in |\Gamma_m|$.

(8.6) Discussions on the Dynkin diagrams.

There does not exist an a priori definition of a Dynkin diagram for an extended affine root system or for a marked extended affine root system, since there does not exist a concept of a Weyl chamber (cf. (3.2) Note 2.) comparing to the case of a finite root system or an affine root system. There are several trials to understand the Dynkin diagrams for such a generalized situation by several authors. (See for instance W. Ebeling [5], F. Knörrer [8], P. Kluitmann [10], Van der Lek [26] for our restricted cases.)

The Dynkin diagrams, which we defined in this note, have several similarities and un-similarities with the classical one for finite or affine root systems. We shall list them in the following.

Unsimilarities

i) The diagram $\Gamma_{R, G}$ depends not only on the roots $R$ but also on the marking $G \subset \text{rad} I$. (cf. (5.4) Appendix)

ii) The number $l(R) + \text{cod}(R, G) + 1$ of the nodes of the diagram is larger than the rank $l(R) + 2$ of the ambient space $F$ in general. The linear dependence relations among the nodes as elements of $F$ are described by the unipotent part of the pre-Coxeter transformation. (cf. (9.6))

iii) The Cartan matrix $(I(\alpha, \beta^\vee))_{\alpha, \beta \in |\Gamma_{R, G}|}$ contains positive numbers $2 = I(\alpha^*, \alpha^\vee)$ for $\alpha \in |\Gamma_m|$ in its off diagonal part. (compare with [33])

(Obviously all these three facts are related to the fact: the high degeneration ($\mu_0 = 2$) of the metric $I$.)

Similarities

i) The lattice $Q(R)$ is generated by $\alpha \in \Gamma_{R,G}$. The group $W_R$ is generated by $w_\alpha$ for $\alpha \in \Gamma_{R,G}$. The set of roots $R$ is equal to $\bigcup_{\alpha \in \Gamma_{R,G}} W_R \alpha$. (9.6) Theorem

ii) Coxeter transformation for $(R, G)$ is naturally defined as a product of $w_\alpha$ for $\alpha \in \Gamma_{R,G}$, which is of finite order, whose eigenvalues describe the degrees of the invariants of $\widehat{W}_R$. (9.7)

iii) Up to isomorphy, the root system $R$ together with the marking $G$ is reconstructed from the datum of $\Gamma_{R,G}$. (9.6)

iv) The diagrams behave naturally with respect to the automorphism of $(R, G)$ and foldings of $(R, G)$. (cf. § 12)

v) The multiplicity of the discriminant (= the square of the fundamental anti-invariant of $\widehat{W}_R$) is equal to $\# |\Gamma_{R,G}|$ (= $l(R) + \text{cod}(R, G) + 1$).

From these phenomenon, one would naturally expect an existence of a theory of infinite dimensional Lie algebra associated to these diagrams. (cf. Slodowy [23])

In fact there exist simple elliptic singularities ([28]) of types called $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8, \tilde{D}_8, \tilde{A}_4$ such that the middle homology groups of their smoothing contain naturally root systems of types $E_6^{(1,1)}, E_7^{(1,1)}, E_8^{(1,1)}, D_8^{(1,1)}$ and $A_4^{(1,1)}$ as the set of vanishing cycles respectively. In these cases, the multiplicity of the discriminant of the unfolding of the singularities is equal to $l(R) + \text{cod}(R, G) + 1$, where $\text{cod}(R, G)$ coincides with the codimension of the unfolding. Then by a suitable choice of paths in the base spaces of the unfolding, one can find a “strongly distinguished basis” for the middle homology group such that their intersection diagrams coincide with the Dynkin diagrams of the corresponding root systems defined in this note. (cf. [31], [34])

In [20] we shall construct a quotient space by the natural action of $\tilde{W}_{R,G}$ (cf. § 11) on a complex half space. The space will be identified with the base space of the universal unfolding of the singularities $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8, \tilde{D}_8, \tilde{A}_4$ for the case $E_6^{(1,1)}, E_7^{(1,1)}, E_8^{(1,1)}, D_8^{(1,1)}, A_4^{(1,1)}$ (cf. also E. Looijenga [12], P. Slodowy [22]). This might give a strong justification for the definition of the Dynkin diagram in this note.

§ 9 Coxeter Transformation I

(Construction of $(R, G)$ from the diagram $\Gamma_{R,G}$)

In this paragraph, we construct the m.e.a.r.s. $(R, G)$ up to an isomorphy from...
the data of its diagram $\Gamma_{R,G}$ ((9.6) Theorem). Here the concept of a (pre-) Coxeter transformation ((9.3) Def.) plays an essential role.

(9.1) A decomposition of $\Gamma_{R,G}$. Let $\Gamma_{R,G}$ be a Dynkin diagram for a m.e.a.r.s. $(R, G)$. The diagram consists of $l(R) + \text{cod}(R, G) + 1$ number of nodes and contains $\text{cod}(R, G)$ number of bonds $\equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv$.

Let us fix a decomposition of the set of $2\text{cod}(R, G)$ nodes connected by such bonds into two subsets $|\Gamma_m|$ and $|\Gamma_m^*|$ so that two nodes at the terminal of any such bond are divided into the two sets. Such decomposition is unique up to isomorphism of the diagram (cf. (8.4) i)). We shall denote by $a^*$ the node in $|\Gamma_m|$ which is connected to the node $a$ of $|\Gamma_m|$ by the bond $\equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv$.

Recall that the complement $\Gamma_{R,G} - \Gamma_m \cup \Gamma_m^*$ decomposes into connected components of diagrams of types $A_i (i = 1, \ldots, r)$, so that we obtain a decomposition of the diagram $\Gamma_{R,G}$.

(9.1.1) $\Gamma_{R,G} = \Gamma_m \cup \Gamma_m^* \cup \bigoplus_{i=1}^{r} \Gamma_{A_i}$.

(9.2) The vector space $\mathring{\mathcal{L}}$ with a metric $\mathring{\ell}$. Define,

1) $\mathring{\mathcal{L}} := \text{the vector space of rank } l(R) + \text{cod}(R, G) + 1 \text{ spanned by } a \in \Gamma_{R,G}.$

To avoid a confusion, the base of $\mathring{\mathcal{L}}$ corresponding to a node $a \in |\Gamma_{R,G}|$ will be denoted by $\mathring{a}$.

2) There exists a symmetric bilinear form $\mathring{\ell}$ on $\mathring{\mathcal{L}}$ s.t.

i) $\mathring{\ell} (\mathring{a}, \mathring{a}) > 0 \text{ for } a \in |\Gamma_{R,G}|.$

ii) $\mathring{\ell}$ satisfies the same rule for $\ell$ stated in (8.2) Def.

Such metric $\mathring{\ell}$ is unique up to a positive constant factor.

3) $\mathring{\ell}$ is positive semi-definite. The radical of $\mathring{\ell}$ has rank $\text{cod}(R, G) + 1$, which is spanned by

$\mathring{\mathcal{R}} := \sum_{a \in \Gamma_{R,G} - \Gamma_m^*} n_a \mathring{a} \text{ and } \mathring{a} - \mathring{a}^* \text{ for } a \in |\Gamma_m|,$

where $n_a$ are the coefficients in (3.3.6).

Proof. By definition $\mathring{a}^* - \mathring{a}$ for $a \in |\Gamma_m|$ belongs to rad $\mathring{\ell}$. The quotient space $\mathring{\mathcal{L}} / \bigoplus_{a \in |\Gamma_m|} R (\mathring{a}^* - \mathring{a})$ may be regarded to be spanned by the diagram $\Gamma_{R,G} - \Gamma_m^*$ for a reduced affine root system, where the $\mathring{\ell}$ induces a positive semi-definite metric, whose radical is spanned by $\mathring{\mathcal{R}}$. q.e.d.
(9.3) Pre-Coxeter transformation.
For a node $a \in |\Gamma_{R,G}|$, let us define a reflection $\hat{w}_a \in GL(\hat{\mathfrak{L}})$,

$$
\hat{w}_a(u) := u - \hat{\mathfrak{L}}(u, \hat{a}^\vee)\hat{a} \quad \text{where} \quad \hat{a}^\vee := 2\hat{a} / \hat{\mathfrak{L}}(\hat{a}, \hat{a}).
$$

Definition. An element $c$ of $O(\hat{\mathfrak{L}}, \hat{\mathfrak{L}})$ is called a pre-Coxeter transformation
for $(R, G)$, if it is a product of $\hat{w}_a$ for $a \in \Gamma_{R,G} - \Gamma_m \cup \Gamma^*_m$ and $\hat{w}_a\hat{w}_a^*$ for $a \in \Gamma_m$ for any ordering of them.

Due to lemma 1 of [1] Ch. V, § 6, every pre-Coxeter transformations are conjugate in the group $\hat{W} := \langle \hat{w}_a : a \in \Gamma_{R,G} \rangle$.

(9.4) Jordan decomposition of $c$.
The following is a key Lemma in this paragraph.

Lemma. Let $c$ be a pre-Coxeter transformation for the diagram $\Gamma_{R,G}$.
Let the Jordan decomposition of $c$ be,

$$
(9.4.1) \quad c = SU.
$$

Then, the set of eigenvalues of the semi-simple part $S$ is given by

$$
(9.4.2) \quad \exp(2\pi \sqrt{-1} m_a / m_{\max}) \quad \text{for} \quad a \in |\Gamma_{R,G}|.
$$

Hence

$$
(9.4.3) \quad S^{(m_{\max} + 1)} = 1.
$$

$$
(9.4.4) \quad (U - 1)^2 = 0, \quad \text{(i.e. log} \ U = U - 1).\n$$

Image$(U - 1) = \sum_{a, \beta \in \Gamma_m} R \left( \frac{\hat{a}^\vee - \hat{a}}{k(\alpha)} - \frac{\hat{\beta}^\vee - \hat{\beta}}{k(\beta)} \right) \subseteq \text{rad} \hat{\mathfrak{L}}$.

Proof. For any $\beta \in |\Gamma_m|$, define

$$
(9.4.6) \quad \hat{b}_\beta := n_\beta \beta + \sum' n_\alpha \hat{a},
$$

$$
(9.4.7) \quad \hat{b}_\beta^* := n_\beta \beta^* + \sum' n_\alpha \hat{a}^*.
$$

Here $\sum'$ means the summation over all nodes $a$ belonging to components $\Gamma_{A_i}$ which is connected to $\beta \in |\Gamma_m|$. Due to (9.2) 3), $\text{rad} \hat{\mathfrak{L}}$ is spanned by $\hat{b} = \sum_{\beta \in |\Gamma_m|} \hat{b}_\beta$ and

by $\hat{b}_\beta^* - \hat{b}_\beta$ for $\beta \in |\Gamma_m|$. 

For each component \( \Gamma_{\Lambda_i} \) of (9.1.1), define a space

\[
\hat{F}_i := \bigoplus_{a \in \Gamma_{\Lambda_i}} R \hat{a} \quad (i=1, \ldots, r),
\]

so that we obtain an orthogonal direct sum decomposition,

\[
\hat{F} = \bigoplus_{i=1}^{r} \hat{F}_i \bigoplus \bigoplus_{a \in \Gamma_{\lambda_1}} R \hat{a} \bigoplus \bigoplus_{a \in \Gamma_{\lambda_1}} R (\hat{a}^* - \hat{a}).
\]

Since the statements of the lemma depends only on the conjugacy class of \( \hat{c} \), we may choose one \( \hat{c} \) as follows.

\[
\hat{c} := \hat{c}_1 \ldots \hat{c}_r \cdot \hat{t},
\]

where

\[
\hat{c}_i := \prod_{a \in \Gamma_{\Lambda_i}} \hat{w}_a \quad (i=1, \ldots, r),
\]

\[
\hat{t} := \prod_{a \in \Gamma_{\lambda_1}} \hat{w}_a \hat{w}_a^*.
\]

To analyze the action of \( \hat{c} \) on \( \hat{F} \), we look at the actions of \( \hat{c}_1, \ldots, \hat{c}_r \) and \( \hat{t} \) on each factor spaces of (9.4.9).

**Assertion 1.**

i) \( \hat{c}_i \big| \bigoplus_{a \in \Gamma_{\lambda_1}} R (a^* - a) \subset \hat{F} \text{ is defined by } u \rightarrow \sum_{a \in \Gamma_{\lambda_1}} \hat{f} (u, a^*)(a^* - a). \)

ii) \( \hat{c}_i \hat{b}_a = \hat{b}_a \) for \( a \in \Gamma_{\lambda_1}, 1 \leq i \leq r, \)

iii) \( \hat{t} = \text{id}_{\hat{F}} - B \)

where

\[
B : \hat{F} \rightarrow \bigoplus_{a \in \Gamma_{\lambda_1}} R (a^* - a) \subset \hat{F} \text{ is defined by } u \rightarrow \sum_{\beta \in \Gamma_{\lambda_1}} \hat{f} (u, \beta^*)(\beta^* - \beta).
\]

**Proof.** i) Trivial, since the spaces \( \hat{F}_1, \ldots, \hat{F}_r \) are orthogonal to each other.

ii) We show only the first formula.

If \( \Gamma_{\Lambda_i} \) and \( a \notin \Gamma_{\lambda_1} \) are disconnected, \( a \) is orthogonal to \( \hat{F}_i \) and hence \( \hat{c}_i \hat{b}_a = \hat{b}_a \). If \( \Gamma_{\Lambda_i} \) and \( a \in \Gamma_{\lambda_1} \) are connected, \( \hat{c}_i \hat{b}_a = \hat{c}_i (\hat{b} - \sum_{\beta \in \Gamma_{\lambda_1}} \hat{b}_\beta) = \hat{b} - \sum_{\beta \in \Gamma_{\lambda_1}} \beta \hat{b} = \hat{b}. \)

iii) Using \( \hat{f} (\hat{a}^*, \hat{a}^* \hat{\beta}) = 2 \) and \( \hat{f} (u, \hat{a}^*) = \hat{f} (u, \hat{a}) \), one computes,
\[ \hat{\omega}_a\hat{\omega}_a^*(u) = u - \hat{\lambda}(u, \hat{\alpha}^\vee)(\hat{\alpha}^* - \hat{\alpha}). \]

Since \(\hat{\alpha}^* - \hat{\alpha} \in \text{rad } \hat{F}\) for \(\alpha \in \Gamma_m\), we obtain,
\[ \hat{\lambda}(u) = u - \sum_{\alpha \in \Gamma_m} \hat{\lambda}(u, \hat{\alpha}^\vee)(\hat{\alpha}^* - \hat{\alpha}). \] q.e.d.

**Assertion 2.** Let the notations be as above. Define

\[ (9.4.11) \quad S := \hat{\gamma}_1 \ldots \hat{\gamma}_r(id_{\hat{F}} - B \cdot \hat{p}_1), \]
\[ (9.4.12) \quad U := \text{id}_{\hat{F}} - B \cdot \hat{p}_2 \]

where \(\hat{p}_1\) and \(\hat{p}_2\) are linear projections from \(\hat{F}\) to the direct factor spaces \(\bigoplus_{i=1}^{r} \hat{F}_i\) and \(\bigoplus_{\alpha \in \Gamma_m} \hat{R}_\alpha\) in (9.4.9) respectively.

Then \(\hat{\gamma} = SU\) gives the Jordan decomposition of \(\hat{\gamma}\).

**Proof.** Using the Assertion 1 one checks immediately \(\hat{p}_1 \cdot B = \hat{p}_2 \cdot B = [B \cdot \hat{p}_2, \hat{\gamma}_1 \ldots \hat{\gamma}_r] = 0\) and \(B(\hat{p}_1 + \hat{p}_2) = B\). Then by definition
\[ SU = \hat{\gamma}_1 \ldots \hat{\gamma}_r(id_{\hat{F}} - B \cdot \hat{p}_1)(id_{\hat{F}} - B \cdot \hat{p}_2) = \hat{\gamma}_1 \ldots \hat{\gamma}_r(id_{\hat{F}} - B) = \hat{\gamma}. \]
\[ US = (id_{\hat{F}} - B \cdot \hat{p}_2) \hat{\gamma}_1 \ldots \hat{\gamma}_r(id_{\hat{F}} - B \cdot \hat{p}_1) \]
\[ = (\hat{\gamma}_1 \ldots \hat{\gamma}_r(id_{\hat{F}} - B \cdot \hat{p}_2) - [B \cdot \hat{p}_2, \hat{\gamma}_1 \ldots \hat{\gamma}_r](id_{\hat{F}} - B \cdot \hat{p}_1)) = \hat{\gamma}_1 \ldots \hat{\gamma}_r(id_{\hat{F}} - B) = \hat{\gamma}. \]

The Unipotency of \(U\) is a direct calculation as,
\[ (U - id_{\hat{F}})^2 = B \cdot \hat{p}_2 \cdot B \cdot \hat{p}_2 = 0. \]

To show the semi-simplicity of \(S\), we show that \(S\) is conjugate to \(\hat{\gamma}_1 \ldots \hat{\gamma}_r\), which is semi-simple since \(\hat{\gamma}_i\) is a Coxeter transformation on \(\hat{F}_i\) for the type \(A_{\ell_i}\) and an identity on the other factors of the decomposition (9.4.9). q.e.d.

**Assertion 3.** Put
\[ T := id_{\hat{F}} - B \cdot \hat{d} \cdot \hat{p}_1 \]

where \(d := (\hat{\gamma}_1 \ldots \hat{\gamma}_r)^{-1} : \bigoplus_{i=1}^{r} \hat{F}_i \longrightarrow \bigoplus_{i=1}^{r} \hat{F}_i\). Then \(T\) is invertible and
\[ T^{-1}ST = \hat{\gamma}_1 \ldots \hat{\gamma}_r. \]
**Proof.** \( T \) is well defined, since \( \hat{c}_i \) does not have eigen value 1 on \( \hat{F}_i \). \( T \) is invertible, since \((T - id)^2 = 0\).

\[
ST = \hat{c}_1 \ldots \hat{c}_r (id - B \cdot p_1)(id - B \cdot d \cdot p_1)
= \hat{c}_1 \ldots \hat{c}_r - B \cdot p_1 - B \cdot d \cdot p_1
= (id - B \cdot d \cdot p_1)\hat{c}_1 \ldots \hat{c}_r + B(d \cdot p_1(\hat{c}_1 \ldots \hat{c}_r - 1) - p_1)
= T \hat{c}_1 \ldots \hat{c}_r.
\]

**q.e.d.**

**Proof of the lemma (9.4).**

(9.4.2) is a direct consequence of (9.4.14), Assertion 1 i) ii) and (8.5) iv) v).

The unipotency (9.4.4) is shown in (9.4.13).

Let us show (9.4.5). Due to (9.4.12), \( \text{Im}(U - 1) = \text{Im}(B \cdot p_2) \).

\[
B(\hat{b}_a) = \sum_{\beta \in \Gamma_m} \hat{f}(\hat{b}_a, \hat{\beta} \cdot \gamma)(\hat{\beta}^* - \hat{\beta})
= \hat{f}(\hat{b}_a, \hat{\alpha} \cdot \gamma)(\hat{\alpha}^* - \hat{\alpha}) + \sum_{\beta \in \Gamma_m} \hat{f}(\hat{b}_a, \hat{\beta} \cdot \gamma)(\hat{\beta}^* - \hat{\beta})
= \hat{f}(\hat{b} - \sum_{\beta \in \Gamma_m} \hat{b}_\beta, \hat{\alpha} \cdot \gamma)(\hat{\alpha}^* - \hat{\alpha}) + \sum_{\beta \in \Gamma_m} \hat{f}(n_a, \hat{\beta} \cdot \gamma)(\hat{\beta}^* - \hat{\beta})
= \sum_{\beta \in \Gamma_m} \hat{f}(\hat{\beta}, \hat{\alpha}) \frac{2}{\max_{\beta} \hat{f}(\hat{\beta}, \hat{\alpha})} (\hat{\beta}^* - \hat{\beta}) - \frac{2}{\max_{\beta} \hat{f}(\hat{\beta}, \hat{\alpha})} (\hat{\alpha}^* - \hat{\alpha})
= \sum_{\beta \in \Gamma_m} \hat{f}(\hat{\beta}, \hat{\alpha}) \frac{I_R: \hat{f}(\hat{\beta}, \hat{\alpha})}{\max_{\beta} \hat{f}(\hat{\beta}, \hat{\alpha})} \left( \frac{\hat{\beta}^* - \hat{\beta}}{k(\beta)} - \frac{\hat{\alpha}^* - \hat{\alpha}}{k(\alpha)} \right).
\]

Here we use a relation \( n_a \hat{f}(\hat{\alpha}, \hat{\alpha}) = k(\alpha) m_a(I_R) \), and \( m_a = \max_{\alpha} \) for \( \alpha \in |\Gamma_m| \).

Using the last expression, it is a straightforward calculation in linear algebra that the fact that \( \Gamma_m \) is a connected linear diagram implies that \( B(\hat{b}_a) \) for \( \alpha \in |\Gamma_m| \) spans the space \( \sum_{\alpha, \beta \in |\Gamma_m|} \text{Re} \left( \frac{\hat{\beta}^* - \hat{\beta}}{k(\beta)} - \frac{\hat{\alpha}^* - \hat{\alpha}}{k(\alpha)} \right) \).

**q.e.d.**

**Note.** The formula (9.4.5) implies that the abstract data of the Dynkin diagram \( \Gamma_{R,G} \) determine automatically the counting constant \( k(\alpha) = (\alpha^* - \alpha) \) for \( \alpha \in |\Gamma_m| \) in the following way.

First, notice that \( \text{Im}(U - 1) \) does not depend on the choice of the pre-Coxeter transformation. ( \( \because \) \( \text{Im}(U - 1) \subset \text{rad} \hat{f} \), so that it is pointwisely fixed by the group \( \hat{W} \)). Through the formula (9.4.5), the relation \( \text{Im}(U - 1) \equiv 0 \) determines the proportion among \( k(\alpha) \) (\( \alpha \in |\Gamma_m| \)). Since g.c.d. \( (k(\alpha) : \alpha \in |\Gamma_m|) = 1 \) (8.5.1)), these data determine \( k(\alpha) \) (\( \alpha \in |\Gamma_m| \)) uniquely.
(9.5) Assertion. Put
\[ \hat{W} := \text{the sub-group of } O(\hat{F}, \hat{I}) \text{ generated by } \hat{w}_a \text{ for } a \in |\Gamma_{R,G}|. \]
\[ \hat{R} := \bigcup_{a \in |\Gamma_{R,G}|} \hat{W} a. \]

Then \( \hat{R} \) is a root system belonging to \( \hat{I} \).

Proof. Recall the definition (1.2) for a root system belonging to \( \hat{I} \). It is almost a routine work to check that \( \hat{R} \) is a root system in the sense of (1.2). (cf. (1. 3) Ex. 4.). q.e.d.

(9.6) We arrived at a goal of this paragraph.

Let \( I_{R,G} \) be the Dynkin diagram for a m.e.a.r.s. \( (R, G) \) w.r.t. a basis \( \{a_0, \ldots, a_t\} \).

Put
\[ F_r := \hat{F} / \text{Im}(U - 1), \]
\[ I_r := \text{the metric on } F_r \text{ induced from } \hat{I}, \]
\[ R_r := \text{the image of } \hat{R} \text{ in } F_r, \]
\[ W_r := \text{the subgroup of } O(F_r, I_r) \text{ induced from } \hat{W}, \]
\[ G_r := \bigoplus_{a \in |I_m|} R(\hat{a}^* - \hat{a})/\text{Im}(U - 1) \text{ (one dimensional subspace of } F_r), \]
\[ Q_r := \bigoplus_{a \in |I_{R,G}|} Z \hat{a} / ((\bigoplus_{a \in |I_{R,G}|} Z \hat{a}) \cap \text{Im}(U - 1)). \]

Here the notations \( \hat{F}, \hat{I}, U, \hat{W}, \hat{R} \) are defined in (9.2) 1), 2), (9.4.1) and (9.5).

Theorem. Let the notations be as above.
1. \( R_r \) is an extended affine root system belonging to \( I_r \), such that \( \hat{W}_{R_r} = \hat{W}_r \), \( Q(R_r) = Q_r \). The subspace \( G_r \) defines a marking of \( R_r \).
2. The marked extended affine root systems \( (R, G) \) and \( (R_r, G_r) \) are isomorphic.

Proof. 1. Due to (9.4.5), \( \text{Im}(U - 1) \) is a two codimensional subspace of \( \text{rad} \hat{I} \), which is defined over \( Q \). (i.e. \( (\bigoplus_{a \in |I_{R,G}|} Z \hat{a}) \cap \text{Im}(U - 1) \) is a full lattice in \( \text{Im}(U - 1) \).) Then the image \( R_r \) of \( \hat{R} \) in the quotient space \( F_r \) is a root system (cf. (1.8) Assertion). \( G_r \) is obviously a rank 1 module defined over \( Q \).
2. Since a change of the choice of the decomposition \( I_m \cup I^*_m \) in (9.1) induces an
isomorphism of the diagram $\Gamma_{R,G}$, it induces an isomorphism of the root system $R_t$. Therefore let us fix a choice of $\Gamma^m$ and $\Gamma^{\ast}_m$ as done in (8.1) (8.2).

Define a linear map $F \rightarrow F$ by $\hat{a} \rightarrow a$ for $a \in |\Gamma_{R,G}|$. Since $\frac{a^\ast - a}{k(a)} = a$ for any $a \in |\Gamma_m|$ (cf. (8.2) i)), taking account of (9.4.5), one sees that the map is factorized by an isomorphism $F_t \simeq F$, inducing $G_t \simeq G$. Furthermore this identification induces $I_t = cI$ for some $c > 0$.

By this identification $W_t$ becomes a subgroup of $W_R$ and hence $R_t = \bigcup_{a \in |\Gamma_{R,G}|} W_t a$ is a subset of $R$. We want to show that $R_t \supseteq R$, which implies also $W_t = W_R$.

Let us put $\{a_0, \ldots, a_l\} = |\Gamma_{R,G}| - |\Gamma_m|^\ast$. Then $a_0, \ldots, a_l$ is a basis for the reduced affine root system $R \cap L^{l+1}$ where $L^{l+1} = \bigoplus_{i=0}^{l} R a_i$ (cf. (3.3)). Recalling (6.1.5) and (6.14), we have

$$R = \bigcup_{a \in R \cap L^{l+1}} (a + Zk(a)a) = \bigcup_{i=0}^{l} W_R \cap L^{l+1} (a_i + Zk(a_i)a).$$

Since $W_R \cap L^{l+1} = \langle w_{a_0}, \ldots, w_{a_l} \rangle \subset W_R$, the last expression implies that if $a_i + Zk(a_i)a \subset R$ for $i = 0, \ldots, l$, then $R \subset R_t$.

Let us show the inclusion $\alpha + k(\alpha) \subset R$ for $\alpha \in |\Gamma_{R,G}| - |\Gamma^\ast_m|$ by decent induction on the exponent $m_\alpha$.

If $a \in |\Gamma_m|$, then

$$(w_{a_0} w_{a^\ast})^m(a) = a - 2m(\alpha - \alpha^\ast) = a + 2mk(a)a \quad \text{for } m \in \mathbb{Z}.$$  

$$(w_{a_0} w_{a^\ast})^m(a^\ast) = a^\ast - 2m(\alpha - \alpha^\ast) = a + (2m+1)k(a)a \quad \text{for } m \in \mathbb{Z}.$$  

If $\alpha, \beta$ are nodes of $\Gamma_{R,G}$ which are joint by a bond. Suppose $m_\beta < m_\alpha$. By induction we assume that $\alpha, \alpha^\ast \in R_t$. Then (7.4) Assertion implies,

$$(w_{a_0} w_{a^\ast})^m \beta = \beta - mI(\beta, \alpha^\ast)(a - a^\ast)$$

$$= \beta + mI(\beta, \alpha^\ast)k(a)a$$

$$= \beta - mk(\beta)a \quad \text{for } m \in \mathbb{Z}.$$  

This completes the proof of the theorem.

(9.7) The Coxeter transformation c. Let $a_0, \ldots, a_l$ be a basis for a m.e.a.r.s. $(R, G)$ and $\Gamma_{R,G}$ be the Dynkin diagram for $(R, G)$ w.r.t. the basis (cf. (8.2)).
A Coxeter transformation $c$ for $(R, G)$ w.r.t. the basis $\{a_0, ..., a_l\}$ is an element of $W_R$ defined as a product of $w_aw_{a^*}$ for $a \in |\Gamma_m|$ and $w_a$ for $a \in |\Gamma_{R,G}|-|\Gamma_m|\cup|\Gamma_m^*|$.

As a conclusion of the study in this paragraph, we have:

**Lemma A.** i) The conjugacy class of a Coxeter transformation in $W_R$ depends only on the linear space $L^{l+1} := \bigoplus_{i=0}^{l} R \alpha_i$ and the sign of the generator $a$ of $Q(R) \cap G$ (cf. (2.3.1)), but neither on the order of the product for the expression of $c$ nor on the choice of the Weyl chamber $C$ in choosing a basis for $R/G$ (cf. (3.3)).

The change $a \rightarrow -a$ of the sign of the generator $a$ of $G$ brings the conjugacy class of $c$ to the conjugacy class of $c^{-1}$.

ii) Let $\varphi : (F, G) \rightarrow (F, G)$ be an (outer) automorphism for $(R, G)$ and $c$ be a Coxeter transformation w.r.t. a basis $\{\alpha_0, ..., \alpha_l\}$. Then $\varphi c \varphi^{-1} \in W_R$ is a Coxeter transformation w.r.t. the basis $\{\varphi \alpha_0, ..., \varphi \alpha_l\}$.

iii) A Coxeter transformation $c$ for $(R, G)$ is semi-simple of finite order $l_{\max} + 1$. The set of eigenvalues of $c$ is given by:

$$\exp(2\pi \sqrt{-1} m_a/m_{\max}) \quad \text{for} \quad a \in |\Gamma_{R,G}| - |\Gamma^*_m| \quad \text{and} \quad 1 = \exp(2\pi \sqrt{-10}).$$

**Proof.** i) The conjugacy class of $c$ does not depend on the order of the product to present $c$, due to Lemma 1 [1] Ch. V, § 6.

The change of the sign $a$ to $-a$, induces the change of $a^{*} := a + k(a)(-a)$ (cf. (8.2) i)). Therefore $w_aw_{a^*}$ is changed to $w_{a^*}w_a = (w_aw_{a^*})^{-1}$. Therefore if $c$ is a Coxeter transformation for $a$, then $c^{-1}$ is a Coxeter transformation for the generator $-a$.

ii) Let $\varphi : F \rightarrow F$ be an automorphism of $(R, G)$. Since $\varphi(a^*) = \varphi(a)^*$ (cf. (6.1.4)) and $\varphi w_a \varphi^{-1} = \varphi w_a$ for $a \in R$, if $c$ is a Coxeter transformation w.r.t. a basis $\alpha_0, ..., \alpha_l$, then $\varphi c \varphi^{-1}$ is a Coxeter transformation w.r.t. the basis $\varphi \alpha_0, ..., \varphi \alpha_l$.

iii) Any Coxeter transformation $c$ is the image of the pre-Coxeter transformation $\hat{c}$ by the natural projection $\hat{W} \rightarrow W_R$. Since $\hat{c}$ act as identity on the kernel $\Im(U - 1) \subset \hat{F}$ of the projection $\hat{F} \rightarrow F$, the eigenvalues of $c$ is equal to that of $\hat{c}$ minus $\{1, ..., 1 : k\text{-times}\}$. Then (9.4) Lemma i) implies the assertion. q.e.d.
§ 10. Coxeter Transformation 2
(The existence of regular eigenspaces of a Coxeter transformation)

(10.1) Let \( c \) be a Coxeter transformation of a m.e.a.r.s. \((R, G)\) defined in (9.7). The following Lemma B is the main result of this paper, which plays a crucial role in the construction of flat \( \vartheta \)-invariants (cf. [20]). The proof of the Lemma B is now a straightforward work.

**Lemma B.**

\[ R \cap \text{Im}(c - \text{id}_F) = \phi. \]

**Proof.** Recall that \( c^{\text{max} + 1} = 1 \) ((9.7) Lemma A iii)). Hence \( (c - 1)(c^{\text{max} + c^{\text{max} - 1} + \ldots + 1}) = 0 \). Due to the semi-simplicity of \( c \), \( \text{Im}(c - 1) = \ker(c^{\text{max} + \ldots + 1}) \). Thus what we have to show is that no root \( \alpha \) of \( R \) satisfies the equation,

\[ P(c)\alpha = 0, \quad \text{where} \quad P(c) := c^{\text{max} + c^{\text{max} - 1} + \ldots + 1}. \]

Due to (9.7) Lemma A i) and ii), the statement of the Lemma B does not depend on a choice of a Coxeter transformation. Therefore we take a Coxeter transformation \( c \) which is the image of the pre-Coxeter transformation \( \check{c} \) of (9.4.10) and we use freely the results and notations in (9.1)–(9.6).

Put,

\[ c := c_1 \cdots c_r t \]

where

\[ c_i := \prod_{a \in |\Lambda_i|} w_a \quad (i = 1, \ldots, r), \]

\[ t := \prod_{a \in |\Gamma_m|} w_aw^*_a, \]

\[ F_i := \bigoplus_{a \in |\Lambda_i|} R\alpha \quad (i = 1, \ldots, r), \]

\[ b_a := n_a\alpha + \sum' n_{ab} \beta \quad (\alpha \in |\Gamma_m|). \]

Here \( \prod \) in (10.1.2) means a product for a fixed linear ordering and \( \sum' \) in (10.1.4) means the summation over all nodes of components \( \Gamma_{\Lambda_i} \) which is connected to the \( \alpha \in |\Gamma_m| \).

**Assertion 1.** \( P(c)F_i \equiv 0 \mod G \) for \( i = 1, \ldots, r \).

**Proof.** Due to Assertion 1. iii) in the proof of (9.4), \( t \equiv \text{id}_F \mod G \). Also due
to the same Assertion 1. i), \( c|F_i \equiv c_i|F_i \) and hence \( P(c)|F_i \equiv P(c_i)|F_i \mod G \). On the other hand, \( l_i + 1|l_{max} + 1 \) (8.5 v)) so that \( P_i(c_i) := c_i^t + c_i^{t-1} + \cdots + 1 \) divides \( P(c_i) \). Since \( c_i|F_i \) is a Coxeter transformation for \( A_i \), so that \( P_i(c_i)|F_i = 0 \) and hence \( P(c_i)|F_i = 0 \).

**Assertion 2.** \( P(c)b_a \equiv (l_{max} + 1)b_a \mod G \) for \( a \in |\Gamma_m| \).

**Proof.** Using (9.4) Assertion 1. ii),

\[
P(c)b_a \equiv P(1)b_a = (l_{max} + 1)b_a \quad \text{for} \quad a \in |\Gamma_m|.
\]

**Assertion 3.** *If a root \( a \in R \) satisfies the equation (10.1.1), then \( a = \beta + ma \) where \( \beta \) is a root of \( A_i \) for some \( i \) and \( m \in \mathbb{Z} \).*

**Proof.** One may choose \( b_\beta (\beta \in |\Gamma_m|), a (a \in |\Gamma_{A_i}|, i = 1, \ldots, r) \) and \( a \) as for (rational) basis for the linear space \( F \).

Put

\[
\alpha = \sum_{\beta \in |\Gamma_m|} c_\beta b_\beta + \sum_{i=1}^r \sum_{\beta \in |\Gamma_{A_i}|} c_\beta \beta + da \quad \text{for} \quad c_\beta, d \in \mathbb{Q}.
\]

Apply \( P(c) \) on \( \alpha \) and applying Assertions 1 and 2,

\[
P(c)\alpha \equiv (l_{max} + 1) \sum_{\beta \in |\Gamma_m|} c_\beta b_\beta \mod G.
\]

Since \( b_\beta (\beta \in |\Gamma_m|) \) are linearly independent mod \( G \), \( P(c)\alpha = 0 \) implies \( c_\beta = 0 \) for \( \beta \in |\Gamma_m| \). Therefore if \( \alpha \) of \( \alpha \) is a root of \( R \) with \( P(c)\alpha = 0 \), then \( \sum_{i=1}^r \sum_{\beta \in |\Gamma_{A_i}|} c_\beta \beta \) is an affine root in \( R \cap (\bigoplus_{i=0}^l R_{A_i}) \). Since the set \( \{\beta \in |\Gamma_{R,G}| - |\Gamma_m^*| : \ c_\beta \neq 0 \} \) is connected, there exists some \( i, 1 \leq i \leq r \), so that \( \{\beta : c_\beta \neq 0\} \subset |\Gamma_{A_i}| \).

**q.e.d.**

4. **Notations.** Let the component \( \Gamma_{A_i} \) be connected to \( \beta \in |\Gamma_m| \), so that the diagram at that branch looks like,
where \( t = 1, 2^{\pm 1}, 3^{\pm 1}, 4^{\pm 1} \) or \( \infty \). (For the convenience we use a convention \( \frac{-1}{t} = \frac{1}{x} \) and \( \frac{-1}{t} = \frac{1}{t-1} \) for \( t = 2, 3, 4 \).)

A positive root of \( A_{l_t} \) with respect to this basis \( \beta_1, ..., \beta_{l_t} \) is given by,

\[
\alpha_{u,v} := \beta_u + \beta_{u+1} + ... + \beta_v \text{ for some } u, v \in \mathbb{Z} \text{ with } 1 \leq u \leq v \leq l_t.
\]

Define a Coxeter transformation \( c_t \) for \( A_{l_t} \) by,

\[
c_t := w_{\beta_{l_t}} \cdots w_{\beta_1}.
\]

Direct from the definition, one calculates

\[
\alpha_{u,v} = \begin{cases} \alpha_{u-1,v-1} & \text{for } u > 1, \\ -\alpha_{v,u} & \text{for } u = 1.
\end{cases}
\]

Since all roots \( \alpha_{u,v} \), \( 1 \leq u \leq v \leq l_t \) are conjugate each other the counting \( k(\alpha_{u,v}) \), \( 1 \leq u \leq v \leq l_t \) is a constant ((6.1 Assertion iii)), which we shall denote \( k_i := k(\alpha_{u,v}) \) \((1 \leq u \leq v \leq l_t)\).

**Assertion 5.** The Coxeter transformation \( c \) leaves the space \( F_i \oplus G \) invariant for \( i = 1, ..., r \). We have a formula,

\[
P(c)\alpha_{u,v} = \frac{l_{\text{max}} + 1}{l_i + 1} (v - u + 1)k_i a \text{ for } 1 \leq u \leq v \leq l_i.
\]

**Proof.** Due to (9.4) Assertion 1. i ) iii), \( c|F_i \oplus G = c_i t_\beta|F_i \oplus G \) where \( t_\beta = w_{\beta_{l_i}} \) and \( t_\beta(u) = u - I(\beta, \beta^\vee)k(\beta)a \).

By induction on \( m \), we have,

\[
(c_i t_\beta)^m u = c_i^m u - I(\sum_{n=0}^{m-1} c_i^n u, \beta^\vee)k(\beta)a \text{ for } u \in F_i.
\]

Hence for \( u \in F_i \), we have

\[
***) P_i(c)u := \sum_{m=0}^{l_i} (c_i t_\beta)^m u = \sum_{m=0}^{l_i} c_i^m u - \sum_{n=0}^{l_i} I(\sum_{n=0}^{m-1} c_i^n u, \beta^\vee)k(\beta)a
\]

\[
= -\sum_{m=0}^{l_i} (l_i - n)I(c_i^m u, \beta^\vee)k(\beta)a.
\]

On the other hand using ** in 4., \( I(c_i^m \alpha_{u,v}, \beta^\vee) \) is either \( I(\beta_1, \beta^\vee) \), \(-I(\beta_1, \beta^\vee)\) or 0 according as \( n = u - 1, n = v \) or \( n \neq u - 1, v \). Therefore applying ***) for \( u = \alpha_{u,v} \),
Here in the last step of the calculation, we have applied (7.4) Assertion for \( \alpha := \beta, \beta := \beta_i \) since \( m_{\beta_i} < m_{\beta} = m_{\text{max}} \).

Since \( l_i + 1 \mid l_{\text{max}} + 1 \) and hence \( P(c) = Q(c)P_i(c) \) for some polynomial \( Q(c) \) such that \( Q(1) = \frac{l_{\text{max}} + 1}{l_i + 1} \), we have

\[
P(c)\alpha_{u,v} = \frac{l_{\text{max}} + 1}{l_i + 1}(v - u + 1)k_ia.
\]

q.e.d.

6. Let \( \alpha \in R \) be a root of the form,

\[
\alpha = \alpha_{u,v} + mk_ia \quad \text{for} \quad 1 \leq u \leq v \leq l_i, \quad m \in \mathbb{Z}.
\]

If we have shown that \( P(c)\alpha \neq 0 \), we have completed the proof of the Lemma of (10.1).

Using the formula of Assertion 5,

\[
P(c)\alpha = \frac{l_{\text{max}} + 1}{l_i + 1}(v - u + 1)k_i a + mk_i(l_{\text{max}} + 1)a
\]

\[
= \left( \frac{v - u + 1}{l_i + 1} + m \right)k_i(l_{\text{max}} + 1)a.
\]

Since \( 1 \leq u \leq v \leq l_i \), the value \( \frac{v - u + 1}{l_i + 1} \) lies in the interval \((0,1)\).

Hence \( \frac{v - u + 1}{l_i + 1} + m \neq 0 \) for any \( m \in \mathbb{Z} \).

End of the proof of Lemma B.

§ 11. Coxeter Transformation 3

(The generator of the extension \( \widetilde{W}_{R,G} \) of \( W_R \)).

Let \( \widetilde{W}_{R,G} \) be the hyperbolic extension of \( W_R \) (cf. (1.18) or (11.1)) and \( \widetilde{c} \) be a hyperbolic Coxeter transformation ((11.2) Def.). We show that \( \widetilde{W}_{R,G} \) is a central extension of \( W_R \) by an infinite cyclic group which is generated by \( \widetilde{c}^{l_{\text{max}} + 1} \). (11.3 Lemma C).

(11.1) Let \((R, G)\) be a m.e.a.r.s.. Let us recall briefly the notion of a hyperbolic
extension w.r.t. the marking $G$ from (1.17), (1.18).

1. There exists a triple $(\tilde{F}, \tilde{I}, \iota)$ where $\tilde{F} :=$ a vector space of rank $l+3$, $\tilde{I} :=$ a symmetric bilinear form on $\tilde{F}$ and $\iota : F \to \tilde{F}$ is an injective linear map s.t. i) $I = \tilde{I} \cdot \iota$, ii) $\text{rad} \tilde{I} = \iota G$.

Such triple is unique up to an isomorphism so that we fix one and regard $\iota$ as an inclusion map.

2. Denote by $\tilde{w}_a \in O(\tilde{F}, \tilde{I})$ the reflection of $a \in R$ as an element in $\tilde{F}$. Denote by $\tilde{W}_{R,G}$ the group generated by $\tilde{w}_a (a \in R)$ and call it the hyperbolic extension of $W_R$.

3. The restriction map $\eta_*^*$ of the action of $\tilde{W}_{R,G}$ from $\tilde{F}$ to the subspace $F$ induces a short exact sequence,

$$1 \to \tilde{K}_G \to E_G \to \tilde{W}_{R,G} \to \tilde{W}_R \to 1,$$

where

$$E_G : F \otimes F/G \to \text{End}(\tilde{F}),$$

$$E_G \left( \sum_i f_i \otimes g_i \right)(u) := u - \sum_i f_i I(g_i, u)$$

is the Eichler Siegel presentation (cf. (1.17.2)) and

$$\tilde{K}_G := E_G^{-1}(\tilde{W}_{R,G}) \cap M_G$$

is a lattice of $M_G := G \otimes (\text{rad} I/G)$ (cf. (1.18) Lemma).

(11.2) Let $\{a_0, \ldots, a_l\} \subseteq \mathbb{Z}$ be a basis for $(R, G)$ and $\Gamma_{R,G}$ be the Dynkin diagram for $(R, G)$ w.r.t. the basis.

**Definition.** A hyperbolic Coxeter transformation $\tilde{c}$ for $(R, G)$ w.r.t. the basis $\{a_0, \ldots, a_l\}$ is an element of $\tilde{W}_{R,G}$ defined as a product of $\tilde{w}_a \tilde{w}_a^*$ for $a \in |\Gamma_m|$ and $\tilde{w}_a$ for $a \in |\Gamma_{R,G}| - |\Gamma_m| \cup |\Gamma_m^*|$.

Same argument as (9.7) Lemma A proves,

**Assertion.** i) The conjugacy class of a hyperbolic Coxeter transformation $\tilde{c}$ in $\tilde{W}_{R,G}$ depends only on the linear space $\bigoplus_{i=0}^l R a_i$ and the choice of a generator $a$ of $Q(R) \cap G$, but neither on the order of the product to present $\tilde{c}$ nor on the Weyl chamber $C$ in choosing basis for $R/G$ (cf. (3.3)).
ii) Let \( \varphi : F \cong F \) induce an automorphism of \( (R, G) \), whose orthogonal extension is denoted by \( \tilde{\varphi} : \tilde{F} \to \tilde{F} \). If \( \tilde{c} \) is a hyperbolic Coxeter transformation w.r.t. a basis \( \{a_0, \ldots, a_1\} \), then \( ad^*_{\tilde{c}} := \tilde{\varphi} \tilde{c} \tilde{\varphi}^{-1} \in \tilde{W}_{R,G} \) is a hyperbolic Coxeter transformation w.r.t. the basis \( \{\varphi a_0, \ldots, \varphi a_1\} \).

iii) \( \tilde{c}^\{t\}_{\text{max}+1} \in E_G(K_G) \).

(11.3) The main purpose of this paragraph is to show the following Lemma C.

**Lemma C.** Let the notations be as above.

i) \( \tilde{c}^\{t\}_{\text{max}+1} = (I_R : I)^{\frac{t_{\text{max}}+1}{m_{\text{max}}}} E_G(a \otimes b) \).

ii) \( (I_R : I)^{\frac{t_{\text{max}}+1}{m_{\text{max}}}} a \otimes b \) generates \( K_G \) as a cyclic group.

i.e. The element \( \tilde{c}^\{t\}_{\text{max}+1} \) generates the cyclic group \( \ker(p_\ast) \).
(Here \( a, b \) are \( \mathbb{Z} \)-basis of \( Q(R) \cap \text{rad}I \)).

**Note.** Let us choose the basis \( a, b \) of \( Q(R) \cap \text{rad}I \) as in (2.3.1) (2.3.2). Then as an element of \( M_G : G \otimes (\text{rad}I/G) \), \( a \otimes b \) is unique up to a sign. The sign of the element is determined by the realization of the Dynkin diagram \( \Gamma_{R,G} \) as a set of roots of \( R \) in the following way.

i) The choice of a basis \( \{a_0, \ldots, a_1\} \) determines the sign of \( b \) by (3.3.6).

ii) The choice of \( \Gamma_m \) and \( \Gamma_m^\ast \) determines the sign of \( a \) by (8.2) i) ii).

(11.4) **Proof of** i) of Lemma C.

Let us denote by \( \tilde{I}_R \) the bilinear form \( (I_R : I) \tilde{I} \). First let us show a formula,

\[
(11.4.1) \quad (\tilde{c} - 1)\xi + \tilde{I}_R(\xi, b) \frac{1}{m_{\text{max}}} a \in (c - 1)F^{t+2} \quad \text{for} \quad \xi \in \tilde{F}^{t+3}.
\]

Note that the formula (11.4.1) does not depend on the choice of \( \tilde{c} \) in the conjugacy class in \( \tilde{W}_{R,G} \), since the change of \( \tilde{c} \) by \( w^{-1} \tilde{c} w \) for a \( w \in \tilde{W}_{R,G} \) induces,

\[
((w^{-1} \tilde{c} w) - 1)\xi + \tilde{I}_R(\xi, b) \frac{a}{m_{\text{max}}} = w^{-1}((\tilde{c} - 1)\xi + \tilde{I}_R(\xi, b) a/ m_{\text{max}}) + w^{-1}(\tilde{c} - 1)(w - 1)\xi \in w^{-1}(c - 1)F^{t+2} = (w^{-1}cw - 1)F^{t+2}.
\]

**Case** \( \text{cod}(R, G) = l + 1 \). This is the case when \( \Gamma_{R,G} = \Gamma_m \cup \Gamma_m^\ast \). Since \( \tilde{c} = \)
\[ \prod_{a \in I_m} w_a w_{a*}, \text{ similarly as in (9.4) Assertion 1 iii),} \]

\[ \tilde{c}(u) = u - \sum_{a \in I_m} \bar{I}(u, a) \frac{2}{I(a, a)} (a* - a) \]

\[ = u - \frac{1}{m_{\text{max}}} \bar{I}_R(u, \sum_{a \in I_m} n_a a) a. \]

(Here we used the facts \( m_a = \frac{I_R(a, a)}{2k(a)} ((7.1.1)) \), \( m_{\text{max}} = m_a \) and \( a* - a = k(a) a \)
((8.2) i ) for \( a \in |I_m|.) \]

**Case** \( \text{cod}(R, G) < l + 1. \) Since \( I_{R,C} = \Gamma_m \cup I_m \neq \phi \), let us take a node \( a_0 \) of the diagram \( \Gamma_{A_i} \) for some \( i \) so that \( a_0 \) lies at terminal of the diagram. Take one \( \tilde{c} \) by \( \tilde{c} := \tilde{w}_{a_0} d \), where \( d \) is a product of \( \tilde{w}_a \tilde{w}_{a*} \) for \( a \in |I_m| \) and \( w_a \) for \( a \in |I_{R,C}| - |I_m| \cup I_m \cup \{a_0\} \).

Since (11.4.1) is obviously true for \( \xi \in F^{l+2} \), we have only to show (11.4.1) for one \( \xi \in \overline{F}^{l+2} \) s.t. \( \xi \in F^{l+2} \). We may choose one such \( \xi \) satisfying \( \bar{I}(\xi, a_0) \neq 0, \bar{I}(\xi, a) = 0 \) for \( a \in |I_{R,C}| a \neq a_0 \), so that \( d(\xi) = \xi \). Then

\[ \tilde{c}(\xi) = \tilde{w}_{a_0}(\xi) = \xi - \bar{I}(\xi, a_0) a_0 \]

\[ = \xi - \bar{I}(\xi, a_0) \left( a_0 - \frac{k(a_0)}{l_{i+1}} a \right) - \frac{2I_R(\xi, a_0)}{I_R(a_0, a_0)} \frac{k(a_0)}{l_{i+1}} a. \]

On one hand, applying **Assertion 5** in the proof of (10.1) **Lemma** for \( u = v = 1, \)
\[ P(c) \left( a_0 - \frac{k(a_0)}{l_{i+1}} a \right) = 0. \] Therefore the semi-simplicity of \( c \) implies that the second term of \( \tilde{c}(\xi) \) \( a_0 - \frac{k(a_0)}{l_{i+1}} a \) belongs to \( (c-1)F^{l+2} \).

On the other hand noting \( \bar{I}(\xi, b) = I(\xi, \sum_a n_a a) = n_a I(\xi, a_0), (l_{i+1}) m_{a_0} = m_{\text{max}} ((8.5) iv)) \) and \( m_{a_0} = \frac{I_R(a_0, a_0)}{2k(a_0)} n_{a_0} \), the last term of \( \tilde{c}(\xi) \) becomes,

\[ \frac{2I_R(\xi, a_0)}{I_R(a_0, a_0)} \frac{k(a_0)}{l_{i+1}} a = I_R(\xi, b) \frac{2k(a_0)}{n_a I_R(a_0, a_0)} \frac{1}{l_{i+1}} a = \frac{I_R(\xi, b)}{m_{\text{max}}} a. \]

This completes the proof of (11.4.1).

Applying \( P(\tilde{c}) = \tilde{c}^{t_{\text{max}}} + \cdots + 1 \) on (11.4.1), one obtains,

\[ (\tilde{c}^{t_{\text{max}}+1} - 1) \xi = -I_R(\xi, b) \frac{t_{\text{max}}+1}{m_{\text{max}}} a \text{ for } \xi \in \overline{F}. \]

This is nothing but i ) of **Lemma C**. q.e.d.
(11.5) For the proof of ii) of Lemma C, we need to recall a map \( r : \tilde{\mathcal{W}}_{R, G} \rightarrow M_G \) from (1.20).

Let \( \{a_0, \ldots, a_l\} \) be a basis for \( (R, G) \) s.t. the images of \( a_1, \ldots, a_l \) in \( R/\text{rad} \ I \) are basis for \( R/\text{rad} \ I \).

Put

\[
L := \bigoplus_{i=1}^{l} Ra_i, \quad H := Rb, \quad b = \sum_{i=0}^{l} n_i a_i, \quad G := R a,
\]

so that we fix a direct sum decomposition,

(11.5.1) \[ F = L \oplus H \oplus G. \]

Then, depending on the decomposition (11.5.1), we have introduced a map,

\[ r : \tilde{\mathcal{W}}_{R, G} \rightarrow M_G, \]

(cf. (1.20.1)), such that

\[ r(g) = E_G^{-1}(g) \quad \text{for} \quad g \in E_G(\tilde{K}_G), \]

\[ r(\tilde{\mathcal{W}}_{R, G}) \subset \sum_{a, \beta \in R} \mathbb{Z} \text{int}(a, \beta) a_G \otimes \beta_H, \]

where

a) we denote by \( a = a_L + a_H + a_G \) the decomposition (11.5.1) for \( a \in R \).

b) \( \text{int}(a, \beta) \) for \( a, \beta \in R \) is a positive number defined by,

\[
\text{int}(a, \beta) := \begin{cases} 
2/I(a, a) & \text{if } a = \beta, \\
\text{g.c.d.} \left\{ \frac{\prod_{j=1}^{k-1} I(a_j, a_{j+1})}{\prod_{j=1}^{k} I(a_j, a_j)} : a_1, \ldots, a_k \in R \text{ s.t. } a_1 = a, a_k = \beta \right\}.
\end{cases}
\]

(See (1.20) Assertions 1-3 for more details and for the proofs about the map \( r \).)

(11.6) A proof of ii) of Lemma C.

1. By definition of \( \tilde{K}_G \) and i) of Lemma C we have the following commutative diagram.

\[ \begin{array}{ccc}
\{(c^{i_{\text{max}}+1})^n : n \in \mathbb{Z}\} & \subset & E_G(\tilde{K}_G) \\
\downarrow E_G & & \downarrow r \\
\mathbb{Z}(I_R: I) & \subset & \tilde{K}_G \subset r(\tilde{\mathcal{W}}_{R, G}) \subset M_G.
\end{array} \]
Thus if we have shown an inclusion relation,

\[ r(\hat{W}_{R,G}) \subseteq \mathbb{Z}(I_R : I)^{l_{\max} + 1}_{m_{\max}} \alpha \otimes b, \]

we get \( K_G = \mathbb{Z}(I_R : I)^{l_{\max} + 1}_{m_{\max}} \alpha \otimes b \) which proves ii) of the Lemma C.

For the purpose it is sufficient to show,

\[ \text{int}(\alpha, \beta)_{\alpha \otimes \beta_R} \subseteq \mathbb{Z}(I_R : I)^{l_{\max} + 1}_{m_{\max}} \alpha \otimes b \quad \text{for} \quad \alpha, \beta \in R, \]
due to (11.5) ii).

2. First let us calculate \( \text{int}(\alpha, \beta) \). As we see below, it is easy to see that the function \( \text{int}(\alpha, \beta) \) depends only on the quotient root system \( R/\text{rad} \, I \).

**Formula**

\[
(11.6.1) \quad \text{int}(\alpha, \beta) = t \frac{I_R : I}{\text{g.c.d.}(\frac{I_R(\alpha, \alpha)}{2}, \frac{I_R(\beta, \beta)}{2})},
\]

where

\( t = 1 \) for all cases except the following 1)-4).

\( t = 2 \)

1) \( R/\text{rad} \, I \) is of type \( A_1 \) and \( \alpha \neq \beta \).

2) \( R/\text{rad} \, I \) is of type \( B_1(l \geq 2) \) and \( \alpha \neq \beta \) are short roots of \( R \).

3) \( R/\text{rad} \, I \) is of type \( C_1(l \geq 2) \) and \( \alpha \neq \beta \) are long roots of \( R \).

4) \( R/\text{rad} \, I \) is of type \( BC_1(l \geq 2) \) and either \( \alpha \neq \beta \) are short roots of \( R \), or \( \alpha \neq \beta \) are long roots of \( R \), or \( \alpha \neq \beta \) are middle roots of \( R \) for \( l = 2 \).

A sketch of a proof for (11.6.1). Since \( \frac{I_1(\alpha, \alpha)}{2} \text{int}(\alpha, \beta), \frac{I_1(\beta, \beta)}{2} \text{int}(\alpha, \beta) \in \mathbb{Z} \),
g.c.d. \( \left( \frac{I_1(\alpha, \alpha)}{2}, \frac{I_1(\beta, \beta)}{2} \right) \) \( \text{int}(\alpha, \beta) \in \mathbb{Z} \). This implies (11.6.1) for an integer \( t \). If there exists a sequence \( \alpha = \alpha_1, \ldots, \alpha_k = \beta \) s.t. \( I(\alpha_i, \alpha_{i+1}) \neq 0 \) \( i = 1, \ldots, k-1 \), \( I(\alpha_i, \alpha_{i+1}) \leq I(\alpha_k, \alpha_{k}) \leq \cdots \leq I(\alpha_1, \alpha_{k}) \) and \( \alpha_i \neq \alpha_{i+1} \) mod \( \text{rad} \, I \) \( i = 1, \ldots, k-1 \), then

\[
2^k \prod_{j=1}^{k-1} I(\alpha_j, \alpha_{j+1}) = \frac{2}{\bigg( \prod_{j=1}^{k-1} I(\alpha_j, \alpha_{j+1}) \bigg)} = \pm \frac{2}{I(\alpha, \alpha)} \quad \text{and} \quad \text{therefore} \quad \text{int}(\alpha, \beta) =
\]
so that \( t = 1 \) in this case. This will cover almost all cases except \( R/\text{rad } I \) is of type either one of \( A_1, B_t, C_t, BC_t \). Some more careful study of the cases \( A_1, B_t, C_t, BC_t \) gives the results of the formula.

3. In the following we give a table of \( \frac{l_{\max} + 1}{m_{\max}} \) for all types of m.e.a.r.s.'s.

<table>
<thead>
<tr>
<th>Type</th>
<th>( A^{(1,1)}_1 )</th>
<th>( A^{(1,1)}_1 )</th>
<th>( B^{(1,1)}_t )</th>
<th>( B^{(2,1)}_t )</th>
<th>( B^{(2,2)}_t )</th>
<th>( C^{(1,1)} )</th>
<th>( C^{(1,2)} )</th>
<th>( C^{(2,1)} )</th>
<th>( C^{(2,2)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( l_{\max} + 1 )</td>
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<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>( m_{\max} )</td>
<td>1</td>
<td>1</td>
<td>4</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
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</table>

<table>
<thead>
<tr>
<th>Type</th>
<th>( B^{(2,2)}_t )</th>
<th>( C^{(1,1)} )</th>
<th>( BC^{(1,1)} )</th>
<th>( BC^{(2,1)} )</th>
<th>( BC^{(2,3)}(1) )</th>
<th>( BC^{(2,3)}(2) )</th>
<th>( D^{(1,1)}_4 )</th>
<th>( E^{(1,1)}_6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( l_{\max} + 1 )</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>( m_{\max} )</td>
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<td>2</td>
<td>4</td>
<td>2</td>
<td>4</td>
<td>2</td>
<td>2</td>
<td>3</td>
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</table>

<table>
<thead>
<tr>
<th>Type</th>
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<th>( E^{(1,1)}_6 )</th>
<th>( F^{(1,1)}_4 )</th>
<th>( F^{(1,2)}_4 )</th>
<th>( F^{(2,2)}_4 )</th>
<th>( G^{(1,1)}_2 )</th>
<th>( G^{(1,3)}_2 )</th>
<th>( G^{(2,1)}_2 )</th>
<th>( G^{(2,3)}_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( l_{\max} + 1 )</td>
<td>4</td>
<td>6</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>( m_{\max} )</td>
<td>4</td>
<td>6</td>
<td>6</td>
<td>4</td>
<td>3</td>
<td>6</td>
<td>3</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

4. To show \( \circ \circ \) is a straight forward work now. Using the descriptions of \( R \) in (5.3) and the formula (11.6.1), one shows that \( \text{int}(\alpha, \beta) \alpha_0 \otimes \beta_R \in \mathbb{Z} (I_R : I) - \alpha \otimes b \) for \( \alpha, \beta \in R \) of type \( P^{(1,1)} \) and that \( \text{int}(\alpha, \beta) \alpha_0 \otimes \beta_R \in \mathbb{Z} (I_R : I) \alpha \otimes b \) for \( \alpha, \beta \in R \) of type \( P^{(t_1, t_2)} \) with \( \max (t_1, t_2) > 1 \).

The root systems of types \( A^{(1,1)}_1, B^{(2,2)}_t \) and \( C^{(1,1)}_t \) are explained from 1), 2) and 3) of the formula (11.6.1).

This completes the proof of (11.3) Lemma C.

(11.7) Note. Let \( \alpha_t = \alpha, \alpha_2 = \alpha + mb, \alpha_3 = \alpha + na, \alpha_4 = \alpha + na + mb \) belong to \( R \).

Put \( g = \bar{w}_{\alpha_1} \cdot \bar{w}_{\alpha_2} \cdot \bar{w}_{\alpha_3} \cdot \bar{w}_{\alpha_4} \in \bar{W}_{R,0} \).

Then \( g = E_\sigma((I_R : I) m \cdot na \otimes b) \in \ker(\rho_\sigma) \).
§ 12. Foldings of Dynkin Diagrams

In this paragraph, we study foldings of Dynkin diagrams for marked extended affine root systems. Precisely, we introduce two types; folding and mean folding (cf. (12.2) Def.)

All Dynkin diagrams with multiple bonds are obtained by foldings of Dynkin diagrams with only simple bonds: \( A^{(1,1)}, D^{(1,1)}, E^{(1,1)} \) (12.4)). In case of finite or affine root systems, which are studied by P. Slodowy [21], T. Yano [27] and others, only one of the two types of foldings is enough to produce all diagrams, whereas in the case of m.e.a.r.s.’s, both types of foldings are necessary to obtain all diagrams.

In this way, we arrive a hierarchy relation among m.e.a.r.s.’s, where the exceptional types \( A^{(1,1)*}, B^{(2,2)*}, C^{(1,1)*}, BC^{(2,2)}(1), BC^{(2,2)}(2) \) form naturally one group (cf. (12.5)).

(12.1) Let \((R, G)\) be a m.e.a.r.s. belonging to \( I \). Let \( \{\alpha_0, ..., \alpha_l\} \) be a basis of \((R, G)\) and \( \Gamma \) be the Dynkin diagram for the affine root system \( R \cap \bigoplus_{i=0}^{l} R\alpha_i \) (cf. (6.4)).

Put
\[
\text{Aut}(R, \text{rad } I) := \{\varphi \in \text{Aut}(R) : \varphi|\text{rad } I = \text{id}_{\text{rad } I}\}.
\]

**Lemma.** Assume that \((R, G)\) is not of exceptional type.

Then there exists a faithful representation,

\[
(12.1.1) \quad r : \text{Aut}(\Gamma) \longrightarrow \text{Aut}(R, \text{rad } I),
\]

such that the action of \( r(\text{auto}(\Gamma)) \) on \( F \) leaves the subspace \( \bigoplus_{i=0}^{l} R\alpha_i \) invariant.

**Proof.** We construct \( r \) explicitly as follows.

Let us regard \( \text{Aut}(\Gamma) \) as a subgroup of the permutation group of \( \{0, ..., l\} \) of the indexes of \( \alpha_0, ..., \alpha_l \).

Under the assumption of Lemma we discluded the types \( A^{(1,1)*}, B^{(2,2)*}, C^{(1,1)*} \). Then we check easily,

\[
k(\alpha_i) = k(\alpha_{\sigma(i)}) \quad (i=0, ..., l),
\]

\[
n_i = n_{\sigma(i)} \quad (i=0, ..., l),
\]
where $k$ is the counting function ((6.1)) and $n_i$ is the coefficients in (3.3). Hence, by the definition of exponents (7.1.1), we have,

$$m_i = m_{\sigma(i)} \quad (i = 0, ..., l),$$

so that $\sigma$ preserves the subset $|\Gamma|^m$ of $|\Gamma|$. Therefore the action of $\sigma \in \text{Aut}(\Gamma)$ can be extended to $\Gamma_{R,G}$ (cf. (8.2)) by

$$\sigma(\alpha_i^t) = \alpha_{\sigma(i)}^t \quad \text{for} \quad \alpha_i \in |\Gamma|^m.$$

Then there exists a unique linear map $r_\sigma : F \cong F$ such that $r_\sigma(\alpha_i) = a_{\sigma(i)}$, $r_\sigma(\alpha_i^t) = \alpha_{\sigma(i)}^t (0 \leq i \leq l)$ and $r_\sigma(a) = a$, $r_\sigma(b) = b$ for $a$, $b$ the basis of $\text{rad } I$. (\therefore) The linear dependence relations among $\alpha_i$'s, $\alpha_i^t$'s, $a$ and $b$ are generated by $a_i - a_i^t = k(\alpha_i)a$ $(0 \leq i \leq l)$ (cf. (8.2)) and $b = \sum_{i=0}^l n_i \alpha_i$ (cf. (3.3.6)). Since $k(\alpha_{\sigma(i)}) = k(\alpha_i)$, $n_{\sigma(i)} = n_i$ $(0 \leq i \leq l)$, the same relations hold for $a_{\sigma(i)}$'s, $\alpha_{\sigma(i)}^t$'s, $a$ and $b$.

Let us show that $r_\sigma$ induces an automorphism of $R$.

$$r_\sigma(R) = r_\sigma\left(\bigcup_{t=0}^l \bigcup_{w<w_0} \left\{ r_{\sigma(i)} + Zk(\alpha_i)a \right\} \right)$$

$$= \bigcup_{t=0}^l \bigcup_{w<w_0} \left\{ r_{\sigma(i)} + Zk(\alpha_i)a \right\}$$

$$= R \quad \text{(cf. (6.4) Lemma II.)}$$

It is obvious that $\sigma \mapsto r_\sigma$ is a representation and is faithful. q.e.d.

**Note.** Under the assumption of the assertion, the (12.1.1) induces,

$$\text{Aut}(\Gamma) \simeq \text{Aut}(\Gamma_{R,G})/2^{\text{cod}(R,G)},$$

where $2^{\text{cod}(R,G)}$ denotes the group generated by the transposition of $\alpha$ and $\alpha^*$ for $\alpha \in |\Gamma|^m$ (cf. (8.4) i)).

(12.2) Let the notations and the assumptions be as in (12.1). Let $H$ be a subgroup of $\text{Aut}(\Gamma)$, which acts on $F$ through (12.1.1).

Define an invariant subspace of $F$,

$$F^H : = \{ x \in F : r_\sigma x = x \text{ for } \sigma \in H \}.$$ 

Clearly by definition, $\text{rank}(F^H) = \#(\left| I^r / H \right| + 1$ and the sign of $I|F^H$ is $(\#(\left| I^r / H \right| - 1, 2, 0)$.
Define two mappings,

\[
\begin{align*}
\text{Tr}_H^H & : F^H \rightarrow F^H, \quad a \mapsto \sum_{\beta \in H^\alpha} \beta, \\
\text{Tr}_H^H & : F^H \rightarrow F^H, \quad a \mapsto (\# H^a)^{-1} \sum_{\beta \in H^a} \beta.
\end{align*}
\]

(12.3) From now on we shall assume the following for the group \( H \).

* There exists at least one node of \( \Gamma \) so that \( H \) is contained in the isotropy subgroup of \( \text{Aut}(\Gamma) \) of the point.

Under the assumption we have:

\[\text{Lemma 1.} \quad \text{There exist extended affine root systems} \ R_H^H \text{ and} \ R^H \text{ in} \ F^H \text{ belonging to} \ I[F^H] \text{ such that the set} \ \text{Tr}_H^H[\Gamma_{R,G}] \text{ and} \ \text{Tr}_H^H[\Gamma_{R,G}] \text{ form Dynkin diagrams for} \ (R^H, G) \text{ and} \ (R_H, G) \text{ respectively.}
\]

Together with the structure of the Dynkin diagram, we shall denote by \( \text{Tr}_H^H[\Gamma_{R,G}] \) and \( \text{Tr}_H[\Gamma_{R,G}] \) the sets, \( \text{Tr}_H^H[\Gamma_{R,G}] \) and \( \text{Tr}_H[\Gamma_{R,G}] \) respectively.

2. The isomorphism classes of \( (R^H, G) \) and \( (R_H, G) \) depend only on \( H \subset \text{Aut}(\Gamma) \) but not on the choice of basis \( \alpha_0, \ldots, \alpha_i. \)

3. \( (R^H, G) \) and \( (R^H, G) \) are dual of each other.

Proof. The proof of Lemma will be done in the following 7 steps. The precise meaning of Lemma 1 will be explained in the steps 4 and 5.

Since each of the steps are rather elementary, we do not give details of the calculations.

1. In the next (12.4), we shall list all groups \( H \) satisfying the assumption *).

As a result of the listing, we have the following.

i) \( I(\alpha, \beta) = 0 \) for all \( \alpha, \beta \in H^\gamma \) s.t. \( \alpha, \beta \in \Gamma_{R,G} \).

ii) \( H \) acts transitively on the set of pairs \( (\alpha, \beta) \in H^\gamma \times H^\delta \) with \( I(\alpha, \beta) = 0 \) (for \( \gamma, \delta \in \Gamma_{R,G} \)).

2. \( Q(\text{Tr}_H^H[\Gamma_{R,G}]) \) and \( Q(\text{Tr}_H[\Gamma_{R,G}]) \) are full lattices of \( F^H \).

Since \( \text{Tr}_H \) is a linear map, we have

\[
\text{Tr}_H Q(R) = Tr_H Q([\Gamma_{R,G}]) = Q(\text{Tr}_H[\Gamma_{R,G}])
\]
3. Using $i$ $ii$), one computes the lengths, intersection numbers and the duals of the elements of $\text{Tr}^H|\Gamma_{R,c}|$ and $\text{Tr}_H|\Gamma_{R,c}|$ as follows.

\begin{enumerate}
  \item $I(\text{Tr}^H\gamma, \text{Tr}^H\gamma) = \# H_{\gamma} I(\gamma, \gamma) \neq 0$ for $\gamma \in |\Gamma_{R,c}|$,
  \item $I(\text{Tr}_H\gamma, \text{Tr}_H\gamma) = (\# H_{\gamma})^{-1} I(\gamma, \gamma) \neq 0$ for $\gamma \in |\Gamma_{R,c}|$,
  \item $I(\text{Tr}^H\gamma, (\text{Tr}^H\delta)\gamma) = \frac{\# H(\gamma, \delta)}{\# H_{\gamma}} I(\gamma, \delta) \in \mathbb{Z}$ for $\gamma, \delta \in |\Gamma_{R,c}|$,
  \item $I(\text{Tr}_H\gamma, (\text{Tr}_H\delta)\gamma) = \frac{\# H(\gamma, \delta)}{\# H_{\gamma}} I(\gamma, \delta) \in \mathbb{Z}$ for $\gamma, \delta \in |\Gamma_{R,c}|$.
  \item $(\text{Tr}^H\gamma)\gamma = \text{Tr}_H(\gamma)\gamma$ for $\gamma \in |\Gamma_{R,c}|$,
  \item $(\text{Tr}_H\gamma)^* = \text{Tr}^H(\gamma)^*$ for $\gamma \in |\Gamma_{R,c}|$.
\end{enumerate}

By definition of $\text{Tr}^H$, $\text{Tr}_H$ directly,

\begin{enumerate}
  \item $\text{Tr}^H(\gamma)^* - \text{Tr}^H\gamma = (\# H_{\gamma})k(\gamma) a$ for $\gamma \in |\Gamma|$,
  \item $\text{Tr}_H(\gamma)^* - \text{Tr}_H\gamma = k(\gamma) a$ for $\gamma \in |\Gamma|$.
\end{enumerate}

4. The sets $\text{Tr}^H|\Gamma_{R,c}|$ and $\text{Tr}_H|\Gamma_{R,c}|$ have naturally the structure of m.e.a.r. s.'s.

By this statement we mean the followings.

\begin{enumerate}
  \item On the sets $\text{Tr}^H|\Gamma|$ and $\text{Tr}_H|\Gamma|$ in $F^H$ we define structures of Dynkin diagrams according to the rule (8.2) ii) using the intersection numbers 3. iii), iv).
  \item Then one verifies that they become diagrams for affine root systems, denoted by $\text{Tr}^H\Gamma$ and $\text{Tr}_H\Gamma$ respectively.
  \item On the diagrams $\text{Tr}^H\Gamma$ and $\text{Tr}_H\Gamma$ we define the counting weights using 3 vii), viii) as follows.
\end{enumerate}

\[
\begin{cases}
  k(\text{Tr}^H a) := (H \cdot k)^{-1} \# H a \cdot k(a) & \text{for } a \in |\Gamma|, \\
  k(\text{Tr}_H a) := k(a) & \text{for } a \in |\Gamma|.
\end{cases}
\]

Here

\[(H \cdot k) := \text{g.c.d.}\{\# H a \cdot k(a) : a \in |\Gamma|\}.
\]

Then one verifies that the weighted diagrams $(\text{Tr}^H\Gamma, (k(\text{Tr}^H a))_a)$ and $(\text{Tr}_H\Gamma, \ldots)$.
(k(Tr_Hα))_a) belong to the table of (6.5) for weighted diagrams of m.e.a.r.s.'s.

iii) The coefficients n (3.3.6) for the diagrams Tr^H \Gamma and Tr_H^H \Gamma are calculated as follows.

\[
\begin{align*}
n_{Tr^H a} & := n_a & \text{ for } a \in |\Gamma|, \\
n_{Tr_H^H a} & := (H \cdot n)^{-1} H a \cdot n_a & \text{ for } a \in |\Gamma|.
\end{align*}
\]

Here (H \cdot n) = \text{g.c.d.}\{#H a \cdot n_a : a \in |\Gamma|\}.

By putting \(b^H := \sum n_{Tr^H a} \cdot Tr^H a\) and \(b_H := \sum n_{Tr_H^H a} \cdot Tr_H^H a\), we have

\[b^H = b, \quad b_H = (H \cdot n)^{-1} b\]

iv) We normalize the metric \(I\) in the following so that they define a minimal even lattice structure on \(Q(Tr^H \Gamma)\) and \(Q(Tr_H^H \Gamma)\). (cf. 3. i), ii).

\[
\begin{align*}
I^H & := (H \cdot I_R)^{-1} I_R, \\
I_H & := (H^{-1} \cdot I_R)^{-1} I_R.
\end{align*}
\]

Here

\[
(H \cdot I_R) := \text{g.c.d.}\{#H a \cdot I_R(a, a) / 2 : a \in |\Gamma|\},
\]

\[
(H^{-1} \cdot I_R) := \text{g.c.d.}\{(#H a)^{-1} \cdot I_R(a, a) / 2 : a \in |\Gamma|\}.
\]

v) Using the preceding ii), iii) and iv), the exponents for (Tr^H \Gamma, (k(Tr^H a))_a) and (Tr_H^H \Gamma, (k(Tr_H^H a))_a) are calculated as follows.

\[
\begin{align*}
m_{Tr^H a} & := m_a \frac{(H \cdot k)}{(H \cdot I_R)} & a \in |\Gamma|, \\
m_{Tr_H^H a} & := m_a \frac{1}{(H \cdot n)(H^{-1} \cdot I_R)} & a \in |\Gamma|.
\end{align*}
\]

vi) Particularly v) implies,

\[
\begin{align*}
(Tr^H \Gamma)_m &= Tr^H \Gamma_m, \\
(Tr_H^H \Gamma)_m &= Tr_H^H \Gamma_m. \quad \text{(Recall Definition (8.1)).}
\end{align*}
\]

vii) Together with the fact ii), we get,

\[
\begin{align*}
(Tr^H \Gamma)_m^* &= Tr^H \Gamma_m^*, \\
(Tr_H^H \Gamma)_m^* &= Tr_H^H \Gamma_m^*.
\end{align*}
\]

Here we put \(a^H := (H \cdot k)a\) and \(a_H := a\).
viii) The preceding i)~vii) altogether imply that the set $\Gamma^{H}\mid_{\Gamma^{R,G}}$ and $\Gamma_{H}\mid_{\Gamma^{R,G}}$ form Dynkin diagrams for some m.e.a.r.s.'s (cf. (8.2) for the definition), which we shall denote by $\Gamma^{H}\mid_{\Gamma^{R,G}}$ and $\Gamma_{H}\mid_{\Gamma^{R,G}}$ respectively.

5. Let us show that there exist m.e.a.r.s.'s, say $R^{H}$ and $R_{H}$ in $F^{H}$ belonging to $I\mid F^{H}$, so that $\Gamma^{H}\mid_{\Gamma^{R,G}}$ and $\Gamma_{H}\mid_{\Gamma^{R,G}}$ are the Dynkin diagrams for them respectively.

Put

$$R^{H} := \bigcup_{\gamma \in I\mid \Gamma_{R,G}} \langle \gamma, \alpha \rangle \in \Gamma^{H} | \gamma \rangle \Gamma_{R,G},$$

$$R_{H} := \bigcup_{\gamma \in I\mid \Gamma_{R,G}} \langle \gamma, \alpha \rangle \in \Gamma_{H} | \gamma \rangle \Gamma_{R,G}. $$

Then 1 and 2 iii) iv) v) vi) imply that $R^{H}$ and $R_{H}$ are root systems belonging to $I\mid F^{H}$ (cf. (1.3) Ex. 4). Since $I\mid F^{H}$ is positive semi-definite s.t. $\text{rad}(I\mid F^{H}) = \text{rad} I$, $R^{H}$ and $R_{H}$ are extended affine root systems by definition. The space $G := \mathbb{R}a$ defines markings for $R^{H}$ and $R_{H}$.

Let $(R^{H}, G^{H})$ and $(R_{H}, G_{H})$ be m.e.a.r.s.'s associated to the diagrams $\Gamma^{H}\mid_{\Gamma^{R,G}}$ and $\Gamma_{H}\mid_{\Gamma^{R,G}}$ respectively. (The constructions of them is given in (9.6)). Then, there exists natural isomorphisms

$$(R^{H}, G^{H}) \simeq (R_{H}, G^{H}),$$

and

$$(R_{H}, G_{H}) \simeq (R_{H}, G).$$

Proof. Define a map $\hat{F}_{R^{H}} : = \bigoplus_{\alpha \in \Gamma^{H} | \gamma \rangle \Gamma_{R,G}} \mathbb{R}a \rightarrow F^{H}$ by $\hat{a} \mapsto \alpha$. Then due to (9.4.5) and 3 vii), 4 vii), it is factorized by an isomorphism $(\hat{F}_{R^{H}})/\text{Image}(U - 1) \simeq F^{H}$, inducing $G^{H} := \mathbb{R}a \simeq G := \mathbb{R}a$.

Then the map induces a bijection,

$$R_{H} \simeq R^{H}$$

by definitions of them.

6. Since the Dynkin diagrams $\Gamma^{H}\mid_{\Gamma^{R,G}}$ and $\Gamma_{H}\mid_{\Gamma^{R,G}}$ are determined only by $H \subset \text{Aut}(I)$ (cf. 3. iii) iv)), the isomorphism classes of $(R^{H}, G)$ and $(R_{H}, G)$ are determined by $H$.

7. The formula 3 vi), vi) imply that, $(R^{H})^{\gamma} = (R^{\gamma})_{H}$ and $(R_{H})^{\gamma} = (R^{\gamma})_{H}$. 


These complete the proof of (12.3) **Lemma**.

**Definition.** Let the notations and the assumptions be the same as in the above Lemma.

We call the diagram $\text{Tr}^H \Gamma_{R,G}$ the folding of $\Gamma_{R,G}$ and $\text{Tr}_H \Gamma_{R,G}$ the mean folding of $\Gamma_{R,G}$ by the group $H$.

We shall not make no distinction between two (mean) foldings by $H$ and $H'$, if the decomposition of the set $|\Gamma_{R,G}|$ into the orbits of $H$ and that of $H'$ coincides each other.

(12.4) The following are the complete list of foldings and mean foldings of Dynkin diagrams for m.e.a.r.s.'s.

The orbits of the nodes of $\Gamma_{R,G}$ by the action of $H$ are drawn as nodes lying in the vertical position in the diagram. Thus if the action of $H$ is obvious from the drawn picture, we have not explicitly mentioned about the group $H$.

If two subgroups $H, H' \subset \text{Aut}(\Gamma')$ satisfying the assumption * of Lemma, commute each other and $H \cap H' = \{1\}$, then $\text{Tr}^H$ and $\text{Tr}_H$ commute. Let us denote by $\text{Tr}^H_{H'}$ the product $\text{Tr}^H \text{Tr}_H = \text{Tr}_H \text{Tr}^H$ and by $\to$ the correspondence of diagrams.

In each figure, the diagrams in antipodal positions are dual of each other.

**Table 1.** ($l \geq 2$)

\begin{align*}
\text{Tr}^{2}_2 & \quad \text{Tr}^{2}_2 \\
B^{[2,2)}_l & \quad A^{[2,1]}_{l,1} & \quad C^{[1,1]}_l
\end{align*}
In the table 2, $\mathcal{Z}_2$ means the group generated by an involution which fixes the nodes on the right terminals of the diagrams and $\mathcal{Z}_2'$ means the group generated by the involution which fixes the nodes on the left terminals of the diagrams.

In case $l=1$, we formally define $B_1^{(2,2)} := A_1^{(1,1)}$, $C_1^{(1,1)} := A_1^{(1,1)}$. 
Table 3. \((l \geq 2)\)

In the table 3, \(Z_2\) means the group generated by the transposition of the nodes on the right terminals of the diagrams and \(Z_2'\) means the group generated by the transposition of the nodes on the left terminals of the diagrams.
In the table 4, $\mathbb{Z}_2$ means a group generated by an involution which fixes the nodes in the right terminals of the diagrams and $\mathbb{Z}_2'$ means a group generated by a (product of two) transposition (s) which exchanges nodes on the left terminals of the diagrams.
Table 5.

\[ \begin{align*}
C^{(1,3)} & \xrightarrow{\text{Tr}_{z_2}} A^{(1,1,1)} & \xrightarrow{\text{Tr}_{z_2}} B^{(2,1,1)} \\
\text{Tr}_{z_2} & \xrightarrow{\text{Tr}_{z_2}} & \text{Tr}_{z_2} \\
\text{Tr}_{z_2} & \xrightarrow{\text{Tr}_{z_2}} & \text{Tr}_{z_2} \\
BC^{(2,4)} & \xrightarrow{\text{Tr}_{z_2} \times z_2} & \text{Tr}_{z_2} \\
4 & \xrightarrow{\text{Tr}_{z_2} \times z_2} & \text{Tr}_{z_2} \\
\text{Tr}_{z_2} & \xrightarrow{\text{Tr}_{z_2}} & \text{Tr}_{z_2} \\
C^{(1,3)} & \xrightarrow{\text{Tr}_{z_2}} & A^{(1,1,1)} \\
2 & \xrightarrow{\text{Tr}_{z_2}} & A^{(1,1,1)} \\
\end{align*} \]

Table 6.

\[ \begin{align*}
G^{(2,3)} & \xrightarrow{\text{Tr}_{z_2}} & D^{(1,1)} \\
3 & \xrightarrow{\text{Tr}_{z_2}} & 3 \\
\end{align*} \]

Table 7.

\[ \begin{align*}
G^{(2,3)} & \xrightarrow{\text{Tr}_{z_3}} & E^{(1,1)} \\
3 & \xrightarrow{\text{Tr}_{z_3}} & 3 \\
\end{align*} \]
Hierarchy among Dynkin diagrams.

The tables in (12.4) classify the diagrams for marked extended affine root systems into the following four groups.

I. The Dynkin diagrams, which have no multiple bonds.

\[ A_l^{(1,1)} \ (l \geq 1), \ B_l^{(1,1)} \ (l \geq 4), \ E_6^{(1,1)}, \ E_7^{(1,1)}, \ E_8^{(1,1)}. \]

These diagrams are characterized as diagrams which can not be expressed neither as a folding nor as a mean folding of some other diagrams.

II. The Dynkin diagrams of type \( P^{(t_1, t(P))} \) (for \( t(P) > 1, t_1 \mid t(P) \)).

\[ B_l^{(1,2)} \ (l \geq 3), \ B_l^{(2,2)} \ (l \geq 2), \ C_l^{(1,2)} \ (l \geq 2), \ C_l^{(2,2)} \ (l \geq 3), \]
\[ BC_l^{(2,4)} \ (l \geq 1), \ F_l^{(1,2)}, \ F_l^{(2,2)}, \ G_2^{(3,3)}. \]

These diagrams are characterized as diagrams which can be obtained by a folding of the diagrams of the group I.

III. The Dynkin diagrams of type \( P^{(t_1,1)} \) (for \( t(P) > 1, t_1 \mid t(P) \)).

\[ B_l^{(1,1)} \ (l \geq 3), \ B_l^{(2,1)} \ (l \geq 2), \ C_l^{(1,1)} \ (l \geq 2), \ C_l^{(2,1)} \ (l \geq 3), \]
These diagrams are characterized as diagrams, which can be obtained by mean foldings of the diagrams of the group $I$.

IV. The Dynkin diagrams of exceptional types.

$A_{i}^{[1,1]e}, B_{i}^{[2,2]e} (i \geq 2), C_{i}^{[1,1]e} (i \geq 2), BC_{i}^{[2,2]}(1) (i \geq 2), BC_{i}^{[2,2]}(2) (i \geq 1)$.

These diagrams are characterized as diagrams, which can be obtained by foldings of the diagrams of the group $III$ and also by mean foldings of the diagrams of the group $II$. In the other words, diagrams of this group can be obtained from the diagrams of the group $I$ by a succession of a folding and a mean folding which are commutative.

As a summary, let us give a table of hierarchy relations among the Dynkin diagrams.

(12.6) Assertion. Let us give formulae for calculating the tier numbers and the exponents, for the foldings and mean foldings.

i) The total tier number.

$$t(Tr^H \Gamma_{R,G}) = t(\Gamma_{R,G}) \frac{1}{(H^{-1} \cdot I_R)(H \cdot I_R)}.$$
The first tier number.

\[ t_1(T_{\mathcal{H}G}) = t_1(\Gamma_{R,G}) \frac{1}{(H^{-1} \cdot I_R)(H \cdot I_R^\vee)}. \]

ii) The second tier number.

\[ t_2(T_{\mathcal{H}G}) = t_2(\Gamma_{R,G}) \frac{1}{(H \cdot k)(H^{-1} \cdot I_R^\vee)}, \]
\[ t_2(T_{\mathcal{H}G}) = t_2(\Gamma_{R,G}) \frac{(H \cdot k^\vee)}{(H \cdot I_R^\vee)}. \]

iii) The exponents

\[ m_{T_{\mathcal{H}G}} = m_{\mathcal{G}} \frac{(H \cdot k)}{(H \cdot I_R)}, \]
\[ m_{T_{\mathcal{H}G}} = m_{\mathcal{G}} \frac{1}{(H \cdot n)(H^{-1} \cdot I_R)}. \]

Proof. Similarly as in the proof of (12.3) Lemma we calculate as follows.

\[ k^\vee((T_{\mathcal{H}G})^\vee) = k^\vee(a^\vee), \]
\[ k^\vee((T_{\mathcal{H}G})^\vee) = (H \cdot k^\vee)^{-1} H \cdot a^\vee, \]
\[ n^\vee((T_{\mathcal{H}G})^\vee) = (H \cdot n^\vee)^{-1} H \cdot n^\vee, \]
\[ I_{(H \cdot n^\vee)^\vee} = I_{(H \cdot n^\vee)^\vee}, \]
\[ I_{(H \cdot n^\vee)^\vee} = I_{(H \cdot n^\vee)^\vee}. \]

Then applying these to the formulae for tier numbers (4.2) iii), (4.5.1) and (6.3.1) and for the exponents (7.1.1), we obtain the formulae in Assertion. q.e.d.

(12.7) Corollary. Let the notations be as before. Then we have the following proportionality.

\[ \frac{m_{T_{\mathcal{H}G}}}{m_{\mathcal{G}}} = \frac{t_2(T_{\mathcal{H}G} \Gamma_{R,G})}{t_2(\Gamma_{R,G})} \quad \text{for} \quad \alpha \in \Gamma, \]
\begin{equation}
\frac{m^*_R \mathfrak{v}}{m^*_a} = \frac{t_1(T^{\mathfrak{v}}_R \Gamma_R \mathfrak{v}, G)}{t_1(\Gamma_R \mathfrak{v}, G)} \text{ for } \mathfrak{v} \in |\Gamma|.
\end{equation}

\textbf{Proof.} Use ii), iii) and iv) of the Assertion. One may also check the proportionality, directly from the tables of (12.4).

These proportionalities are the last statements of this paper.

\textbf{References}

Added in Proof.

1. The complete intersection of quadric cones defines a sequence of singularities, called $\tilde{D}_t$ (see F. Knörrer [8]). Recently in a private letter [34], W. Ebeling has informed to the author that he found a strongly distinguished basis of the middle homology group of the Milnor fiber of the singularities, which are intersecting in the form of the diagram as defined in (8.2) Definition of the present paper and that the Milnor's monodromies of the singularities are identified with the Coxeter transformations as defined in (9.7). (cf. also [31])

2. It might be worthwhile to notice that the Lemmas A, B of the present paper is an analog of a result of Coleman [36], who has calculated systematically the Betti numbers (exponents) for simple Lie groups, by showing the existence of regular eigen vectors for a Coxeter transformation. Another analog of the result for the case of indefinite root systems will be shown in a forthcoming paper.