Fourier Integral Operators in Gevrey Class on $\mathcal{R}^n$ and the Fundamental Solution for a Hyperbolic Operator

By

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Introduction

Consider a hyperbolic operator

\[ L = D_t^m + \sum_{j=0}^{m-1} \sum_{|\alpha| \leq m-j} a_{j,\alpha}(t, x) D_j D_x^\alpha \text{ on } [0, T] \]

with constant multiplicity, where $a_{j,\alpha}(t, x)$ are functions in the Gevrey class of order $d(>1)$, that is, they satisfy

\[ |\partial_t^j \partial_x^\alpha a_{j,\alpha}(t, x)| \leq C M^{-k(1+|\beta|)k} |\xi|^{m-|\alpha|} \text{ for } (t, x) \in [0, T] \times \mathcal{R}^n. \]

The purpose of the present paper is to construct the fundamental solution $E_\alpha(t, s)$ of the Cauchy problem

\[ \begin{cases} L u = 0 & t > 0, \\ \partial_t u(0) = g_{j, s} & j = 0, 1, \ldots, m-1, \end{cases} \]

and obtain the result on the propagation of singularities for a solution $u(t)$ of (2).

To investigate the above problem we introduce the following symbol classes as subclasses of a symbol class $\mathcal{S}^m$ studied in [12]. In the following we tacitly use the notation in [12].

Definition $\mathcal{S}$. i) We say that a symbol $\rho(x, \xi) (\in \mathcal{S}^m)$ belongs to a class $S^{\alpha, \beta}_m$ if

\[ |\rho^{(\alpha)}(x, \xi)| \leq C M^{-k(1+|\beta|)k} |\xi|^{m-|\alpha|} \]

hold with constants $C$ and $M$ independent of $\alpha$ and $\beta$.  

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ii) We say that a symbol \( p(x, \xi) \) belongs to a class \( S_{G(d,1)} \) if \( p(x, \xi) \) belongs to \( S_{G(d)} \) and satisfies for constants \( C, M \) and \( \mu \) independent of \( \alpha, \beta \)

\[
|p_{\beta \alpha}^{(0)}(x, \xi)| \leq CM^{-(|\alpha|+|\beta|)}\alpha!\beta!\xi^{m-|\alpha|} \quad \text{for} \quad |\xi| \geq \mu.
\]

iii) We say that a symbol \( p(x, \xi) (\in S^{-\infty}) \) belongs to a class \( \mathcal{R}_{G(d)} \) if for any \( \alpha \) there exists a constant \( C_\alpha \) such that

\[
|p_{\beta \alpha}^{(0)}(x, \xi)| \leq C_\alpha M^{-(|\beta|+N)}\beta!|\xi|^{m-|\alpha|} \quad \text{for} \quad |\xi| \geq \mu.
\]

hold for any \( \beta \) and \( N \) with a constant \( M \) independent of \( \alpha, \beta \) and \( N \).

**Definition (T).** Let \( \mathcal{A} \) be a subset of an Euclidian space \( \mathbb{R}^n \). We say that a symbol \( p(t, x, \xi) \) in \( \mathcal{A} \times \mathbb{R}^{2n} \) belongs to \( M_1(\mathcal{S}^n_{G(d)}) \) if for any \( \alpha \) and \( \beta \) \( p_{\beta \alpha}^{(0)}(t, x, \xi) \) is a \( C^1 \)-function and for any fixed \( t \in \mathcal{A} \) the symbol \( \partial_t^\alpha p(t, x, \xi) (|\gamma| \leq \gamma) \) satisfies (3) with \( C \) and \( M \) independent also of \( t \). We also set \( M_1(\mathcal{S}^n_{G(d)}) = \bigcap_t M_1(S^m_{G(d)}) \).

In the same way we define the classes \( M_1(\mathcal{S}^m_{G(d,1)}), M_1(\mathcal{R}_{G(d)}) \) and \( M_1(\mathcal{R}_{G(d)}) \), which correspond to \( S_{G(d,1)}^m \) and \( \mathcal{R}_{G(d)} \). Using these symbol classes we reduce the problem (2) to the problem

\[
\begin{aligned}
\mathcal{L}U &= 0, \\
U(0) &= G
\end{aligned}
\]

for the perfectly diagonalized operator

\[
\mathcal{L}_0 = D_t - \begin{bmatrix}
\lambda_1(t, X, D_x) \mathcal{I}_1 & 0 \\
0 & \ddots & \ddots \\
& \ddots & \lambda_r(t, X, D_x) \mathcal{I}_r \\
B_1(t) & 0 & \ddots & + R_s(t) \\
0 & B_r(t)
\end{bmatrix}
\]

under the condition that (1) is a hyperbolic operator with constant multiplicity (c.f. Proposition 3.4). Here, \( \lambda_i(t, x, \xi) \) belong to \( M_1(\mathcal{S}^m_{G(d)}) \), \( \mathcal{I}_i \) is an identity matrix, \( B_j(t) \) are \( l_j \times l_j \) matrices of pseudo-differential operators with symbols in \( M_1(\mathcal{S}^m_{G(d)}) \) (\( 0 \leq r \leq (r-1)/r \)) and
$R_\varepsilon(t)$ is a matrix of pseudo-differential operators with symbols in $\mathcal{R}_{G(d)}$. Note that from (5), for any $t$, $R_\varepsilon(t)$ maps a class $\mathcal{E}'$ of distributions with compact supports to a class $\gamma^d$ of functions in the Gevrey class of order $d$. This result shows that in order to study the problem (2) for (1) it is sufficient to construct the fundamental solution $E(t, s)$ for the operator

$$\mathcal{L} = D_t - \lambda(t, X, D_x) + b(t, X, D_x)$$

on $[0, T]$ with $\lambda(t, x, \xi) \in M^0_\sigma(S_{G(d)})$ and $b(t, x, \xi) \in M^0_\sigma(S_{G(d)})$ $(0 \leq \sigma \leq 1/d)$. So, what we have to do is the construction of the fundamental solution $E(t, s)$ for $\mathcal{L}$.

Now, we give our main theorem in this paper.

**Theorem 1.** Assume $\lambda(t, x, \xi) \in M^0_\sigma(S_{G(d)})$ is real-valued and $b(t, x, \xi) \in M^0_\sigma(S_{G(d)})$ for some $0 \leq \sigma \leq 1/d (< 1)$. Then, the fundamental solution $E(t, s)$ of (8) can be written in the form

$$E(t, s) = [I + \sum_{\nu=1}^{\infty} W_\nu(t, s)] I_\phi(t, s) + R(t, s)$$

for $0 \leq t, s \leq T_*$

for a small $T_*$. In (9) $\sum_{\nu=1}^{\infty} W_\nu(t, s)$ is a series of pseudo-differential operators $W_\nu(t, s)$ with symbols $w_\nu(t, s; x, \xi)$ satisfying for $j = 0, 1$

$$|\partial_t^i \partial_x^j D_\xi^k w_\nu(t, s; x, \xi)| \leq (C_0 |t - s|^{|\nu|+1}) M^{-\left(\left|\alpha\right| + \left|\beta\right|\right)} a_{t} b_{t} < \xi, \nu_{0} > ^{-|\alpha|}$$

with constants $C_0$ and $M$ independent of $\alpha, \beta, and \nu$; $I_\phi(t, s)$ is a Fourier integral operator with the phase function $\phi(t, s; x, \xi)$, where $\phi(t, s; x, \xi)$ is a solution of

$$\left\{ \begin{array}{l}
\partial_t \phi = \lambda(t, x, \nabla_x \phi), \\
\phi|_{t=s} = x \cdot \xi;
\end{array} \right.$$ 

and $R(t, s)$ is a pseudo-differential operator with symbol $r(t, s; x, \xi)$ in $M^1_{i,s}(\mathcal{R}_{G(d)})$.

Since the symbol of $R(t, s)$ belongs to $M^1_{i,s}(\mathcal{R}_{G(d)})$, $R(t, s)$ maps $\mathcal{E}'$ to $\gamma^d$ for any fixed $t, s$. Hence we can call $R(t, s)$ a regularizer. From Theorem 1 and Proposition 1.3 we easily obtain

**Theorem 2.** Assume that $\lambda(t, x, \xi)$ in (8) is homogeneous for large
Then, for a solution \( u(t) \) of
\[
\begin{align*}
\mathcal{L} u(t) &= 0, \\
u(0) &= g,
\end{align*}
\]
we have
\[
WF_{G_0}(u(t)) = \{(q(t, y, \eta), \rho p(t, y, \eta)); (y, \eta) \in WF_{G_0}(g) \text{ for large } |\eta|, \rho > 0 \},
\]
where \( WF_{G_0}(u) \) is a wave front set of \( u \) in the Gevrey class of order \( d \) (see Definition 1.2) and \( \{q(t, y, \eta), \rho(t, y, \eta)\} \) is a solution of
\[
\begin{align*}
\left\{ \begin{array}{l}
dq = -\nabla_x \lambda(t, q, \rho), \\
q|_{t=0} = y, \quad \rho|_{t=0} = \eta.
\end{array} \right.
\end{align*}
\]

This result is also obtained by Mizohata [17]. He has showed it by using the energy method, not by constructing the fundamental solution. For the parametrix of \( \mathcal{L} \) Lascar reported in [15] that he constructed it, but the author has not known his detailed proof. Another result concerning the construction of the parametrix is reported in [2] and the propagation of singularities for a solution of (2) is studied in [19] and [25].

From Theorems 1, 2 and Proposition 3.4 we obtain

**Corollary 3.** In (1) we assume
\[
\tau^m + \sum_{j=0}^{m-1} \sum_{|\alpha|=j} a_{j,\alpha}(t, x) \xi^\alpha \tau^j = \prod_{j=1}^{\kappa} (\tau - \lambda_j(t, x, \xi))^{m_j} \text{ for } |\xi| \geq 1
\]
and \( \sigma \equiv \max \{(m_j - 1)/m_j\} \leq 1/d \). Then, the fundamental solution \( E_\sigma(t, s) \) can be constructed in the form
\[
E_\sigma(t, s) = \sum_{j=1}^{\kappa} \sum_{\nu=0}^{\infty} W_{j,\nu}(t, s) I_{\phi_j}(t, s) + R(t, s),
\]
where \( W_{j,\nu}(t, s), I_{\phi_j}(t, s) \) and \( R(t, s) \) satisfy the similar properties to those in Theorem 1. Moreover, let \( \{q_j(t, y, \eta), \rho_j(t, y, \eta)\} \) be a solution of (14) with \( \lambda = \lambda_j \). Then, we have for a solution of (2)
\[
\bigcup_{j=0}^{m-1} WF_{G_0}(\partial_t^j u(t)) = \bigcup_{j=1}^{\kappa} \{(q_j(t, y, \eta), \rho p_j(t, y, \eta)); (y, \eta) \in \bigcup_{j=0}^{m-1} WF_{G_0}(g_j) \text{ for large } |\eta|, \rho > 0 \}.
\]
In Section 3 we study the above result under a weaker condition: There exist regularly hyperbolic operators $L_1, L_2, \ldots, L_r$ such that $L$ has a form

\[(17) \quad L = L_1 L_2 \cdots L_r + \sum_{j=0}^{m-q} a_j(t, X, D_x) D_x^j\]

with $a_j(t, x, \xi) \in M_t(S^{d-\frac{j}{2}-1}_0) \quad (1 \leq q \leq r)$. This formulation is based on the work in [16], where the authors proved $\gamma^d$-well-posedness for $d \leq r/(r-q)$ in the case that $a_j(t, X, D_x)$ are differential operators. The number $r/q$ is called the irregularity in [8]. For the case of constant multiplicity, Ohya [22] also proved the $\gamma^d$-well-posedness and in [5] Ivrii gave the necessary and sufficient condition for (1) to be $\gamma^d$-well-posed. Under Ivrii's condition we can also reduce (2) for (1) to (6) for (7) and get Corollary 3.

The construction of the fundamental solution $E(t, s)$ (the proof of Theorem 1) is performed by the way employed in [13], [14] and [23]. There, the authors construct $E(t, s)$ for the $C^\infty$-case by using the successive approximation after solving an eiconal equation (11). The key point of their proof is obtaining a sharp estimate of multi-products of Fourier integral operators. Since we assume $d > 1$, we can use cut functions in the Gevrey class and can improve their estimate to the Gevrey class. This enables us to prove Theorem 1. In [6] and [7] Kajitani has constructed the fundamental solution for a hyperbolic system with coefficients in the Gevrey class of order $d$ by solving transport equations and using the asymptotic sum of amplitude functions. His fundamental solution $E(t, s)$ has the form similar to (15) and the regularizer $R(t, s)$ in his $E(t, s)$ is an integral operator with a kernel in the Gevrey class of order $2d-1$. So, from his $E(t, s)$ we get (16) in the case $d=1$, but we cannot obtain (16) for the case $d > 1$. In our construction, since we do not solve transport equations and hence we do not use the asymptotic sum, the regularizer $R(t, s)$ becomes an integral operator with a kernel in the Gevrey class of order $d$ and get (16) for the case $d > 1$.

The outline of the present paper is the following: In Section 1 we give a class of Fourier integral operators and a result on wave front sets. In Section 2, after showing the result on products of Fourier integral operators, conjugate Fourier integral operators and pseudo-differential operators we obtain a sharp estimate of multi-
products of Fourier integral operators. Since we need tedious calculation to obtain the former results, we devote their proofs to Section 4. In Section 3 we prove Theorem 1 and show the way of reducing the problem (2) to the problem (6).

In Section 5 we prove a sharp estimate of symbols of multi-products of pseudo-differential operators, which is also used in Section 2. For the proof we follow the discussions in Section 1 of [23]. There, to obtain the key estimate we divide the multi-product of \( \nu + 1 \) pseudo-differential operators into \( 2^\nu \) terms by using cut functions depending on a parameter \( \varepsilon \) (see (1.57) of [23]). But, in our case we cannot use such a decomposition since we cannot find a suitable \( \varepsilon \) to obtain our estimate, especially to find a suitable "radius of convergence". So, we employ a different method of the division into \( 2^\nu \) terms. Then, we obtain the desired estimate for our case.

The final section, Section 6, is devoted to the proof of Proposition 3.4 on the perfect diagonalization. This is a version of the one in [10] for the Gevrey class. Since we use the asymptotic sum for products of pseudo-differential operators, we use the class \( S_{G(\mathcal{d},1)}^m \), not the class \( S_{G(\mathcal{d})}^m \), and the discussion in [1]. Then, the discussions in [10] work well and we can obtain Proposition 3.4.

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§ 1. Definitions and Wave Front Sets

Throughout this paper the constant \( d \) denotes a number larger than 1. To define Fourier integral operators we will introduce a class \( \mathcal{P}_{G(d)}(\tau) \) of phase functions as follows:

**Definition 1.1.** Let \( 0 \leq \tau < 1 \). We say that a phase function \( \phi(x, \xi) \) belongs to a class \( \mathcal{P}_{G(d)}(\tau) \) if \( \phi(x, \xi) \) belongs to \( \mathcal{P}(\tau) \) and for \( J(x, \xi) = \phi(x, \xi) - x \cdot \xi \) the estimate

\[
|J(\beta)| (x, \xi) | \leq \tau M^{-(\langle \alpha \rangle + |\beta|)} \alpha! \beta! \langle \xi \rangle^{1-|\alpha|}
\]

hold for a constant \( M \) independent of \( \alpha, \beta \). We also set

\[
\mathcal{P}_{G(d)} = \bigcup_{0 \leq \tau < 1} \mathcal{P}_{G(d)}(\tau).
\]

**Remark 1.** We say that for \( \phi_{\theta} \in \mathcal{P}_{G(d)}(\tau_\theta) \) \( 0 \leq \tau_\theta < 1 \) the set \( \{ \phi_{\theta} \}_{\theta \in \Theta} \) is bounded in \( \mathcal{P}_{G(d)} \) if \( \tau_\theta \leq \bar{\tau}_\theta \) for a constant \( \bar{\tau}_\theta \) independent of \( \theta \).
and $J_\theta(x, \xi) \equiv \phi_\theta(x, \xi) - x \cdot \xi$ satisfies (1.1) with $\tau = \tau_\theta$ and a constant $M$ independent of $\theta$.

Remark 2. In the same way we define bounded sets in $S^\infty_{G(d)}$ and $S^\infty_{G(w)}$ as follows: We say that for $p_\theta \in S^\infty_{G(d)}$ the set $\{p_\theta\}_{\theta \in \Theta}$ is bounded in $S^\infty_{G(d)}$ if we can take constants $C$ and $M$ in (3) independent also of the parameter $\theta \in \Theta$, and we say that for $p_\theta \in S^\infty_{G(d)}$ the set $\{p_\theta\}_{\theta \in \Theta}$ is bounded in $S^\infty_{G(w)}$ if we can take constants $C_\alpha$ and $M$ in (5) independent also of $\theta \in \Theta$.

Let $\phi(x, \xi)$ be a phase function in $\mathcal{P}_{G(d)}$. Then, a Fourier integral operator $P_\phi = p_\phi(X, D_x)$ with the phase function $\phi(x, \xi)$ and a symbol $p(x, \xi) \in S^\infty_{G(d)}$ is defined by

\[
P_\phi u(x) = C_\phi^{-1} \int \int \phi(\theta(x, \eta) - x \cdot \xi) p(x, \xi) u(x') dx' d\xi
\]

for $u \in \mathcal{S}$, where $\mathcal{S}$ is the Schwartz space of rapidly decreasing functions on $\mathbb{R}^n$ and the right hand side of (1.2) is the oscillatory integral defined in [12] (Chap. 10). Following [12] we denote the set of such Fourier integral operators by $S^\infty_{G(d), \phi}$. If $\phi = x \cdot \xi$, the set $S^\infty_{G(d), \phi}$ is the one of pseudo-differential operators. In this case we write $S^\infty_{G(d), \phi}$ simply by $S^\infty_{G(d)}$. Similarly, we define a Fourier integral operator $P_\phi$ with the phase function $\phi(x, \xi)$ and a symbol $p(x, \xi) \in S^\infty_{G(w)}$ by (1.2) and denote a class of such Fourier integral operators by $S^\infty_{G(w), \phi}$. Corresponding to this class we write a class of pseudo-differential operators as $\mathcal{P}_{G(d), \phi} = \{p(X, D_x) : p(x, \xi) \in S^\infty_{G(d)}\}$, since no confusion occurs between the class of symbols and that of pseudo-differential operators. Remark that the following holds: If $p(x, \xi)$ belongs to $S^\infty_{G(d)}$ and a real symbol $\phi(x, \xi)$ satisfies (1.1) then $\phi^{I(x, \eta)} p(x, \xi)$ also belongs to $S^\infty_{G(d)}$. This fact shows that $\mathcal{P}^{\infty}_{G(d), \phi} = S^\infty_{G(d)}$ for all $\phi \in \mathcal{P}_{G(d)}$, which corresponds to (2.6) of [9]. So, we may use mainly the class $S^\infty_{G(d)}$ among the class $S^\infty_{G(d), \phi}$, $\phi \in \mathcal{P}_{G(d)}$. Denote by $\gamma^d(M)$ the class of functions $u(x)$ satisfying

$$|\partial^\alpha u(x)| \leq CM^{-|\alpha|}\delta$$

and denote $\gamma^d = \bigcup_{M>0} \gamma^d(M)$. Then, the operator in $S^\infty_{G(d)}$ maps a class $\mathcal{E}'$ of distributions with compact supports to a class $\gamma^d$. In this sense, we call the operators in $S^\infty_{G(d)}$ regularizers.
The following definition coincides with that of $\text{WF}_L(u)$ for $L = \{L_k = (k + 1)^4\}$ in [4].

**Definition 1.2.** Let $u \in \mathcal{E}'$. We say that a point $(x^o, \xi^o)$ of $T^*(\mathbb{R}^n) \setminus \{0\}$ does not belong to the wave front set $\text{WF}_{G^{(\omega)}}(u) \subset T^*(\mathbb{R}^n) \setminus \{0\}$ of $u$, if there exist a conic neighborhood $\Gamma$ of $\xi^o$ and a function $\chi(x)$ in $\mathbb{R}^d$ with $\chi(x^o) \neq 0$ such that the Fourier transform $\mathcal{F}[\chi u](\xi)$ of $\chi(x)u(x)$ satisfies for any $N$

\begin{equation}
|\xi|^N |\mathcal{F}[\chi u](\xi)| \leq CM^{-N}N!^d \quad \text{for} \quad \xi \in \Gamma
\end{equation}

with constants $C$ and $M$ independent of $N$.

Concerning the wave front set $\text{WF}_{G^{(\omega)}}(u)$ of the Gevrey class the following holds.

**Proposition 1.3.** Let a phase function $\phi(x, \xi) \in \mathcal{G}^{(\omega)}$ be homogeneous for large $|\xi|$. Then, for a Fourier integral operator $P_{\phi}$ with a symbol $p(x, \xi) \in \mathcal{S}^{(\omega)}$ the relation

\begin{equation}
\text{WF}_{G^{(\omega)}}(P_{\phi}u) \subseteq \{(x, \rho P_{\phi} \phi(x, \xi)) ; (P_{\phi} \phi(x, \xi), \xi) \in \text{WF}_{G^{(\omega)}}(u) \quad \text{for large} \quad |\xi|, \rho > 0\}
\end{equation}

holds for $u \in \mathcal{E}'$.

**Proof.** We may assume $u \in \mathcal{E}$ by the similar result for the $C^\omega$-case. Suppose that the points $(x^o, \xi^o)$ and $(\eta^o, \eta^o)$ satisfy $\xi^o = P_{\phi} \phi(x^o, \eta^o)$, $\eta^o = P_{\phi} \phi(x^o, \eta^o)$, $(\eta^o, \eta^o) \in \text{WF}_{G^{(\omega)}}(u)$ and $|\eta^o| \geq C_1$. From the definition there exist a function $\chi_1(x)$ in $\mathbb{R}^d$ satisfying $\chi = 1$ in a neighborhood of $\eta^o$ and a conic neighborhood $\Gamma_1$ of $\eta^o$ such that (1.3) holds with $\chi = \chi_1$ and $\Gamma = \Gamma_1$. Take $\phi(\xi) \in \mathcal{S}^{(\omega)}$ satisfying supp$\phi \subset \Gamma_1$ and $\phi = 1$ in a conic neighborhood of $\eta^o$, and take a function $\chi_2(x)$ in $\mathbb{R}^d$ such that $\chi_2(x^o) \neq 0$ and supp$\chi_2 \subset \{x ; \chi_1(P_{\phi} \phi(x, \eta)) = 1$ for all $\eta\}$. Using these functions we divide $\mathcal{F}[\chi_2 P_{\phi} u](\xi)$ into three parts:

\begin{equation}
\mathcal{F}[\chi_2 P_{\phi} u](\xi) = \int e^{i(-x \cdot \xi + \phi(x, \eta))} \chi_2(x) p(x, \eta) \phi(\eta) \mathcal{F}[\chi u](\eta) d\eta dx
\end{equation}

$$+ \int e^{i(-x \cdot \xi + \phi(x, \eta))} \chi_2(x) p(x, \eta)(1 - \phi(\eta)) \mathcal{F}[\chi u](\eta) d\eta dx$$
For a fixed $\alpha$ we estimate $\xi^\alpha f_j(\xi)$, $j=1, 2, 3$, individually. First, we estimate $\xi^\alpha f_1(\xi)$. Note that we can write

$$e^{-i\phi(x, \xi)} \partial_x^\alpha_x D_{\xi} e^{i\phi(x, \xi)} = \sum_{k=0}^{[\frac{\alpha}{2}]} \phi_{j, k}(x, \xi)$$

with the property

$$|\partial_x^\alpha_x \phi_{j, k}(x, \xi)| \leq CM^{-\|\beta\|+|\alpha|} |\partial_x^\beta_x \phi^\alpha_x|^d |\partial_x^d \phi^{1-k-|\alpha|}|.$$ 

Using (1.6) we write

$$\xi^\alpha f_1(\xi) = \sum_{\alpha' + \alpha'' = \alpha} \sum_{k=0}^{[\frac{\alpha'}{2}]} \int e^{-(x' \xi + \phi(x, \eta))} \phi_{j, k}(x, \eta)$$

$$\times D_{\xi}^\alpha \left( \chi_3(x) \phi^\alpha_x \right) \psi(\eta) F [\chi_3 u](\eta) d\eta dx.$$

Then, from (1.3) for $u(x)$ we have

$$|\xi^\alpha f_1(\xi)| \leq CM^{-|\alpha|} \alpha!^d$$

if we take new constants $C$ and $M$ independent of $\alpha$. Next, we estimate $\xi^\alpha f_2(\xi)$. If we take an appropriate conic neighborhood $\Gamma_2$ of $\xi^0$, the relation

$$|\xi - \partial_x^\alpha \phi(x, \eta)| \geq C(|\xi| + |\eta|)$$

holds on the support of the integrand of $f_2(\xi)$. So, if we set

$$L_1 = i |\xi - \partial_x^\alpha \phi(x, \eta)|^{-2} (\xi - \partial_x^\alpha \phi(x, \eta)) \cdot \partial_x$$

we have from $L_1 e^{-(x \xi + \phi(x, \eta))} = e^{-(x \xi + \phi(x, \eta))}$

$$\xi^\alpha f_2(\xi) = \xi^\alpha \int e^{-(x \xi + \phi(x, \eta))} (L_1)^{|\alpha|} \chi_3(x) \phi^\alpha_x \phi(\eta)$$

$$\times (1 - \psi(\eta)) \mathcal{F} [\chi_3 u](\eta) d\eta dx.$$

Hence, we have

$$|\xi^\alpha f_2(\xi)| \leq CM^{-|\alpha|} \alpha!^d$$

for $\xi \in \Gamma_2$.

For $f_3(\xi)$ we write

$$\xi^\alpha f_3(\xi) = \sum_{\alpha' + \alpha'' + \alpha'' = \alpha} \frac{\alpha!}{\alpha!^d} \sum_{k=0}^{[\frac{\alpha}{2}]} e^{-i x \xi} \chi_2(\alpha) \chi_3(\eta)$$

$$\times (O_1 - \int e^{i(\phi(x, \eta) - \chi_3(\eta))} \phi_{j, k}(x, \eta) \phi^\alpha_x(\eta)$$

$$\times (1 - \chi_3(\eta)) \phi(\eta) d\eta dx.$$
\[ L_2 = -i |\mathcal{F}_x \phi(x, \eta) - y |^{-2}(\mathcal{F}_x \phi(x, \eta) - y) \cdot \mathcal{F}_y \]

we write

\[ \xi^a f_2(\xi) = \sum_{\alpha + \beta = \alpha} \frac{\alpha!}{\alpha! \beta!} \sum_{k=0}^{\lceil \alpha \rceil} e^{-i\xi \cdot \xi_k} \chi_{\alpha}(x) \]
\[ \times [O_j - \int e^{i(\phi(x, y) - y \cdot \phi)(L_{x})^{k+l(m)+n+1}} [\phi_{\theta, k}(x, \eta) \]
\[ \times \phi_{0, \gamma}(x, \eta)](1 - \chi_j(y)) u(y) dy \cdot \xi^a f_2(\xi, \eta) \cdot dx, \]

where \( l(m) = [\max(m, 0)] \). This implies

\[ |\xi^a f_2(\xi)| \leq CM^{-|a|} a!^d. \]

Consequently, we have for any \( \alpha \)

\[ |\xi^a \mathcal{F}[\mathcal{F}_x P_{\phi, \mu}](\xi)| \leq CM^{-|a|} a!^d \text{ for } \xi \in \Gamma \]

from (1.5) and (1.8) -- (1.10). This means \((x^\circ, \xi^\circ) \in \text{WF}_{G(\theta)}(P_{\phi, \mu})\).

Q. E. D.

\section{Multi-Products of Fourier Integral Operators}

In this section we will obtain a sharp estimate for the symbols of multi-products of Fourier integral operators. For simplicity we denote for \( \phi \in \mathcal{P}_{G(\theta)} \)

\[ L_\mathcal{P}_{G(\theta)}(\phi) = \{ \phi_\theta(X, D_x) + \phi_\theta(X, D_x) \} \]

that is, symbolically \( L_\mathcal{P}_{G(\theta)}(\phi) = S_{G(\theta), \theta} + \mathcal{R}_{G(\theta), \phi} \). If \( \phi(x, \xi) = x \cdot \xi \) we denote \( L_\mathcal{P}_{G(\theta)}(\phi) \) simply by \( L_\mathcal{P}_{G(\theta)} \).

For a sequence \( \{ \phi_j \} \) of phase functions \( \phi_j(x, \xi) \in \mathcal{P}_{G(\theta)}(\tau_j) \) we consider multi-products

\[ Q_{\nu+1} = P_{1, \phi_1} P_{2, \phi_2} \cdots P_{\nu+1, \phi_{\nu+1}} \]

of Fourier integral operators \( P_{j, \phi_j} \) in \( L_\mathcal{P}_{G(\theta)}(\phi_j) \) with \( \sigma \geq 0 \). We put following assumptions:

(A.1) There exists a small constant \( \tau^0 \) such that \( \sum_{j=1}^{m} \tau_j \leq \tau^0 \). If we set \( J_j(x, \xi) = \phi_j(x, \xi) - x \cdot \xi \), \( \{ J_j/\tau_j \} \) is bounded in \( S_{G(\theta)} \).

(A.2) If we write \( P_{j, \phi_j} = \phi_\mathcal{P}_{j, \phi_j}(X, D_x) + \phi_\mathcal{R}_{j, \phi_j}(X, D_x) \in S_{G(\theta), \phi_j} + \mathcal{R}_{G(\theta), \phi_j} \) the set \( \{ \phi_j(x, \xi) \} \) is bounded in \( S_{G(\theta)} \) and \( \{ \phi_j(x, \xi) \} \) is bounded in \( \mathcal{R}_{G(\theta), \phi_j} \).

The result we want to show in this section is the following:
Theorem 2.1. We assume (A.1) and (A.2). Then, the multi–product (2.1) of Fourier integral operators $P_{i, \phi}$ is a Fourier integral operator $Q_{\nu+1, \Phi_{\nu+1}}$ in $L_{G(d)}^{(\nu+1),r}(\Phi_{\nu+1})$ with a phase function $\Phi_{\nu+1}(x, \xi)$ in $\mathcal{P}_{G(d)}$, and is represented by the form

$$Q_{\nu+1, \Phi_{\nu+1}} = q_{\nu+1}(X, D_{x})I_{\Phi_{\nu+1}} + \tilde{q}_{\nu+1}(X, D_{x})I_{\Phi_{\nu+1}}$$

for the symbols $q_{\nu+1}(x, \xi)$ and $\tilde{q}_{\nu+1}(x, \xi)$ satisfying

$$|q_{\nu+1}^{(\alpha)}(x, \xi)| \leq C_{0}M^{-(|\alpha|+|\beta|)}\beta!\xi^{(\nu+1)\sigma-|\alpha|},$$

$$|\tilde{q}_{\nu+1}^{(\alpha)}(x, \xi)| \leq C_{0}C_{\alpha}M^{-(|\beta|+N)}\xi^{N+|\alpha|}$$

for any $N$, where the constants $C_{0}$ and $M$ are independent of $\nu$, $\alpha$, $\beta$, $N$ and the constant $C_{\alpha}$ is independent of $\nu$, $\beta$ and $N$.

In (2.2) the operator $I_{\phi}$ for $\phi \in \mathcal{P}_{G(d)}$ is the Fourier integral operator with the phase function $\phi(x, \xi)$ and the symbol 1.

We will prove Theorem 2.1 after some preparations. First, we give the product formulae between Fourier integral operators, conjugate Fourier integral operators and pseudo–differential operators, whose proofs are given in Section 4.

Proposition 2.2. The following inclusion formulae hold.

$$S_{G(d)}^{m} \times S_{G(d)}^{m'} \subset L_{G(d)}^{m+m'}(\phi),$$

$$S_{G(d)}^{m} \cdot S_{G(d)}^{m'} \subset L_{G(d)}^{m+m'}(\phi),$$

$$\mathcal{R}_{G(d)} \cdot L_{G(d)}^{m}(\phi) \subset \mathcal{R}_{G(d)}, \quad L_{G(d)}^{m}(\phi) \cdot \mathcal{R}_{G(d)} \subset \mathcal{R}_{G(d)}.$$

Remark 1. It is easy to see from (2.5) – (2.7)

$$L_{G(d)}^{m} \times L_{G(d)}^{m'}(\phi) \subset L_{G(d)}^{m+m'}(\phi),$$

$$L_{G(d)}^{m}(\phi) \cdot L_{G(d)}^{m'} \subset L_{G(d)}^{m+m'}.$$
Denote by $I_{\phi}$ the conjugate Fourier integral operator with the phase function $\phi(x, \xi) \in \mathcal{P}_{G(\theta)}$ and the symbol 1. Then, we have

**Proposition 2.3.** The following relations hold.

\[(2.10) \quad L_{G(\theta)}^0(\phi) \cdot I_{\phi} \subset L_{G(\theta)}^0, \quad I_{\phi} \cdot L_{G(\theta)}^0(\phi) \subset L_{G(\theta)}^0.\]

**Remark.** The inclusion mappings in (2.10) are bounded in the similar sense to Remark 2 of Proposition 2.2.

For the multi-products of phase functions we have

**Proposition 2.4.** Let $\phi_{j}(x, \xi) \in \mathcal{P}_{G(\theta)}(\tau_j), j = 1, 2, \ldots$. Assume (A.1). Then, the multi-product $\Phi_{\nu+1}(x, \xi) = \phi_1 \# \cdots \# \phi_{\nu+1}(x, \xi)$ defined in [14] belongs to $\mathcal{P}_{G(\theta)}(c_0 \tau_{\nu+1}^{\nu+1} = \tau_1 + \cdots + \tau_{\nu+1})$ with some constant $c_0$ independent of $\nu$.

**Proof.** Let $\{X^0_j, \varepsilon^j\}_{j=1}^\nu(x, \xi)$ be the solution of

\[(2.11) \quad \begin{align*}
X^j &= V^j \phi_{\nu+1}(X^0_j, \varepsilon^j), \\
\varepsilon^j &= V^j \phi_{\nu+1}(X^0_j, \varepsilon^j),
\end{align*}
\]

Then, by the induction on $N$ we can prove from the method of the proof of Theorem 1.7' in [14] for constants $C_1$ and $M$ independent of $\alpha, \beta$ and $\nu$. This implies

\[(2.12) \quad \sum_{j=1}^\nu \left\{ \zeta^\alpha ||D^\alpha \delta^\beta (X^j - X^{j-1})|| + \zeta^\beta ||D^\alpha \delta^\beta (\varepsilon^j - \varepsilon^{j-1})|| \right\} \leq C_{1} \tau_{\nu+1} M^{-((|\alpha| + |\beta|) \alpha) \beta }.
\]

Since $\Phi_{\nu+1}(x, \xi) = \phi_1 \# \cdots \# \phi_{\nu+1}(x, \xi)$ is defined by

\[(2.13) \quad \Phi_{\nu+1}(x, \xi) = \sum_{j=1}^\nu (\phi_j(X^0_j, \varepsilon^j) - X^0_j \cdot \varepsilon^j) + \phi_{\nu+1}(X^0, \xi)\]

\[(X^0 = x),\]

we get $\Phi_{\nu+1}(x, \xi) \in \mathcal{P}_{G(\theta)}(c_0 \tau_{\nu+1}^{\nu+1})$ with an appropriate constant $c_0$.

Q. E. D.

**Proposition 2.5.** Let $\phi_j(x, \xi) \in \mathcal{P}_{G(\theta)}(\tau_j), j = 1, 2$. Assume $\tau_1 + \tau_2$...
is small enough. Then, we have

\begin{equation}
I_{\phi_1}I_{\phi_2} \in L^0_G(\phi_1 \# \phi_2).
\end{equation}

Remark. For \( \phi_{j, \theta} \in \mathcal{P}_{G(\theta)}(\tau_{j, \theta}) \), \( j = 1, 2 \), we denote \( I_{\phi_{1, \theta}}I_{\phi_{2, \theta}} = p_{\theta, \phi_{1, \theta}}(X, D_2) + \tilde{p}_{\theta, \phi_{2, \theta}}(X, D_2) \) \( (\phi_{\theta} = \phi_{1, \theta} \# \phi_{2, \theta}) \). Then, if \( \tau_{1, \theta} + \tau_{2, \theta} \leq \tau_0 \) for a \( \tau_0 \) independent of \( \theta \) and the sets \( \{ \phi_{j, \theta} \}_{\theta \in \Theta} \) \( (j = 1, 2) \) are bounded in \( \mathcal{P}_{G(\theta)} \), the sets \( \{ p_{\theta} \}_{\theta \in \Theta} \) and \( \{ \tilde{p}_{\theta} \}_{\theta \in \Theta} \) of the corresponding symbols \( p_{\theta}(x, \xi) \) and \( \tilde{p}_{\theta}(x, \xi) \) are bounded in \( S^0_{G(\theta)} \) and \( R_{G(\theta)} \), respectively.

We postpone the proof of this proposition to Section 4.

For the multi-products of pseudo-differential operators we have

**Theorem 2.6.** Let \( P_j = p_{j}^0(X, D_2) + \tilde{p}_j(X, D_2) \in L^0_G (j = 1, 2, \ldots) \) with \( \sigma \geq 0 \). Assume

\begin{equation}
|p^0_{j}(x, \xi)| \leq C_\sigma M^{-[(|\alpha| + |\beta|)1]} \alpha! \beta! \xi^{-|\alpha|},
\end{equation}

\begin{equation}
|\tilde{p}^0_{j}(x, \xi)| \leq C_\sigma M^{-[(|\beta| + N)1]} \beta! N! \xi^{-|\alpha| - N}
\end{equation}

for any \( N \), where the constants \( C_\sigma \) and \( M \) are independent of \( j, \alpha, \beta, \) \( N \) and the constant \( C_\sigma \) is independent of \( j, \beta, N \). Then, the multi-product \( Q_{v+1} = P_1 P_2 \cdots P_{v+1} \) has the form \( Q_{v+1} = q_{v+1}^0(X, D_2) + \tilde{q}_{v+1}(X, D_2) \) with the properties

\begin{equation}
|q^0_{v+1}(x, \xi)| \leq A^v C^v_{\sigma} M_{1}^{-[(|\alpha| + |\beta|)1]} \alpha! \beta! \xi^{-|\alpha| - |\alpha|},
\end{equation}

\begin{equation}
|\tilde{q}^0_{v+1}(x, \xi)| \leq A^v \tilde{C}^v_{\sigma} M_{1}^{-[(|\beta| + N)1]} \beta! N! \xi^{-|\alpha| - N}
\end{equation}

for any \( N \).

Here, \( A \) and \( M_1 \) are constants determined only by the dimension \( n \) and \( M \), \( C^v_{\sigma} = \max_{\alpha \leq n+1} |C_{\alpha}, C_{\nu}| \) and the constants \( C^v_{\sigma} \) are determined by \( n, \alpha \) and \( C_{\sigma} \). All the constants \( A, M_1 \) and \( C^v_{\sigma} \) are independent of \( \nu \).

Since the proof is rather long, we will give it in Section 5. As in [23] we get two corollaries.

**Corollary 2.7** (cf. Theorem 3 of [23]). Let \( P = \tilde{p}^0(X, D_2) + \tilde{p}(X, D_2) \in L^0_G \) with

\begin{equation}
\begin{cases}
|p^0_{j}(\phi, \xi)| \leq C_\sigma M^{-[(|\alpha| + |\beta|)1]} \alpha! \beta! \xi^{-|\alpha|},
|\tilde{p}^0_{j}(\phi, \xi)| \leq C_\sigma M^{-[(|\beta| + N)1]} \beta! N! \xi^{-|\alpha| - N}
\end{cases}
\end{equation}

for any \( N \). Assume that \( C_\sigma \) and \( \max_{\alpha \leq n+1} C_{\alpha} \) are small enough. Then the inverse \( Q_{v} \) of
$I - P$ exists in $L^0_{G(d)}$ and is represented by the Neumann series $Q = \sum_{\nu=0}^{\infty} P^\nu$.

**Corollary 2.8** (cf. Proposition 2.2 of [23]). Let $\phi(x, \xi) \in \mathcal{P}_{G(d)}(\tau)$. If $\tau$ is small enough, there exist pseudo-differential operators $R$ and $R'$ in $L^0_{G(d)}$ such that

$$\begin{align*}
I_\theta RI_\phi &= I, \\
I_\theta R'I_\phi &= I.
\end{align*}$$

(2.20)

Now, we are prepared to prove Theorem 2.1. Since we can prove it by the parallel way to the proof of Theorem 1 in [23], we will give only the sketch of the proof. Set $\Phi_j = \phi_j \# \phi_2 \# \cdots \# \phi_j$. Then, as in Proposition 2.3 and Lemma 2.10 in [23] we can find, by using Proposition 2.2, Proposition 2.3, Proposition 2.5 and Corollary 2.8, pseudo-differential operators $P_j$ in $L^0_{G(d)}$ such that

$$\begin{align*}
P_{1,1} &= P_1 I_{\phi_2}, \\
P_{2,1} &= P_2 I_{\phi_3}, \\
&\vdots \\
P_{j-1,2} &= P_{j-1} I_{\phi_j} (j \geq 2; \Phi_1 = \phi_1).
\end{align*}$$

From this we have

$$\begin{align*}
Q_{u+1} &= P_1(I_{\phi_2} P_2 I_{\phi_3} P_3 \cdots P_{u+1} I_{\phi_{u+1}}) \\
&= P_1 P_2(I_{\phi_2} P_3 I_{\phi_3} P_4 \cdots P_{u+1} I_{\phi_{u+1}}) \\
&= \cdots \\
&= P_1 P_2 \cdots P_{u+1}(I_{\phi_1} P_2 \cdots P_{u+1} I_{\phi_{u+1}}) \\
&= P_1 P_2 \cdots P_{u+1} I_{\phi_{u+1}}.
\end{align*}$$

Combining this with Theorem 2.6 and (2.8) we get the theorem.

§ 3. Fundamental Solution for a Hyperbolic Operator

We will construct the fundamental solution for $\mathcal{L}$ of (8). For the proof we will solve an eiconal equation

$$\begin{align*}
\partial_t \phi &= \lambda(t, x, P x, \phi), \\
\phi|_{t=s} &= x \cdot \xi.
\end{align*}$$

(3.1)

**Proposition 3.1.** Assume that $\lambda(t, x, \xi)$ is a real symbol in $M^r_1(S^1_{G(d)})$. Then, there exists a constant $T_0$ such that the solution $\phi(t, s; x, \xi)$ exists uniquely in $\{(t, s) ; 0 \leq t, s \leq T_0 \}$ and belongs to $\mathcal{P}_{G(d)}(\varepsilon_0 |t - s|)$ for a
constant $\varepsilon_0$ independent of $t$ and $s$.

**Proof.** We follow the proof of Theorem 3.1 in [9] combined with the idea in §1 of Chap. XI of [24]. Let $(q, p) (t, s; y, \eta)$ be a solution of

$$
\begin{align*}
\frac{dq}{dt} &= -F_q \lambda(t, q, p), \\
\frac{dp}{dt} &= F_p \lambda(t, q, p) \\
q |_{t=s} &= y, \\
p |_{t=s} &= \eta.
\end{align*}
$$

Then, by the method of the proof of Lemma 3.1 in [9] we obtain

$$
\begin{align*}
|\partial^\alpha_q \partial^\beta_p (q - y)| &\leq C |t-s| M^{-(|\alpha|+|\beta|)} \alpha! \beta! \langle \xi \rangle^{-|\alpha|} \\
|\partial^\alpha_q \partial^\beta_p (p - \eta)| &\leq C |t-s| M^{-(|\alpha|+|\beta|)} \alpha! \beta! \langle \xi \rangle^{-|\alpha|}
\end{align*}
$$

if $0 \leq t, s \leq T_0$ for a small $T_0$. From (3.2) there exists an inverse function $Y(t, s; x, \xi)$ of $x = q(t, s; Y, \xi)$ for $0 \leq t, s \leq T_0$ if $T_0 (\leq T_0')$ is small, and it satisfies

$$
|\partial^\alpha_q \partial^\beta_p (Y - x)| \leq C |t-s| M^{-(|\alpha|+|\beta|)} \alpha! \beta! \langle \xi \rangle^{-|\alpha|}.
$$

Set $\phi(t, s; x, \xi) = p(t, s; Y(t, s; x, \xi), \xi)$ and

$$
\phi(t, s; x, \xi) = x \cdot \xi + \int_0^t \lambda(\theta, x, \phi(\theta, s; x, \xi)) d\theta.
$$

Then, $\phi(t, s; x, \xi)$ is a solution of (3.1) by the similar discussions in §1 of Chap. XI of [24] and it belongs to $\mathcal{G}_{\phi(\xi, t-s)}$ with a constant $\varepsilon_0$ independent of $t$ and $s$. Q. E. D.

Now, we prove Theorem 1. Let $I_\phi(t, s)$ be the Fourier integral operator with the phase function $\phi(t, s; x, \xi)$ and the symbol 1. Operate $\mathcal{L}$ in (8) to $I_\phi(t, s)$. Then, we can prove by the similar way to the proof of (2.5)

$$
\mathcal{L} I_\phi(t, s) = P_\phi(t, s)
$$

with $P_\phi(t, s)$ in $L^g_{\phi(\xi, t-s)}(\phi(t, s))$ for any $t$ and $s$. Now, we seek $E(t, s)$ in the form

$$
E(t, s) = I_\phi(t, s) + \int_0^t I_\phi(t, \theta) W(\theta, s) d\theta.
$$

Then, $W(t, s)$ must satisfy

$$
P_\phi(t, s) - i W(t, s) + \int_0^t P_\phi(t, \theta) W(\theta, s) d\theta = 0.
$$

Set
\[ W_v(t, s) = -iP_\varphi(t, s), \]
\[ W_v(t, s) = -i\int_s^t P_\varphi(t, \theta) W_{v-1}(\theta, s) d\theta \quad (\nu \geq 2). \]

Then, in the formal sense \( W(t, s) = \sum_{\nu=1}^\infty W_v(t, s) \) is a solution of (3.7). From (3.8), \( W_v(t, s) \) for \( \nu \geq 2 \) has the form
\[ W_v(t_0, s) = (-i)^i \prod_{j=0}^{t_0} t_1 P_\varphi(t_1, t_2) \cdots P_\varphi(t_{v-1}, s) dt_{v-1} \cdots dt_0. \]

Hence, substituting \( W(t, s) = \sum_{\nu=1}^\infty W_v(t, s) \) with (3.9) into (3.6), the fundamental solution \( E(t, s) \) can be written formally in the form
\[ E(t, s) = I_\varphi(t, s) - i\int_s^t I_\varphi(t, s) P_\varphi(t_0, s) dt_0 \]
\[ + \sum_{\nu=2}^\infty (-i)^i \prod_{j=0}^{t_0} t_1 P_\varphi(t_1, t_2) \cdots P_\varphi(t_{v-1}, s) dt_{v-1} \cdots dt_0. \]

In what follows we give the precise meaning for (3.10). From 3° of Theorem 2.3 in [14] we have \( \varphi(t, t_0) \# \varphi(t_0, t_1) \# \cdots \# \varphi(t_{v-1}, s) = \varphi(t, s) \). Hence, if \( T_\varphi(\leq T_0) \) is small, from Theorem 2.1 there exist symbols \( \tilde{w}_v(t, \tilde{t}; x, \xi) \) in \( M^0(t, t_0, s; x, \xi) \) and \( \tilde{w}_v(t, \tilde{t}; x, \xi) \) in \( M^0(t, t_0, s; x, \xi) \) such that for \( j = 0, 1 \)
\[ |\partial_t^\alpha \partial_x^\beta D_t^\gamma \tilde{w}_v(t, \tilde{t}; x, \xi) | \leq C_0 M^{-(|\alpha|+|\beta|) + \delta}[\tilde{t}^0(t) \tilde{t}^1(t) \cdots \tilde{t}^n(t) \tilde{\xi}^0 \tilde{\xi}^1 \cdots \tilde{\xi}^n(t)]^{-|\alpha|}, \]
\[ |\partial_t^\alpha \partial_x^\beta D_t^\gamma \tilde{w}_v(t, \tilde{t}; x, \xi) | \leq C_0 M^{-(|\beta|+\delta)} \delta \tilde{t}^0(t) \tilde{t}^1(t) \cdots \tilde{t}^n(t) \tilde{t}^{n+1}(t) \tilde{\xi}^{-|\alpha|} \]
for any \( N \) and
\[ I_\varphi(t, s) P_\varphi(t_0, s) \cdots P_\varphi(t_{v-1}, s) \]
\[ = \tilde{w}_v(t, \tilde{t}; x, \xi) + \tilde{w}_v(t, \tilde{t}; x, \xi) \]
\[ \tilde{w}_v(t, \tilde{t}; x, \xi) | \]

Here, we have applied Theorem 2.1 noting that the order of the above operator becomes \( w_0 \) because the order of \( I_\varphi(t, s) \) is zero. Define
\[ \tilde{w}_v(t, s; x, \xi) = (-i)^i \int_s^t \cdots \int_s^{t_0} \tilde{w}_v(t, \tilde{t}; x, \xi) dt_{v-1} \cdots dt_0 \quad (t_0 = t) \]
and
\[ \tilde{w}_v(t, s; x, \xi) = (-i)^i \int_s^t \cdots \int_s^{t_0} \tilde{w}_v(t, \tilde{t}; x, \xi) dt_{v-1} \cdots dt_0. \]

Then, they satisfy for \( j = 0, 1 \)
\[ |\partial_t^\alpha \partial_x^\beta D_t^\gamma \tilde{w}_v(t, s; x, \xi) | \]

Take $T_o \leq T_0^*$ such that $T_o C_0 < 1$. Then, the series $\sum_{n=1}^{\infty} \hat{r}_n(t, s; x, \xi)$ converges and we can see the sum $\hat{r}(t, s; x, \xi)$ belongs to $M_{i, i} (G_{G(d)})$ if $0 \leq t, s \leq T_o$. Note that the series $\sum_{n=1}^{\infty} \omega_n(t, s; x, \xi)$ has a meaning as an operator from $\hat{r}^d$ into itself if $a d < 1$ and as an operator from $\hat{r}^d(M)$ into $\hat{r}^d$ for a small $M$ if $a d = 1$. Summing up the above results, we obtain Theorem 1.

**Remark.** In the expression (9) of Theorem 1 we set $R(t, s) = \hat{r}(t, s; x, D_x) I_d(t, s)$. This belongs to $G_{G(d)}, \Phi_i(t, s) = G_{G(d)}$ we see that $R(t, s)$ belongs to $G_{G(d)}$.

Next, we consider a hyperbolic system

$$ (3.11) \quad L = D_t - \begin{pmatrix} \lambda_1(t, X, D_x) & 0 \\ \vdots & \ddots & \ddots \\ 0 & \cdots & \lambda_l(t, X, D_x) \end{pmatrix} + (b_{jk}(t, X, D_x)) $$

with the diagonal principal part, where $\lambda_j(t, x, \xi)$ are real symbols in $M^p_d(S^\delta_d)$ and $b_{jk}(t, x, \xi)$ belong to $M^p_d(S^\delta_d)$ ($0 \leq \sigma \leq 1/d$). Let $\phi_j(t, s; x, \xi)$ be a solution of (3.1) with $\lambda = \lambda_j$. Then, we have by the similar discussions for the proof of Theorem 1

**Theorem 3.2.** The fundamental solution $E(t, s)$ for (3.11) can be represented in the form

$$ (3.12) \quad E(t, s) = \sum_{m=1}^{l} E_{0, m, \Phi_m}(t, s) $$

$$ + \sum_{\nu=1}^{\infty} \sum_{\mu \in \Pi_{\nu+1}} \int_s^{t_{\nu-2}} \cdots \int_s^{t_0} E_{\nu, \mu, \Phi_{\mu}}(t, t_0, \cdots, t_{\nu-1}, s) dt_{\nu-1} \cdots dt_0 $$

$$ + R(t, s) \quad (t = t) $$

when $0 \leq t, s \leq T_o$. Here, $T_o$ is a small constant, $\Pi_{\nu+1} = \{ \mu = (m_1, \cdots, m_{\nu+1}) \}$, $m_j = 1, \cdots, l$ and $\Phi_{\mu}(t, t_0, \cdots, t_{\nu-1}, s) = \Phi_{m_1}(t, t_0) \# \Phi_{m_2}(t_0, t_1) \# \cdots \# \Phi_{m_{\nu+1}}(t_{\nu-1}, s)$.
for \( \mu=(m_1, \ldots, m_{u+1}) \in \Pi_{u+1} \). In (3.12) \( E_{\nu, \mu, \omega}(t, t_0, \ldots, t_{u-1}, s) \) for \( \mu \in \Pi_{u+1} \) is a Fourier integral operator with symbol \( e_{\nu, \mu}(t, t_0, \ldots, t_{u-1}, s; x, \xi) \) satisfying
\[
|\partial_t^a D_x^b e_{\nu, \mu}| \leq C_0 M^{-(|a|+|b|)} a! b! \langle \xi \rangle^{n_a-|a|}
\]
and \( R(t, s) \) is a regularizer with a symbol in \( \mathcal{R}_{G, \omega} \).

From (3.12) we can investigate the propagation of singularities for a solution \( U(t) \) of the Cauchy problem
\[
\begin{align*}
\mathcal{L} U(t) &= 0 \quad 0 < t \leq T_0, \\
U|_{t=0} &= G.
\end{align*}
\]
(3.13)
The details are left in the future. The author is not convinced that the similar result to (3.34) in [13] holds.

In the remainder of this section we give a method of reducing a Cauchy problem
\[
\begin{align*}
Lu &= 0, \\
\partial_t^a u|_{t=0} &= h_j, \quad j=0, 1, \ldots, m-1
\end{align*}
\]
(3.14)
for a hyperbolic operator
\[
L = D_t^n + \sum_{j=0}^{m-1} \sum_{|\alpha|+|\beta|=m} a_{j, \alpha}(t, x) D_x^{\beta} D_t^j
\]
(3.15)
to a Cauchy problem (3.13) for a hyperbolic system \( \mathcal{L} \) of the form (3.11). Here, \( a(t, x) \in M_1(\gamma^d) \) means that it satisfies
\[
|\partial_t^a D_x^b a(t, x)| \leq C_j M_j^{-|a|} a! b! \gamma^d
\]
for constants \( C_j \) and \( M_j \) independent of \( \alpha \). We assume that there exist regularly hyperbolic operators \( L_1, L_2, \ldots, L_r \) such that \( L \) has a form
\[
L = L_1 L_2 \cdots L_r + \sum_{j=0}^{m-q} a_j(t, X, D_x) D_t^j
\]
(3.16)
for \( a_j \in M_1(S_{G, \omega}^{m-q}) \), where \( q \) is an integer satisfying \( 1 \leq q \leq r \).

**Proposition 3.3.** Set \( \sigma = (r-q)/r \). Then, there exists a hyperbolic system \( \mathcal{L} \) of the form (3.11) with \( b_j(t, x, \xi) \) in \( M_1^0(S_{G, \omega}^{m-q}) \) such that the Cauchy problem (3.14) can be reduced to an equivalent Cauchy problem (3.13).

In the following we disregard the contribution of regularizers and
the equality means that it holds modulo regularizers in $\mathcal{R}_{G(\Omega)}$.

**Proof of Proposition 3.3.** We divide the proof into two steps.

1. Denote the order of $L_k$ by $s_k$ and let $\lambda_k(t, x, \xi), j = 1, \ldots, s_k$, be characteristic roots of $L_k$. We may assume $\lambda_k(t, x, \xi) \in M_t(S_{G(\Omega)}^{s_k})$ by multiplying a cut function with respect to $\xi$ if necessary. Since $L_k$ is a regularly hyperbolic operator, there exist $\lambda_k(t, x, \xi) \in M_t(S_{G(\Omega)}^{s_k})$ such that

$$L_k = (D_t - \lambda_{k,1}(t, x, D_x) - \lambda_{k,1}(t, x, D_x)) \cdots$$

$$\times (D_t - \lambda_{k,s_k}(t, x, D_x) - \lambda_{k,s_k}(t, x, D_x))$$

$$+ \sum_{j=0}^{s_k-1} b_{k,j}(t, x, D_x) D_t^j$$

with $b_{k,j} \in M_t(S_{G(\Omega)}^{s_k})$. Denote $\bar{s}_0 = 0, \bar{s}_k = s_1 + \cdots + s_k$ and

$$\bar{a}_j = D_t - \lambda_{k,j-s_k-1}(t, x, D_x) - \lambda_{k,j-s_k-1}(t, x, D_x)$$

if $s_k - 1 < j \leq \bar{s}_k$.

Then, $L$ has the form

$$L = \bar{a}_1 \bar{a}_2 \cdots \bar{a}_m + \sum_{j=2}^{m-1} b_j(t, x, D_x) \bar{a}_j \cdots \bar{a}_m + \sum_{j=0}^{m-q} \bar{b}_j(t, x, D_x) D_t^j$$

with $b_j(t, x, \xi) \in M_t(S_{G(\Omega)}^0)$ and $\bar{b}_j(t, x, \xi) \in M_t(S_{G(\Omega)}^{m-q})$. Set $II = \{0\} \cup \bigcup_{r' = 1}^r II_{r'}$ for

$$II_{r'} = \{J = (j_1, \ldots, j_k) ; \bar{s}_{r-1} + 1 \leq j_1 \leq \cdots < j_k \leq \bar{s}_r - 1\}$$

and denote the number of elements in $II$ by $l_r$, where $\bar{s}_r = s_r + \cdots + s_r$, and $\mu_r(J) = \bar{s}_r - \{\text{the number of elements in} : \{j_1, \ldots, j_k\} \cap \{\bar{s}_{r-1} + 1, \ldots, \bar{s}_r\}\}$ for $J = (j_1, \ldots, j_k)$. Then, by the method of [20] (see also §111 in Appendix of [12]) we can prove that $L$ in (3.17) has the form

$$L = \bar{a}_1 \bar{a}_2 \cdots \bar{a}_m + \sum_{j=2}^{m-1} b_j(t, x, D_x) \bar{a}_j \cdots \bar{a}_m + \sum_{j=0}^{m-q} b_j(t, x, D_x) D_t^j$$

with $b_j(t, x, \xi) \in M_t(S_{G(\Omega)}^{s_k}) (1 \leq j \leq q), b_j(t, x, D_x) \in M_t(S_{G(\Omega)}^{r_k-1}) (q < j \leq r)$ and $b_j(t, x, D_x) \in M_t(S_{G(\Omega)}^{r_k})$, where $II' = \{(j_1, \ldots, j_k) \in II ; k \leq m - r\}$, $\bar{a}^{J'} = \bar{a}_1 \cdots \bar{a}_k$ for $J = (j_1, \ldots, j_k)$ and $\bar{a}^J = I$ for $J = 0$. In fact, first we prove

$$\sum_{j=0}^{m-q} b_j(t, x, D_x) D_t^j = \sum_{J \in II', |J| = r_k-1} \sum_{j=0}^{s_{r_k-1} - q + 1} (\sum_{j=0}^{s_{r_k-1} - q + 1} b_{j,J}(t, x, D_x) D_t^j) \bar{a}^{J'}$$
with \( b_{j,i} \in M_r(S_{\overline{G}_{\alpha,d}}^{1-q-i}) \) and \( b_i^j \in M_r(S^{-d}_{\overline{G}_{\alpha,d}}) \), where \(|J|\) denotes the length \( k \) of \( J=(j_1, \ldots, j_k) \); and next we prove

\[
\sum_{J \in \Pi_r} \sum_{|J|=r-1} \sum_{j=0}^{l_r-1} b_{j,i}^j(t, X, D_x) D_{j}^i \partial^j
\]

with \( b_{j,i} \in M_r(S_{\overline{G}_{\alpha,d}}^{1-q-i}) \) and \( b_i^j \in M_r(S^{-d}_{\overline{G}_{\alpha,d}}) \). Repeating this method we can prove (3.18).

(II) Suppose that a function \( u \equiv u(t, x) \) satisfies \( Lu=0 \). Set for \( J \in \Pi_r \)

\[
\begin{cases}
  u_o = A^{(r-1)\sigma} u & (J=0), \\
  u_j = A^{(r-1)\sigma} \partial^j u & \text{for } |J| \leq m-r, \\
  u_j = A^{(r-1)\sigma} \partial^j u & \text{for } |J| = m-r+k \quad (1 \leq k \leq r-1),
\end{cases}
\]

where \( A=\langle D_x \rangle \). Then, by applying the method of §3 in [11] the \( l \)-dimensional vector \( U=(u_j)_{j \in \Pi} \) satisfies \( \mathcal{L}U=0 \) for a system \( \mathcal{L} \) of the form (3.11). In this way we reduce the problem (3.14) to a problem (3.13). The fact that (3.14) and (3.13) are equivalent is verified by the method in [18] and [11]. This concludes the proof.

Q. E. D.

From this proposition and Theorem 3.2 we can prove that (3.14) is \( \gamma^d \)-correct (or \( \gamma^d \)-well-posed) in the sense of [5] if \( d \leq r/(r-q) \) and we can investigate the propagation of singularities for a solution of the Cauchy problem (3.14). In the case of constant multiplicity we can improve Proposition 3.3 as follows.

**Proposition 3.4.** Assume that the operator (3.15) is a hyperbolic operator with constant multiplicity and its coefficients \( a_{j,i,a}(t, x) \) satisfy

\[
|\partial^k D_x^\alpha a_{j,i,a}(t, x)| \leq CM^{-\frac{(a+|\beta|)k!}{d}} \text{ for } (t, x) \in [0, T] \times R^n_x.
\]

Then, the problem (3.14) is reduced to the problem (3.13) for a perfectly diagonalized operator \( \mathcal{L} \) of the form (7) in the Introduction. Moreover, if the operator (3.15) satisfies (3.16), then the lower order terms \( B_j(t) \)
in (7) are pseudo-differential operators of order $\sigma$ with $\sigma = (r-q)/r$.

We will prove this proposition in Section 6. We note that from Proposition 3.4 we obtain Corollary 3 with $\sigma = \max \{(m_j-1)/m_j\}$ replaced by $\sigma = (r-q)/r$ if (3.15) satisfies (3.16).

Next, we turn to the problem studied in [3] and [21]. Consider a regularly hyperbolic operator

\begin{equation}
L = D_t^2 - \sum_{j,k} b_{jk}(t, x) D_{xj} D_{xk} + b(t, x) D_t + \sum_{j} b_j(t, x) D_{xj} + c(t, x) \quad \text{on } [0, T]
\end{equation}

with continuous coefficients. We assume

\begin{align}
&b_{jk}(t, x), b(t, x), b_j(t, x), c(t, x) \in \mathcal{D} \quad \text{for any fixed } t, \\
&|D_x^a (b_{jk}(t, x) - b_{jk}(s, x))| \leq C|t-s|^a M^{-|a|} \quad (t, s \in [0, T]), \\
&|\sum_{j,k} b_{jk}(t, x) \xi_j \xi_k| \geq \delta |\xi|^2 \quad (\delta > 0).
\end{align}

We show Proposition 3.3 with $\sigma = (r-q)/r$ replaced by $\sigma = 1 - \kappa$. We may assume (3.22) holds for all $t, s \in \mathbb{R}$. Take an even function $\chi(t) \in \mathcal{D}$ such that $\int \chi(t) dt = 1$ and $\chi = 0$ if $|t| \geq 1/2$. We approximate $b_{jk}(t, x)$ by

$$b_{jk}(t, x, \xi) = \int \chi((t-s)\langle \xi \rangle) b_{jk}(s, x) ds \langle \xi \rangle.$$  

Then, from the evenness of $\chi$ the symbols $b_{jk}(t, x, \xi)$ belong to $M^1_\xi(S^0_{G(\delta)}) \cap M^1_\xi(S^{-\infty}_G)$ and

$$b_{jk}(t, x, \xi) - b_{jk}(t, x) \in M^1_\xi(S^{-\infty}_G)$$

hold. Denote

$$\lambda(t, x, \xi) = \{ \sum_{j,k} b_{jk}(t, x, \xi) \xi_j \xi_k\}^{1/2} \frac{\chi'(\xi)}{\chi(\xi)}$$

where $\chi'(\xi)$ is a function in $\mathcal{D}$ satisfying $\chi' = 1$ for $|\xi| \geq 2$ and $\chi' = 0$ for $|\xi| \leq 1$. Then, the operator $L$ can be written in the form

$$L = D_t^2 - \lambda(t, X, D_x)^2 + b(t, X) (D_t - \lambda(t, X, D_x)) + \tilde{b}(t, X, D_x)$$

with $\tilde{b}(t, x, \xi) \in M^0(p(S^0_{G(\delta)}))$. Note that $\partial_t \lambda(t, x, \xi) \in M^0(p(S^0_{G(\delta)}))$. Then, $L$ has the form

$$L = (D_t + \lambda(t, X, D_x) + b(t, X)) (D_t - \lambda(t, X, D_x)) + \tilde{b}'(t, X, D_x)$$

with $\tilde{b}'(t, x, \xi) \in M^0(p(S^0_{G(\delta)}))$. For a function $u = u(t, x)$ we set $u_1 = Au$.
and $u_2 = (D_t - \lambda(t, X, D_x)) u$. Then, if $u$ satisfies $Lu = 0$, $U = u_1, u_2$ satisfies $L^* U = 0$ for a system

$$L^*_1 = D_t - \begin{bmatrix} \lambda(t, X, D_x) & A \\ 0 & -\lambda(t, X, D_x) \end{bmatrix} + \begin{bmatrix} \lambda(t, X, D_x) A A^{-1} & 0 \\ b'(t, X, D_x) A A^{-1} & b(t, X) \end{bmatrix}. $$

$L^*_1$ has a lower order term in $M^g_1(S^1_{\mathbb{G}(d)})$. Next, we set

$$N(t) = \begin{bmatrix} 1 & n(t, X, D_x) \\ 0 & 1 \end{bmatrix}$$

for a symbol $n(t, x, \xi) \in M^g_1(S^1_{\mathbb{G}(d)}) \cap M^g_1(S^1_{\mathbb{G}(d)})$ satisfying $n(t, x, \xi) = -\langle \xi \rangle / (2\lambda(t, x, \xi))$ for $|\xi| \geq 2$. Then, using $N(t)^{-1} = \begin{bmatrix} 1 & -n(t, X, D_x) \\ 0 & 1 \end{bmatrix}$ and $\partial_t (\sigma (N(t))) \in M^g_1(S^1_{\mathbb{G}(d)})$ we obtain

\begin{equation}
(3.24) \quad L^*_1 N(t) = N(t) L
\end{equation}

with

\begin{equation}
(3.25) \quad L = D_t - \begin{bmatrix} \lambda(t, X, D_x) & 0 \\ 0 & -\lambda(t, X, D_x) \end{bmatrix} + (b_{jk}(t, X, D_x)) 
\end{equation}

$b_{jk} \in M^g_1(S^1_{\mathbb{G}(d)})$.

In this way, we can prove Proposition 3.3 with $L$ in (3.25).

\section*{§ 4. Calculus of Products of Fourier Integral Operators}

The end of this section is to prove Proposition 2.2, Proposition 2.3 and Proposition 2.5. To begin with, we prove

\textbf{Lemma 4.1.} i) Suppose that a double symbol $r(x, \xi, x', \xi')$ satisfies

\begin{equation}
|r^{(a, a')}_{(\beta, \beta')}(x, \xi, x', \xi')| \leq CM^{-(|\alpha| + |a'| + |\beta| + |\beta'|)}
\end{equation}

\begin{equation}
\times \alpha!\alpha'^!\beta!\beta'^! <\xi'>^m - |\alpha| - |a'| - N \end{equation}

with constants $C$ and $M$ independent of $\alpha, \alpha', \beta$ and $\beta'$. Then, the simplified symbol

$$r_L(x, \xi) = O_{-} - \int e^{-ir'\eta r}(x, \xi + \eta, x + y, \xi) dy d\eta$$

of $r(x, \xi, x', \xi')$ belongs to $S^m_{\mathbb{G}(d)}$.

ii) Suppose that a double symbol $r(x, \xi, x', \xi')$ satisfies

\begin{equation}
|r^{(a, a')}_{(\beta, \beta')}(x, \xi, x', \xi')| \leq C_{a, a'} M^{-(|\beta| + |\beta'| + N)}
\end{equation}

\begin{equation}
\times \beta!\beta'^! N! <\xi'>^{-|\alpha| - |a'| - N} \text{ for any } N
\end{equation}
with a constant $M$ independent of $\alpha$, $\alpha'$, $\beta$, $\beta'$, $N$ and a constant $C_{\alpha,\beta}$ independent of $\beta$, $\beta'$ and $N$. Then, the simplified symbol $r_L(x, \xi)$ of $r(x, \xi, x', \xi')$ belongs to $\mathcal{R}_{G(d)}$.

Proof. First we assume (4.1). Differentiating $r_L(x, \xi)$ with respect to $x$ and $\xi$, we have

$$r_{L(\alpha)}(x, \xi) = \sum_{a \in \mathbb{Z}^n} \left( \frac{\alpha}{\alpha'} \right) \left( \frac{\beta}{\beta'} \right) \int e^{-i(y \cdot \xi)} r_{L(a, a')}(x, \xi') dy d\eta.$$

Here and in what follows we often omit the notion “$O_{-}$”. Using (6.10) of Chap. 1 in [12] we obtain

$$|r_{L(a)}(x, \xi)| \leq CM^{-\left(\alpha' + |\alpha|\right)} a!^{\alpha} \beta!^{\beta} <\xi>^{m-|\alpha|}$$

for new constants $C$ and $M$ independent of $\alpha$ and $\beta$. This shows $r_L(x, \xi) \in S^m_{G(d)}$. Similarly, we can prove ii). Q. E. D.

Now, we prove Proposition 2.2.

**Proof of Proposition 2.2.** Let $p_1(X, D_x) \in S^m_{G(d)}$ and $p_2(x, D_x) \in S^m_{G(d)}$. Then, from Theorem 2.2. of Chap. 10 in [12] the product $p_1(X, D_x) p_2(x, D_x)$ is a Fourier integral operator with a symbol $q(x, \xi)$ defined by

$$q(x, \xi') = O_\xi \int e^{i\phi} p_1(x, \xi) p_2(x', \xi') \, dx' d\xi,$$

where $\phi = x \cdot \xi - x' \cdot \xi + \phi(x', \xi) - \phi(x, \xi')$. Take a function $\chi(\xi)$ in $\gamma^d$ satisfying

(4.3) \quad $0 \leq \chi \leq 1$, \quad $\chi = 1$ \quad ($|\xi| \leq 2/5$), \quad $\chi = 0$ \quad ($|\xi| \geq 1/2$)

and divide $q(x, \xi)$ into two terms:

(4.4) \quad $q(x, \xi) = q_0(x, \xi) + q_\infty(x, \xi)$,

where

$$q_0(x, \xi') = \int e^{i\phi} p_1(x, \xi) \chi((\xi - \xi') / <\xi'>) p_2(x', \xi') \, dx' d\xi,$$

$$q_\infty(x, \xi') = \int e^{i\phi} p_1(x, \xi) (1 - \chi((\xi - \xi') / <\xi'>)) p_2(x', \xi') \, dx' d\xi.$$

Denote $\tilde{P}_\xi \phi(x, \xi, x') = \int_{\mathbb{R}^n} \phi(x + \theta (x - x'), \xi) d\theta$. Then, $\phi$ has the form $\phi = (x - x') \cdot (\xi - \tilde{P}_\xi \phi(x, \xi, x'))$. Hence, using the change of variables:
y = x' - x, \eta = \xi - \hat{\theta}_x \phi (x, \xi', x') we have

\[ q_0(x, \xi') = \int e^{-iy} p_1(x, \eta + \hat{\theta}_x \phi (x, \xi', x + y)) \times \chi (\eta + \hat{\theta}_x \phi (x, \xi', x + y) - \xi') \bigg/ \langle \xi' \rangle p_2(x + y, \xi') dy d\eta. \]

This formula shows that \( q_0(x, \xi) \) is the simplified symbol of a double symbol

\[ q'(x, \xi, x', \xi') = [p_1(x, \xi) \chi (\langle \xi - \xi' \rangle / \langle \xi' \rangle) p_2(x', \xi')] \big|_{\eta = -\nu + \hat{\theta}_x \phi (x, \xi', x')}, \]

which satisfies (4.1). Hence, from i) of Lemma 4.1 we obtain \( q_0(x, \xi) \in \mathcal{S}^{s+m'}_{G(d)}. \)

Now, we prove \( q_\omega (X, D_x) \in \mathcal{R}_{G(d), \phi} \). This result is equivalent to \( q_\omega (X, D_x) \in \mathcal{R}_{G(d)} \) if we set \( q_\omega (x, \xi) = q_\omega (x, \xi) e^{i(x_0\xi)} \) for \( J(x, \xi) = \phi (x, \xi) - x \cdot \xi \). So, we may prove \( q_\omega (x, \xi) \in \mathcal{R}_{G(d)} \). It has the form

\[ q_\omega (x, \xi') = \int e^{i}\phi \ p_1(x, \xi) (1 - \chi (\langle \xi - \xi' \rangle / \langle \xi' \rangle)) p_2(x', \xi') dx' d\xi, \]

where \( \phi = x \cdot \xi - x' \cdot \xi + \phi (x', \xi') - x \cdot \xi'. \) Set \( \Phi_a(x', \xi; x, \xi') = e^{-i\phi} \partial_{\xi'} e^{i\phi}. \) Then, it satisfies

\[ |D_x^a D_x^b \partial_{\xi} \Phi_a| \leq C_{a, \gamma} M^{-(|\beta| + |\beta'|) |\beta| |\beta'|} \langle x - x' \rangle_a. \]

Using this and \( \Phi_a \phi = \xi - \xi' \) we have

\[ q_\omega (x, \xi') = \sum_{\alpha, \gamma = a} \left( \frac{\alpha}{\gamma} \right) D_x^a \int e^{i\phi} \Phi_a (x', \xi; x, \xi') \times p_1(x, \xi) \partial_{\xi'} \left( (1 - \chi (\langle \xi - \xi' \rangle / \langle \xi' \rangle)) p_2(x', \xi') \right) dx' d\xi, \]

\[ = \sum_{\alpha, \gamma, \gamma', \gamma''} \left( \frac{\alpha}{\gamma} \right) D_x^a \int e^{i\phi} (\xi - \xi') \partial_{\xi'} \left( (1 - \chi (\langle \xi - \xi' \rangle / \langle \xi' \rangle)) p_2(x', \xi') \right) dx' d\xi. \]

Set \( L = -i |\xi - \xi' + \Phi_a \phi (x', \xi')|^{-2} (\xi - \xi' + \Phi_a \phi (x', \xi')) \cdot \Phi_a \) and \( L_2 = (1 + |x - x'|^{-2})^{-1} (1 - i(x - x') \cdot \Phi_a) \). Then, integrating by parts we have

\[ q_\omega (x, \xi') = \sum_{\alpha, \gamma, \gamma', \gamma''} \left( \frac{\alpha}{\gamma} \right) D_x^a \int e^{i\phi} (\Phi_a (x', \xi; x, \xi')) \times p_1(x, \xi) \partial_{\xi'} \left( (1 - \chi (\langle \xi - \xi' \rangle / \langle \xi' \rangle)) p_2(x', \xi') \right) dx' d\xi. \]

In (4.6) we take \( l = |\beta'| + |\alpha'| + N + n + 1 + \lbrack \max (m, 0) + \max (m', 0) \rbrack. \)
Then, we obtain from \[ | -\xi + F_\phi(x, \xi') | \geq C |\xi - \xi' | \geq C <\xi'> \]

(4.7) \[ | q_{\gamma}(\beta)(x, \xi') | \leq C_M^{(|\beta| + N)} \beta! N!^d <\xi'>^{-|\alpha| - N} \] for any \( N \).

This shows \( q_{\alpha} \in \mathcal{R}_{\gamma}(\phi) \). Consequently, we have proved (2.5). Similarly, we can prove (2.6). The above proof is also valid for the proof of (2.7) if we use Lemma 4.1-ii) instead of using Lemma 4.1-i).

This concludes the proof of Proposition 2.2. Q. E. D.

**Remark.** Throughout this section we denote by \( \chi(\xi) \) a function in \( \mathcal{S} \) satisfying (4.3).

Using the method of the proof of (2.12) we can improve Lemma 2.1 in [23] as follows.

**Lemma 4.2.** Let \( \phi(x, \xi) \) belong to \( \mathcal{P}_{\gamma}(\phi) \). Then, we have the following:

i) The inverse function \( \xi = \tilde{\phi}^{-1}_{\xi}(x, \eta, \xi') \) of \( \eta = \int_0^1 \tilde{\phi}_{\xi}(x' + \theta (x - x'), \xi') d\theta \) satisfies (2.3)-a), (2.3)-b) in [23] and

(4.8) \[ | \partial_{\xi} \tilde{\phi}_{\xi} D_{\xi'} D_{\xi'} \tilde{\phi}^{-1}_{\xi}(x, \eta, \xi') | \leq C_M^{-|\alpha| + |\beta| + |\beta'|} \alpha!^d \beta!^d \beta'!^d <\eta>^{-|\alpha|}. \]

ii) The inverse function \( x' = \tilde{\phi}^{-1}_{\xi}(\xi, y', \xi') \) of \( y' = \int_0^1 \tilde{\phi}_{\xi}(x', \xi' + \theta (\xi - \xi')) d\theta \) satisfies

(4.9) \[ | \partial_{\xi} \partial_{\xi'} D_{\xi'} \tilde{\phi}^{-1}_{\xi}(\xi, y', \xi') - y' | \leq C_M^{-|\alpha| + |\alpha'| + |\beta|} \alpha!^d \beta!^d \beta'!^d, \]

(4.10) \[ | \partial_{\xi} \partial_{\xi'} D_{\xi'} \tilde{\phi}^{-1}_{\xi}(\chi(\xi - \xi') \langle \xi' \rangle) (\tilde{\phi}_{\xi}^{-1}(\xi, y', \xi') - y') | \leq C_M^{-|\alpha| + |\alpha'| + |\beta|} \alpha!^d \beta!^d \beta'!^d <\xi'>^{-|\alpha| - |\alpha'|}. \]

Now, we prove Proposition 2.3.

**Proof of Proposition 2.3.** Let \( p_{\phi}(X, D_x) \) be a Fourier integral operator in \( S_{\gamma}(\phi) \). Then, by (1.29)-(1.30) in Chap. 10 of [12] \( p_{\phi}(X, D_x) I_{\phi^*} \) is a pseudo-differential operator with a symbol

\[ q(x, \xi) = O_I \int e^{-iy \cdot x} q'(x, \xi + \eta, x + y) dy d\eta \]

for
\( q'(x, \xi, x') = \{ p(x, \xi) | \det \frac{\partial}{\partial \xi} \hat{p}_x \phi(x, \xi, x') |^{-1} \}_{\xi = \phi^{-1} x, \xi'} \)

where for a vector \( f = (f_1, \ldots, f_n) \) of functions \( f_j(x, \xi) \) \( \frac{\partial}{\partial \xi} f \) denotes \((\frac{\partial f_j}{\partial \xi_k})_{k=1}^n \).

Write
\[
q(x, \xi) = \int e^{-\frac{i}{\gamma} \eta (\eta \langle \xi \rangle)} q'(x, \xi + \eta, x + \eta) dy d\eta + \int e^{-\frac{i}{\gamma} \eta (1 - \chi(\eta \langle \xi \rangle))} q'(x, \xi + \eta, x + \eta) dy d\eta.
\]

Then, using Lemma 4.2-i) we see that the first term belongs to \( S^m \) because of Lemma 4.1-i) and the second term belongs to \( \mathcal{R}_\Delta \) if we use the integration by parts. This shows \( S^m \). Next, we assume \( p(X, D_x) \in \mathcal{R}_\Delta \). Then, the product \( p(X, D_x)I_\phi \) is a pseudo-differential operator \( q(X, D_x) \) with a symbol \( q(x, \xi, x') = p(x, \xi) e^{iJ(x, \xi)} \). Since \( q(x, \xi, x') \) satisfies
\[
| q^{(0)}(x, \xi, x') | \leq C_\alpha M^{-\frac{1}{2} |\alpha| + |\beta| - |\xi|} N^{\frac{1}{2} |\xi'| - |\xi|} - N
\]
for any \( N \), we can prove that its simplified symbol \( q_L(x, \xi) \) belongs to \( \mathcal{R}_\Delta \).

These results show the first formula of (2.10). In the same way we can prove the second formula of (2.10) by using Lemma 4.2-ii).

Q. E. D.

The remainder of this section is devoted to the proof of Proposition 2.5. We divide it into three steps.

(I) We follow the proof of Proposition 2.8 in [23]. Set \( \Phi(x, \xi) = \phi_1 \# \phi_2(x, \xi) \) and set
\[
\psi(x, \xi', \xi) = \phi_1(x, \xi) - x' \cdot \xi + \phi_2(x', \xi') - \Phi(x, \xi').
\]

Then, \( I_{\psi_1} I_{\psi_2} \) is a Fourier integral operator with the phase function \( \Phi(x, \xi) \) and a symbol \( p(x, \xi) \) defined by
\[
(4.11) \quad p(x, \xi') = O_{\gamma} \int e^{i \psi} dx' d\xi.
\]

Set \( \chi_\alpha(\xi, \xi') = 1 - \chi((\xi - \xi')/\langle \xi' \rangle) \) and consider
\[
p_\alpha(x, \xi') = O_{\gamma} \int e^{i \psi} \chi_\alpha(\xi, \xi') dx' d\xi.
\]
For \( \phi' = \phi_1(x, \xi) + \phi_2(x', \xi') - \Phi(x, \xi) \) we have

\[
\delta_i^j D_k^j \phi' = \epsilon^{ij} \sum_{k=0}^{|\beta|} \Psi_{k;a,\beta}(x', \xi; x, \xi').
\]

with symbols \( \Psi_{k;a,\beta}(x', \xi; x, \xi') \) satisfying

\[
|\delta_i^j D_k^j \Phi_{k;a,\beta}| \leq C_{a,\gamma} M^{-|\beta|+|\gamma|} \beta^j \delta^k |x-x'|^{|\alpha|} \xi - \xi'|^k.
\]

Hence, following the way of (2.34) – (2.36) in [23] we obtain

\[
p_{\alpha}(x, \xi') = \sum_{k=0}^{|\beta|} \int e^{ik} p_{\alpha}(x', \xi; x, \xi') d\xi d\xi'
\]

for symbols \( p_{\alpha}(x', \xi; x, \xi') \) satisfying

\[
|\delta_i^j D_k^j p_{\alpha}(x', \xi; x, \xi')| \leq C_{a,\gamma} M^{-|\beta|+|\gamma|} \beta^j \delta^k |x-x'|^{|\alpha|} \xi - \xi'|^k.
\]

On supp \( p_{\alpha}(x', \xi; x, \xi') \) the inequality \(|\xi - \xi'| \geq (2/5) \langle \xi' \rangle \) holds. This implies

\[
|F_{\alpha}(\xi)| \geq \frac{1}{6} |\xi - \xi'| \geq \frac{1}{15} \langle \xi' \rangle
\]

if \( \tau < 1/3 \). Set

\[
L_1 = -i |F_{\alpha}(\xi)|^{-1} F_{\alpha}(\xi) F_{\alpha}',
L_2 = (1 + |F_{\alpha}(\xi)|^{-2})^{-1} (1 - iF_{\alpha}(\xi) F_{\alpha}').
\]

Then, by the integration by parts we have

\[
p_{\alpha}(x, \xi') = \sum_{k=0}^{|\beta|} \int e^{ik} (L_1)^{n+1+k-N} (L_2)^{n+1+|\alpha|} p_{\alpha}(x', \xi; x, \xi') d\xi d\xi'.
\]

Note \( 1 + |F_{\alpha}(\xi)| \geq C <x-x'> \). Then, we obtain from (4.15) and (4.16)

\[
|p_{\alpha}(x, \xi')| \leq C_{a} M^{-|\beta|+|\gamma|} \beta^j \delta^k \langle \xi' \rangle^{-N}
\]

for any \( N \).

This implies

\[
p_{\alpha}(x, \xi) \in \mathcal{R}_{G(\xi)}.
\]

(II) For \( \chi_{F}(\xi, \xi') = \chi((\xi - \xi')/\langle \xi' \rangle) \) we consider

\[
p_{\alpha}(x, \xi') = \sum_{k=0}^{|\beta|} \int e^{ik} \chi_{F}(\xi, \xi') d\xi d\xi'.
\]

Let \( \{ X, \xi \} (x, \xi) \) be the solution of

\[
X = F_{\alpha}(x, \xi),
\]
\[
\xi = F_{\alpha}(x, \xi).
\]
Using a change of variables: \( x' = X(x, \xi') + y, \xi = \Xi(x, \xi') + \eta, \) we write

\[
(4.19) \quad p_0(x, \xi') = O_1 \int \int e^{-i\phi(y, \eta; x, \xi')} \bar{\rho}_0(\eta; x, \xi') \, dy \, d\eta.
\]

Here,

\[
(4.20) \quad \bar{\rho}_0(\eta; x, \xi) = \chi((\Xi(x, \xi) + \eta - \bar{\xi})/\langle \xi \rangle)
\]

and

\[
(4.21) \quad \phi = \phi(y, \eta; x, \xi) = -\phi(x, X(x, \xi) + y; \Xi(x, \xi) + \eta, \xi) = y \cdot \eta - \{\phi_1(x, \Xi + \eta) - X \cdot \eta - \phi_1(x, \Xi)\} - \{\phi_2(X + y, \xi) - y \cdot \Xi - \phi_2(X, \xi)\}.
\]

As in the proof of Theorem 2.3 in [13] we divide \( p_0(x, \xi) \) into two terms

\[
(4.22) \quad p_0(x, \xi) = p_{0,0}(x, \xi) + p_{0,\infty}(x, \xi);
\]

\[
(4.23) \quad p_{0,0}(x, \xi) = O_1 \int \int e^{-i\phi} \bar{\rho}_0(\eta; x, \xi) \chi((\langle \xi \rangle^2 |y|^2 + |\eta|^2)/\langle \xi \rangle^2) \, dy \, d\eta;
\]

\[
(4.24) \quad p_{0,\infty}(x, \xi) = O_1 \int \int e^{-i\phi} \bar{\rho}_0(\eta; x, \xi) \chi((\langle \xi \rangle^2 |y|^2 + |\eta|^2)/\langle \xi \rangle^2) \, dy \, d\eta,
\]

where a positive constant \( \varepsilon \) (\( \lessgtr 1/2 \)) is determined in the step (III). In this step we will prove

\[
(4.25) \quad p_{0,\infty} \in S^0_{G(\delta)} \cap \mathcal{R}_{G(\delta)}.
\]

On the support of the integrand of (4.24) we have

\[
|\partial_x^\alpha D_y^\beta \partial_{\xi}^\delta \phi| \leq CM^{-\langle |\alpha| + |\beta| + |\gamma| - |\delta| \rangle} \alpha! \beta! \gamma! \delta! \langle \langle \xi \rangle \rangle^{-|\alpha| - |\beta| - |\gamma| - |\delta|} \langle \langle \xi \rangle \rangle^{2 - |\alpha| - |\beta| - |\gamma| - |\delta|}.
\]

because of

\[
\phi = y \cdot \eta - \left\{ \int_0^1 \frac{1}{\partial_x^\alpha D_y^\beta \partial_{\xi}^\delta \phi} (x, \Xi + \theta \eta) \, d\theta \cdot \eta - X \cdot \eta \right\} - \left\{ \int_0^1 \frac{1}{\partial_x^\alpha D_y^\beta \partial_{\xi}^\delta \phi} (X + \theta y, \xi) \, d\theta \cdot y - y \cdot \Xi \right\}.
\]

Hence, \( p_{0,\infty}^{(\alpha)}(x, \xi) \) can be written in the form

\[
(4.26) \quad p_{0,\infty}^{(\alpha)}(x, \xi) = \sum_{\kappa=0}^{\infty} \int \int \int e^{-i\phi} p_{0,\infty}^{(\alpha, \kappa, \beta)}(y, \eta; x, \xi) \, dy \, d\eta
\]

for symbols \( p_{0,\infty}^{(\alpha, \kappa, \beta)}(y, \eta; x, \xi) \) satisfying
\[ (4.27) \quad \left| \partial_\xi D_y^2 p_{0,\omega,(k,\alpha,\beta)} \right| \leq CM^{-\left(\alpha + \beta + \gamma + \delta\right)} \alpha! \beta! \gamma! \delta! k! -d \times \left( \langle \xi \rangle |\gamma| + |\eta| \right)^k \langle \xi \rangle^{-|\alpha| - |\gamma|}. \]

Here, we have used the similar discussions to the one in (4.12) – (4.15). From (2.45) in [23] we have

\[ (4.28) \quad \langle \xi \rangle^2 |\mathcal{F}_x \tilde{\phi}|^2 + |\mathcal{F}_y \tilde{\phi}|^2 \geq \frac{1}{18} \langle \xi \rangle |\gamma| + |\eta| \right)^2 \geq \frac{1}{45} \varepsilon \langle \xi \rangle^2 \quad \text{on supp} p_{0,\omega,(k,\alpha,\beta)}. \]

Set \( L_\alpha = i \langle \langle \xi \rangle^2 |\mathcal{F}_x \tilde{\phi}|^2 + |\mathcal{F}_y \tilde{\phi}|^2 \rangle^{-1} \langle \langle \xi \rangle^2 \rangle |\mathcal{F}_x \tilde{\phi} \cdot \mathcal{F}_y \tilde{\phi} + \mathcal{F}_y \tilde{\phi} \cdot \mathcal{F}_y \tilde{\phi} \rangle \) and integrate each term in (4.26) by parts. Then, we have for any \( N \)

\[ (4.29) \quad p_{0,\omega,(k,\alpha,\beta)}(x, \xi) = \sum_{k=0}^{[\alpha+\beta]} \int \! e^{-i\dot{\phi}} (L_\alpha)^{2n+1+k+N} p_{0,\omega,(k,\alpha,\beta)}(y, \eta; x, \xi) dy d\eta. \]

From (4.27) – (4.29) we have

\[ \left| p_{0,\omega,(k,\alpha,\beta)}(x, \xi) \right| \leq CM^{-\left(\alpha + \beta + \gamma + \delta\right)} \alpha! \beta! \gamma! \delta! N! \langle \xi \rangle^{-|\alpha| - |\gamma|}, \]

which shows (4.25).

(III) In this step we consider (4.23). The following lemma originates from the idea in the III)–step of the proof of Theorem 2.3 in [13].

**Lemma 4.3.** Assume \( \tau_1 + \tau_2 \leq 1/4 \). Then, there exist matrices \( F(y, \eta; x, \xi) \), \( F'(y, \eta; x, \xi) \) and \( G(y, \eta; x, \xi) \) such that

\[ (4.30) \quad \begin{cases} \left| \partial_\xi D_x^2 \partial_\eta D_y^2 F \right| \leq CM^{-\left(\alpha + \beta + \gamma + \delta\right)} \alpha! \beta! \gamma! \delta! \langle \xi \rangle^{-1 - |\alpha| - |\gamma|}, \\ \left| \partial_\xi D_x^2 \partial_\eta D_y^2 F' \right| \leq CM^{-\left(\alpha + \beta + \gamma + \delta\right)} \alpha! \beta! \gamma! \delta! \langle \xi \rangle^{-1 - |\alpha| - |\gamma|}, \\ \left| \partial_\xi D_x^2 \partial_\eta D_y^2 G \right| \leq CM^{-\left(\alpha + \beta + \gamma + \delta\right)} \alpha! \beta! \gamma! \delta! \langle \xi \rangle^{-1 - |\alpha| - |\gamma|} \end{cases} \]

when \( |\eta| \leq \langle \xi \rangle / 2 \) and

\[ (4.31) \quad \dot{\phi} = (y + F \eta) \cdot (F' y + G \eta) \]

holds for \( \dot{\phi} \) in (4.21). Moreover, there exists a constant \( \varepsilon (\leq 1/2) \) such that

\[ (4.32) \quad D(y, \eta; x, \xi) \equiv \begin{vmatrix} \frac{\partial}{\partial y} (y + F \eta) & \frac{\partial}{\partial \eta} (y + F \eta) \\ \frac{\partial}{\partial y} (F' y + G \eta) & \frac{\partial}{\partial \eta} (F' y + G \eta) \end{vmatrix} \geq C > 0 \]

holds when \( |y| + |\eta| \langle \xi \rangle^{-1} \leq \varepsilon \).
Admitting this lemma for the moment we continue the proof of Proposition 2.5. In (4.23) we take a constant $\varepsilon$ as the one in the above lemma. Then, by (4.32) we can change the variables from $(y, \eta)$ to $(z, \zeta) = (y + F\eta, F' y + G\eta)$. Let $y = Y(z, \zeta; x, \xi)$ and $\eta = H(z, \zeta; x, \xi)$ be the inverse function of $z = y + F(y, \eta; x, \xi)\eta$, $\zeta = F'(y, \eta; x, \xi)y + G(y, \eta; x, \xi)\eta$. Then, from (4.23) we have

$$p_{0,0}(x, \xi) = O_{\varepsilon} \int e^{-iz\xi} \{ \xi_0(y; x, \xi, \eta) \chi((\xi\eta)^2, |y|^2 + |\eta|^2) \},$$

This shows that $p_{0,0}(x, \xi)$ is the simplified symbol of

$$p'_{0,0}(x, \xi, \xi') = \xi_0(y; x, \xi, \xi') \chi((\xi'\xi)^2, |y|^2 + |\eta|^2) \},$$

Combining (4.33) with (4.17) and (4.25) we obtain (2.14). This concludes the proof of Proposition 2.5.

Proof of Lemma 4.3. Set

$$B \equiv B(\eta; x, \xi) = \int_0^1 (1 - \theta) \frac{\partial}{\partial \eta} \phi_1(x, \xi + \theta \eta) d\theta,$$

$$B' \equiv B'(y; x, \xi) = \int_0^1 (1 - \theta) \frac{\partial}{\partial b} \phi_2(x + \theta y, \xi) d\theta.$$

Then, from (4.21) and (4.18) we have

$$\tilde{\phi} = y \cdot \eta - B\eta \cdot \eta - B'y \cdot y.$$

Hence, the equation (4.31) holds when $F$, $F'$ and $G$ satisfy

$$F' = -B',$$

$$G + FF' = I,$$

$$GF = -B,$$

where $I$ is the identity matrix.

Denote the norm $\max_{j=1}^n |a_{jk}|$ of a matrix $A = (a_{jk})$ by $||A||$. Note that $B$ and $B'$ are symmetric and

$$||B|| \leq 2\tau_1 \langle \xi \rangle^{-1}, \quad ||B'|| \leq \tau_2 \langle \xi \rangle$$

if $|\eta| \leq \langle \xi \rangle / 2$. In the following we assume that the inequality $|\eta| \leq \langle \xi \rangle / 2$.

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\[ F = -B \sum_{l=0}^{\infty} \frac{(2l)!}{(l+1)!!} (B'B)^l. \]

Then, since \(2\tau_1 \tau_2 < 1/4\) the series (4.36) converges from (4.35) and satisfies
\[ FB'F + F + B = 0. \]

Using this matrix \(F\) we set
\[ G = \mathcal{F} + B'F. \]

Then, the matrices \(F, F'\) and \(G\) satisfy (4.34). From (4.36) and (4.37) we also get (4.30) and
\[ \begin{align*}
||F|| &\leq 3\tau_1 \langle \xi \rangle^{-1}, \\
||F'|| &\leq \tau_2 \langle \xi \rangle, \\
||G - \mathcal{F}|| &\leq \tau_1.
\end{align*} \]

Hence, we obtain (4.32) if \(\tau_1 + \tau_2 \leq 1/4\) and \(|y| + |\eta| \langle \xi \rangle^{-1} \leq \varepsilon\) hold for a small constant \(\varepsilon\). Summing up, we get matrices \(F, F'\) and \(G\) satisfying (4.30)–(4.32).

Q. E. D.

§ 5. Multi-Products of Pseudo-Differential Operators

In this section we prove Theorem 2.6. First, we consider
\[ Q_{\nu+1} = p_1(\xi, D_x) p_2(\xi, D_x) \cdots p_{\nu+1}(\xi, D_x) \]
for pseudo-differential operators \(p_j(\xi, D_x)\) in \(S_{G(d)}^j\).

**Proposition 5.1.** Assume that \(p_j(\xi, \xi)\) satisfy (2.15) with constants \(C_0\) and \(M\) independent of \(j, \alpha\) and \(\beta\). Then, the same result as in Theorem 2.6 holds with \(C_0 = C_\nu\).

**Proof.** We follow the discussions in Section 1 of [23]. We divide the proof into three steps.

(1) It is well-known that the symbol \(q_{\nu+1}(x, \xi)\) of \(Q_{\nu+1}\) has the form
\[ \begin{align*}
q_{\nu+1}(x, \xi) &= \mathcal{O} - \int e^{-i \sum_{j=1}^{\nu+1} \beta_j (x + y^j, \xi + \eta^j) d\gamma^j d\gamma^p} \\
&\quad \left(\gamma^0 = \gamma^{\nu+1} = 0\right),
\end{align*} \]
where

\[(5.3) \quad \phi = \sum_{j=1}^{\nu} \gamma^j \cdot (\eta^j - \eta^{j+1}) = \sum_{j=1}^{\nu} (y^j - y^{j-1}) \cdot \eta^j \quad (y^0 = \eta^{\nu+1} = 0)\]

and \(d\eta^\nu = dy^1 \cdots dy^\nu\), \(d\eta^i = dy^1 \cdots dy^i\) for \(\eta^\nu = (y^1, \ldots, y^\nu)\), \(\eta^i = (\eta^1, \ldots, \eta^i)\). For \(j \leq \nu\) we set

\[(5.4) \quad p_j(x, \xi, x') = (1 + |x - x'|^2)^{-(\alpha + 1)} (1 + i(x - x') \cdot \xi)^{\alpha + 1} p_j^0(x, \xi).

Then, from Lemma 1.5 of [23] we have

\[Q_{\nu+1} = p_1(X, D_x, X') \cdots p_\nu(X, D_x, X') p_{\nu+1}(X, D_x),\]

which implies

\[(5.5) \quad q_{\nu+1}(x, \xi) = O_{-\sum_{j=1}^{\nu} e^{-i\phi} \Pi_{j=1}^{\nu} p_j(x + y^{j-1}, \xi + y^j, x + y^j) \times p_{\nu+1}(x + y^\nu, \xi) d\eta^\nu.

From (5.4) we also have

\[(5.6) \quad |p_j(x, \xi, x')| \leq C_{\alpha, M} e^{-C_{\alpha} |x| + |\xi| + \beta |x| + \beta' |\xi|} \times (1 + |x - x'|)^{-(\alpha + 1)}\]

with new constant \(A_1\) and \(M\) independent of \(j\), \(\alpha\), \(\beta\) and \(\beta'\). Here and in what follows the constant \(C_{\alpha}\) denotes the one in (2.15). For \(x = \xi + \chi\) satisfying (4.3) we set

\[(5.7) \quad \chi_0(\xi, \xi') = \chi((\xi - \xi')/(\xi') \cdot \chi_1(\xi, \xi') = 1 - \chi_0(\xi, \xi').\]

Setting

\[K_x = \{ \kappa = (k_1, \ldots, k_\nu) ; k_j = 0, 1\},\]

we divide \(q_{\nu+1}(x, \xi)\) in (5.5) into \(2^\nu\) terms:

\[(5.8) \quad q_{\nu+1}(x, \xi) = \sum_{\kappa \in K_x} q_{\nu+1}(x, \xi),\]

where \(q_{\nu+1}(x, \xi)\) is a simplified symbol of

\[(5.9) \quad q_{\nu+1}(x^0, \xi^0, \xi^{\nu+1}) = \prod_{j=1}^{\nu} \chi_j(\xi^j, \xi^{j+1}) \prod_{j=1}^{\nu} p_j(x^{j-1}, \xi^j, x^j) p_{\nu+1}(x^\nu, \xi^{\nu+1})\]

\[(x^0 = (x^1, \ldots, x^\nu), \xi^{\nu+1} = (\xi^1, \ldots, \xi^{\nu+1}).\]

(II) First, we consider \(q_{\nu+1}(x^0, \xi^0, \xi^{\nu+1})\) for \(\kappa^0 = (0, 0, \ldots, 0) \in K_x\). On \(\text{supp} q_{\nu+1}(x^0, \xi^0, \xi^{\nu+1})\) the relations \((1/2) \langle \xi^{\nu+1} \rangle \leq \langle \xi^j \rangle \leq 2 \langle \xi^{\nu+1} \rangle\) hold for any \(j\). Hence, \(q_{\nu+1}(x^0, \xi^0, \xi^{\nu+1})\) satisfies
with a constant $M$ replaced by a smaller constant $M$, where $|\alpha^{v+1}| = |\alpha_1| + \cdots + |\alpha^{v+1}|$, $|\beta^v| = |\beta_1| + \cdots + |\beta^v|$, $\alpha^{v+1} = \alpha_1^{v+1} \cdots \alpha^{v+1}$ and $\beta^v = (\beta_1, \ldots, \beta^v)$. Differentiating $q_{v+1, (\alpha^v), L}(x, \xi)$ with respect to $x$ and $\xi$ we have

\begin{equation}
|\partial_\xi^j \partial_x^l q_{v+1, (\alpha^v), L}(x, \xi)| 
\leq C_{v+1} (2^{n+1} A_1)^{-v} (\alpha^{v+1}) \alpha^{v+1} \beta^v \prod_{j=1}^v (1 + |\xi^j - x^j|)^{-v}.
\end{equation}

for constants $A_2$ and $M_2$ independent of $a$, $\beta$, and $p$. Hence, applying Proposition 1.7 in [23] we have

\begin{align*}
|r_L(x, \xi) - \langle r(x, \xi) \rangle| &\leq A_0^v B,
\end{align*}

that is,
Combining this with (5.11) we get
\[
|D_x D_y q_{\nu+1}(x, \xi)| 
\leq \sum_{\alpha, \beta, \gamma} \frac{\alpha! \beta! \gamma!}{\alpha^{\nu+1} \beta! \gamma!} C_{\nu+1} A_0 A_{\gamma} M_2^{-\nu|\alpha| - |\beta|} 
\times \frac{1}{\gamma^{\nu+1} \beta! \gamma!} \langle \xi \rangle^{\nu \sigma - |\alpha|} 
\leq (A_0 A_{\gamma})^\nu C_{\nu+1} (M_2/2)^{-\nu|\alpha|} \alpha! \beta! \gamma! 
\times \bigg\{ \frac{1}{\gamma^{\nu+1} \beta! \gamma!} \langle \xi \rangle^{\nu \sigma - |\alpha|} \bigg\} 
\leq (2^{2\nu} A_0 A_{\gamma})^\nu C_{\nu+1} (M_2/2)^{-\nu|\alpha|} \alpha! \beta! \gamma! 
\langle \xi \rangle^{\nu \sigma - |\alpha|}.
\]
Consequently, if we set \( q_{\nu+1}(x, \xi) = q_{\nu+1}(x, \xi) \), it satisfies (2.17).

(III) We will prove for \( \nu \neq \nu^0 \)

(5.13)
\[
|D_x D_y q_{\nu+1}(x, \xi)| 
\leq A^\nu C_{\nu+1} C_{\nu+1} M_1^{-\nu|\beta| + N} \beta! (N + [\nu \sigma]) \langle \xi \rangle^{\nu \sigma} |
\]
for any \( N \) with constants \( A, C_\nu \) and \( M_1 \) satisfying the same conditions as in Theorem 2.6. Then, setting \( q_{\nu+1}(x, \xi) = \sum_{\kappa \in K_{\nu}} q_{\nu+1}(x, \xi) \) we obtain the desired symbols \( q_{\nu+1}(x, \xi) \) and \( q_{\nu+1}(x, \xi) \).

In the following we fix \( \kappa = (k_1, \ldots, k_\nu) \in K_{\nu} \) \( (\kappa \neq \nu^0) \) and prove (5.13). We change the variables in

\[
q_{\nu+1}(x, \xi) = \sum_{\kappa \in K_{\nu}} q_{\nu+1}(x, \xi) \]

as \( z^j = y^j - y^{j-1} \) \( (j = 1, \ldots, \nu; y^0 = 0) \). Then, we have

\[
q_{\nu+1}(x, \xi) = \sum_{\kappa \in K_{\nu}} q_{\nu+1}(x, \xi) \]

where \( z^j = z^j + \cdots + z^j \) and \( d \xi = d z^1 \cdots d z^\nu \). Take a sequence \( \{ \mu_j \}_{j=1}^\nu \) and a sequence \( \{ \mu_j \}_{j=0}^\nu \) of non-negative integers \( \mu_j \) satisfying

\[
(5.14)
\]
where \( j = 1, 2, \ldots, \nu + 1, \)
Set \( J^0 = \{ j ; k_j = 0 \} \cup \{ \nu + 1 \} \), \( J^1 = \{ j ; k_j = 1 \} \), \( J = \max \{ j ; j \in J^1 \} \) and \( l = \sum_{j \in J^0} \mu_j + [m_{\nu + 1} + 1] \). Then, using the integration by parts we have

\[
q_{\nu + 1, (\omega)}(x, \xi) = \exp \left\{ -i \sum_{j=1}^{\nu} z^j \cdot \gamma_j^j \right\} \prod_{j \in J^1} \left\{ -i \left| \gamma_j^0 \right| -2 \gamma_j^0 \cdot F_{\gamma_j^0} \right\}^{\nu + 1} q_{\nu + 1, (\omega)}(x, \xi + \gamma_j^1, x + \xi^0, \xi) \, d\xi \, d\gamma_j^0
\]

where \( m_{\nu + 1} = m_1 + \cdots + m_{\nu + 1} \) and

\[
q_{\nu + 1, (\omega)}(x^0, x^\nu, \xi^{\nu + 1}) = \Pi_{j \in J^0} \left\{ -i (\xi_j^0 - \xi_j^{\nu + 1}) \cdot (\gamma_j^0 + \cdots + \gamma_j^{\nu + 1}) \right\}^{\nu + 1}
\]

Note that on \( \supp q_{\nu + 1, (\omega)}(x^0, x^\nu, \xi^{\nu + 1}) \) we have

\[
\frac{1}{2} \langle \xi^{\nu + 1} \rangle \leq \langle \xi \rangle \leq 2 \langle \xi^{\nu + 1} \rangle \quad \text{if } j \in J^0,
\]

\[
\left| \xi_j^0 - \xi_j^{\nu + 1} \right| \leq \frac{2}{5} \langle \xi^{\nu + 1} \rangle \quad \text{if } j \in J^1.
\]

Hence from (5.9) and (5.6) we have for \( |\gamma_j| \leq n + 1 \) (\( j = 1, \cdots, \nu \))

\[
|\partial_x^1 \cdots \partial_x^{\nu} q_{\nu + 1, (\omega)}(x^0, x^\nu, \xi^{\nu + 1})|
\]

\[
\leq C_{\nu + 1} A_2 P M_2^{-\nu(N+1)} \left( \sum_{j \in J^0} \mu_j + l + N \right)^{\nu + 1}
\]

\[
\times \Pi_{j=1}^{\nu + 1} \langle \xi_j^0 \rangle \Pi_{j \in J^1} \left\{ \langle \xi_j^0 \rangle \left| \xi_j^0 - \xi_j^{\nu + 1} \right| - 1 \right\}^{\nu_j + 1}
\]

\[
\times \Pi_{j=1}^{\nu} \langle \xi_j^0 \rangle^{\nu_j + 1} \langle \xi_j^{\nu + 1} \rangle^{N + m_{\nu + 1} - l - N}
\]

\[
\times \Pi_{j=1}^{\nu} (1 + |x_j^{l+1} - x_j^0|) - (n+1)
\]

\[
\leq C_{\nu + 1} A_2 P M_2^{-\nu(N+1)} 5^{(\nu+1)n+1} (N + [n + 1])^{\nu + 1} \Pi_{j=1}^{\nu + 1} \langle \xi_j^0 \rangle^{\nu_j + 1}
\]

\[
\times \Pi_{j=1}^{\nu} (1 + |x_j^{l+1} - x_j^0|) - (n+1)
\]

\[
\leq C_{\nu + 1} A_2 C_0 A_0 P M_2^{-N} 24^N (N + [n])^{\nu + 1} \Pi_{j=1}^{\nu + 1} \langle \xi_j^0 \rangle^{\nu_j + 1}
\]

\[
\times \Pi_{j=1}^{\nu} (1 + |x_j^{l+1} - x_j^0|) - (n+1)
\]

with constants \( A_2, A_3, C_0 \) and \( M_2 \) independent of \( \nu \) and \( N \). This
means that \( q_{\nu+1}(x^0, x^\nu, \xi^{\nu+1}) \) satisfies (1.32) in [23] with \( B=A^\nu \times C_{\nu+1}C_0M_2^{-N}2^{2N}(N+[\nu\sigma])^{d}, \delta=0 \) and a sequence \( \{m_j\} \) defined by (5.14). Hence applying Proposition 1.7 in [23] we obtain

\[
|q_{\nu+1}(x, \xi)| \leq A_0^\nu A_3^{\nu+1}C_0M_2^{-N}2^{2N}(N+[\nu\sigma])^{d}\langle \xi \rangle^{\nu+1}\langle \xi \rangle^{-N-\nu+1}.
\]

Here, we remark that in [23] we have used only the condition "\[\sum_{j=1}^{k} m_j \leq 1/2 \] for any \( k \)" for the proof of Proposition 1.7 in [23] instead of using a stronger condition (1.12) in [23]. So, we obtain (5.13) with constants \( A=A_0A_3 \) and \( M_1=2^{-d}M_2/5 \) for the case \( \alpha=\beta=0 \). Similarly we obtain (5.13) for the other cases. This proves Proposition 5.1. Q. E. D.

Next, we consider a multi-product

\[ p_1(X, D_x) p_2(X, D_x) \cdots p_{\nu+1}(X, D_x) \]

of pseudo-differential operators \( p_1(X, D_x) \), assuming that at least one factor \( p_1(X, D_x) \) belongs to \( \mathcal{R}_{G(\delta)} \).

**Proposition 5.2.** In (5.15) we assume that

\[ |\rho^{(\nu)}_{\beta}(x, \xi)| \leq C_{\nu}M^{-|\beta|}2^{\nu}\langle \xi \rangle^{\nu-|\alpha|}\]

and there exists a number \( l \in \{1, 2, \ldots, \nu+1\} \) such that \( p_1(x, \xi) \) satisfies

\[ |\rho^{(\nu)}_{\beta}(x, \xi)| \leq C_{\nu}M^{-(|\beta|+N)}2^{\nu}N^{l}\langle \xi \rangle^{-|\alpha|+N} \]

for any \( N \), where \( M \) is independent of \( \alpha, \beta, j, N \) and \( C_{\alpha} \) is independent of \( \beta, j, N \). Set \( C_{\nu}=\max_{|\alpha|=\nu+1} C_{\alpha} \). Then, the symbol \( q_{\nu+1}(x, \xi) \) of multi-product (5.15) satisfies (2.18).

**Proof.** As in (5.4) we set for \( j(\leq \nu) \)

\[ p_j(x, \xi, x') = (1+|x-x'|)_{-(\nu+1)}(1+i(x-x') \cdot p_j)_{\nu+1}p_j(x, \xi). \]

Then, we have

\[
q_{\nu+1}(x, \xi) = \int \cdots \int e^{-i\phi} \rho^{\nu}_{\beta}(x+y^\nu, \xi + \eta^\nu, x+y^\nu) \times p_{\nu+1}(x+y^\nu, \xi + \eta^\nu) d\eta^\nu d\phi
\]

\[
= \sum_{\kappa \in K_{\nu}} q_{\nu+1}(\kappa, x, \xi) \quad (\nu=0)
\]

for
Here, $\phi$ is a function in (5.3) and $\chi_k(\xi, \xi')$, $k=0, 1$, are defined by (5.7). Note that the following holds: There exists a constant $A_1$ independent of $j$, $\alpha$, $\beta$, $\beta'$ and $N$ such that we have for constants $C^\alpha$ with $C_0^0=1$

$$|p_j^{(\psi)}(x, \xi, x')| \leq A_1 C^\alpha M^{-|\beta|+|\beta'|+1} \beta! \beta'! \langle \xi \rangle^{|\alpha|-|\alpha'|}$$

$$\times (1 + |x-x'|)^{-(\alpha+1)}$$

and

$$|p_j^{(\psi)}(x, \xi, x')| \leq A_1 C^\alpha M^{-|\beta|+|\beta'|+1} \beta! \beta'! \langle \xi \rangle^{|\alpha|-|\alpha'|}$$

$$\times (1 + |x-x'|)^{-(\alpha+1)}$$

if $p_j(x, \xi)$ satisfies (5.17). Hence, by the same way as in (III) in the proof of the preceding proposition we obtain

$$|\partial_x^\kappa D_x^\kappa q_{j+1}(x, \xi)|$$

$$\leq A_1 C^\alpha M^{-|\beta|+|\beta'|+1} \beta! \beta'! \langle \xi \rangle^{|\alpha|-|\alpha'|}$$

$$\times (1 + |x-x'|)^{-(\alpha+1)}$$

for any $N$ for $\kappa \neq \kappa'$. It is clear that $q_{j+1}, \psi, \xi)$ also satisfies (5.18). This shows that the symbol $q_{j+1}(x, \xi)$ of (5.15) satisfies (2.18).

Q. E. D.

Now, we prove Theorem 2.6.

Proof of Theorem 2.6. Assume that $P_j=p_j^0(X, D_x)+p_j^1(X, D_x)$ for $p_j^0(x, \xi)$ and $p_j^1(x, \xi)$ satisfying (2.15) and (2.16). Write $Q_{j+1}=P_1P_2\cdots P_{j+1}$ as

$$Q_{j+1}=P_1^0P_2^0\cdots P_{j+1}^0$$

$$+ P_1^1P_2^0\cdots P_{j+1}^0$$

$$+ P_1^0P_2^1\cdots P_{j+1}^0$$

$$+ \cdots$$

$$+ P_1^0P_2^0\cdots P_{j+1}^0$$

$$+ P_1^0P_2^0\cdots P_{j+1}^0$$

where $P_j=p_j^0(X, D_x)$ and $P_j^1(X, D_x)$. Note that $p_j(x, \xi)(=\sigma(P_j))=p_j^0(x, \xi)+p_j^1(x, \xi)$ satisfies (5.16). Hence, applying Proposition 5.1 to the first term in (5.19) and Proposition 5.2 to the other terms in
(5.19), we obtain \(Q_{\nu+1} = q_{\nu+1}^{0}(X, D_x) + q_{\nu+1}(X, D_x)\) for the symbols \(q_{\nu+1}^{0}(X, \xi)\) and \(q_{\nu+1}(X, \xi)\) satisfying (2.17) and (2.18). This concludes the proof. Q. E. D.

§ 6. Perfect Diagonalization

In this section we give a proof of Proposition 3.4 by showing a method of the perfect diagonalization for a first order hyperbolic system with constant multiplicity. We will begin with giving the product formula for pseudo-differential operators with symbols in \(S_{G(d,1)}^{m}\).

**Proposition 6.1.** Let \(p_j(x, \xi)\) belong to \(S_{G(d,1)}^{m_j}\) \((j = 1, 2)\). Then, there exist symbols \(q^0(x, \xi)\) in \(S_{G(d,1)}^{m_1+m_2}\) and \(q(x, \xi)\) in \(a_{GW}\) such that for \(P_j = p_j(X, D_x)\) the equation

\[
P_1 P_2 = q^0(X, D_x) + q(X, D_x)
\]

holds. Moreover, there exist constants \(C, M\) and \(\mu\) such that

\[
|\partial_\xi D^\alpha_x (q^0(x, \xi) - \sum_{|\gamma| < N} \frac{1}{\gamma!} p^{(\gamma)}(x, \xi) p_{2(\gamma)}(x, \xi))| 
\leq CM^{-|\alpha| - |\beta|} |\xi|^{-|\alpha|} N^{|\alpha|} \xi^{|\beta|} \phi^{|\xi|^2} M_{\xi}^{m_1 + m_2 - N - 2}|\xi|
\]

for \(|\xi| \geq \mu\) hold for any \(N\).

We will prove Proposition 6.1 after some preparations. Let \(\mu^0\) be a constant satisfying for \(j = 1, 2\)

\[
|p_j^{(\gamma)}(x, \xi)| \leq CM^{-|\alpha| - |\beta|} |\xi|^{|m_j - |\alpha|} \phi^{|\xi|^2} M_{\xi}^{m_1 + m_2 - 2N - |\alpha|}
\]

for \(|\xi| \geq \mu^0\).

Define

\[
q_\nu(x, \xi) = \sum_{|\gamma| = \nu} \frac{1}{\gamma!} p^{(\gamma)}(x, \xi) p_{2(\gamma)}(x, \xi).
\]

Then, \(q_\nu(x, \xi)\) satisfy for new constants \(C\) and \(M\) independent of \(\alpha, \beta,\) and \(\nu\)

\[
|q_\nu^{(\gamma)}(x, \xi)| \leq CM^{-|\alpha| - |\beta| + |\gamma|} |\xi|^{|m_j - |\alpha|} \phi^{|\xi|^2} M_{\xi}^{m_1 + m_2 - 2N - |\alpha|}
\]

for \(|\xi| \geq \mu^0\).

Taking account of (6.5) we investigate the following lemma.
Lemma 6.2. Assume that \( q_{\nu}(x, \xi) \in S_{0}^{-\infty}(\nu = 0, 1, \cdots) \) satisfy
\[
|q_{\nu}(x, \xi)\| \leq CM^{-(|\alpha| + |\beta| + \nu)\alpha!}N^d\|\xi\|^{m - |\alpha|} \quad \text{for} \quad |\xi| \geq \mu
\]
with constants \( C, M \) and \( \mu \) independent of \( \alpha, \beta \) and \( \nu \). Then, there exists a symbol \( q(x, \xi) \) in \( S_{0}^{\infty}(\alpha, \beta, \nu) \) such that the inequalities
\[
|\partial_{\xi}^{\beta} D_{x}^{\alpha}(q(x, \xi) - \sum_{\nu=0}^{N-1} q_{\nu}(x, \xi))| \\
\leq CM_{2}^{-1/2(\beta + \nu)\alpha!}N^d\|\xi\|^{m - |\alpha|} \quad \text{for} \quad |\xi| \geq \mu + 1
\]
hold for any \( N \), where the constants \( C_{1} \) and \( M_{1} \) are independent of \( \alpha, \beta \) and \( N \).

Proof. Let \( \{s_{\nu}\}_{\nu=0}^{\infty} \) be a sequence of complex numbers satisfying
\[
||\{s_{\nu}\}||^{2} \equiv \sum_{\nu} |s_{\nu}|^{2}M_{2}^{2\nu} < \infty
\]
for a constant \( M_{2} \). Then, from the discussions in [1], pp. 314–317, we can find a function \( \phi(t) \) such that the inequalities
\[
|\partial_{t}^{\nu} \phi(t)| \leq C||\{s_{\nu}\}||M_{3}^{-k!\nu!} |t|^{-k} \quad (t \neq 0), \\
|\partial_{t}^{\nu} \phi(t) - \sum_{\nu=0}^{N-1} \frac{\nu!}{\nu!} s_{\nu}| \leq C||\{s_{\nu}\}||M_{3}^{-1/2(\beta + \nu)\alpha!}N^d |t|^{-k} \quad (t \neq 0)
\]
hold with constants \( C \) and \( M_{3} \) independent of \( k, \beta \) and \( N \). Apply this result to a sequence
\[
s_{\nu}(x, \xi) = q_{\nu}(x, \xi) \langle \xi \rangle^{\nu} \psi|_{\xi}
\]
with parameters \( x \) and \( \xi \). Note that from (6.6) we have for new constants \( C \) and \( M \)
\[
||\{s_{\nu}\}||^{2} \equiv \sum_{\nu} |s_{\nu}|^{2}M_{2}^{2\nu} < \infty
\]
if we take an appropriate constant \( M_{2} \) in (6.8). Then, we can find a function \( \phi(t; x, \xi) \) satisfying
\[
|\partial_{t}^{\nu} \gamma_{t} D_{x}^{\alpha}(\phi(t; x, \xi))| \leq CM_{4}^{-1/2(\beta + \nu)\alpha!}N^d |\xi|^{m - |\alpha|} |t|^{-k} \quad (t \neq 0),
\]
\[
|\partial_{t}^{\nu} \gamma_{t} D_{x}^{\alpha} \phi(t; x, \xi) - \sum_{\nu=0}^{N-1} \frac{\nu!}{\nu!} s_{\nu}(x, \xi)| \\
\leq CM_{4}^{-1/2(\beta + \nu)\alpha!}N^d |\xi|^{m - |\alpha|} |t|^{-k}
\]
for \( |\xi| \geq \mu, t \neq 0 \).

Take a function \( \chi_{\mu}(\xi) \) in \( \gamma^{d} \) satisfying \( \chi_{\mu} = 1 \) for \( |\xi| \geq \mu + 1 \) and \( \chi_{\mu} = 0 \) for \( |\xi| \leq \mu \). Then, the desired symbol \( q(x, \xi) \) is obtained by setting
\[
q(x, \xi) = \phi(\langle \xi \rangle^{-1}; x, \xi) \chi_{\mu}(\xi),
\]
since the property \( q(x, \xi) \in S^m_{\omega d, d} \) follows from (6.9) and the property (6.7) follows from (6.10).

Q. E. D.

Now, we prove Proposition 6.1.

**Proof of Proposition 6.1.** For a sequence \( \{q_v(x, \xi)\} \) of symbols defined by (6.4) we apply Lemma 6.2. Then, we can find a symbol \( q^0(x, \xi) \) in \( S^m_{\omega d, d} \) satisfying (6.2). Note that for the symbol \( q(x, \xi) \) defined by

\[
q(x, \xi) = O_* \int e^{-i\xi \cdot \eta} p_1(x, \xi + \eta) p_2(x + y, \xi) dyd\eta
\]

we have

\[
P_1P_2 \xi = q(X, D_x).
\]

So, the proof is completed if we prove that the symbol

\[
q(x, \xi) = q(x, \xi) - q^0(x, \xi)
\]

belongs to \( \mathcal{R}_{\omega d} \).

Let \( \chi(\xi) \) be the function in \( \gamma^d \) satisfying (4.3). We write \( q(x, \xi) \) as

\[
(6.11) \quad \tilde{q}(x, \xi) = \check{r}(x, \xi) + \check{r}'(x, \xi),
\]

where

\[
(6.12) \quad \check{r}(x, \xi) = O_* \int e^{-i\xi \cdot \eta} p_1(x, \xi + \eta) \chi(\eta/\langle \xi \rangle) p_2(x + y, \xi) dyd\eta - q^0(x, \xi),
\]

\[
(6.13) \quad \check{r}'(x, \xi) = O_* \int e^{-i\xi \cdot \eta} p_1(x, \xi + \eta) (1 - \chi(\eta/\langle \xi \rangle)) p_2(x + y, \eta) dyd\eta.
\]

It is easy to see

\[
(6.14) \quad \check{r}'(x, \xi) \in \mathcal{R}_{\omega d}
\]

from the similar discussions in the proof of Proposition 2.2 in §4.

For the proof of

\[
(6.15) \quad \check{r}(x, \xi) \in \mathcal{R}_{\omega d}
\]

we fix an integer \( N \) and divide \( \check{r}(x, \xi) \) into three terms

\[
(6.16) \quad \check{r}(x, \xi) = \sum_{|\gamma| \leq N} \frac{1}{\gamma!} p_1^{(\gamma)}(x, \xi) p_2^{(\gamma)}(x, \xi)
\]

\[
+ \sum_{|\gamma| < N} \sum_{|\theta| = 1} \frac{1}{\gamma!} \left[ \int_0^1 (1 - \theta)^{|\gamma|} \int_0^1 e^{-i\xi \cdot \eta} p_1^{(\gamma)}(x, \xi + \eta) dyd\eta \right]
\]
where

\[ r_N^1(x, \xi) = \sum_{|\gamma| < N} \frac{1}{\gamma!} p_1^{(\gamma)}(x, \xi) p_2^{(\gamma)}(x, \xi) - q^0(x, \xi), \]

\[ r_N^2(x, \xi) = \sum_{|\gamma| < N} \sum_{|\beta| = 1} \frac{1}{\beta!} \int \left\{ \int e^{-iy\eta} p_1^{(\gamma)}(x, \xi + \eta) \right. \]
\[ \times \chi_{\Omega}(\eta/\langle \xi \rangle) \langle \xi \rangle^{-1} \left. \int p_2^{(\gamma)}(x + \theta y, \xi) dy d\eta \right\} d\theta, \]

\[ r_N^3(x, \xi) = N \sum_{|\gamma| < N} \frac{1}{\gamma!} \int \left\{ \int e^{-iy\eta} p_1^{(\gamma)}(x, \xi + \eta) \right. \]
\[ \times \chi_{\Omega}(\eta/\langle \xi \rangle) p_2^{(\gamma)}(x + \theta y, \xi) dy d\eta \right\} d\theta. \]

Then, from (6.2) we have

\[ |r_N^{(a)}(\xi, \xi)| \leq CM^{-((|\alpha| + |\beta| + N)\alpha + \beta + N \delta d \langle \xi \rangle^{m_1 + m_2 - N - |\alpha|}} \]

for \( |\xi| \geq \mu. \)

In the integrand of each term in \( r_N^2(x, \xi) \) the inequality \( (2/5) \langle \xi \rangle \leq |\eta| \leq \langle \xi \rangle/2 \) holds. Hence, integrating by parts we have

\[ r_N^3(x, \xi) = \sum_{|\gamma| < N} \sum_{|\beta| = 1} \frac{1}{\beta!} \int \left\{ \int e^{-iy\eta} p_1^{(\gamma)}(x, \xi + \eta) \right. \]
\[ \times \chi_{\Omega}(\eta/\langle \xi \rangle) \langle \xi \rangle^{-1} \left. \times (-i |\eta|^{-2} \eta \cdot \nabla_p)^{N-1} p_2^{(\gamma)}(x + \theta y, \xi) \right\} dy d\eta \right\} d\theta, \]

which implies

\[ |r_N^{(a)}(\xi, \xi)| \leq CM^{-((|\alpha| + |\beta| + N)\alpha + \beta + N \delta d \langle \xi \rangle^{m_1 + m_2 - N - |\alpha|}} \]

for \( |\xi| \geq \mu'. \)

if we take a constant \( \mu' \geq \mu \) satisfying

\[ |\xi + \eta| \geq \mu' \text{ when } |\xi| \geq \mu \text{ and } |\eta| \leq \langle \xi \rangle/2 \]

for the constant \( \mu' \) in (6.3). Using (6.19) and (6.3) we also have

\[ |r_N^{(a)}(\xi, \xi)| \leq CM^{-((|\alpha| + |\beta| + N)\alpha + \beta + N \delta d \langle \xi \rangle^{m_1 + m_2 - N - |\alpha|}} \]

for \( |\xi| \geq \mu' \).

Summing up (6.16), (6.17), (6.18) and (6.20) we obtain for \( |\xi| \geq \mu' \)

\[ |r_N^{(a)}(\xi, \xi)| \leq CM^{-((|\alpha| + |\beta| + N)\alpha + \beta + N \delta d \langle \xi \rangle^{m_1 + m_2 - N - |\alpha|}} \]

for any \( N \).
From the definition (6.12) the inequalities (6.21) hold also for
$|\xi| \leq \mu'$. This proves (6.15).

**Remark.** We will write $q^0(x, \xi)$ satisfying (6.1) and (6.2) as
$\sigma_0(P_1 P_2)$.

Now, we turn to the proof of Proposition 3.4. In the following
we use the symbol class $I^+_{r}(S^m_{G(d,1)})$ defined by the following: We say
that a symbol $p(t, x, \xi)$ belongs to $I^+_{r}(S^m_{G(d,1)})$ if $p(t, x, \xi)$ belongs
to $\cap M_{r}^1(S^m_{G(d,1)})$ and satisfies

$$
|\partial_t^k p_{(t)}(t, x, \xi)| \leq CM^{-(k+|\alpha|+|\beta|)}k!\alpha!\beta!|\xi|^{n-|\alpha|}
$$

for $(t, x, \xi) \in [0, T] \times R_{x, \xi}$, $|\xi| \geq \mu$

with constants $C$, $M$ and $\mu$ independent of $k$, $\alpha$ and $\beta$. Since the
operator $L$ of (3.15) is of constant multiplicity, in view of (3.19)
we may assume that its distinct characteristic roots $\lambda_1(t, x, \xi), \ldots,$
$\lambda_h(t, x, \xi)$ satisfy $\lambda_j(t, x, \xi) \in I^+_{r}(S^m_{G(d,1)})$ and

$$
|\lambda_j(t, x, \xi) - \lambda_k(t, x, \xi)| \geq C|\xi|^j (C>0, j \neq k)
$$

by modifying them in $[0, T] \times R_{x} \times \{|\xi| \leq 1\}$. Assume (3.16). Then,
using Proposition 6.1 and the discussions in the proof of Proposition
3.3 we can reduce the problem (3.14) to the problem (3.13) with

$$
\mathcal{L} = D_t - \mathcal{D}(t) + \left( B_{j k}(t) \begin{pmatrix} j \downarrow 1, \ldots, h \downarrow \end{pmatrix} + R(t),
$$

where

$$
\mathcal{D}(t) = \begin{pmatrix}
\lambda_1(t, X, D_x) \mathcal{J}_{l_1} & 0 \\
& \cdot \\
& \cdot \\
& 0 & \lambda_h(t, X, D_x) \mathcal{J}_{l_h}
\end{pmatrix}
$$

$(l_1 + \cdots + l_h = l)$,

$B_{jk}(t)$ are $l_j \times l_k$ matrices of pseudo-differential operators with symbols
in $I^+_{r}(S^m_{G(d,1)}$, $\sigma = (r-q)/r$, and $R(t)$ is a regularizer in $\mathcal{A}_{G(d)}$, which
means that $R(t)$ is an $l \times l$ matrix of pseudo-differential operators
with symbols in $M_{r}^1(\mathcal{A}_{G(d)})$. So, the proof of Proposition 3.4 is
completed if we apply the following theorem.
Theorem 6.3. Let $\mathcal{L}$ be a hyperbolic operator of the form (6.23), where $\mathcal{D}(t)$ is defined by (6.24) with real symbols $\lambda_j(t, x, \xi) \in \Gamma_0^1(S_{G(d,\omega)}^1)$ satisfying (6.22), $B_{jk}(t)$ are $l_j \times l_k$ matrices of pseudo-differential operators with symbols in $\Gamma_0^1(S_{G(d,\omega)}^1)$ $(0 \leq \sigma \leq 1/d)$ and $R(t)$ is a regularizer in $\mathcal{R}_{G(d)}$. Then, there exists a matrix $P(t)$ of pseudo-differential operators with symbols in $\Gamma_0^1(S_{G(d,\omega)}^1)$ such that

\begin{equation}
\mathcal{L}(\mathcal{I} + P(t)) = (\mathcal{I} + P(t))\mathcal{L}_\sigma
\end{equation}

holds for a perfectly diagonalized operator

\begin{equation}
\mathcal{L}_\sigma = D_\sigma - \mathcal{D}(t) + \begin{pmatrix}
B_1(t) & 0 \\
\cdot & \cdot \\
0 & B_h(t)
\end{pmatrix} + R_\sigma(t)
\end{equation}

and the inverse $Q(t)$ of $\mathcal{I} + P(t)$ exists with $Q(t) \in \mathcal{L}_{0(G(d))}$ for any $t$, where in (6.26) $B_j(t)$ are $l_j \times l_j$ matrices of pseudo-differential operators with symbols in $\Gamma_0^1(S_{G(d,\omega)}^1)$ and $R_\sigma(t)$ is a regularizer in $\mathcal{R}_{G(d)}$.

For the proof we will give two lemmas.

Lemma 6.4. Let $\mathcal{L}$ be a hyperbolic operator of the form (6.23). Then, under the assumptions in Theorem 6.3 there exists a matrix $P^1(t)$ of pseudo-differential operators with symbols in $\Gamma_0^1(S_{G(d,\omega)}^1)$ such that for a matrix $B^1(t)$ with $\sigma(B^1(t)) \in \Gamma_0^1(S_{G(d,\omega)}^1)$ and a regularizer $R^1(t)$ in $\mathcal{R}_{G(d)}$

\begin{equation}
\mathcal{L}(\mathcal{I} + P^1(t)) = (\mathcal{I} + P^1(t))\mathcal{L}_1 + B^1(t) + R^1(t)
\end{equation}

holds with $\mathcal{L}_1$ of the form

$$
\mathcal{L}_1 = D_1 - \mathcal{D}(t) + F^\sigma(t),
$$

where

$$
F^\sigma(t) = \text{diag}[F_1^\sigma(t), \ldots, F_h^\sigma(t)]
$$

for $l_j \times l_j$ matrices $F_j^\sigma(t)$ of pseudo-differential operators with symbols in $\Gamma_0^1(S_{G(d,\omega)}^1)$. Here, $\text{diag}[F_1^\sigma(t), \ldots, F_h^\sigma(t)]$ means

$$
\text{diag}[F_1^\sigma(t), \ldots, F_h^\sigma(t)] = \begin{pmatrix}
F_1^\sigma(t) & 0 \\
\cdot & \cdot \\
0 & F_h^\sigma(t)
\end{pmatrix}
$$
Proof. We construct $F^0(t)$ and $P^1(t)$ satisfying

\begin{equation}
(6.28) \quad \sigma(F^0(t)) + \sigma(D(t)) \sigma(P^1(t)) - \sigma(P^1(t)) \sigma(D(t)) = \sigma(B(t)) + \sigma(B(t)) \sigma(P^1(t)) - \sigma(P^1(t)) \sigma(F^0(t))
\end{equation}

modulo $\Gamma^d_t(S^d_G,1)$, where $B(t) = (B_{jk}(t))$. Then, from Proposition 6.1 the property (6.28) yields (6.27).

Define pseudo-differential operators $F^{(0)}(t)$ and $P^{(0)}(t)$ by

\[ F^{(0)}(t) = \text{diag}[B_{11}(t), \ldots, B_{hh}(t)] \]

and

\[ \sigma(P^{(0)}(t)) = (\sigma(P^{(0)}(t))_{jk})_{j \atop \downarrow k}^{1 \atop \downarrow 1}, \ldots, h \]

with

\[
\begin{aligned}
\sigma(P^{(0)}(t))_{jj} &= 0, \\
\sigma(P^{(0)}(t))_{jk} &= \frac{1}{\lambda_j(t) - \lambda_k(t)} \sigma(B_{jk}(t)) \quad (j \neq k).
\end{aligned}
\]

Here, we denote the $(j, k)$ blocks of $\sigma(P^{(0)}(t))$ by $\sigma(P^{(0)}(t))_{jk}$. Then, $\sigma(F^{(0)}(t))$ belongs to $\Gamma^d_t(S^d_G,1)$ and $\sigma(P^{(0)}(t))$ belongs to $\Gamma^d_t(S^d_G,1)$. Hence, if $\sigma \leq 1/2$, the property (6.28) holds by setting $F^0(t) = F^{(0)}(t)$ and $P^1(t) = P^{(0)}(t)$. This proves the lemma in the case of $\sigma \leq 1/2$. For the case of $\sigma > 1/2$ we take an integer $N$ satisfying $N \geq (2\sigma - 1) / (1 - \sigma)$ and define $F^{(\nu)}(t)$ and $P^{(\nu)}(t)$ ($\nu = 1, \ldots, N$) inductively by $\sigma(F^{(\nu)}(t)) = \text{diag}[\sigma(F^{(\nu)}(t))_{11}, \ldots, \sigma(F^{(\nu)}(t))_{hh}]$ with

\[ \sigma(F^{(\nu)}(t))_{jj} = \sum_{k=1}^{\nu} \sigma(B_{jk}(t)) \sigma(P^{(\nu-1)}(t))_{kj} \quad (\in \Gamma^d_t(S^d_G,1)) \]

and $\sigma(P^{(\nu)}(t)) = (\sigma(P^{(\nu)}(t))_{jk})$ with

\[
\begin{aligned}
\sigma(P^{(\nu)}(t))_{jj} &= 0, \\
\sigma(P^{(\nu)}(t))_{jk} &= \frac{1}{\lambda_j(t) - \lambda_k(t)} \left\{ \sum_{k'=1}^{\nu} \sigma(B_{jk'}(t)) \sigma(P^{(\nu-1)}(t))_{k'k} \\
&\quad + \sum_{\nu' + \nu'' = \nu - 1} \sigma(P^{(\nu')}_{jk'}(t)) \sigma(P^{(\nu'')}_{k'k}(t)) \right\}_{kk} \quad (\in \Gamma^d_t(S^d_G,1)) \quad (j \neq k).
\end{aligned}
\]

Then, we obtain the desired operators $F^0(t)$ and $P^1(t)$ by setting

\[
\begin{aligned}
F^0(t) &= F^{(0)}(t) + F^{(1)}(t) + \cdots + F^{(N)}(t), \\
P^1(t) &= P^{(0)}(t) + P^{(1)}(t) + \cdots + P^{(N)}(t).
\end{aligned}
\]

Q. E. D.

Lemma 6.5. Let $P^1(t)$ be as in the preceding lemma. Then, there
exists a matrix $Q^1(t)$ of pseudo-differential operators with symbols in $\Gamma^d(S^0_d(x,1))$ such that
\[(6.29) \quad (\mathcal{J} + P^1(t))Q^1(t) = \mathcal{J} + R(t)\]
holds for a regularizer $R(t)$ in $\mathcal{R}_{G(d)}$.

**Proof.** Since $\sigma < 1$, there exists a constant $\mu$ such that
\[
|\det (\mathcal{J}_r + \sigma (P^1(t)) (x, \xi))| \geq C > 0 \quad \text{for} \quad |\xi| \geq \mu.
\]
Take a function $\chi_\mu(\xi)$ in $\mathcal{S}$ satisfying $\chi_\mu(\xi) = 0$ if $|\xi| \leq \mu$ and $\chi_\mu(\xi) = 1$ if $|\xi| \geq \mu + 1$. Define matrices $q^{[\nu]}(t, x, \xi)$ ($\nu = 0, 1, \ldots$) of symbols by
\[
q^{[0]}(t, x, \xi) = (\mathcal{J}_r + \sigma (P^1(t)) (x, \xi))^{-1} \chi_\mu(\xi),
\]
\[
q^{[\nu]}(t, x, \xi) = - \sum_{\nu' + \nu'' = \nu} \frac{1}{\nu'} q^{[0]}(t, x, \xi) \partial_{i} \partial_{j} (\sigma (P^1(t)) (x, \xi)) \times D_{i} q^{[\nu']} (t, x, \xi) \quad (\nu \geq 1).
\]
Then, $q^{[\nu]}(t, x, \xi)$ belongs to $\Gamma^d(S^0_d(x,1))$ and satisfy
\[
|\partial_{i} \partial_{j} D_{i} q^{[\nu]}(t, x, \xi)| 
\leq CM^{-\frac{1}{2}(|a| + |\beta| + \nu) + 1} |\partial_{i} \partial_{j} \sigma (P^1(t)) (x, \xi)|^{-\nu - |a|} \quad \text{for} \quad |\xi| \geq \mu + 1
\]
with constants $C$ and $M$ independent of $k, \alpha, \beta$ and $\nu$. Hence, applying Lemma 6.2 we can find $q^1(t, x, \xi)$ in $\Gamma^d(S^0_d(x,1))$ satisfying for constants $C_i, M_i$ and $\mu_i$
\[(6.30) \quad |\partial_{i} \partial_{j} D_{i} q^{1}(t, x, \xi)| = \sum_{\mu=0}^{N-1} q^{[\nu]}(t, x, \xi) \quad (\nu = 0, 1, \ldots, N)
\]
with $C_i, M_i$ independent of $k, \alpha, \beta$ and $\nu$. Hence, applying Lemma 6.2 we can find $q^1(t, x, \xi)$ in $\Gamma^d(S^0_d(x,1))$ satisfying for constants $C_i, M_i$ and $\mu_i$
\[
(6.30) \quad |\partial_{i} \partial_{j} D_{i} q^{1}(t, x, \xi)| = \sum_{\mu=0}^{N-1} q^{[\nu]}(t, x, \xi) \quad (\nu = 0, 1, \ldots, N)
\]
with $C_i, M_i$ independent of $k, \alpha, \beta$ and $\nu$. Hence, applying Lemma 6.2 we can find $q^1(t, x, \xi)$ in $\Gamma^d(S^0_d(x,1))$ satisfying for constants $C_i, M_i$ and $\mu_i$

Using (6.30) and Proposition 6.1 we get (6.29) with $Q^1 = q^1(t, x, D_x)$ by the usual method.

Q. E. D.

Define $B^2(t) = (B^2_{j_1}(t), \ldots, B^2_{j_k}(t))$ by
\[
B^2(t) = Q^1(t) B^1(t).
\]

Then, from (6.27) we obtain
\[(6.31) \quad \mathcal{L} (\mathcal{J} + P^1(t)) = (\mathcal{J} + P^1(t)) \mathcal{L}_2 + R^2(t)
\]
for a regularizer $R^2(t)$ in $\mathcal{R}_{G(d)}$, where $\mathcal{L}_2$ denotes the operator
\[(6.32) \quad \mathcal{L}_2 = D_t - \mathcal{D}(t) + F^1(t) + B^2(t).
\]
In (6.32) we may assume $\sigma(B^2(t)) \in \Gamma^d(S^0_d(x,1))$ by means of Proposi-
tion 6.1. Moreover, replacing $F^*(t)$ by
\[ F^*(t) + \text{diag}[B_{11}^2(t), \ldots, B_{hh}^2(t)], \]
we may assume that in (6.32) the diagonal blocks of $B^2(t)$ are zero. Now, we are prepared to prove Theorem 6.3. The following discussions originate from those in [10] and in §IV of Appendix in [12].

**Proof of Theorem 6.3.** First, we construct $P^2(t)$ and $F(t)$ with $\sigma(P^2(t)) \in \Gamma_1^d(S_{G(d, 2)}^{-1})$ and $\sigma(F(t)) \in \Gamma_1^d(S_{G(d, 2)}^{-1})$ such that
\[ \mathcal{L}_2(J + P^2(t)) = \langle J + P^2(t), (D_i - \partial_j(t) + F^*(t) + F(t)) + R^3(t) \rangle \]
holds, where the blocks of $F(t)$ are zero except diagonal blocks and $R^3(t)$ is a regularizer in $R_{G(d)}$. Set
\[ \Sigma = \left\{ \left[ P^{\nu}(t) \right]_{\nu=0}^{\infty} ; \sigma(P^{[\nu]}(t)) = \left( (\sigma(P^{[\nu]}(t)))_{jk} \right)_{j=1, \ldots, h} \right\}, \]
where $\sigma(P^{[\nu]}(t))_{jk} = 0$, $\sigma(P^{[\nu]}(t))_{jk} \in \Gamma_1^d(S_{G(d, 2)}^{-1}) (j \neq k)$.

Here, the notation $\sigma(P^{[\nu]}(t))_{jk}$ has the same meaning as in the proof of Lemma 6.4. For $\left[ P^{\nu}(t) \right]_{\nu=0}^{\infty} \in \Sigma$ we define $\left[ F^{[\nu]}(t) \right]$ and $\left[ A^{[\nu]}(t) \right]$ as follows:
\[ \sigma(F^{[\nu]}(t)) = \text{diag}[\sigma(F^{[\nu]}(t))_{11}, \ldots, \sigma(F^{[\nu]}(t))_{hh}] \]
with
\[ \sigma(A^{[\nu]}(t)) = \sigma(B^2(t)), \]
\[ \sigma(A^{[\nu]}(t)) = \langle \sigma(A^{[\nu]}(t)) \rangle_{jk} \quad (\nu \geq 1) \]
with
\[ \left\{ \begin{array}{l}
\sigma(A^{[\nu]}(t))_{jk} = 0, \\
\sigma(A^{[\nu]}(t))_{jk}
\end{array} \right. \]
\[ = \sum_{k=1}^{h} \sum_{|\gamma| + \nu = \mu - 1} \frac{1}{\gamma!} \partial^\gamma_i (\sigma(B^{[\nu]}(t))) D_k^i (\sigma(P^{[\nu]}(t)))_{k/k} \\
+ D_k^i (\sigma(P^{[\nu]}(t)))_{jk} \\
- \sum_{|\gamma| + \nu = \mu - 2} \frac{1}{\gamma!} \partial^\gamma_i (\sigma(P^{[\nu]}(t)))_{jk} D_k^i (\sigma(F^{[\nu]}(t)))_{kh} \\
- \sum_{|\gamma| + \nu = \mu - 1} \frac{1}{\gamma!} \partial^\gamma_i \left[ J_j(t) \partial_j - \sigma(F^*_j(t)) \right] D_k^i (\sigma(P^{[\nu]}(t)))_{jk}
\]
regarding the third term in the right hand side of (6.38) as zero in case \( \nu=1 \). Next, we solve equations for unknowns \( \sigma(\hat{P}^{[\nu]}(t)) = (\sigma(\hat{P}^{[\nu]}(t))_{jk})(j, k=1, \ldots, h) \):

\[
(6.39) \quad \sigma(\hat{P}^{[\nu]}(t))_{jj} = 0, \\
(6.40) \quad \sigma(\hat{P}^{[\nu]}(t))_{jk} = \frac{1}{\lambda_j(t) - \lambda_k(t)} \sigma(F^j(t)) \sigma(\hat{P}^{[\nu]}(t))_{jk} \\
+ \sigma(\hat{P}^{[\nu]}(t))_{jk} \left\{ - \frac{1}{\lambda_j(t) - \lambda_k(t)} \sigma(F^j(t)) \right\} \\
= \frac{1}{\lambda_j(t) - \lambda_k(t)} \sigma(A^{[\nu]}(t))_{jk} \quad \text{for large } |\xi| (j \neq k),
\]

where \( A^{[\nu]}(t) \) are defined by (6.36)—(6.38). Set

\[
\begin{cases}
g_{jk}(t, x, \xi) = \frac{1}{\lambda_j(t) - \lambda_k(t)} \sigma(F^j(t)), \\
h_{jk}(t, x, \xi) = - \frac{1}{\lambda_j(t) - \lambda_k(t)} \sigma(F^j(t)) \quad (j \neq k).
\end{cases}
\]

From (6.22) and \( \sigma(F^j(t)) \in \Gamma_t^\xi(S^{[\nu]}_{G_{(d,1)})}) \) for \( \sigma<1 \) there exists a constant \( \mu \) such that the matrix norms \( ||g_{jk}(t, x, \xi)|| \) and \( ||h_{jk}(t, x, \xi)|| \) of \( g_{jk}(t, x, \xi) \) and \( h_{jk}(t, x, \xi) \) satisfy

\[
||g_{jk}(t, x, \xi)|| \leq 1/4, \quad ||h_{jk}(t, x, \xi)|| \leq 1/4 \quad \text{for } |\xi| \geq \mu.
\]

Hence, the solutions \( \sigma(\hat{P}^{[\nu]}(t))_{jk} \) of (6.40) are given by

\[
(6.41) \quad \sigma(\hat{P}^{[\nu]}(t))_{jk} = \sum_{k=0}^{\infty} \sum_{k'=0}^{\infty} \binom{\kappa}{k} \binom{\kappa}{k'} g_{jk}(t)^{\kappa'} \times \frac{1}{\lambda_j(t) - \lambda_k(t)} \sigma(A^{[\nu]}(t))_{jk} h_{jk}(t)^{\kappa - \kappa'}
\]

for \( |\xi| \geq \mu \). We modify \( \sigma(\hat{P}^{[\nu]}(t))_{jk} \) in \([0, T] \times R^d_{x} \times \{|\xi| \leq \mu+1\}) \) such that \( \sigma(\hat{P}^{[\nu]}(t))_{jk} \) belong to \( \Gamma_t^\xi(S^{[\nu]}_{G_{(d,1)})}) \). Then, we obtain a solution \( \{\sigma(\hat{P}^{[\nu]}(t))\} \) of (6.39)—(6.40) in \( \Sigma \).

Now, we define a mapping \( \mathcal{F} \) from \( \Sigma \) to \( \Sigma \) by \( \mathcal{F}(\{P^{[\nu]}(t)\}) = \{\hat{P}^{[\nu]}(t)\} \). From the definition the operator \( \hat{P}^{[\nu]}(t) \) is determined only by \( P^{[0]}(t), \ldots, P^{[\nu-1]}(t) \). So, by the induction on \( \nu \) we can see that the fixed point \( \{P^{[\nu]}(t)\} \) of \( \mathcal{F} \) exists uniquely in \( \Sigma \). Assume that the fixed point \( \{P^{[\nu]}(t)\} \) of \( \mathcal{F} \) satisfies

\[
(6.42) \quad ||\hat{\partial}_t^\xi \hat{\partial}_t^\nu D_\sigma(\tilde{P}^{[\nu]}(t))|| \leq CM^{-|\kappa| - |\nu| + |\kappa'| + |\nu'|} \sigma^{\kappa} \sigma^{\kappa'} \lambda^{|\nu|} \lambda^{|\nu'|} \langle \xi \rangle^{-1 - \nu - |\nu'|} \quad \text{for } |\xi| \geq \mu
\]
with constants $C, M$ and $\mu$ independent of $k, \alpha, \beta$ and $\nu$. Then, from Lemma 6.2 there exists a symbol $\tilde{p}^*(t, x, \xi)$ in $\Gamma^1_\gamma(S^-_{0, (d, 1)})$ such that

$$
|\partial_t^k \partial_x^\alpha D_x^\beta (\tilde{p}^*(t, x, \xi)) - \sum_{\beta < |x|} \sigma (\tilde{P}^{(2)}_\gamma (t)) (x, \xi)| \leq CM^{-(k+|x|+|\xi|+N-|\nu|)} \quad \text{for } |\xi| \geq \mu,
$$

with new constants $C, M$ and $\mu$. Then, setting $P^2(t) = \tilde{p}^*(t, X, D_x)$ and

$$
F(t) = \text{diag}[F_1(t), \ldots, F_{k+1}(t)]
$$

with $\sigma (F_j(t))$ the $(j, j)$ block of $\sigma (B^2(t) P^2(t))$, we have

$$
F(t) + (D(t) - F^a(t)) P^2(t) - P^2(t) (D(t) - F^a(t)) = B^2(t) + B^2(t) P^2(t) + P_1(t) - P^2(t) F(t) - R^3(t),
$$

for a regularizer $R^3(t)$ in $\mathcal{R}_{G, 0}$, where $P^2(t)$ is a matrix of pseudo-differential operators with symbol $D_\sigma (P^2(t))$. This is nothing else but (6.33).

In order to prove that the fixed point $\{P^\infty_\gamma (t)\}$ of $\mathcal{F}$ satisfies (6.42), we define following [1] a formal norm $|||\{q^{(2)}(t)\}, M|||^{(m)}$ for a sequence $\{q^{(2)}(t)\}_{m=0}^\infty$ with $q^{(2)}(t) \equiv q^{(2)}(t, x, \xi) \in \Gamma^1_\gamma(S^-_{0, (d, 1)})$ by

$$
|||\{q^{(2)}(t)\}, M|||^{(m)} = \sum_{k, \alpha, \beta, \nu} C_{k, \alpha, \beta} \sup_{t, x, \xi} \left| \left| \partial_t^k \partial_x^\alpha D_x^\beta q^{(2)}(t, x, \xi) \right| \right| \cdot \mu^{k+|\alpha|+|\beta|+2\nu},
$$

where $\mu$ is some constant,

$$
C_{k, \alpha, \beta} = \frac{2 (2n)^{-\nu}!}{(|\alpha|+\nu)! (k+|\beta|+\nu)!},
$$

and $|| \cdot ||$ denotes the matrix norm. For a symbol $q(t, x, \xi) \in \Gamma^1_\gamma(S^-_{0, (d, 1)})$ we denote $||q(t), M|||^{(m)} = |||q^{(2)}(t)\rangle\rangle, M|||^{(m)}$ by setting $q^{(2)}(t) = q(t)$ and $q^{(2)}(t) = 0$ ($\nu \geq 1$). Then, from the assumptions there exists a constant $M$ such that the following hold if we take $\mu$ in (6.43) sufficiently large:

$$
|||\sigma (B^2(t)), M|||^{(0)} \leq C_1,
$$

$$
|||\lambda_j(t) - \lambda_k(t)\rangle\rangle^{-1}, M|||^{(0)} \leq C_2 \quad (j \neq k),
$$

$$
|||g_{jk}(t), M|||^{(0)} \leq 1/3,
$$

$$
|||h_{jk}(t), M|||^{(0)} \leq 1/3,
$$

and
(6.48) \[ \sum_{k,a,b} C^\nu_{k,a,b} \sup_{|x| \geq 1} |\frac{\partial^k}{\partial x^k} D^\nu_j (\lambda_j (t) \kappa_j - \sigma (F^j (t)))| \langle \xi \rangle^{-1+|a|} \times M^{k+|a|+|b|-1} \leq C_3 \]

for \( C^\nu_{k,a,b} \) defined by (6.44) with \( \nu = 0 \).

Now, suppose that \( \{ P^0 (t) \} \in \Sigma \) satisfies

\[
\| \| \sigma (P^0 (t)) \| \|^{-1} < \infty.
\]

Let \( \{ F^0 (t) \}_{t=0}^\infty \), \( \{ A^0 (t) \}_{t=0}^\infty \) and \( \{ \tilde{P}^0 (t) \}_{t=0}^\infty \) be defined by (6.34) — (6.40). Then, from Lemma 1.2 of [1] we have

\[
(6.49) \quad \| \| \sigma (F^0 (t)) \| \|^{-1} \leq C_1 \| \sigma (P^0 (t)) \|, \quad M \| \|^{-1},
\]

and from (6.46) — (6.47) and (6.41)

\[
(6.50) \quad \| \| \sigma (A^0 (t)) \| \|^{-1} \leq C_1 + C_2 M \| \| \sigma (P^0 (t)) \|, \quad M \| \|^{-1} + M \| \sigma (P^0 (t)) \|, \quad M \| \|^{-1}
+ M \| \| \sigma (P^0 (t)) \|, \quad M \| \|^{-1} \| \sigma (P^0 (t)) \|, \quad M \| \|^{-1}
+ 2 C_2 M \| \| \sigma (P^0 (t)) \|, \quad M \| \|^{-1}.
\]

Hence, we have

\[
(6.51) \quad \| \| \sigma (\tilde{P}^0 (t)) \| \|^{-1} \leq \sum_{\varepsilon=0}^\infty \sum_{\varepsilon'=0}^\infty \varepsilon \left( \frac{1}{3} \right) \varepsilon' \left( \frac{1}{3} \right) C_2 \| \sigma (A^0 (t)) \|, \quad M \| \|^{-1},
\]

with some constants \( C_4 \) and \( C_5 \). So, if we set for a constant \( C^0 \)

\[
\Sigma_0 = \{ \{ P^0 (t) \}_{t=0}^\infty \in \Sigma ; \quad \| \| \sigma (P^0 (t)) \| \|^{-1} \leq C^0 \},
\]

we see that the mapping \( \mathcal{F} \) maps \( \Sigma_0 \) into \( \Sigma_0 \) if \( M \) is sufficiently small. Moreover, if we go over the proof of (6.52) once again, we see that the restriction of \( \mathcal{F} \) to \( \Sigma_0 \) is a contraction if we take \( M \) satisfying

\[
C_4 M + 2 C_5 C^0 M < 1.
\]

This implies that the fixed point \( \{ P^0_0 (t) \} \) of \( \mathcal{F} \)

belongs to \( \Sigma_0 \), which means \( \{ P^0_0 (t) \} \) satisfies (6.42). Summing up, we have found \( P^0 (t) \) and \( F^0 (t) \) satisfying (6.33).

In (6.33) we set

\[
B^0_0 (t) = F^0_0 (t) + F (t).
\]

Then, \( B^0_0 (t) \) has a form
and from (6.31) and (6.33) we have

\[
B^0(t) = \begin{pmatrix}
B_1(t) & 0 \\
& \ddots \\
& & B_n(t)
\end{pmatrix}
\]

with a regularizer \( P(\xi; \mu) \). For a positive number \( \mu \) we take a function \( \chi_\mu(\xi) \) in \( S^d(\mathbb{R}^d) \) satisfying \( \chi_\mu(\xi) = 0 \) if \( |\xi| \leq \mu \) and \( \chi_\mu(\xi) = 1 \) if \( |\xi| \geq \mu + 1 \) and let \( P(t; \mu) \) be the pseudo-differential operator with the symbol

\[
\sigma(P(t; \mu)) = (\sigma(P_1(t)) + \sigma(P_2(t)) + \sigma_\xi(P_1(t) P_2(t))) \chi_\mu(\xi).
\]

Then, from (6.53) we have

\[
\mathcal{L}(\mathcal{S} + P(t; \mu)) = (\mathcal{S} + P(t; \mu))(D_t - \mathcal{D}(t) + B^0(t)) + R(t; \mu),
\]

for a regularizer \( R(t; \mu) \) in \( \mathcal{R}_G(d) \) depending on a parameter \( \mu \). Since the order of \( \sigma(P_1(t)) + \sigma(P_2(t)) + \sigma_\xi(P_1(t) P_2(t)) \) is less than zero, \( \sigma(P(t; \mu)) \) satisfies the first inequality of (2.19) with an arbitrary small constant \( C_0 \) if \( \mu \) tends to the infinity. Therefore, we can take sufficiently large constant \( \mu^0 \) such that the inverse \( Q(t) \) of \( \mathcal{S} + P(t; \mu^0) \) exists. Now, we set \( P(t) = P(t; \mu^0) \) and \( R_0(t) = Q(t) R(t; \mu^0) \). Then, we obtain (6.25). This concludes the proof of Theorem 6.3.

Q. E. D.

Finally, we give some remarks concerning the inverse \( Q(t) \) of \( \mathcal{S} + P(t) \) in Theorem 6.3. In order to apply Theorem 6.3 to Proposition 3.4 it is sufficient that \( Q(t) \) belongs to \( L^0_G(d) \) for any \( t \). But, we can improve the result “the inverse \( Q(t) \) has the form \( Q(t) = q^0(t, X, D_x) + \tilde{q}(t, X, D_x) \) with symbols \( q^0(t, x, \xi) \in S^0(\mathbb{R}^d, \mathbb{D}) \) and \( \tilde{q}(t, x, \xi) \in \mathcal{R}_G(d) \) for any \( t \)” in the following way: “the inverse \( Q(t) \) has the form \( Q(t) = q^0(t, X, D_x) + \tilde{q}(t, X, D_x) \) with symbols \( q^0(t, x, \xi) \in S^0(\mathbb{R}^d, \mathbb{D}) \) and \( \tilde{q}(t, x, \xi) \in \mathcal{R}_G(d) \) for any \( t \)” This result is proved by applying the following property (*) and the discussions in proving Corollary 2.7 since \( \sigma(P(t; \mu)) \) belongs to \( P^0_t(S^{-1}_G(d, \mathbb{D})) \).

(*) In Theorem 2.6 we assume furthermore that \( p_0^0(x, \xi) \) belong to
$S_{(d,1)}^a$ and satisfy
\[ |p_j^{(a)}(x, \xi)| \leq C_\alpha M^{-\langle |\alpha| + |\beta| \rangle} |\alpha| |\beta| \langle \xi \rangle^{\sigma - |\alpha|} \quad \text{for} \quad |\xi| \geq \tilde{\mu} \]
with constants $C_\alpha$, $M$ and $\tilde{\mu}$ independent of $\alpha$, $\beta$ and $j$. Then, the multi-product $Q_{\nu+1} = P_1 \cdots P_{\nu+1}$ of $P_j = p_j(X, D_x)$ has the form $Q_{\nu+1} = q_{\nu+1}^0(X, D_x) + q_{\nu+1}^1(X, D_x)$ with symbols $q_{\nu+1}^0(x, \xi)$ satisfying
\[ |q_{\nu+1}^0(x, \xi)| \leq A^\nu C_\alpha^\nu M_1^{-\langle |\alpha| + |\beta| \rangle} |\alpha| |\beta| \langle \xi \rangle^{\sigma - |\alpha|} \quad \text{for} \quad |\xi| \geq \tilde{\mu}_1 \]
and $q_{\nu+1}^1(x, \xi)$ satisfying (2.18). Here, the constants $A$, $M_1$ and $\tilde{\mu}_1$ are independent of $\nu$.

The above discussions also give another proof of Lemma 6.5. In fact, we can prove (6.29) by setting $Q^1(t) = (\mathcal{F} + P^1(t) \chi_\nu(D_x))^{-1}$ for large constant $\mu$, where $\chi_\nu(\xi)$ is the function used in the last part of the proof of Theorem 6.3.

References


