On the Algebraic $K$-Cohomology of Lens Spaces

*Dedicated to Professor Nobuo Shimada on his 60th birthday*

By

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§ 1. Introduction

Let $F_q$ be a finite field of order $q=p^d$ and $b_{F_q}$ be the 0-connected spectrum of algebraic $K$-theory for $F_q$. Then the homotopy groups of $b_{F_q}$ are

$$
\pi_{2k}(b_{F_q}) = 0,
$$

$$
\pi_{2k-1}(b_{F_q}) = \begin{cases} Z/(q^k-1) & \text{if } k>0, \\ 0 & \text{if } k\leq 0. \end{cases}
$$

Let $l$ be an odd prime number ($l \neq p$) and $L^n(l)$ the standard $2n+1$ dimensional lens space $S^{2n+1}/(Z/l)$. We write $L^n(l)$ for its $2n$-skeleton.

The cohomology groups $b_{F_q}(L^n(l))$ were studied by G. Nishida [6] in a special case. The purpose of the paper is to determine the cohomology group $b_{F_q}(L_0^n(l))$. The main theorem is Theorem 5.2.

This paper is organized as follows:
In Section 2 we state the splitting of algebraic $K$-theory for a finite field. In Sections 3 and 4, we study the topological $K$-group of a lens space and its generators. In the last section we state the main theorem and prove it.

§ 2. Splitting of $b_{F_q}$

Denote by $A$ the ring of $l$-adic integers $Z_l$. By $X_A$ we denote the $l$-adic completion of a spectrum $X$. Let $K$ (resp. $bu$) be the periodic (resp. 1-connected) spectrum which represents topological $K$-theory.
Definition 2.1. Let $\rho \in A$ be a primitive $(l-1)$-th root of unity. Then for $1 \leq i \leq l-1$ we define $\Phi_i: K_{l-1} \to K_{l-1}$ by

$$\Phi_i = \frac{1}{l-1} \sum_{i=1}^{l-1} \rho^{-mi} \psi^m,$$

where $\psi^k$ is the Adams operation.

Then splitting of topological $K$-theory is as follows (see [1]):

Theorem 2.2. We have

(i) $\sum_{i=1}^{l-1} \Phi_i = id$,

(ii) $\Phi_i \Phi_j = \begin{cases} \Phi_i & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$,

(iii) $K_{l-1}(-) \cong \bigoplus_{i=1}^{l-1} \Phi_i K_{l-1}(-)$ and

(iv) $b_{l-1}(-) \cong \bigoplus_{i=1}^{l-1} \Phi_i b_{l-1}(-)$.

Then $\Phi_i K_{l-1}(-)$ (resp. $\Phi_i b_{l-1}(-)$) is a generalized cohomology theory and we write $G_i$ (resp. $g_i$) for its spectrum. Theorem 2.2 (iii) and (iv) imply

(iii)' $K_{l-1} \cong \bigvee_{i=1}^{l-1} G_i$ and

(iv)' $b_{l-1} \cong \bigvee_{i=1}^{l-1} g_i$.

Let $\iota_i: g_i \to b_{l-1}$ and $\pi_i: b_{l-1} \to g_i$ be the canonical inclusion and projection of the splitting (iv)'.

Let $b_{F_q}$ be the 0-connected spectrum which represents the algebraic $K$-theory for a finite field $F_q$. By Fiedorowicz-Priddy [3], we know that $(b_{F_q})_A$ is the homotopy fibre of $1-\psi^q: b_{l-1} \to b_{l-1}$ where $\psi^q$ is the Adams operation.

Definition 2.3. Let $(1-\psi^q)_i: g_i \to g_i$ be the composition $g_i \xrightarrow{\iota_i} b_{l-1} \xrightarrow{1-\psi^q} b_{l-1} \xrightarrow{\pi_i} g_i$ and $g_{F_q,i}$ the homotopy fibre of $(1-\psi^q)_i$. Let $r$ be the least positive integer such that $q^r \equiv 1 \pmod{l}$.

Then we have the splitting of $(b_{F_q})_A$ as follows (see [4]):

Theorem 2.4.

(i) $(b_{F_q})_A \cong \bigvee_{\substack{1 \leq i \leq l-1 \atop i=0 \pmod{r}}} g_{F_q,i}$,
\[ \pi_{2k}(g_{F_x}) = 0, \]
\[ \pi_{2k-1}(g_{F_x}) = \begin{cases} 
\mathbb{Z}/(q^k - 1) \otimes A & \text{if } k > 0 \text{ and } k \equiv i \pmod{l-1} \\
0 & \text{otherwise},
\end{cases} \]

§ 3. Topological K-Group of Lens Spaces

Let \( \eta \) be the canonical complex line bundle of \( L^n(l) \) and put \( x = \eta - 1 \). By [1] we denote the greatest integer which is less than or equal to \( t \).

Then the topological \( K \)-group of lens spaces is as follows:

**Theorem 3.1.** (Kambe [5]) Let \( M_i \) (\( 1 \leq i \leq l-1 \)) be a cyclic group generated by \( x^i \) of order \( d_i^{(b)} = I \left( (n+1-i)/(l-1) \right) \). Then

(i) \( K^0(L_0(l)) = \mathbb{Z}[x]\langle (1+x)^i - 1, x^{*+1} \rangle \)
\[ \cong \mathbb{Z} \oplus M_1 \oplus M_2 \oplus \cdots \oplus M_{l-1} , \]
(ii) \( K^1(L_0(l)) = 0. \)

We define the filtration \( F^k \tilde{K}^0(L_0(l)) \) of \( \tilde{K}^0(L_0(l)) \) as follows:

**Definition 3.2.** Let \( L_0(l) \to L_g(l) \) (\( 0 \leq k \leq n \)) be the canonical inclusion, then we put
\[ F^k \tilde{K}^0(L_0(l)) = \begin{cases} 
\tilde{K}^0(L_0(l)) & \text{if } k < 0 , \\
\ker (\tilde{K}^0(L_0(l)) \to \tilde{K}^0(L_g(l))) & \text{if } 0 \leq k < n , \\
0 & \text{if } k \geq n .
\end{cases} \]

Since Atiyah-Hirzeburch spectral sequence of \( K^*(L_0(l)) \) collapses, we have:

**Corollary 3.3.** The bu-cohomology group of lens space is

(i) \( \tilde{b}^k(L_0(l)) = F^k \tilde{K}^0(L_0(l)), \)
(ii) \( \tilde{b}^{k+1}(L_0(l)) = 0. \)

By the cohomology exact sequence of the fibration \( (b_{F_x})_A \to bu_A \xrightarrow{1 - \psi^q} bu_A \) and \( \tilde{b}^k_{F_x}(L_0(l)) \cong (b_{F_x})_A^* (L_0(l)) \), we have the following exact sequence:
\[ 0 \to \tilde{b}^k_{F_x}(L_0(l)) \to \tilde{b}^k_{F_x}(L_0(l)) \xrightarrow{(1 - \psi^q)^*} \tilde{b}^{k+1}_{F_x}(L_0(l)) \to 0 . \]

The Bott periodicity and (3.3) imply the diagram
\[ \begin{array}{ccc}
\tilde{b}^{2k}_{F_x}(L_0(l)) & \xrightarrow{(1 - \psi^q)^*} & \tilde{b}^{2k}_{F_x}(L_0(l)) \\
\| & \| & \| \\
F^k \tilde{K}^0(L_0(l)) & \xrightarrow{F^k \tilde{K}^0(L_0(l))} & F^k \tilde{K}^0(L_0(l))
\end{array} \]
commutes. Then we have

**Proposition 3.4.** The $b_{p^k}$-cohomology group of lens space is

(i) $\tilde{B}_{p^k}^2(L^n(l)) \cong \text{Ker} \left( \left( 1 - \frac{1}{q^k} \psi^q \right)^* : F^*\tilde{K}^0(L^n(l)) \to F^*\tilde{K}^0(L^n(l)) \right)$,

(ii) $\tilde{B}_{p^k}^{2k+1}(L^n(l)) \cong \text{Coker} \left( \left( 1 - \frac{1}{q^k} \psi^q \right)^* : F^*\tilde{K}^0(L^n(l)) \to F^*\tilde{K}^0(L^n(l)) \right)$.

Recall that $K^0(L^n(l)) = \mathbb{Z}[x]/((1+x)^l-1, x^{s+1})$ and $(\psi^q)^*x = (1+x)^q-1$. To the generators $x, x^2, \ldots, x^{l-1}$ of $\tilde{K}^0(L^n(l))$, the action of $\left( 1 - \frac{1}{q^k} \psi^q \right)^*$ is

$$\left( 1 - \frac{1}{q^k} \psi^q \right)^* x^i = x^i - \frac{1}{q^k} \{ (x+1)^q-1 \}^i.$$ 

§ 4. On Generators of $\tilde{K}^0(L^n(l))$

To compute the kernel and cokernel of $\left( 1 - \frac{1}{q^k} \psi^q \right)^*$, we define new generators of $\tilde{K}^0(L^n(l))$.

**Definition 4.1.** We define the element $\xi_i$ ($1 \leq i \leq l-1$) of $\tilde{K}^0(L^n(l))$ by $\xi_i = \Phi_i(x)$ and put $N_i = \text{Im} \left( \Phi_i : \tilde{K}^0(L^n(l)) \to \tilde{K}^0(L^n(l)) \right)$.

Then we have

**Theorem 4.2.** Let $a^{(n)}_l$ be the integer defined in Theorem 3.1, then

(i) $K^0(L^n(l)) \cong A \oplus N_1 \oplus N_2 \oplus \cdots \oplus N_{l-1}$,

(ii) $N_i$ is a cyclic group generated by $\xi_i$ of order $a^{(n)}_l$.

**Proof** of (i) is clear by Theorem 2.2. To prove (ii) we need following two lemmas.

By $y_i (i \in \mathbb{Z}/l)$ we denote the element of $\tilde{K}^0(L^n(l))$ such that $y_i = (1+x)^i-1$. This notation is well defined since $(1+x)^l=1$.

When $k$ is an element of $\mathbb{Z}$ or $A$ we write $\bar{k}$ for the mod $l$ reduction of $k$. Then $(\psi^q)^*x = y_{\bar{k}}$. Thus we can regard that the Adams operation $(\psi^q)^* : \tilde{K}^0(L^n(l)) \to \tilde{K}^0(L^n(l))$ is defined for $k \in \mathbb{Z}/l$.

Let $k$ be an element of $A$ such that $k \equiv 0 \pmod{l}$. Then there exists one and only one element $m$ of $A$ such that $m^{l-1} = 1$ and $k \equiv m \pmod{l}$. We write $\bar{k}$ for the element $m$. Then we have

**Lemma 4.3.** $\Phi_i(y_k) = \bar{k}^i \xi_i$ for $1 \leq k \leq l-1$.

**Proof.** By definition
\[
\Phi_i(y_k) = \frac{1}{l-1} \sum_{m=1}^{l-1} \rho^{-mi} \psi^m \psi^k(x) = \frac{1}{l-1} \sum_{m=1}^{l-1} \rho^{-mi} \psi^m \bar{k}(x) .
\]

Since \(\{\rho \tilde{k}, \rho^{2} \tilde{k}, \cdots, \rho^{l-1} \tilde{k}\} = \{\rho, \rho^{2}, \cdots, \rho^{l-1}\}\), we have
\[
\Phi_i(y_k) = \frac{1}{l-1} \sum_{m=1}^{l-1} \rho^k \rho^{-mi} \psi^m(x) = \frac{1}{l-1} \sum_{m=1}^{l-1} m^{-i} \{(1+x)^m - 1\} .
\]

Since \(\psi^k\) commutes with \(\Phi_i\) we have the following corollary:

**Corollary 4.4.** \(\psi^k(\xi_i) = \bar{k}^i \xi_i\) for \(1 \leq i \leq l-1\).

**Lemma 4.5.** Let \(\bar{\Phi}_i(x)\) be the mod 1 reduction of \(\Phi_i(x)\) and put \(\bar{\Phi}_i(x) = c_1 x + c_2 x^2 + \cdots + c_{l-1} x^{l-1}\) (\(c_k \in \mathbb{Z}/l\)). Then

(i) \(c_k = 0\) for \(1 \leq k < i\), and
(ii) \(c_i \neq 0\).

**Proof.** Since \(\{\rho^1, \rho^2, \cdots, \rho^{l-1}\} = \{1, 2, \cdots, l-1\}\) we have
\[
\bar{\Phi}_i(x) = \frac{1}{l-1} \sum_{m=1}^{l-1} m^{-i} \{(1+x)^m - 1\} .
\]

Inductively we define \(f_k(x)\) by \(f_0(x) = \bar{\Phi}_i(x)\) and \(f_k(x) = (1+x) \frac{d}{dx} f_{k-1}(x)\). Then, for \(1 \leq k \leq i\)
\[
f_k(x) = \frac{1}{l-1} \sum_{m=1}^{l-1} m^{-i+k}(1+x)^m .
\]

Therefore
\[
k^1 c_k = f_k(0) = \frac{1}{l-1} \sum_{m=1}^{l-1} m^{-i+k}
\]
\[
= \begin{cases} 
0 & \text{if } 1 \leq k < i \\
1 & \text{if } k = i
\end{cases}
\]

**Proof of Theorem 4.2(ii).** It is clear that \(\bar{K}^0(L^i(l))\) is generated by \(y_1, y_2, \cdots, y_{l-1}\). Then \(N_i = \Phi_i(\bar{K}^0(L^i(l)))\) is generated by \(\Phi_i(y_1), \Phi_i(y_2), \cdots, \Phi_i(y_{l-1})\). By Lemma 4.3, \(N_i\) is a cyclic group generated by \(\xi_i = \Phi_i(x)\). By Lemma 4.5, the mod 1 reduction of \(\xi_i\) is
\[
\bar{\xi}_i = c_1 x^l + c_1 x^{l-1} + \cdots + c_{l-1} x^{l-i-1} (c_i \neq 0) .
\]

Theorem 3.1 implies that the order \(a_k^{(n)}\) of \(x^k\) \((1 \leq k \leq l-1)\) has the following two properties:
(i) \(a_1^{(n)} \geq a_2^{(n)} \geq \cdots \geq a_{l-1}^{(n)}\) and
(ii) \(a_j^{(n)} a_k^{(n)} = l\) or 1 if \(j < k\).
Therefore the order of $\xi_i$ is that of $x'$. This completes the proof of Theorem 4.2 (ii).

§ 5. $b_{F_q}$-Cohomology of Lens Spaces

Let $\nu_i: \mathbb{A} \to \mathbb{Z} \cup \{\infty\}$ be the $l$-adic valuation, that is $\nu_i(\lambda)$ is the largest integer $\nu$ such that $l^\nu$ divides $\lambda$ where $\lambda$ is an element of $\mathbb{A}$.

**Lemma 5.1.** Let $\xi_i$ be the element of $\tilde{R}^0(L^0_0(l))$ defined in (4.1), then

\begin{enumerate}
  \item \((1 - \frac{1}{q^i} q^i)^* \xi_i = \left(1 - \frac{q^i}{q^i}\right) \xi_i\),
  \item \(1 - \frac{q^i}{q^i}\) is a unit of $\mathbb{A}$ if $i \equiv k \pmod{r}$, and
  \item $\nu_i \left(1 - \frac{q^i}{q^i}\right) = \nu_i(q^r - 1) + \nu_i(k)$ if $i \equiv k \pmod{r}$.
\end{enumerate}

**Proof.** By Corollary 4.4, (i) is clear. (ii) holds since $q^k \equiv q^i \equiv q^i \pmod{l}$. To prove (iii), assume $i \equiv k \pmod{r}$. Since $q^i = q^k$ and $\nu_i \left(1 - \frac{q}{q}\right) \geq 1$, we have

$$\nu_i \left(1 - \frac{q^i}{q^i}\right) = \nu_i \left(1 - \left(\frac{q}{q}\right)^k\right) = \nu_i \left(1 - \frac{q}{q}\right) + \nu_i(k).$$

On the other hand

$$q^r - 1 = q^r - q^r = (q - q) (q^{r-1} + q^{r-2} q + \cdots + q^1),$$

where

$$q^{r-1} + q^{r-2} q + \cdots + q^1 \equiv q^{r-1} + q^{r-1} + \cdots + q^1 = q^{r-1} \equiv 0 \pmod{l}.$$ 
Therefore

$$\nu_i \left(1 - \frac{q}{q}\right) = \nu_i(q - q) = \nu_i(q^r - 1).$$

This completes the proof of the lemma.

**Theorem 5.2.** The $b_{F_q}$-cohomology of lens space is

$$\tilde{b}_{F_q}^n(L^0_0(l)) \cong \bigoplus_{1 \leq i \leq l-1} \mathbb{Z} / (l^i \mathbb{Z})$$

where $m_i \ (for \ i \equiv k \pmod{r})$ is the integer defined as follows:

$$m_i = \begin{cases} 
\min \{\nu_i(a_i^{(n)}), \nu_i(q^r - 1) + \nu_i(k)\} & \text{if } k \leq 0, \\
\min \{\nu_i(a_i^{(n)}) - \nu_i(a_i^{(r)}), \nu_i(q^r - 1) + \nu_i(k)\} & \text{if } 0 < k < n, \\
0 & \text{if } k \geq n.
\end{cases}$$
Proof. Since \( \{a_i^k\xi_i\}_{1 \leq i \leq l-1} \) is a basis of \( F^k\tilde{K}^0(L^0(l)) \), it is easy from Proposition 3.4, Theorem 4.2 and Lemma 5.1.

References


