Cohomology of (H, C)-Groups

Dedicated to Professor Shigeo Nakano on his 60th birthday

By

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Introduction

In this paper we consider an n-dimensional connected complex Lie group G without nonconstant holomorphic functions (Such a Lie group is called an (H, C)-group). In the previous paper [8] we found a sufficient condition for $H^p(G, \mathcal{O}_G)$ to be finite-dimensional ($p \geq 1$), using the resolution: $0 \rightarrow \mathcal{O}_G \rightarrow \mathcal{A}^{0,0} \rightarrow \mathcal{A}^{0,1} \rightarrow \cdots$ where $\mathcal{A}^{p,q}$ denotes the sheaf of germs of real analytic (p, q)-forms on G. It was not possible to find a necessary and sufficient condition for $H^p(G, \mathcal{O}_G)$ to be finite-dimensional by the method of the paper [8]. Roughly speaking the cause of the above unsuccess is that the resolution by the sheaves of germs of real analytic forms is not good enough to calculate the $\partial$ cohomology groups of G.

The purpose of this paper is to establish the cohomology groups $H^p(G, \mathcal{O}_G)$ of an (H, C)-group G ($p \geq 1$), using some number theoretical property of G. It is known that every (H, C)-group G has a structure of $C^\ast$-principal bundle $\pi : G \rightarrow T^q$ over a q-dimensional complex torus $T^q (p + q = n)$ ([14]). We take the subsheaf $\mathcal{H}$ of $\mathcal{A}^{0,0}$ so that $\mathcal{H} : = \{ f \in \mathcal{A}^{0,0} ; f$ is holomorphic along each fiber of $\pi \}$. First we shall prove a cohomology vanishing theorem for the sheaf $\mathcal{H}$ on G in Section 2. Using the sheaf $\mathcal{H}$, we shall get the resolution:

$$0 \rightarrow \mathcal{O}_G \rightarrow \mathcal{H}^{0,0} \rightarrow \cdots \rightarrow \mathcal{H}^{0,q} \rightarrow 0$$
of $\partial_G$ in Section 3 to calculate the $\partial$ cohomology groups $H^p(G, \partial_G)$ ($p \geq 1$). In Section 4 we shall find a necessary and sufficient condition for $H^p(G, \partial_G)$ to be finite-dimensional and calculate the dimension of $H^p(G, \partial_G)$ (Theorem 4.3). We can regard an $(H, C)$-group $G$ as a quotient group $C^*/\Gamma$ by a discrete subgroup $\Gamma$. The above necessary and sufficient condition is expressed by a Diophantine inequality with respect to the subgroup $\Gamma$ of $C^*$. Unless the condition is fulfilled for $\Gamma$, then by Theorem 4.3, there exists $j$ ($1 \leq j \leq q$) such that $H^j(G, \partial_G)$ is infinite-dimensional. Further we shall prove that $H^p(G, \partial_G)$ is not Hausdorff for all $p$ ($1 \leq p \leq q$) (Theorem 4.4). By the theorems in Section 4 the cohomology groups $H^p(G, \partial_G)$ of an $(H, C)$-group $G$ are completely determined by some number theoretical property of $G$ and we have a classification of all $(H, C)$-groups as follows. Let $C^*/\Gamma$ be an $n$-dimensional $(H, C)$-group. If $\Gamma$ is generated by $R$-linearly independent vectors $v_1, \ldots, v_{n+q}$, then $C^*/\Gamma$ is called an $(H, C)$-group of rank $n+q$ ([11]). Let $\mathcal{F}^{n,q}$ be the set of all $n$-dimensional $(H, C)$-groups of rank $n+q$. Then

$$\mathcal{F}^{n,q} = \{ C^*/\Gamma \in \mathcal{F}^{n,q} ; \dim H^p(C^*/\Gamma, \partial) < \infty, p \geq 1 \}$$

$$\cup \{ C^*/\Gamma \in \mathcal{F}^{n,q} ; H^p(C^*/\Gamma, \partial) \text{ is not Hausdorff for any } p \text{ satisfying } 1 \leq p \leq q \} \quad \text{(disjoint).}$$

The author is very grateful to Prof. S. Nakano who raised the question in 1975 whether the cohomology groups $H^p(G, \partial_G)$ for an $(H, C)$-group $G$ are finite-dimensional.

§ 1. Preliminaries

In this paper we consider an $n$-dimensional connected complex Lie group $G$ without nonconstant holomorphic functions. Such a Lie group $G$ is said to be a toroid group or an $(H, C)$-group ([5], [9], [11]). We recall that $G$ is abelian and then $G$ is isomorphic onto $C^*/\Gamma$ for some discrete subgroup $\Gamma$ of $C^*$ as a Lie group ([11]). We may assume that $\Gamma$ is generated by $R$-linearly independent vectors $e_1, \ldots, e_m, v_1 = (v_{11}, \ldots, v_{1n}), \ldots, v_q = (v_{q1}, \ldots, v_{qn})$ of $C^*$ ($1 \leq q \leq n$), where $e_j$ is the $j$-th unit vector of $C^*$. Since every holomorphic function on $G=C^*/\Gamma$ is constant, $[v_1, \ldots, v_q]$ must satisfy the condition:

$$\max \{|\sum_{j=1}^q v_{ij} m_j - m_{n+1}| ; 1 \leq i \leq q\} > 0$$
for all \( m = (m_1, \ldots, m_n, m_{n+1}, \ldots, m_{n+q}) \in \mathbb{Z}^{n+q} - \{0\} \) ([9], [11]). Since \( \text{Im} \; v_i := (\text{Im} \; v_{i1}, \ldots, \text{Im} \; v_{in}) \), \( \ldots \), \( \text{Im} \; v_q := (\text{Im} \; v_{q1}, \ldots, \text{Im} \; v_{qn}) \) are \( R \)-linearly independent, we may assume \( \text{det} \; [\text{Im} \; v_{ij}; 1 \leq i, j \leq q] \neq 0 \) without loss of generality. Throughout this paper we assume that \( G = C^\ast/G \) and \( G \) denotes the discrete subgroup satisfying the above assumption and (1.1). Further we use the notations:

\[
K_{m,i} := \sum_{j=1}^{s} v_{ij} m_j - m_{n+i} \quad \text{and} \quad K_m := \max \{|K_{m,i}|; 1 \leq i \leq q\}
\]

for \( m \in \mathbb{Z}^{n+q} \). Then from (1.1) we have

\[
(1.2) \quad K_m > 0 \quad \text{for all} \quad m \in \mathbb{Z}^{n+q} - \{0\}.
\]

We denote the projection \( C^\ast \ni (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_q) \in C^q \) by \( \pi_q : C^\ast \longrightarrow C^q \). Let \( e_i^\ast := \pi_q(e_i), \quad v_i^\ast := \pi_q(v_i) \) for \( 1 \leq i \leq q \) and \( \Gamma^\ast := \pi_q(G) \). Since \( e_i^\ast, \; v_i^\ast \) are \( R \)-linearly independent, we have a \( q \)-dimensional complex torus \( T^q = C^q/\Gamma^\ast \).

We recall the following proposition due to [14].

**Proposition 1.1.** The projection \( \pi_q : C^\ast \longrightarrow C^q \) induces the \( C^{n-q} \) principal bundle \( \pi_q : C^\ast/G \ni z + \Gamma \mapsto \pi_q(z) + \Gamma^\ast \in T^q \) over \( T^q \).

We put

\[
\begin{align*}
\alpha_{ij} &= \begin{cases} \text{Re} \; v_{ij} & (1 \leq i \leq q, \; 1 \leq j \leq n) \\ 0 & (q+1 \leq i \leq n, \; 1 \leq j \leq n) \end{cases} \\
\beta_{ij} &= \begin{cases} \text{Im} \; v_{ij} & (1 \leq i \leq q, \; 1 \leq j \leq n) \\ \delta_{ij} & (q+1 \leq i \leq n, \; 1 \leq j \leq n) \end{cases}
\end{align*}
\]

\([\gamma_{ij}; 1 \leq i, j \leq n] := [\beta_{ij}; 1 \leq i, j \leq n]^{-1} \) and \( v_i := \sqrt{-1} \; e_i \) for \( q+1 \leq i \leq n \). Since \( \{e_1, \ldots, e_n, v_1, \ldots, v_q\} \) are \( R \)-linearly independent, we have an isomorphism

\[
\phi : C^\ast \ni (x_1, \ldots, x_n) \mapsto (t_1, \ldots, t_{2n}) \in R^{2n}
\]

as a real Lie group, where \( (x_1, \ldots, x_n) = \sum_{i=1}^{n} (t_i e_i + t_{n+i} v_i) \). Then we obtain the relations

\[
(1.3) \quad t_j = x_j - \sum_{k=1}^{n-j} y_k \gamma_{kj} \alpha_{ij} \quad \text{and} \quad t_{n+j} = \sum_{i=1}^{n} y_i \gamma_{ij}
\]

for \( 1 \leq j \leq n \), where \( z_i = x_i + \gamma_{ij} y_j \) for \( 1 \leq i \leq n \). \( \phi \) induces the isomorphism \( \phi^\ast : C^\ast/G \cong T^{n+q} \times R^{n-q} \) as a real Lie group, where \( T^{n+q} \) is a \( n+q \)-dimensional real torus. Henceforth we identify \( C^\ast/G \) with the real Lie group \( T^{n+q} \times R^{n-q} \) and use the real coordinate system \( \langle t_1, \ldots, t_{2n} \rangle \) according to the need. We make the following change of
Then we can regard \( (\zeta_0, \ldots, \zeta_n) \) as a local coordinate system of \( C^n/\Gamma \) and we have global vector fields

\[
\frac{\partial}{\partial \zeta_i} = \sum_{j=1}^n \beta_{ij} \frac{\partial}{\partial \xi_j}
\]

and \((0, 1)\)-forms

\[
d\zeta_i = \sum_{j=1}^n \gamma_{ji} d\xi_j \quad (1 \leq i \leq n)
\]
on \( C^n/\Gamma \). It follows from (1.3) that

\[
(1.4) \quad \frac{\partial}{\partial \zeta_i} = \frac{1}{2} \left( \sum_{j=1}^n \beta_{ij} \frac{\partial}{\partial t_j} - \sqrt{-1} \sum_{j=1}^n \alpha_{ij} \frac{\partial}{\partial t_j} + \sqrt{-1} \frac{\partial}{\partial \tau_{n+i}} \right).
\]

Then for \( q+1 \leq i \leq n \) we have

\[
(1.5) \quad \frac{\partial}{\partial \zeta_i} = \frac{1}{2} \left( \frac{\partial}{\partial t_i} + \sqrt{-1} \frac{\partial}{\partial \tau_{n+i}} \right).
\]

Let \( \mathcal{A} \) be the sheaf of germs of (complex valued) real analytic functions on \( C^n/\Gamma \) and

\[
\mathcal{H}_0 := \left\{ f \in \mathcal{A} \mid \frac{\partial f}{\partial \zeta_i} = 0 \quad q+1 \leq i \leq n \right\}.
\]

Let \( f \in H^0(C^n/\Gamma, \mathcal{A}) \). Then we have the following Fourier expansion of \( f \):

\[
(1.6) \quad f(t_0, \ldots, t_{2n}) = \sum_{m \in \mathbb{Z}^{n+q}} c^m(t') \exp 2\pi \sqrt{-1} \langle m, t' \rangle,
\]

where \( t' = (t_0, \ldots, t_{n+q}) \in T^{n+q}, \quad t' = (t_{n+q+1}, \ldots, t_{2n}) \in R^{n-q}, \quad m = (m_0, \ldots, m_{n+q}) \in Z^{n+q}, \quad \langle m, t' \rangle = \sum_{i=0}^{n+q} m_i t_i \) and \( c^m(t') \) is real analytic in \( t' \in R^{n-q} \) for any \( m \in Z^{n+q} \). We put

\[
f^m(t) := c^m(t') \exp 2\pi \sqrt{-1} \langle m, t' \rangle.
\]

It follows from (1.4), (1.5) and (1.6) that

\[
(1.7) \quad \frac{\partial f^m}{\partial \zeta_i} = \begin{cases} \pi (\sum_{j=1}^n \nu_{ij} m_j - m_{n+i}) f^m = \pi K_{m,i} f^m, & 1 \leq i \leq q \\ \sqrt{-1} \left( \frac{\partial c^m(t')}{\partial \tau_{n+i}} + 2\pi m_i c^m(t') \right) \exp 2\pi \sqrt{-1} \langle m, t' \rangle, & q+1 \leq i \leq n. \end{cases}
\]

Furthermore suppose \( f \in H^0(C^n/\Gamma, \mathcal{H}) \). Since \( \frac{\partial f}{\partial \zeta_i} = 0 \quad (q+1 \leq i \leq n) \), we have, by (1.6) and (1.7),

\[
(1.8) \quad f(t) = \sum_{m \in \mathbb{Z}^{n+q}} c^m \exp (-2\pi \sum_{i=q+1}^{n+q} m_i t_{n+i}) \exp 2\pi \sqrt{-1} \langle m, t' \rangle,
\]

where \( c^m \) is complex constant for any \( m \in Z^{n+q} \).
§ 2. Cohomology Groups with Coefficients in the Sheaf \( \mathcal{H} \)

Let \( M \) be a paracompact real analytic manifold and \( \mathcal{A}_M \) be the sheaf of germs of real analytic functions on \( M \). By the result of [4] we can regard \( M \) as a closed real analytic submanifold of a complex manifold \( N \) and \( M \) has a Stein neighbourhood basis \( \{ U_i ; i \in I \} \) in \( N \). Since \( \text{ind. lim} \{ H^p(U_i, \mathcal{O}_N) ; U_i \supset M \} = 0 \), we have \( H^p(M, \mathcal{A}_M) = 0 \) for \( p \geq 1 \) ([10]).

In this section we treat cohomology groups as the following type. Let \( \mathcal{F} \) be the sheaf \( \{ f(z, t) \in \mathcal{A}_{C^k \times R} ; \frac{\partial f}{\partial z} = 0 \} \) on \( C \times R \). We wish to consider whether \( H^p(C \times R, \mathcal{F}) \) vanishes for \( p \geq 1 \). Using a power series expansion of a function \( f \in \mathcal{A}_{C^k \times R} \) we can prove that a homomorphism \( \frac{\partial}{\partial z} : \mathcal{A}_{C^k \times R} \rightarrow \mathcal{A}_{C^k \times R} \) is surjective. Then we have an exact sequence \( 0 \rightarrow \mathcal{F} \rightarrow \mathcal{A}_{C^k \times R} \rightarrow \mathcal{A}_{C^k \times R} \rightarrow 0 \). From this exact sequence and the lemma of [6, Lemma 2, p. 25], we can regard \( H^p(C \times R, \mathcal{F}) \) as \( H^p(C, \mathcal{O}_E) \), where \( \mathcal{O}_E \) is the sheaf of germs of holomorphic functions with values in the locally convex space \( E := H^0(R, \mathcal{A}_R) \). Since \( E \) admits no structures of Frechet spaces, then we cannot apply the result of [1] and [2] to \( H^p(C \times R, \mathcal{F}) \). And since \( C \times R \) has no Stein neighbourhood bases in \( C \times C \), then we cannot prove the vanishing of \( H^p(C \times R, \mathcal{F}) \) by the same method of the proof of the theorem: \( H^p(M, \mathcal{A}_M) = 0 \). To get our purpose in this section, we must investigate a property of Stein open neighbourhood of \( C^k \times C^l \) in \( C^k \times C^l \).

We will use the following notations in the rest of this paper. For an \( m \)-tuple \( \xi = (\xi_1, \ldots, \xi_m) \), \( ||\xi|| := \max \{ ||\xi_i|| ; 1 \leq i \leq m \} \). And the notation \{equalities and inequalities involving functions \( h_1, \ldots, h_m \)\} denotes the set of all points in the intersection of the domains of definition of \( h_1, \ldots, h_m \) satisfying the given equalities and inequalities.

Lemma 2.1. Let \( \pi : S \rightarrow C^k \times C^l \) be a (unramified Riemann) domain of holomorphy over \( C^k \times C^l \) \((k, l \geq 1)\), \( \Delta_\pi := \{(w_1, \ldots, w_l) \in C^l ; |w_j - a_j| < r_j, 1 \leq j \leq l\} \), where \( r = (r_1, \ldots, r_l) \), \( r_j > 0 \) and \( (a_1, \ldots, a_l) \in C^l \) and let \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_l) \) for \( \varepsilon_j \geq 0 \) \((1 \leq j \leq l)\). Further assume there exist an open subset \( V_1 \) of \( S \) and \( \delta > 0 \) such that \( \pi|_{V_1} \) is biholomorphic into \( C^k \times C^l \) and \( \pi(V_1) \supset (C^k \times \Delta_\pi) \cup \{||z|| < \delta\} \times \Delta_{r+\delta} \), where \( \Delta_{r+\delta} := \)
Then there exists an open subset $V_2$ of $S$ with $V_1 \subset V_2$ such that $\pi|_{V_2}$ is biholomorphic into $C^k \times C^l$ and $\pi(V_2) \supset C^k \times \Delta_{r+\epsilon}$.

**Proof.** We may assume $a_1 = \cdots = a_l = 0$. Let $f \in H^\infty(S \setminus \emptyset)$. Then $f$ can be expanded in the power series $f(x) = \sum_{\nu, \mu} a_{\nu\mu} (z \circ \pi(x))^\nu (w \circ \pi(x))^\mu$, where $(z \circ \pi(x))^\nu = (z_1 \circ \pi(x))^{\nu_1} \cdots (z_k \circ \pi(x))^{\nu_k}$ and $(w \circ \pi(x))^\mu = (w_1 \circ \pi(x))^{\mu_1} \cdots (w_l \circ \pi(x))^{\mu_l}$. Then the power series $F(x, w) := \sum_{\nu, \mu} a_{\nu\mu} z^\nu w^\mu$ converges in $(C^k \times \Delta_r) \cup \{||z|| < \delta\} \times \Delta_{r+\epsilon}$. We put $D_d := \{(||z|| < d) \times \{||w|| < r_j, 1 \leq j \leq l\}\} \cup \{||z|| < \delta\} \times \{(||w|| < r_j + \epsilon_j, 1 \leq j \leq l\}\}$ for $d > \delta$. The envelope of holomorphy of $D_d$ is the smallest logarithmically convex complete Reinhardt domain $\hat{D}_d := \{||z|| < d, \quad ||w|| < r_j + \epsilon_j, \quad \log ||w|| - \log r_j < \frac{\log d - \log ||z||}{\log d - \log \delta} (\log (r_j + \epsilon_j) - \log r_j)\}$ which contains $D_d$ (for instance see [13]). Since $F$ converges in $D_d$ for all $d > \delta$, then $F$ can be continued holomorphically in $\hat{D}_d$ for any $d > \delta$. We take any point $(x, w) \in C^k \times \Delta_{r+\epsilon}$. Then we can find a sufficiently large positive number $d_0$ such that $\log ||w|| - \log r_j < \frac{\log d_0 - \log ||z||}{\log d_0 - \log \delta} (\log (r_j + \epsilon_j) - \log r_j)$. Then $(x, w) \in \hat{D}_{d_0}$. This implies that $F$ converges in $C^k \times \Delta_{r+\epsilon}$. Since $\pi : S \rightarrow C^k \times C^l$ is a domain of holomorphy, we find an open subset $V_2$ of $S$ satisfying the statements of the lemma (for instance see [6, Theorem 18, p. 55]).

The following lemma asserts that $C \times R$ admits no Stein open neighbourhood bases in $C \times C$. For instance we take an open neighbourhood $V := \{(z, w) \in C^2; \quad |\text{Im} w| < (1 + |z|)^{-1}\}$ of $C \times R$ in $C \times C$. Then we cannot find a Stein open subset $V^*$ so that $C \times R \subset V^* \subset V$.

**Lemma 2.2.** Let $I_j := \{w_j \in C; \quad \text{Im} w_j = 0, \quad a_j < \text{Re} w_j < b_j\}$, where $-\infty \leq a_j < b_j \leq \infty$ (1 $\leq j \leq l$), $I := I_1 \times \cdots \times I_l \subset R^l \subset C^l$ and $f$ a holomorphic function in a neighbourhood of $C^k \times I$ in $C^k \times C^l$. Then there exists a Stein open neighbourhood $V$ of $I$ in $C^l$ such that $f$ can be continued holomorphically to $C^k \times V$.

**Proof.** First we assume $l = 1$. Then $I = \{w \in C; \quad \text{Im} w = 0, \quad a < \text{Re} w < b\}$, where $-\infty \leq a < b \leq \infty$. We have an open and connected neighbourhood $D$ of $C^k \times I$ in $C^k \times C$ so that $f$ is holomorphic in $D$. Let $\pi : \hat{D} \rightarrow C^k \times C$ be the envelope of holomorphy of $D$ which is given
by a Riemann domain over \( C^k \times C \). Then there exists a holomorphic injection \( j : D \longrightarrow \hat{D} \) such that \( \pi \circ j \) is identity and the mapping \( H^o(\hat{D}, \emptyset , g) \rightarrow H^o(D, \emptyset , g) \) is an isomorphism. Put \( U := j(D) \), for \( \varepsilon = \pm 1 \) \( \hat{D}_1 := \{ (z, w) \in D ; a < \text{Re} w < b \} \). Then \( \pi : \hat{D}_1 \longrightarrow C^k \times C \) is a domain of holomorphy. We identify \( (z, w) \in D \) and \( x \in \hat{D} \) if \( j(z, w) = x \). Then for \( \varepsilon = \pm 1 \) we get Riemann domains

\[
\pi_{\varepsilon} : G^\varepsilon : \hat{D}_1 \cup D^\varepsilon \longrightarrow C^k \times C,
\]

where \( \pi_{\varepsilon}(x) := \pi(x) \) if \( x \in \hat{D}_1 \) and \( \pi_{\varepsilon}(z, w) := (z, w) \) if \( (z, w) \in D^\varepsilon \).

Now we consider the case \( \varepsilon = 1 \). We put

\[
G^1 := G^1 \cap \pi_{1}^{-1}(C^k \times [a + t < \text{Re} w < b - t]) \text{ for } 0 < t < (b - a)/2.
\]

Since \( G^1 \) is \( \rho_t \)-convex in the sense of [3], \( G^1 = \bigcup_{0 < t < (b - a)/2} G^t_1 \) is a domain of holomorphy. We take \( x_0 \in U \) with \( \pi_1(x_0) = (z^0, w^0) \) for some \( w^0 \in I \). We put

\[
\tau_0 := 1/2 \text{ min } \{ w^0 - a, b - w^0 \} > 0
\]

\[
P_0 := C^k \times \{ |w - w^0 - \sqrt{-1} \tau_0| < \tau_0 \}.
\]

Then we have \( P_0 \subset D^1 \subset G^1 \). And there exist \( \delta_1 > 0 \) and an open subset \( U^1 \subset G^1 \) such that \( \pi_1|_{U^1} \) is biholomorphic into \( C^k \times C \) and \( \pi_1(U^1) \supset \{ ||z|| < \delta \} \times \{ |w - w^0 - \sqrt{-1} \tau_0| < \tau_0 + \delta \} \). By Lemma 2. 1 we have an open subset \( U_2 \) of \( G^1 \) with \( x_0 \in U_2 \) so that \( \pi_1|_{U_2} \) is biholomorphic into \( C^k \times C \) and

\[
\pi(U_2) \supset C^k \times \{ |w - w^0 - \sqrt{-1} \tau_0| < \tau_0 + \delta \}.
\]

Then \( \pi(\hat{D}_1 \cap U_2) \supset C^k \times \{ |w - w^0| < \delta_0 \} \). Applying the above method to the case \( \varepsilon = -1 \), we get \( \delta_1 > 0 \) and an open subset \( U_3 \) of \( G^1 \) with \( x_0 \in U_3 \) so that \( \pi_{-1}|_{U_3} \) is biholomorphic and \( \pi(\hat{D}^{-1}_1 \cap U_3) \supset C^k \times \{ |w - w^0| < \delta_0 \} \). Then there exists an open neighbourhood \( U_4 \) of \( x_0 \) in \( \hat{D} \) such that \( \pi|_{U_4} \) is biholomorphic into \( C^k \times C \) and \( \pi(U_4) \supset C^k \times \{ |w - w^0| < \text{min} \{ \delta_0, \delta_2 \} \} \). This means that \( f \) can be continued holomorphically to \( C^k \times V^* \) for some open neighbourhood \( V^* \) of \( I \) in \( C \). Then we complete the proof in the case \( l = 1 \). We can prove the assertion of the lemma for \( l \geq 3 \) similarly to the case \( l = 2 \). Then we shall only prove the lemma in the case \( l = 2 \). Let \( I := I_1 \times I_2 = \)}
{(w₁, w₂); Im w₁=0, a₁<Re w₁<b₁, i=1, 2} ⊆ R² ⊆ C², f a holomorphic function in a neighbourhood E of C⁺×I in C⁺×C² and ˜E the envelope of holomorphy of E. In general ˜E is given by a Riemann domain over C⁺×C². Using the same technique of the proof in the case l=1, we may treat ˜E as a univalent domain of holomorphy in C⁺×C² which contains E without loss of generality. We take (t₁₀, t₂₀) ∈ I₁ × I₂ and δ>0 satisfying δ<MIN{tᵢ₀−aᵢ, bᵢ−tᵢ₀; i=1, 2} and {||x||<δ} × {||wᵢ−tᵢ₀||<δ, i=1, 2} ⊆ E. Let I₁₀ := R ∩ {wᵢ∈C; |wᵢ−tᵢ₀|<δ} (i=1, 2), Iᵢ := Iᵢ₀ ∩ {||wᵢ−tᵢ₀||<δ/3} and ˜E(t₀):={(z, w₁)∈C⁺×C; (z, w₁, t₀)∈ ˜E, Re w₁∈I₁₀} for t₀∈I₂₀. And we put

\[ ˜E(t₀)⁺:= ˜E(t₀) ∪ C⁺× \{w₁; ε Im w₁≥0, Re w₁∈I₁₀\} \]

for ε=±1. Then ˜E(t₀)⁺ is a domain of holomorphy for s=±1 and t₀∈I₂₀. We have C⁺×{||wᵢ−tᵢ−\sqrt{-1}(δ/3)||<δ/3} ⊆ ˜E(t₀)⁺ and {||x||<δ} × {||wᵢ−tᵢ−\sqrt{-1}(δ/3)||<2δ/3} ⊆ ˜E(t₀)⁺ for t₀∈I₂₀ and tᵢ∈Iᵢ. It follows from Lemma 2.1 that

\[ C⁺×{||wᵢ−tᵢ−\sqrt{-1}(δ/3)||<2δ/3} ⊆ ˜E(t₀)⁻¹ \]

for t₀∈I₁₀ and tᵢ∈Iᵢ. Similarly we have

\[ C⁺×{||wᵢ−tᵢ+\sqrt{-1}(δ/3)||<2δ/3} ⊆ ˜E(t₀)⁻¹ \]

for t₀∈I₂₀ and tᵢ∈Iᵢ. We put V₁₀ := {w₁; |Im w₁|<δ/3, Re w₁∈I₁₀}. Then we have C⁺×V₁₀×I₁₀⊂ ˜E. We set

\[ ˜E₁:={(z, w₁, w₂)∈ ˜E; w₁∈V₁₀, Re w₂∈I₂₀}, \]

\[ ˜E₁⁺:= ˜E₁ ∪ C⁺×V₁₀× \{w₂; ε Im w₂≥0, Re w₂∈I₂₀\} \]

for ε=±1. Since C⁺×V₁₀×{w₂; ||wᵢ−tᵢ−\sqrt{-1}(εδ/2)||<δ/2} ⊆ ˜E₁⁺ and {||x||<δ} × V₁₀×{w₂; ||wᵢ−tᵢ−\sqrt{-1}(εδ/2)||<δ} ⊆ ˜E₁⁻ (ε=±1), it follows from Lemma 2.1 that

\[ C⁺×V₁₀×{||wᵢ−tᵢ||<δ/2} ⊆ ˜E. \]

Since (t₁₀, t₂₀) is an arbitrary point of I, we find an open subset V of I in C² so that f can be continued holomorphically to C⁺×V. I has a Stein neighbourhood basis in C². Then we may regard V as a Stein open subset of C² ([4]).

**Lemma 2.3.** Let I be as in Lemma 2.2 and f a holomorphic function in a neighbourhood of C⁺⁺×I in C⁺⁺×C¹. Then there exists a Stein open neighbourhood V of I in C¹ such that f can be continued
holomorphically to $C^k \times V$.

Proof. It is sufficient to prove the lemma in the case $k=2$. Let $a_v(t) := 1/(2\pi \sqrt{\text{-}1})^{2} \int_{|z_1|=1} \int_{|z_2|=1} \frac{f(z_1, z_2, t)}{z_1^{v_1+1} z_2^{v_2+1}} \, dz_1 dz_2$ for $v = (v_1, v_2) \in \mathbb{Z}^2$ and $t \in I$. And we use the notations: $\Sigma = \Sigma_{v_1, v_2 = 0}, \Sigma_{\frac{1}{2}} = \Sigma_{v_1, v_2 < 0}, \Sigma_{\frac{1}{3}} = \Sigma_{v_1 < 0, v_2 = 0}, \Sigma_{\frac{1}{4}} = \Sigma_{v_1 < 0, v_2 < 0}$ and $f_k(z_1, z_2, t) := \sum_{v} a_v(t) z_1^{v_1} z_2^{v_2}$. Then we can apply Lemma 2.2 to each $f_i$. For instance we take $f_i^* (z, \zeta_2, t) := f_i (z^{-1}, \zeta_2^{-1}, t)$ which is holomorphic in $(\zeta_1, \zeta_2) \in C^2$.

By Lemma 2.2 $f_i^*$ can be continued to $C^2 \times V$ for some Stein open neighbourhood $V$ of $I$ in $C^1$. Since $f = f_1 + f_2 + f_3 + f_4$, we get the proof of the lemma.

Let $\pi_q : C^n / \Gamma \equiv (z_1, \ldots, z_n) + \Gamma \longrightarrow (z_1, \ldots, z_n)$ and $\pi^* \in T^q = C^q / \Gamma^*$ be the $C^* \cdot q$-principal bundle over $T^q$ as in Proposition 1.1. Since $\alpha_{ij} = 0$ for $1 \leq j \leq n, q+1 \leq i \leq n$ and $\gamma_{ij} = 0$ for $q+1 \leq i \leq n, 1 \leq j \leq q$, from (1.3) it follows that $t_j = x_j - \sum_{i=1}^n \sum_{i=1}^q y_i \gamma_{ij} \alpha_{ij}$ and $t_{n+j} = \sum_{i=1}^q y_i \alpha_{ij}$ for $1 \leq j \leq q$. This relation induces an isomorphism $\sigma : T^q \equiv (z_1, \ldots, z_q) + \Gamma^* \longrightarrow (\exp 2\pi \sqrt{-1} t_1, \ldots, \exp 2\pi \sqrt{-1} t_q)$ which is holomorphic in $(\zeta_1, \zeta_2) \in C^2$.

By Lemma 2.2 $f_i^*$ can be continued to $C^2 \times V$ for some Stein open neighbourhood $V$ of $I$ in $C^1$. Since $f = f_1 + f_2 + f_3 + f_4$, we get the proof of the lemma.

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By Lemma 2.2 $f_i^*$ can be continued to $C^2 \times V$ for some Stein open neighbourhood $V$ of $I$ in $C^1$. Since $f = f_1 + f_2 + f_3 + f_4$, we get the proof of the lemma.

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By Lemma 2.2 $f_i^*$ can be continued to $C^2 \times V$ for some Stein open neighbourhood $V$ of $I$ in $C^1$. Since $f = f_1 + f_2 + f_3 + f_4$, we get the proof of the lemma.
\[ \{ j = (e_0, \ldots, e_q) ; e_i = \pm 1 \} \] and for \( j = (e_0, \ldots, e_q) \in J \) \( U_j : = \{ \exp 2\pi \sqrt{-1} t_0, \ldots, \exp 2\pi \sqrt{-1} t_q, \exp 2\pi \sqrt{-1} t_{n+1}, \ldots, \exp 2\pi \sqrt{-1} t_{n+q} \} \subset T^{2q}; \ -1/2 \leq t_0, t_{n+1} \leq 1/2, t_{n+i} \neq e_i/4, t_i \neq e_i/4 \text{ for } 1 \leq i \leq q \}. \] Then we have open coverings \( \mathcal{U} : = \{ C^{*+q} \times U_j ; j \in J \} \) and \( \mathcal{S} : = \{ \phi^{-1}(C^{*+q} \times U_j) ; j \in J \} \) of \( C^{*+q} \times T^{2q} \) and \( C^* / \Gamma \), respectively.

**Proposition 2.4.** Let \( H^p(\mathcal{S}, \mathcal{F}) \) be the \( p \)-th Čech cohomology group of the covering \( \mathcal{S} \) of \( C^* / \Gamma \). Then

\[ H^p(\mathcal{S}, \mathcal{F}) = 0 \quad \text{for } p \geq 1. \]

**Proof.** We have an isomorphism \( \phi^* : H^p(\mathcal{U}, \mathcal{F}) \longrightarrow H^p(\mathcal{S}, \mathcal{F}) \) by (2.1). Then we may prove \( H^p(\mathcal{U}, \mathcal{F}) = 0 \) for \( p \geq 1 \). We regard \( T^{2q} \) as the closed analytic submanifold \( \{ (w_0, \ldots, w_q) ; \ |w_i| = 1, 1 \leq i \leq 2q \} \) of \( C^q \). Let \( C \) be any connected component of \( U_{i_0} \cap \ldots \cap U_{i_p} \) in \( C^q \) and \( \phi \) be the biholomorphic mapping \( \phi : V \ni (w_0, \ldots, w_q) \longmapsto (\exp \frac{2\pi}{T} w_0, \ldots, \exp \frac{2\pi}{T} w_q) \in C^q \). We take \( \{ c_{i_0}, \ldots, c_{i_p} \} \in \mathcal{Z}^p(\mathcal{U}, \mathcal{F}) \). In virtue of such mapping \( \phi \) and by Lemma 2.3 there exist a Stein and connected open neighbourhood \( U^*_{i_0} \cap \ldots \cap U^*_{i_p} \) of \( U_{i_0} \cap \ldots \cap U_{i_p} \) in \( C^q \) and a unique holomorphic function \( c_{i_0} \cdots c_{i_p} \) in \( C^{*+q} \times U^*_{i_0} \cap \ldots \cap U^*_{i_p} \) such that \( c_{i_0} \cdots c_{i_p} | C^{*+q} \times U^*_{i_0} \cap \ldots \cap U^*_{i_p} \). Since each \( U_j \) admits a Stein neighbourhood basis in \( C^q \), we can choose a Stein neighbourhood \( U^*_{i_0} \cap \ldots \cap U^*_{i_p} \) so that \( U_{i_0} \cap \ldots \cap U_{i_p} \subset U^*_{i_0} \cap \ldots \cap U^*_{i_p} \). We take \( \varepsilon \) (\( 0 < \varepsilon < 1 \)) satisfying \( A_\varepsilon = \{ |w_i| < 1 + \varepsilon, 1 \leq i \leq 2q \} \subset \cup U_{i_0} \). Then we have \( \{ c_{i_0} \cdots c_{i_p} \} \subset C^{*+q} \times U^*_{i_0} \cap \ldots \cap U^*_{i_p} \cap A_\varepsilon \). Since \( C^{*+q} \times A_\varepsilon \) is a Stein open set, there exists \( \{ \delta \{ d_{i_0} \cdots d_{i_p} \} \} \subset C^{*+q} \times U^*_{i_0} \cap \ldots \cap U^*_{i_p} \cap A_\varepsilon \). This completes the proof.

Let \( f \in H^p(C^* / \Gamma, \mathcal{F}) = Z^p(\mathcal{S}, \mathcal{F}) \). From (1.8) we have \( f(t) = \sum_{m \in \mathbb{Z}^{n+q}} c^m \exp \left( -2\pi \sum_{i=q+1}^{n+q} m_i t_{n+i} \right) \exp 2\pi \sqrt{-1} \langle m, t \rangle \),

(2.2) \[ f = \phi^{-1}(\xi, \eta) = \sum_{m \in \mathbb{Z}^{n+q}} c^m \xi_1^{m_{q+1}} \cdots \xi_{n-q}^{m_{n+1}} \eta_{q+1}^{m_{q+1}} \cdots \eta_{2q}^{m_{2q}} \]

for \( (\xi, \eta) \subset C^{*+q} \times T^{2q} \). Observing the proof of Proposition 2.4 and by Lemma 2.3, we have the following

**Proposition 2.5.** There exists \( \varepsilon > 0 \) such that the Laurent series
expansion:
\[
\sum_{m=0}^{n+q} c^m z_1^{m+1} \cdots z_{n-q}^{m+1} \cdots w_0^m w_{n+1}^m \cdots w_{2q}^m
\]
which induced by (2.2) converges for all \((\xi, \omega) \in C^{*n-\delta} \times \Delta_v\), where \(\Delta_v = [1-\epsilon < |w_i| < 1+\epsilon]\).

§ 3. A Special Resolution of the Structure Sheaf \(\mathcal{O}\) of \(C^n/\Gamma\)

Let \(U\) be an open subset of \(C^l\), \(\mathcal{A}\) be the sheaf of germs of real analytic functions on \(C^{*k} \times U\). We consider the sheaf \(\mathcal{V} := \{f \in \mathcal{A} : \frac{\partial f(z, \omega)}{\partial z_j} = 0, 1 \leq j \leq k\}\), where \((z, \omega) \in C^{*k} \times U\) and \(\mathcal{V}^{0,p} := \{(1/p!) \sum_{\sum_{i_p = 1}} f_{i_1} \cdots i_p d\omega_{i_1} \wedge \cdots \wedge d\omega_{i_p} : f_{i_1} \cdots i_p \in \mathcal{V}\}\) for \(0 \leq p \leq l\). Here any form treated in the rest of this paper is written skew-symmetrically in all indices.

Let \(f \in H^0(C^{*k} \times U, \mathcal{V})\) and \(\omega^0 = (\omega^0_1, \ldots, \omega^0_l) \in U\). By Lemma 2.3 there exists an open neighbourhood \(U^0\) of \(\omega^0\) in \(U\) such that \(f\) has the following expansion:
\[
(f(z, \omega) = \sum_{\alpha} \sum_{\beta} a_{\alpha, \beta} z^{\alpha}(\omega - \omega^0)^{\beta} \] which converges for all \((z, \omega) \in C^{*k} \times U^0\), where \((\omega - \omega^0)^{\alpha} = (\omega_1 - \omega^0_1)^{\alpha_1} \cdots (\omega_l - \omega^0_l)^{\alpha_l}\) and \((\omega - \omega^0)^{\beta} = (\omega_1 - \omega^0_1)^{\beta_1} \cdots (\omega_l - \omega^0_l)^{\beta_l}\).

Lemma 3.1. Let \(f = 1/p! \sum_{\sum_{i_p = 1}} f_{i_1} \cdots i_p d\omega_{i_1} \wedge \cdots \wedge d\omega_{i_p} \in H^0(C^{*k} \times U, \mathcal{V}^{0,p})\) with \(\partial f = 0\) \((p \geq 1)\). For any \(\omega^0 \in U\) choose an open neighbourhood \(U^0\) of \(\omega^0\) so that any \(f_{i_1} \cdots i_p\) can be expanded in \(C^{*k} \times U^0\) as in (3.1). Then there exists \(g^0 \in H^0(C^{*k} \times U^0, \mathcal{V}^{0,p-1})\) such that \(\partial g^0 = f\).

Proof. Let \(m\) be the least integer such that the explicit representation of \(f\) in coordinate form involves only the conjugate differentials \(d\omega_1, \ldots, d\omega_m\). The proof will be by induction on \(m\). First we consider \(m = p\). Then \(f = f_{12} \cdot p d\omega_1 \wedge \cdots \wedge d\omega_p\) and we have an expansion \(f_{12} \cdot p = \sum \alpha_{\alpha, \beta} z^{\alpha}(\omega - \omega^0)^{\beta} \) in \(C^{*k} \times U^0\) as in (3.1). Since \(\partial f = 0\), \(f_{12} \cdot p\) must be holomorphic in \(\omega_{p+1}, \ldots, \omega_l\). Putting \(\xi_{12} \cdot p-1 = \sum \alpha_{\alpha, \beta}(\beta_{p+1} \cdots \beta_l) z^{\alpha}(\omega - \omega^0)^{\beta}(\omega_{p+1} - \omega^0_{p+1}) \wedge \cdots \wedge d\omega_{p+1} \wedge \cdots \wedge d\omega_{p-1}\), \(\xi_{12} \cdot p-1\) is also holomorphic in \(\omega_{p+1}, \ldots, \omega_l\) and \(\partial g = f\). Using the standard argument for the Dolbeault lemma (for instance
see [6, the proof of Theorem 3, p. 27]), we can complete the proof.

Observing the proof of Proposition 2.4, we have the following lemma.

**Lemma 3.2.** Let \( \{U_i\} \) be a locally finite open covering of \( U \subset \mathbb{C}^l \) and each \( U_i \) is a rectangular open subset \( \{a_j^{(i)} < \text{Re} \ w_j < b_j^{(i)}, c_j^{(i)} < \text{Im} \ w_j < d_j^{(i)} : 1 \leq j \leq l \} \) for some \( a_j^{(i)}, b_j^{(i)}, c_j^{(i)} \) and \( d_j^{(i)} \in \mathbb{R} \). Take the open covering \( \mathcal{U} := \{C^k \times U_i\} \) of \( C^k \times U \). Then \( H^p(\mathcal{U}, \mathcal{O}^{-1}) = 0 \) for \( p \geq 1 \), \( 0 \leq s \leq l \).

By Lemmata 3.1 and 3.2 we have the following lemma.

**Lemma 3.3.** Let \( U \) be a Stein open subset of \( \mathbb{C}^l \) and \( f \in H^0(\mathbb{C}^k \times U, \mathcal{O}^{-1}) \) with \( \partial f = 0 \). Then there exists \( g \in H^0(\mathbb{C}^k \times U, \mathcal{O}^{-1}) \) such that \( \partial g = f \).

**Proof.** By Lemma 3.1 we have a Stein covering \( \{U_i\} \) and \( g_i \in H^0(\mathbb{C}^k \times U_i, \mathcal{O}^{-1}) \) with \( \partial g_i = f \). We put \( h^{(i)}_{t_0} := g_i - g_i^0 \). Then \( \partial h^{(i)}_{t_0} = 0 \). Further by Lemma 3.1 we get \( g_i^{(1)} \in H^0(\mathbb{C}^k \times U_i^1, \mathcal{O}^{-1}) \) with \( \partial g_i^{(1)} = h^{(i)}_{t_0} \), where we use the notation \( U_{i_0, \ldots, i_s} := U_{i_0} \cap \ldots \cap U_{i_s} \). We set \( \{h^{(1)}_{t_0}\} := \partial \{g^{(1)}_{t_0}\} \). Then \( \partial h^{(2)}_{t_0} = 0 \). Inductively we find sequences \( \{g^{(s)}_{t_0, \ldots, i_s}\} \in C^s(\mathcal{U}, \mathcal{O}^{-1}) \) for \( 1 \leq s \leq p \) and \( \{h^{(s)}_{t_0, \ldots, i_s}\} \in C^s(\mathcal{U}, \mathcal{O}^{-1}) \) for \( 1 \leq s \leq p \), \( \mathcal{U} := \{C^k \times U_i\} \) so that \( \partial h^{(p)}_{t_0, \ldots, i_p} = 0 \), \( \{h^{(p)}_{t_0, \ldots, i_p}\} = \delta \{g^{(p)}_{t_0, \ldots, i_p}\} \) and \( \delta g^{(p)}_{t_0, \ldots, i_p} = h^{(p)}_{t_0, \ldots, i_p} \). Since \( \{h^{(p)}_{t_0, \ldots, i_p}\} \in Z^p(\mathcal{U}, \mathcal{O}) \) and \( \mathcal{U} \) is a Stein covering of the Stein open set \( C^k \times U \), then there exists \( \{f^{(p)}_{t_0, \ldots, i_p}\} \in C^p(\mathcal{U}, \mathcal{O}) \) such that \( \partial h^{(p)}_{t_0, \ldots, i_p} = \delta \{f^{(p)}_{t_0, \ldots, i_p}\} \). Then \( \{f^{(p)}_{t_0, \ldots, i_p}\} \in Z^p(\mathcal{U}, \mathcal{O}) \). By Lemma 3.2 we get \( \{f^{(p-2)}_{t_0, \ldots, i_p-2}\} \subset C^{p-2}(\mathcal{U}, \mathcal{O}^{-1}) \) so that \( \{g^{(p-1)}_{t_0, \ldots, i_p-1} - f^{(p-1)}_{t_0, \ldots, i_p-1}\} \in Z^{p-1}(\mathcal{U}, \mathcal{O}^{-1}) \). We have \( \delta \{g^{(p-1)}_{t_0, \ldots, i_p-1} - f^{(p-1)}_{t_0, \ldots, i_p-1}\} = \delta \{g^{(p-2)}_{t_0, \ldots, i_p-2}\} = \delta \{f^{(p-1)}_{t_0, \ldots, i_p-1}\} = \delta \{f^{(p-2)}_{t_0, \ldots, i_p-2}\} \). Then \( \{g^{(p-2)}_{t_0, \ldots, i_p-2} - f^{(p-2)}_{t_0, \ldots, i_p-2}\} \in Z^{p-2}(\mathcal{U}, \mathcal{O}^{-1}) \). Repeating the above argument, finally we find \( \{f^{(p)}\} \subset B^p(\mathcal{U}, \mathcal{O}^{-1}) \) so that \( \partial h^{(p)}_{t_0} = g_i - g_{i_0} = \delta f^{(p)}_{t_0} - \partial f^{(p)}_{t_0} \). We put \( g := g_i - \partial f^{(p)}_{t_0} \). Then \( g \in H^0(\mathbb{C}^k \times U, \mathcal{O}^{-1}) \) and \( \partial g = f \).

Now we need the sheaf \( \mathcal{H} = \{f \in \mathcal{A} : \frac{\partial f}{\partial s_j} = 0 \quad q + 1 \leq j \leq n \} \) on \( C^* / \Gamma \) defining in Section 1. Further we consider the sheaf

\[
\mathcal{H}^{0,p} := \{1/p! \sum_{1 \leq i_1, \ldots, i_p \leq q} f_{i_1, \ldots, i_p} \frac{\partial^p}{\partial s_{i_1} \ldots \partial s_{i_p}} \wedge \ldots \wedge \frac{\partial^p}{\partial s_{i_1} \ldots \partial s_{i_p}} ; f_{i_1, \ldots, i_p} \in \mathcal{H} \}.
\]
of germs of $\mathcal{H}$-forms of type $(0, p)$ which involves only the differentials $d\zeta_0, \ldots, d\zeta_p$.

**Proposition 3.4.** The sequence

\[ 0 \to \mathcal{H}^0(C^n/\Gamma, \mathcal{O}) \to \mathcal{H}^0(C^n/\Gamma, \mathcal{O}) \to \cdots \to \mathcal{H}^0(C^n/\Gamma, \mathcal{O}) \to 0 \]

is exact. And $H^p(C^n/\Gamma, \mathcal{O})$ is isomorphic to the quotient space \{ $f$ : $f \in H^0(C^n/\Gamma, \mathcal{H}^{0, q})$, $\partial f = 0$ \} / \{ $\tilde{g}$ : $g \in H^0(C^n/\Gamma, \mathcal{H}^{0, p-1})$ \} for $p \geq 1$.

**Proof.** We can regard $(\zeta_1, \ldots, \zeta_n)$ as a local coordinate system of $C^n/\Gamma$. Let $f$ be a germ belonging to a stalk $H_f$ at $\zeta^0 \in C^n/\Gamma$. $f$ has an expansion

\[ f = \sum a_{\alpha} \zeta^\alpha \left( \partial \zeta_1, \ldots, \partial \zeta_n \right) \left( \zeta_1 - \zeta_1^0 \right)^{a_1} \cdots \left( \zeta_q - \zeta_q^0 \right)^{a_q} \left( \zeta_1 - \zeta_1^0 \right)^{a_1} \cdots \left( \zeta_q - \zeta_q^0 \right)^{a_q} \]

which is similar to (3.1) and converges in a small neighbourhood $U^0$ of $\zeta^0$ in $C^n/\Gamma$, where $a_{\alpha}$ is holomorphic in $\zeta^0$. Applying the method of the proof of Lemma 3.1 to this expansion of $f \in \mathcal{H}_\zeta$, we can prove the exactness of the sequence of the proposition. We put $\text{Ker} \delta_k := \text{Ker} \{ \partial : \mathcal{H}^{0, k+1} \to \mathcal{H}^{0, k} \}$ and $\text{Im} \delta_k := \text{Im} \{ \partial : \mathcal{H}^{0, k+1} \to \mathcal{H}^{0, k} \}$. Then we have $\text{Im} \delta_k = \text{Ker} \delta_{k+1}$ and the short exact sequences

\[ \text{Im} \delta_k \to \mathcal{H}^{0, k} \to \text{Ker} \delta_k \to 0 \]

for $0 \leq k \leq q$.

Let $\mathfrak{N} = \{ \phi^{-1}(C^{*n-q} \times U_j) \}$ be the same locally finite covering of $C^n/\Gamma$ as in Proposition 2.4. Since $\phi^{-1}(C^{*n-q} \times U_j)$ is biholomorphic to $C^{*n-q} \times U_j$, it follows from Lemma 3.3 that

\[ \partial : C^p(\mathfrak{N}, \mathcal{H}^{0, k}) \to C^p(\mathfrak{N}, \text{Im} \delta_k) \]

is an epimorphism. Then we have an exact sequence

\[ 0 \to C^p(\mathfrak{N}, \text{Im} \delta_k) \to C^p(\mathfrak{N}, \mathcal{H}^{0, k}) \to C^p(\mathfrak{N}, \text{Ker} \delta_k) \to 0. \]

From (3.2) there exists a long exact sequence

\[ 0 \to H^0(\mathfrak{N}, \text{Ker} \delta_k) \to H^0(\mathfrak{N}, \mathcal{H}^{0, k}) \to H^0(\mathfrak{N}, \text{Im} \delta_k) \to H^1(\mathfrak{N}, \text{Ker} \delta_k) \to H^1(\mathfrak{N}, \mathcal{H}^{0, k}) \to \cdots \]

Using this exact sequence and the result of Proposition 2.4, we have $H^s(\mathfrak{N}, \mathcal{H}^{0, k}) = 0$ for $s \geq 1$, $H^p(C^n/\Gamma, \mathcal{O}) = H^p(\mathfrak{N}, \mathcal{O}) = H^p(\mathfrak{N}, \text{Im} \delta_k)$ and $H^{p-k}(\mathfrak{N}, \text{Ker} \delta_k) = H^{p-k-1}(\mathfrak{N}, \text{Im} \delta_k)$ for $0 \leq k \leq p - 1$. Then we obtain

\[ H^p(C^n/\Gamma, \mathcal{O}) \cong H^1(\mathfrak{N}, \text{Ker} \delta_{p-k}) \cong H^p(\mathfrak{N}, \text{Im} \delta_{p-k}) / \text{Im} \{ \partial : H^p(\mathfrak{N}, \mathcal{H}^{0, p-k-1}) \to H^p(\mathfrak{N}, \text{Im} \delta_{p-k-1}) \}. \]

This coincides with the quotient space asserted in this proposition.
Remark. By Proposition 3.4 we obtain $H^p(C^*/\Gamma, \mathcal{O}) = 0$ for $p \geq q+1$. This comes from the result of [7] directly, since we showed in [7] that $C^*/\Gamma$ is strongly $(q+1)$-complete in the sense of Andreotti and Grauert.

§ 4. $\tilde{\partial}$ Cohomology Groups of $(H, C)$-Groups

Let $f \in H^0(C^*/\Gamma, \mathcal{O})$. By (1.8) we have the Fourier expansion: $f = \sum_{m \in \mathbb{Z}^{n+q}} c^m \exp(-2\pi \sum_{i=q+1}^n m_i t_{n+i}) \exp 2\pi \sqrt{-1} \langle m, t' \rangle$. For $m = (m_1, \ldots, m_n+q) \in \mathbb{Z}^{n+q}$ we put $m' = (m_1, \ldots, m_q, m_{n+1}, \ldots, m_{n+q})$, $m^* = (m_{q+1}, \ldots, m_n)$, $||m'|| = \max\{|m_i|, |m_{n+i}|; 1 \leq i \leq q\}$ and $||m^*|| = \max\{|m_j|; q+1 \leq j \leq n\}$. Then we have the following

Lemma 4.1. The following conditions on a sequence $\{c^m \in C; m \in \mathbb{Z}^*\}$ are equivalent.

(a) The Fourier expansion $\sum_{m \in \mathbb{Z}^{n+q}} c^m \exp(-2\pi \sum_{i=q+1}^n m_i t_{n+i}) \exp 2\pi \sqrt{-1} \langle m, t' \rangle$ converges to a function in $H^0(C^*/\Gamma, \mathcal{O})$.

(b) There exists $\varepsilon > 0$ such that for all $a > 0$
$$C(a) := \sup \{|c^m| \exp(\varepsilon ||m'|| + a ||m^*||); m \in \mathbb{Z}^{n+q}\} < \infty.$$  

Proof. We first prove $(a) \implies (b)$. Put $f(t) := \sum_{m \in \mathbb{Z}^{n+q}} c^m \exp(-2\pi \sum_{i=q+1}^n m_i t_{n+i}) \exp 2\pi \sqrt{-1} \langle m, t' \rangle \in H^0(C^*/\Gamma, \mathcal{O})$, $(w_1, \ldots, w_q)$:
$$= (\exp 2\pi \sqrt{-1} t_1, \ldots, \exp 2\pi \sqrt{-1} t_q, \exp 2\pi \sqrt{-1} t_{n+1}, \ldots, \exp 2\pi \sqrt{-1} t_{n+q}) \in \{ |w_i| = 1, 1 \leq i \leq 2q \} \subset C^q \text{ and } \xi_i := \exp 2\pi \sqrt{-1} (t_i + \sqrt{-1} t_{n+i}) \subset C^*$, $1 \leq i \leq n-q$. Then by Proposition 2.5 we have $\delta > 0$ so that $f^*(\xi, w) := \sum_{m \in \mathbb{Z}^{n+q}} c^m \xi^m w^m$ is holomorphic in $(\xi, w) \in C^{*\times q} \times [1 - \delta < |w_j| < 1 + \delta]$, where $\xi^m = \xi_1^{m_1+1} \cdots \xi_{n-q}^{m_{n-q}}, w^m = w_1^{m_1} \cdots w_q^{m_q} w_{n+1}^{m_{n+1}} \cdots w_{n+q}^{m_{n+q}}$. Put $\varepsilon := 1/2 \min\{-\log (1 - \delta), \log (1 + \delta)\}$. Then for any $a > 0$
$$\sup \{|c^m| \xi_1^{m_1+1} \cdots \xi_{n-q}^{m_{n-q}} |w_1|^{m_1} \cdots |w_q|^{m_q} |w_{n+1}|^{m_{n+1}} \cdots |w_{n+q}|^{m_{n+q}}; m \in Z^{n+q}, \exp(-a) \leq |\xi_i| \leq \exp a, \exp(-\varepsilon) \leq |w_i| \leq \exp \varepsilon\} < \infty.$$ Conversely assume (b) holds. Then $\sum_{m \in \mathbb{Z}^{n+q}} c^m \xi^m w^m$ converges uniformly on every compact subset of $C^{**\times q} \times [\exp(-\varepsilon) < |w_j| < \exp \varepsilon]$. This implies (a).

For $m \in \mathbb{Z}^{n+q}$ we use the notation: $||m^*|| = \max\{|m_i|; 1 \leq i \leq n\}$.

Lemma 4.2. The following conditions (0) and (1) are equivalent.

...
(0) For any \( \varepsilon > 0 \) there exists a positive number \( a = a(\varepsilon) \) such that
\[
\sup_{m \neq 0} \exp \left( -\varepsilon \| m' \| - a \| m' \| \right) / K_m < \infty.
\]

(1) There exists \( a > 0 \) such that
\[
\sup_{m \neq 0} \exp \left( -a \| m^* \| \right) / K_m < \infty.
\]

Proof. Since \( K_m = \max \left( \sum_{j=1}^q \Re v_{ij} m_j - m_{n+i} \right)^2 + \sum_{j=1}^q \Im v_{ij} m_j \right)^2 \) and the \( q \times q \)-matrix \( [\Im v_{ij}; 1 \leq i, j \leq q] \) is non-singular, we can find \( C_1, C_2 > 0 \) so that \( S := \{ m; m \neq 0, \| K_m \| \leq 1 \} \subset \{ m; m \neq 0, \| m' \| \leq C_1 \), \( C_2 \| m^* \| \} \). We shall show that the statement (1) is equivalent to the following

(*) There exists \( a > 0 \) such that \( \sup_{m \neq 0} \exp \left( -a \| m^* \| \right) / K_m < \infty. \)

Since \( \| m^* \| \leq \| m^* \| \), the implication (*) \( \Rightarrow \) (1) is trivial. Assume (1) holds. We have \( \| m^* \| \leq \| m' \| + \| m^* \| \leq C_1 + (C_2 + 1)\| m^* \| \) for \( m \in S \). We put \( b := (C_2 + 1)a > 0 \). Then \( \sup_{m \in S} \exp \left( -b \| m^* \| \right) / K_m \leq \exp \left( bC_1 / (C_2 + 1) \right) \) \( \sup_{m \in S} \exp \left( -a \| m^* \| \right) / K_m < \infty \). This implies (*) holds. We prove (0) \( \Rightarrow \) (*). Assume (0) holds. We get \( a > 0 \) such that \( \sup_{m \neq 0} \exp \left( -\| m' \| - a \| m^* \| \right) / K_m < \infty \). We have \( \exp \left( -(C_1 + C_2 \| m^* \| - a \| m^* \| \right) / K_m \leq \exp \left( -\| m' \| - a \| m^* \| \right) / K_m \) for \( m \in S \). This implies the statement (*) holds. The implication (*) \( \Rightarrow \) (0) is trivial.

Remark. The condition (0) depends on our assumption that \( \det [\Im v_{ij}; 1 \leq i, j \leq q] \neq 0 \). But the condition (1) is independent on that.

Let \( \rho = 1/\rho ! \sum_{1 \leq i_1 \cdots i_p \leq q} \rho_{i_1 \cdots i_p} d\xi_{i_1} \wedge \cdots \wedge d\xi_{i_p} \in H^p(C^\infty / \Gamma, \mathcal{H}^{p,q}) \). We expand each \( \rho_{i_1 \cdots i_p} \) as in (1.8): \( \rho_{i_1 \cdots i_p} = \sum_{m \in \mathbb{Z}^{n+q}} b^n_{i_1 \cdots i_p} \exp \left( -2\pi \sum_{i=t+1}^m m_i t_{n+i} \right) \) \( \exp 2\pi \sqrt{-1} \left< m, t' \right> \). We put \( \rho^n_{i_1 \cdots i_p} = b^n_{i_1 \cdots i_p} \exp \left( -2\pi \sum_{i=t+1}^m m_i t_{n+i} \right) \) \( \exp 2\pi \sqrt{-1} \left< m, t' \right> \). Then \( \rho = 1/\rho ! \sum_{1 \leq i_1 \cdots \leq i_p \leq q} \rho^n_{i_1 \cdots i_p} \). Suppose \( \rho \) is \( \delta \)-exact. Namely there exists a \( (0, p-1) \)-form \( \lambda = \sum_{m \in \mathbb{Z}^{n+q}} \lambda^m \in H^0(C^\infty / \Gamma, \mathcal{H}^{0,p-1}) \) such that \( \rho = \delta \lambda \). Then we have \( \rho^m = \delta \lambda^m \) for any \( m \in \mathbb{Z}^{n+q} \). We write \( \lambda^m = 1/(p-1)! \sum_{1 \leq i_1 \cdots \leq i_p \leq q} \lambda^n_{i_1 \cdots i_p} \) \( \exp 2\pi \sqrt{-1} \left< m, t' \right> \). The equation \( \rho^m = \delta \lambda^m \) implies
Combining (4.1) with (1.7), we have for any $m \in \mathbb{Z}^{\ast+q}$

\[(4.2)\]

\[b_{i_1, \ldots, i_p}^m = \sum_{k=1}^p (-1)^{k+1} \frac{\partial^2 \rho_{i_1, \ldots, i_k}^m}{\partial \xi_{k, i_k}^m}.\]

Now suppose $\phi = \sum_{m \in \mathbb{Z}^{\ast+q}} \phi_m \in H^0(C^n/\Gamma, A^{p,q})$ is $\delta$-closed. From $\delta \phi = 0$ and (4.2) it follows that

\[(4.3)\]

\[\pi \sum_{k=1}^p (-1)^{k+1} K_{m,i_k} \phi_{i_1, \ldots, i_k}^m = 0,\]

where we denote $\phi_m = \frac{1}{p!} \sum_{1 \leq i_1 < \cdots < i_p \leq q} \phi_{i_1, \ldots, i_p}^m \exp(-2\pi \sum_{s=1}^m t_{i_s+i}) \exp \frac{2\pi i}{\sqrt{-1}} \langle m, t \rangle \prod_{1 \leq i_1 < \cdots < i_p \leq q} d\xi_{i_1}^m \wedge \cdots \wedge d\xi_{i_p}^m.$

We put

\[i(m) := \min \{ i ; |K_{m,i}| = K_{m,i} \leq i \leq q \}\]

and the indices $(i(m), i_1, \ldots, i_p)$ in the place of $(i_1, \ldots, i_{p+1})$ of the formula (4.3), then we have

\[(4.4)\]

\[\pi K_{m,i(m)} \phi_{i_1, \ldots, i_p}^m = \pi \sum_{s=1}^p (-1)^{s+1} K_{m,i_s} \phi_{i_1, \ldots, i_i}^m \]

Since $K_{m,i(m)} > 0$ for $m \neq 0$ by (1.2), we can put

\[(4.5)\]

\[\phi_m = \frac{1}{p!} (p-1)! \sum_{1 \leq i_1 < \cdots < i_p \leq q} c_{i_1, \ldots, i_p}^m / K_{m,i(m)} \exp(-2\pi \sum_{s=q+1}^m t_{s+i}) \exp \frac{2\pi i}{\sqrt{-1}} \langle m, t \rangle \prod_{1 \leq i_1 < \cdots < i_p \leq q} d\xi_{i_1}^m \wedge \cdots \wedge d\xi_{i_p}^m\]

for $m \neq 0.$ Observing (4.2), it follows from (4.4) and (4.5) that

\[(4.6)\]

\[\phi_m = \delta \phi_m \text{ for any } m \neq 0.\]

\textbf{Remark.} For any $\delta$-closed $(0, p)$-form $\phi = \sum \phi_m$ we have always a formal solution $\sum_{m \neq 0} \phi_m$ for the $\delta$-equation $\delta \sum_{m \neq 0} \phi_m = \sum_{m \neq 0} \phi_m$ by (4.6).

Here we need to topologize $H^0(C^n/\Gamma, A^p).$ Let $A^p(R)$ be the vector space of real analytic functions on $R.$ We regard $R$ as a closed real analytic submanifold of $C$ under the natural inclusion. We take a compact subset $K$ of $R$ and an open and connected neighbourhood $U_j$ of $K$ in $C,$ $1 \leq j \leq \infty$ satisfying $U_{j-1} \subset U_j$ and $\bigcap_j U_j = K.$ We denote by $\mathcal{H}(U_j)$ the space of bounded holomorphic functions on $U_j,$ $j \geq 1.$ Put $||f|| := \sup \{|f(z)| : z \in U_j\}, f \in \mathcal{H}(U_j).$ This norm makes $\mathcal{H}(U_j)$ into a Banach space.
By the inductive limit: $\mathcal{A}(K) = \operatorname{ind} \lim \mathcal{H}(U_i)$ we regard $\mathcal{A}(K)$ as a $(D, F, S)$-space. The restriction mapping $\mathcal{A}(K_2) \rightarrow \mathcal{A}(K_1)$ induces the projective limit: $\mathcal{A}(R) = \operatorname{proj} \lim \mathcal{A}(K)$. It is known that the above locally convex topology on $\mathcal{A}(R)$ is complete and semi-Montel. Similarly to the topology of $\mathcal{A}(R)$ we can make the vector space $H^0(C^*/\Gamma, \mathcal{A})$ into a locally convex space. Then $H^0(C^*/\Gamma, \mathcal{A})$ is regarded as a closed subspace of $H^0(C^*/\Gamma, \mathcal{A})$ and itself a locally convex space. And we have the locally convex topology of $H^0(C^*/\Gamma, \mathcal{A})$, induced by $H^0(C^*/\Gamma, \mathcal{A})$. Further by Proposition 3.4 we have the locally convex topology of $H^p(C^*/\Gamma, \mathcal{O})$, using the quotient topology.

The following theorem gives a characterization of an $(H, C)$-group $C^*/\Gamma$ whose cohomology groups $H^p(C^*/\Gamma, \mathcal{O})$ $(p \geq 1)$ are finite-dimensional.

**Theorem 4.3.** Let $C^*/\Gamma$ be an $(H, C)$-group, where $\Gamma$ is generated by $\{e_i, \ldots, e_m, v_1, \ldots, v_q\}$, $K_{m,i} = \sum_{j=1}^o v_im_j - m_{i+1}$ $(1 \leq i \leq q)$ and $K_m := \max \{ |K_{m,i}|; 1 \leq i \leq q\}$ for $m \in \mathbb{Z}^+$. Then the following statements (1), (2), (3) and (4) are equivalent.

1. There exists $a > 0$ such that $
\sup_{m \neq 0} \exp(-a||m^*||)/K_m < \infty,$
where $||m^*|| = \max \{ |m_i|; 1 \leq i \leq n\}$.

2. $\dim H^p(C^*/\Gamma, \mathcal{O}) = \begin{cases} \frac{q!}{(q-p)! \cdot p!} & \text{if } 1 \leq p \leq q \\ 0 & \text{if } p > q. \end{cases}$

3. $\dim H^p(C^*/\Gamma, \mathcal{O}) < \infty$ for any $p \geq 1$.

4. $\delta(H^0(C^*/\Gamma, \mathcal{A}))$ is a closed subspace of $H^0(C^*/\Gamma, \mathcal{A})$ for any $p \geq 1$.

**Proof.** Assume (1) holds. Then by Lemma 4.2 we may suppose that the statement (0) of Lemma 4.2 holds. We take a $\delta$-closed form $\phi = 1/p! \sum_{m \in \mathbb{Z}^+} \sum_{i_1 \leq \ldots \leq i_p \leq m} \phi_{i_1 \ldots i_p} \delta_{e_{i_1} \ldots} \ldots \delta_{e_{i_p}} \in H^0(C^*/\Gamma, \mathcal{A})$, where $\phi_{i_1 \ldots i_p} = c_{i_1 \ldots i_p}$, $i_p \exp(-2\pi i \sum_{i=1}^n m_{i+1}) \exp(2\pi i - 1 < m, t')$. By Lemma 4.1 there exists $\varepsilon_0 > 0$ such that for any $a > 0 \ C(a) = \sup \{|c_{i_1 \ldots i_p}| \exp(\varepsilon_0 ||m^*|| + a ||m^*||); m \in \mathbb{Z}^+\} < \infty$ $(1 \leq i_1, \ldots, i_p \leq q)$. By the statement (0) of Lemma 4.2 we find $a_0 > 0$ such that
Then for any $a > 0$, $m \neq 0$ and $1 \leq i_1, \ldots, i_p \leq q$, \( |c_{i_1}^{m_n} \exp(\varepsilon_0/2||m'|| |a_0||m'^||)|/K_m \leq C_0\). This means that $\Sigma_{m=0} \phi^m$ given by (4.5) converges to a $(0, p-1)$-form $\psi$ in $H^0(C^*/\Gamma, \mathcal{H}^{0,p-1})$. And by (4.6) we have $\phi - \delta \psi = \phi^0 + \Sigma_{m=0} (\phi^m - \tilde{\phi}^m) = \sum_{1 \leq i_1, \ldots, i_p \leq q} c_{i_1}^{m_n} d\zeta_{i_1} \wedge \ldots \wedge d\zeta_{i_p}$. This shows (2) holds.

It is obvious that (2) $\implies$ (3) $\implies$ (4). Finally we prove (4) $\implies$ (1).

By Lemma 4.2 we may prove that (4) implies the statement (0) of Lemma 4.2 instead of (4) $\implies$ (1). Suppose that $\{K_m; m \in Z^{s+q}\}$ doesn't satisfy the statement (0) of Lemma 4.2. Then there exists $\varepsilon_0 > 0$ such that we can choose $\{m_v; \nu \geq 1 \in Z^{s+q} - [0]\}$ satisfying exp( $-\varepsilon_0 ||m_v'|| \nu ||m_v'^||)/K_m \geq \nu$ for any $\nu \geq 1$. We put

$$
\delta_n = \begin{cases} 
\exp(-\varepsilon_0 ||m_v'|| \nu ||m_v'^||)/K_{m_v} & \text{if } m = m_v \text{ for some } \nu \geq 1, \\
0 & \text{otherwise} 
\end{cases}
$$

and $\phi_n := \delta_n \exp(-2\pi \sum_{j=1}^{s+q} m_j \lambda_{n+1})$ exp $2\pi \sqrt{-1} \langle m, t' \rangle$ for any $m \in Z^{s+q}$. Since $\delta \phi_n = \sum_{j=1}^{s+q} \pi K_{m_v} \exp(-\varepsilon_0 ||m_v'|| \nu ||m_v'^||)/K_{m_v} \exp 2\pi \sqrt{-1} \langle m, t' \rangle d\zeta_j$, if $m = m_v$ for some $\nu \geq 1$ and $|K_{m_v}/K_{m}| \leq 1$, then $\Sigma_n \delta \phi_n$ converges to a form $\phi \in H^0(C^*/\Gamma, \mathcal{H}^{0,1})$. By the choice of the sequence $\{m_v\}$, $\Sigma_n \phi_n$ cannot converge to any function in $H^0(C^*/\Gamma, \mathcal{H}^{0,0})$. Suppose $\phi = \delta \lambda$ for some $\lambda = \sum_n \lambda_n \in H^0(C^*/\Gamma, \mathcal{H}^{0,0})$, then $\lambda = \phi_n$ for $m \neq 0$. It is a contradiction. Then $\phi = \lim_{N \to \infty} \delta (\Sigma_n ||m|| < \theta \phi_n)$ belongs not to $\delta (H^0(C^*/\Gamma, \mathcal{H}^{0,0}))$, but to the closure of $\delta (H^0(C^*/\Gamma, \mathcal{H}^{0,0}))$ in $H^0(C^*/\Gamma, \mathcal{H}^{0,0})$. This contradicts the statement (4).

By the above proof of the implication (4) $\implies$ (1), if $\{K_m; m \in Z^{s+q}\}$ doesn't satisfy the statement (1) of Theorem 4.3, then $H^1(C^*/\Gamma, \emptyset)$ is a non-Hausdorff locally convex space and then infinite-dimensional. Further in the above situation we shall prove that $H^p(C^*/\Gamma, \emptyset)$ are also non-Hausdorff spaces for all $p$ satisfying $2 \leq p \leq q$.

**Theorem 4.4.** Every $(H, C)$-group $C^*/\Gamma$ satisfies either of the following statements (a) and (b).

(a) $H^p(C^*/\Gamma, \emptyset)$ is finite-dimensional for any $p$.

(b) $H^p(C^*/\Gamma, \emptyset)$ is a non-Hausdorff locally convex space for any $p$ satisfying $1 \leq p \leq q$. 


Further the statement (b) is equivalent to the following

(c) \( \sup_{m \neq 0} \exp(-a||m^*||)/K_m = \infty \) for any \( a > 0 \),

where \( ||m^*|| = \max \{|m_i|; 1 \leq i \leq n\} \).

Proof. By Lemma 4.2 and Theorem 4.3, we must prove (b) holds on the assumption that \( \{K_m\} \) doesn't satisfy the statement (0) of Lemma 4.2. We choose \( \epsilon_i > 0 \), the sequence \( \{m_n\} \) and \( \delta^m \) as in the proof of (4) \( \Rightarrow (1) \) in Theorem 4.3. We can find \( i_0 \) so that \( 1 \leq i_0 \leq q \) and \( \sup \{\nu; |K_{m_{i_0}}| = K_{m_{i_0}} = \infty \} \). We may assume \( i_0 = q \) without loss of generality. We take a \((0, p - 1)\)-form

\[ \phi^m := \delta^m \exp(-2\pi \sum_{i=q+1}^n m_i t_{q+i}) \exp 2\pi \sqrt{-1} \langle m, t \rangle d\xi_1 \wedge \ldots \wedge d\xi_{p-1}. \]

Then \( \sum_m \phi^m \) cannot converge to any \((0, p - 1)\)-form in \( H^0(C^n/\Gamma, \mathcal{H}^{0, p-1}) \). On the other hand \( \sum_m \delta\phi^m \) converges to a \((0, p)\)-form \( \phi = \sum_m \sum_{i=p}^n \pi K_{m_i} \delta^m \exp(-2\pi \sum_{i=q+1}^n m_i t_{q+i}) \exp 2\pi \sqrt{-1} \langle m, t \rangle d\xi_1 \wedge d\xi_2 \wedge \ldots \wedge d\xi_{p-1} \neq 0 \). Suppose \( \phi = \delta\lambda \) for some \( \lambda = \sum_m \lambda^m \in H^0(C^n/\Gamma, \mathcal{H}^{0, p-1}) \). Then \( \delta\phi^m = \delta\lambda^m \). We write

\[ \lambda^m = 1/(p-1)! \sum_{i_1, \ldots, i_p} b_{i_1, \ldots, i_p}^m \exp(-2\pi \sum_{i=q+1}^n m_i t_{i}) \exp 2\pi \sqrt{-1} \langle m, t \rangle d\xi_1 \wedge \ldots \wedge d\xi_{p-1}. \]

Comparing the term of \( \delta\phi^m \) with that of \( \delta\lambda^m \) involving only the exterior differential \( d\xi_1 \wedge \ldots \wedge d\xi_{p-1} \) of type \((0, p)\), we have

\[ \pi K_{m_{i_1}} b_{i_1, \ldots, i_p}^m + \sum_{i=q+1}^n (-1)^i K_{m_{i_1}} b_{i_1, \ldots, i_{p-1}} = \pi K_{m_{i_1}} \delta^m \]

Then \( \delta^m = b_{i_1, \ldots, i_p} \) for \( \nu \geq 1 \). Since \( \sup \{\nu; |K_{m_{i_1}}| = K_{m_{i_1}} = \infty \} \), we can choose a subsequence \( \{m_n^\nu\} \) of \( \{m_n\} \) so that \( |K_{m_{i_1}}| \leq 1 \) for any \( 1 \leq i \leq q \) and that

\[ \lim_{\nu \to \infty} (b_{i_1, \ldots, i_p}^m + \sum_{i=q+1}^n (-1)^i b_{i_1, \ldots, i_{p-1}}^m K_{m_{i_1}}^\nu / K_{m_{i_1}}^\nu = 0. \]

This contradicts that \( \lim \delta^m = \infty \). Hence \( \phi \) belongs not to \( \delta(H^0(C^n/\Gamma, \mathcal{H}^{0, p-1})) \) but to the closure of \( \delta(H^0(C^n/\Gamma, \mathcal{H}^{0, p-1})) \).

Remark. When the author was making the preprint for this paper, he got the following information which was given by S. Takeuchi. Independently C. Vogt [15] showed in his Dissertation that the statements (a) and (b) are equivalent.

(a) There exist \( C > 0 \) and \( a > 0 \) such that \( K_m \geq C \exp(-a||m^*||) \).
(b) \( \dim H^0(C^n/\Gamma, 0) < \infty \).
By Theorem 4.3 and 4.4 we have the following

**Corollary 4.5.** The statements (1), (2), (3) and (4) in Theorem 4.3 are equivalent to each of the following statements (5) and (6).

(5) For some \( p \ (1 \leq p \leq q) \) \( \dim H^p(C^n/\Gamma, \mathcal{O}) < \infty. \)

(6) For some \( p \ (1 \leq p \leq q) \) \( \delta(H^p(C^n/\Gamma, \mathcal{H}^{0,b-1})) \) is a closed subspace of \( H^p(C^n/\Gamma, \mathcal{H}^{0,b}) \).

**Remark.** We constructed an example of an \( (H, C) \)-group \( C^n/\Gamma \) so that \( H^1(C^n/\Gamma, \mathcal{O}) \) is not Hausdorff ([7]). By Corollary 4.5 we can show \( H^p(C^n/\Gamma, \mathcal{O}) \) are not Hausdorff for this \( (H, C) \)-group \( C^n/\Gamma \) \((2 \leq p \leq q)\).

**References**


**Supplementary notes:**

After this paper was submitted, the referee informed the author the following results.
By another method L. C. Piccinini showed implicitly that $C \times R$ admits no Stein neighbourhood bases in $C \times C$ in the article:

[16] Non surjectivity of $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ as an operator on the space of analytic functions on $R^3$, \textit{Lecture Notes of the Summer College on Global Analysis}, Trieste, August 1972,

and C. Vogt has published the paper:


In [17] he showed independently that the statements (1) and (2) of Theorem 4.3 of this paper are equivalent.