A Remark on the Segal-Becker Theorem

Dedicated to Professor Minoru Nakaoka on his 60th birthday

By
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§ 1. Introduction

Let $CP^\infty$ be the infinite dimensional complex projective space and $BU$ the classifying space of stable complex vector bundles. Then there is the natural inclusion $j: CP^\infty \to BU$ and the structure map of the infinite loop space structure defined by the Bott periodicity $\xi: Q(BU) \to BU$ where $Q(\ ) = \text{Colim}_n \Omega^n\Sigma^n(\ )$. Let $\lambda: Q(CP^\infty) \to BU$ be the composition $\xi \circ Q(j)$. The results of Segal [8] and Becker [2] show us that there exists a map $s: BU \to Q(CP^\infty)$ such that $\lambda \circ s = \text{id}$.

The main result of this paper is to show that one can take $s$ satisfying that $s \circ j \simeq \text{inclusion}: CP^\infty \to Q(CP^\infty)$.

To show this, we will use the results of Brumfiel–Madsen [4] for the evaluation of the transfer map.

§ 2. The Construction of the Splitting

Let $U(n)$ be the unitary group and $T^n$ its maximal torus. Let $NT^n$ be the normaliser of $T^n$ in $U(n)$. We also define homogeneous spaces of $U(2n)$:

$$E_n = U(2n)/U(n) \quad \text{and} \quad E'_n = U(n+1)/U(n).$$

Then the construction of splitting in [2] can be reformulated as follows.

Let $r: Q(X_+) \to Q(X)$ be the map induced by the canonical projection and $a: Q(X) \to Q(X_+)$ the right adjoint of $r$. Let $t_n: E_n/U(n)_+$

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$\longrightarrow Q(E_n/NT^n_*)$ the Becker–Gottlieb transfer ([2], [3]) associated with the smooth fiber bundle

$$U(n)/NT^n \longrightarrow E_n/NT^n \longrightarrow E_n/U(n).$$

$E_n/T^n$ has the action of $NT^n/T^n = \Sigma_n$ which sends $eT^n$ to $enT^n$ where $e \in E_n$ and $n \in NT^n$. $(X)^n$ is also a $\Sigma_n$–space by the permutation of the coordinates.

Since the elements of $E_n$ can be considered as the $n$–frames in $C^{2n}$, we define a $\Sigma_n$–equivariant map

$$h_n : E_n/T^n \longrightarrow (CP^{2n-1})^n$$

by corresponding each vector to its representative element in $CP^{2n-1}$.

Also, since $E_n/T^n \longrightarrow E_n/NT^n$ is a principal $\Sigma_n$–bundle, there is a $\Sigma_n$–equivariant map

$$\epsilon_n : E_n/T^n \longrightarrow E\Sigma_n$$

which covers the classifying map of this principal bundle where $E\Sigma_n$ is the contractible free $\Sigma_n$–space. Thus we obtain a map

$$k_n = (\epsilon_n \times h_n) / \Sigma_n : E_n/NT^n \longrightarrow (E\Sigma_n \times (CP^{2n-1})^n) / \Sigma_n.$$

There is also the Barratt–Quillen map

$$w_n : (E_n \times (X))^n / \Sigma_n \longrightarrow Q(X_+).$$

Notice that the composition $X \overset{i_1}{\longrightarrow} (E\Sigma_n \times (X))^n / \Sigma_n \overset{\epsilon_n}{\longrightarrow} Q(X_+)$ is homotopic to the composition $X \overset{\text{incl.}}{\longrightarrow} Q(X) \overset{\epsilon}{\longrightarrow} Q(X_+)$ where $i_1$ is the map defined by the equation

$$i_1(x) = (\ast_{E_n}, (x, \ast_x, \ast_x, \cdots, \ast_x)) \text{ for } x \in X.$$

So the following Lemma is clear.

**Lemma 2.1.** The composition

$$CP^n = E_n / S^1 \longrightarrow E_n / NT^n \overset{w_n \ast k_n}{\longrightarrow} Q(CP^{2n-1})$$

is homotopic to the composition $CP^n \longrightarrow CP^n \overset{\text{incl.}}{\longrightarrow} Q(CP^{2n-1})$.

**Remark.** One can easily show that the composition $w_n \circ k_n : E_n / NT^n \longrightarrow Q(CP^{2n-1})$ agrees with the composition of the Kahn–Priddy pre-transfer $t : E_n / NT^n \longrightarrow Q(E_n / NT^n \times S^n)$ associated with the $n$–fold covering $E_n / NT^n \times S^n \longrightarrow E_n / NT^n$ and the map $Q(E_n / NT^n \times S^n) \longrightarrow Q(CP^{2n-1})$ which is induced from the quotient map. (Compare
Now we are ready to define the splitting $s$. Let us consider the composition

$$s_n : E_n/U(n) \to E_n/U(n) + t_n \to Q(E_n/NT^*_+) \to Q(CP^{2n-1})$$

where $w_n \circ k_n$ is the pointed extension of $w_n \circ k_n$ and $\zeta$ is the structure map of the infinite loop space $Q(CP^{2n-1})$. As in [2] and [9], $t_n$ is compatible with $n$. So, since all the constructions are compatible with $n$, by taking the limit, we obtain $s : BU \to Q(CP^n)$.

§ 3. The Proof of the Main Result

By virtue of (2.1), we have only to prove that the diagram

$$E_n'\to E_n'/S^1_+ \xrightarrow{\text{incl.}} Q(E_n'/S^1_+)$$

$$E_n/U(n) + t_n \to Q(E_n/NT^*_+)$$

commutes up to homotopy where the vertical maps are induced from the inclusion $E_n' \to E_n$.

We need the evaluation of the transfer.

**Proposition 3.1.** The following diagram is homotopy commutative;

$$E_n/T^*_+ \to E_n/NT^*_+$$

where the maps with no name are induced from the canonical projections.

This proposition is a corollary of Brumfiel and Madsen [4]. (See Theorem 3.5 of [4].)

Since the diagram

$$E_n'/S^1_+ \to Q(E_n'/S^1_+)$$

$$E_n/T^*_+ \xrightarrow{\text{incl.}} Q(E_n/T^*_+)$$

$$E_n/NT^*_+ \xrightarrow{\text{incl.}} Q(E_n/NT^*_+)$$
commutes up to homotopy, we get the main result:

\[ CP^n - E_n/S^1 \xrightarrow{\text{incl}} E_n/U(n) \xrightarrow{s_n} Q(CP^n) \]

commutes up to homotopy.

Thus \( s \circ j \) is homotopic to the canonical inclusion as an element of \( \lim_n \text{Map}(CP^n, Q(CP^n)) \). Then \( \lambda \circ s \circ j \) is homotopic to \( j \) on the finite skeleton. So one can easily show that \( \lambda \circ s : BU \to BU \) induces identities on the \( K \)-homology groups and on the \( K \)-cohomology groups, by using the fact that \( s \) is an \( H \)-map. (See [9].) Thus our \( s \) is a splitting.

Let \( P^m(\ ) \) be the \( m \)-th term of the cohomology defined by \( Q(CP^n) \). Then we have the Milnor exact sequence

\[ 0 \to \lim_i P^{-1}(CP^n) \to P^0(CP^n) \to \lim_i P^0(CP^n) \to 0. \]

As in [5], one can easily prove that \( P^{-1}(CP^n) \) is finite. So \( \lim_i - \) term vanishes and we have the main theorem:

**Theorem 3.2.** The composition

\[ s \circ j : CP^\infty \to BU \to Q(CP^\infty) \]

is homotopic to the canonical inclusion.

**References**


