Quasi-Invariant Measures on the Orthogonal Group over the Hilbert Space

By

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§ 1. Introduction

Let $H$ be a real separable Hilbert space and $O(H)$ be the orthogonal group over $H$. In this paper, we shall discuss left, right or both translationally quasi-invariant probability measures on a $\sigma$-field $\mathcal{B}$ derived from the strong topology on $O(H)$. Invariant (rather than quasi-invariant) measures have been considered by several authors. For example in [3], [7] and [4] such measures were constructed as suitable limits of Haar measures on $O(n)$ by methods of Schmidt's orthogonalization or of Cayley transformation. And in [6] some approach based on Gaussian measures on infinite-dimensional linear spaces was attempted. However these measures are defined on larger spaces rather than $O(H)$ and invariant under a sense that “$O(H)$ acts on these spaces.” This is reasonable, because it is impossible to construct measures on $O(H)$ which are invariant under all translations of elements of $G$, if $G$ is a suitably large subgroup of $O(H)$. For example, let $e_1, \cdots, e_n, \cdots$ be a c.o.n.s. in $H$, and for each $n$ consider a subgroup consisting of $T \subseteq O(H)$ which leaves $e_p$ invariant for all $p \geq n$. We may identify this subgroup with $O(n)$. Put $O_0(H) = \bigcup_{n=1}^{\infty} O(n)$. Then $O_0(H)$-invariant finite measure does not exist on $O(H)$. (See, [6]). However replacing invariance with quasi-invariance, the above situation becomes somewhat different. One but main purpose of this paper is to indicate this point. We will show that “there does not exist any $\sigma$-finite $G$-quasi-invariant measure on $\mathcal{B}$, as far as $G$ acts transitively on the unit sphere $S$ of $H$. While $O_0(H)$-quasi-invariant probability measures certainly exist.

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We can construct one of them by the Schmidt's orthogonalization method using a suitable family of measures on $H$.” In the remainder parts, we will state basic properties, especially ergodic decomposition of $O_0(H)$-quasi-invariant probability measures. These arguments are carried out in parallel with them for quasi-invariant measures on linear spaces. (See, [5]).


Let $e_1, \cdots, e_n, \cdots$ be an arbitrarily fixed c.o.n.s. in $H$, and define a metric $d(\cdot, \cdot)$ on $O(H)$ such that $d(U, V) = \sum_{n=1}^{\infty} 2^{-n} ||Ue_n - Ve_n|| + ||U^{-1}e_n - V^{-1}e_n||$, where $|| \cdot ||$ is the Hilbertian norm on $H$. A map $U \in O(H) \rightarrow ((Ue_1, \cdots, Ue_n, \cdots), (U^{-1}e_1, \cdots, U^{-1}e_n, \cdots)) \in H^\omega \times H^\omega$ is a into homeomorphism from $(O(H), d)$ to $H^\omega \times H^\omega$ equipped with the product-topology. Hence $(O(H), d)$ is a separable metric space. The topology derived from $d$ coincides with the strong topology on $O(H)$, so $(O(H), d)$ is a topological group and $\mathfrak{B}$ is a $\sigma$-field generated by open sets of $(O(H), d)$. Moreover, since inverse terms $||U^{-1}e_n - V^{-1}e_n||$ are added to the definition of $d$, $(O(H), d)$ is a complete metric space and therefore a Polish space. Now let $\mu$ be a measure on $\mathfrak{B}$ and $T \in O(H)$. We shall define measures $L_T \mu (R_T \mu)$ by $L_T \mu (B) = \mu (T^{-1}B)$ ($R_T \mu (B) = \mu (B \cdot T^{-1})$) for all $B \in \mathfrak{B}$, and call them left translation (right translation) of $\mu$ by $T$, respectively. If for a fixed subgroup $G \subset O(H)$, $L_T \mu (R_T \mu)$ is equivalent to $\mu$, $L_T \mu \equiv \mu$, for all $T \in G$, $\mu$ is said to be left (right) $G$-quasi-invariant, respectively. Left and right $G$-quasi-invariant measures are defined in a similar manner. Since results for right $G$-quasi-invariant measures are formally derived from them for left $G$-quasi-invariant measures, we shall omit the “right” case for almost everywhere.

Theorem 1. There does not exist any left (right) $G$-quasi-invariant $\sigma$-finite measure on $\mathfrak{B}$, as far as $G$ acts transitively on the unit sphere $S$ of $H$.

Proof. Suppose that it would be false, and let $\mu$ be a such one of left $G$-quasi-invariant measures. As $(O(H), d)$ is a Polish space, there exists a sequence of compact sets $\{K_n\}$ of $(O(H), d)$ such
that $\mu(K_n) > 0$ for $n = 1, 2, \cdots$ and $\mu(\cap_{n=1}^\infty K_n^c) = 0$. From the assumption, we have $0 < \mu(gK_n) = \mu(\cup_{n=1}^\infty K_n \cap gK_n)$ for all $g \in G$, and therefore $K_n \cap gK_1 \neq \phi$ for some $n$. It follows that $G \subseteq \cup_{n=1}^\infty K_n K_1^{-1}$. Take an $e \in S$ and consider a continuous map $f: U \in (O(H), d) \mapsto Ue \in S$. Then we have $S = f(G) = \cup_{n=1}^\infty f(K_n) K_1^{-1} \subseteq S$. Hence $S$ is a $\sigma$-compact set. However it is impossible in virtue of Baire's category theorem.

Q. E. D.

§ 3. A Construction of $O_0(H)$-Quasi-Invariant Measures

As it was stated in the Introduction, let us form $O_0(H)$ from an arbitrarily fixed c.o.n.s. $e_1, \cdots, e_n, \cdots$. For the purpose of the above title, it is enough to regard $H$ as $\ell^2$ and the above base as $e_n = (0, \cdots, \delta, 0 \cdots) \in \ell^n$. First we shall consider left quasi-invariant probability measures, and shall state some lines of the construction. Let $\bar{x} = (x_1, \cdots, x_n, \cdots)$ be a sequence of $\ell^2$. If they are linearly independent, we have an orthonormal system $G(x_1), G(x_1, x_2), \cdots, G(x_1, \cdots, x_n), \cdots$, operating on $x_1, \cdots, x_n, \cdots$ Schmidt's orthogonalization process. Moreover, they form a c.o.n.s., if a subspace $L(x)$ spanned by $x_1, \cdots, x_n, \cdots$ is dense in $H$. And then we can define an orthogonal operator $U(x)$ on $\ell^2$ as $e_n \mapsto G(x_1, x_2)$ for all $n$. Now for each $T \in O(\ell^2)$, we shall define a map $\bar{T}$ on the $\ell^2$-sequence space $(\ell^2)^\omega$ such that $\bar{T}(x_1, \cdots, x_n, \cdots) = (Tx_1, \cdots, Tx_n, \cdots)$. Then it is easy to see that $L_T \circ U = U \circ \bar{T}$, namely $TU(x) = U(\bar{T}x)$. Hence one of left $O_0(H)$-quasi-invariant measures $\lambda$ on $B$ is defined as $\lambda(B) = \overline{\nu}(x \mid U(x) \in B)$ for all $B \in \mathcal{B}$, if we can construct a probability measure $\nu$ on the usual Borel field $\mathcal{B}((\ell^2)^\omega)$ on $(\ell^2)^\omega$ satisfying following three properties,

(a) $x_1, x_2, \cdots, x_n, \cdots$ are linearly independent for $\nu$-a.e. $x = (x_1, \cdots, x_n, \cdots)$,

(b) "$L(x)$ is dense in $\ell^2$" holds for $\nu$-a.e. $\bar{x}$,

(c) $\bar{T} \nu \ (\bar{T} \nu(B) = \nu(\bar{T}^{-1}(B))$ for all $B \in \mathcal{B}((\ell^2)^\omega)$ is equivalent to $\nu$ for all $T \in O_0(\ell^2)$.

Now let $\rho$ be a probability measure on $\mathcal{B}(\mathbb{R}^1)$ which is equivalent to the Lebesgue measure and satisfies $\int_0^1 t^{-2} d\rho(t) = 1$. 1-dimensional Gaussian measures with mean 0 and variance $c$ will be denoted by $g_c$. And take positive sequences $\{v_n\}_{n=2}^\infty$ and $\{c_n\}_{n=2}^\infty$ such that
Then for each $n$, a measure of product-type $\mu_n = g_{x_n} \times \cdots \times g_{x_n} \times f \times g_{x_{n+1}} \times \cdots \times g_{x_j} \times \cdots$ is defined on $\mathfrak{B}(\ell^2)$, in virtue of the choice of $\{c_n\}$. Moreover, from the rotational-invariance of $g_{x_n} \times \cdots \times g_{x_n}$, $\mu_n$ is $O(n-1)$-invariant for all $n$. Now let us consider a measure of product-type $\bar{\mu} = \mu_1 \times \cdots \times \mu_n \times \cdots$ on $\mathfrak{B}((\ell^2)^\infty)$. It is fairly easy that $\bar{\mu}$ satisfies (a). Since for all $n$ and for all $T \in O(n-1)$, we have

$$\bar{T} \bar{\mu} = T \mu_1 \times \cdots \times T \mu_{n-1} \times T \mu_n \times \cdots \times T \mu_j \times \cdots$$

$$= T \mu_1 \times \cdots \times T \mu_{n-1} \times \mu_n \times \cdots \times \mu_j \times \cdots$$

$$= \mu_1 \times \cdots \times \mu_n \times \cdots = \bar{\mu},$$

so $\bar{\mu}$ satisfies (c) too. We shall consider for (b). Let $\langle \cdot, \cdot \rangle$ be the scalar product on $\ell^2$. Then,

$$\int_{\ell^2} \langle y, e_n \rangle^{-1} y - e_n \rangle^2 d\mu_n(y)$$

$$= \int \langle y, e_n \rangle^{-2} \sum_{j,n} \langle y, e_j \rangle^2 d\mu_n(y)$$

$$= (n-1) \nu_n + \sum_{j=n+1}^\infty c_j^2,$$

it follows that

$$\int \langle x, e_n \rangle^{-2} \sum_{x=1}^\infty \langle x, e_n \rangle^{-1} x_n - e_n \rangle^2 d\bar{\mu}(\bar{x})$$

$$= \sum_{x=1}^\infty \nu_n + \sum_{j=n+1}^\infty c_j^2,$$

Hence putting

$$E = \{ \bar{x} \mid \sum_{x=1}^\infty \langle x, e_n \rangle^{-1} x_n - e_n \rangle^2 < 1 \},$$

we have $\bar{\mu}(E) > 0$. Thus for all $\bar{x} \in E$, $L(\bar{x})$ is dense in $\ell^2$ by the following lemma.

**Lemma 1.** Suppose that $\sum_{x=1}^\infty \| t_n - e_n \|^2 < 1$ for a sequence $\{t_n\} \subset \ell^2$. Then a subspace spanned by $t_1, \cdots, t_n, \cdots$ is dense in $\ell^2$.

**Proof.** By the assumption, we can define an operator $A$ such that $Ae_n = t_n$ for all $n$ and $I - A$ is a Hilbert-Schmidt operator whose Hilbert-Schmidt norm is strictly less than 1. Hence we have $\|I - A\|_{sp} < 1$. It implies $A$ is an isomorphic operator. Consequently, $Ae_1, \cdots, A e_n, \cdots$ span a dense linear subspace. Q.E.D.

At the same time we shall prove the measurability of the set
\[ \{ x \mid L(x) \text{ is dense} \} \equiv F. \] Consider a set \((\ell^2)^{\omega} \times S \supseteq \Omega \equiv \{ (\bar{x}, a) \mid \langle x_n, a \rangle \geq 0 \text{ for all } n \}\) and let \( p \) be a projection to the first coordinate. It is evident that \( F^c = p(\Omega) \), and the later is a Souslin set. Therefore \( F \) is universally-measurable. As \( \tilde{\mu}(F) \geq \tilde{\mu}(E) > 0 \), so we can put \( \tilde{\nu}(B) = \frac{\tilde{\mu}(B \cap F)}{\tilde{\mu}(F)} \) for all \( B \in \mathcal{B}((\ell^2)^\omega) \). Clearly, \( \tilde{\nu} \) satisfies (a) and (b). Moreover (c) is also satisfied, because \( F \) is an invariant set for all \( T, T \in O(\ell^2) \). By the above, there exist left \( O_0(H) \)-quasi-invariant probability measures on \( \mathcal{B} \). Next, if we wish to construct left and right \( O_0(H) \)-quasi-invariant measures, we shall prepare such \( \tilde{\nu}_1 \) and \( \tilde{\nu}_2 \) and form a product-measure \( \tilde{\nu}_1 \times \tilde{\nu}_2 \) on \((\ell^2)^{\omega} \times (\ell^2)^{\omega} \). Then for \( \tilde{\nu}_1 \times \tilde{\nu}_2 \)-a.e. \((\bar{x}, \bar{y}) \), \( U(\bar{x}, \bar{y}) : G(x_1, \cdots, x_n) \mapsto G(y_1, \cdots, y_n) \) \((n = 1, 2 \cdots)\) is an orthogonal operator on \( \ell^2 \) which satisfies \( U(T\bar{x}, S\bar{y}) = SU(\bar{x}, \bar{y}) T^{-1} \) for all \( T, S \in O(\ell^2) \). It follows by similar arguments that a measure \( \lambda = U(\bar{S}_1 \times \bar{S}_2) \) on \( \mathcal{B} \) is a left and right \( O_0(\ell^2) \)-quasi-invariant probability measure.

\section*{§ 4. Basic Results and Ergodic Decomposition of \( O_0(H) \)-Quasi-Invariant Measures}

From now on, we put \( \mathfrak{A}_n = \{ E \in \mathcal{B} \mid T \cdot E = E \text{ for all } T \in O(n) \} \),
\[ \mathfrak{B}_n = \{ E \in \mathcal{B} \mid T \cdot E \cdot S = E \text{ for all } T, S \in O(n) \} \quad (n = 1, 2, \cdots), \]
\[ \mathfrak{A}_\infty = \{ E \in \mathcal{B} \mid T \cdot E = E \text{ for all } T \in O_0(H) \} \]
and
\[ \mathfrak{B}_\infty = \{ E \in \mathcal{B} \mid T \cdot E \cdot S = E \text{ for all } T, S \in O_0(H) \}. \]
Then we have \( \mathfrak{A}_1 \supset \cdots \supset \mathfrak{A}_n \supset \cdots, \mathfrak{B}_1 \supset \cdots \supset \mathfrak{B}_n \supset \cdots, \cap_{n=1}^{\infty} \mathfrak{A}_n = \mathfrak{A}_\infty \) and \( \cap_{n=1}^{\infty} \mathfrak{B}_n = \mathfrak{B}_\infty \). \( \mathfrak{A}_\infty (\mathfrak{B}_\infty) \) plays an essential role for left (left and right) \( O_0(H) \)-quasi-invariant measures.

**Lemma 2.** (a) Let \( \mu \) be a left \( O_0(H) \)-quasi-invariant probability measure on \( \mathcal{B} \), and let \( E \in \mathcal{B} \) satisfy \( \mu(E \ominus T \cdot E) = 0 \) for all \( T \in O_0(H) \). Then there exists an \( E_0 \in \mathfrak{A}_\infty \) such that \( \mu(E \ominus E_0) = 0 \).

(b) Let \( \mu \) be a left and right \( O_0(H) \)-quasi-invariant probability measure on \( \mathcal{B} \), and let \( E \in \mathcal{B} \) satisfy \( \mu(E \ominus T \cdot E \cdot S) = 0 \) for all \( T, S \in O_0(H) \). Then there exists an \( E_0 \in \mathfrak{B}_\infty \) such that \( \mu(E \ominus E_0) = 0 \).

**Proof.** (a) Put \( f_n(U) = \sum_{T \in O(n)} \chi_E(T \cdot U) \) for all \( U \in O(n) \), where \( dT \) is the normal-
ized Haar measure on $O(n)$ and $\chi_E$ is the indicator function of $E$. Then $f_n(U)$ is an $O(n)$-invariant function and
\[
|f_n(U) - \chi_E(U)| \, d\mu(U)
\leq \int |\chi_E(T \cdot U) - \chi_E(U)| \, dT \, d\mu(U)
= \int_{O(n)} \mu(E \ominus T^{-1} \cdot E) \, dT = 0.
\]
Hence we have $f_n(U) = \chi_E(U)$ for $\mu$-a.e. $U$. Put $f(U) = \lim f_n(U)$, if the limit exists and $f(U) = 0$, otherwise. Since $f(U)$ is $O_0(H)$-invariant, so putting $E_0 = \{ U \mid f(U) = 1 \}$, it holds $\mu(E \ominus E_0) = 0$.

(b) It is carried out in a similar manner, only changing the integral into $\int_{O(n) \times O(n)} \chi_E(T \cdot U \cdot S) \, dT \, dS$.

**Q. E. D.**

**Lemma 3.** Let $\mu$ be a left $O_0(H)$-quasi-invariant probability measure on $\mathcal{B}$. Then for any $B \in \mathcal{B}$ there exists a countable set $\{ T_n \}_{n=1}^\infty \subset O_0(H)$ such that $\hat{B} = \bigcup_{n=1}^\infty T_n \cdot B$ satisfies $\mu(T \cdot \hat{B} \ominus \hat{B}) = 0$ for all $T \in O_0(H)$. If $\mu$ is a left and right $O_0(H)$-quasi-invariant probability measure, then it holds $\mu(T \cdot \hat{B} \cdot S \ominus \hat{B}) = 0$ for all $T, S \in O_0(H)$, replacing the above set with $\hat{B} = \bigcup_{n=1}^\infty T_n \cdot B \cdot S_n$ for some $\{ T_n \}, \{ S_n \} \subset O_0(H)$.

**Proof.** As $L_\mu^1(\mathcal{B}(O(H)))$ is separable, we can take a countable dense set $\{ \chi_{T \cdot B}(U) \}_{n=1}^\infty$ of $\{ \chi_{T \cdot B}(U) \}_{T \in O_0(H)}$ in the left case and $\{ \chi_{T_n \cdot B \cdot S_n}(U) \}_{n=1}^\infty$ of $\{ \chi_{T \cdot B \cdot S}(U) \}_{T, S \in O_0(H)}$ in the left and right case. It is easily checked that $\bigcup_{n=1}^\infty T_n \cdot B$ and $\bigcup_{n=1}^\infty T_n \cdot B \cdot S_n$ are desired ones respectively.

**Q. E. D.**

**Proposition 1.** Two left $O_0(H)$-quasi-invariant probability measures $\mu$ and $\nu$ are equivalent, if and only if $\mu \cong \nu$ on $\mathcal{A}_\infty$. In the case of left and right $O_0(H)$-quasi-invariant measures, it is necessary and sufficient that they are equivalent on $\mathcal{B}_\infty$.

**Proof.** The necessity is obvious. So let $\mu$ and $\nu$ be left $O_0(H)$-quasi-invariant and suppose that they are not equivalent, for example, $\mu(B) > 0$ and $\nu(B) = 0$ for some $B \in \mathcal{B}$. Then applying Lemma 3 for $\mu$, there exists $\{ T_n \} \subset O_0(H)$ such that $\hat{B} = \bigcup_{n=1}^\infty T_n \cdot B$ satisfies $\mu(T \cdot \hat{B} \ominus \hat{B}) = 0$ for all $T \in O_0(H)$. Clearly we have $\nu(T \cdot \hat{B} \ominus \hat{B}) = 0$ for all
Thus applying Lemma 2 for \( \lambda=2^{-1}(\mu+\nu) \), there exists an \( A \in \mathfrak{B} \) such that \( \lambda(A \ominus \hat{B})=0 \). It follows that \( \mu(A)=\mu(\hat{B})>0 \) and \( \nu(A)=\nu(\hat{B})=0 \). Therefore \( \mu \) and \( \nu \) are not equivalent on \( \mathfrak{A}_\infty \). The left and right case is discussed in a similar way. \( \text{Q.E.D.} \)

Now we shall introduce a notion of ergodicity. A left (left and right) \( O_{0}(H) \)-quasi-invariant probability measure \( \mu \) is said to be left (left and right) \( O_{0}(H) \)-ergodic, if \( \mu(A)=1 \) or 0 for every subset \( A \in \mathfrak{B} \) satisfying \( \mu(T \cdot A \ominus A)=0 \) for all \( T \in O_{0}(H) \) (\( \mu(T \cdot A \ominus A)=0 \) for all \( T, S \in O_{0}(H) \)), respectively. In virtue of Lemma 2, it is equivalent that \( \mu \) takes only the values 0 or 1 on \( \mathfrak{A}_\infty(\mathfrak{B}_\infty) \), respectively.

Corollary. Two left (left and right) \( O_{0}(H) \)-ergodic measures are equivalent, if and only if they agree on \( \mathfrak{A}_\infty(\mathfrak{B}_\infty) \), respectively.

Proposition 2. Let \( \mu \) and \( \nu \) be left \( O_{0}(H) \)-quasi-invariant probability measures on \( \mathfrak{B} \), and put \( \lambda(B)=\int_{g \in O(H)} \mu(Bg)\,d\nu(g) \) for all \( B \in \mathfrak{B} \). Then \( \lambda \) is left and right \( O_{0}(H) \)-quasi-invariant. Moreover, if \( \mu \) and \( \nu \) are left \( O_{0}(H) \)-ergodic, then \( \lambda \) is left and right \( O_{0}(H) \)-ergodic.

Proof. Let \( S \in O_{0}(H) \). Then we have \( \lambda(B)=0 \Leftrightarrow \mu(Bg)=0 \) for \( \nu \)-a.e. \( g \Leftrightarrow \mu(Bg)=0 \) for \( L_\nu \)-a.e. \( g \Leftrightarrow \lambda(B \cdot S)=\int \mu(Bg)\,dL_\nu(g)=0 \). This shows that \( \lambda \) is right \( O_{0}(H) \)-quasi-invariant. Left \( O_{0}(H) \)-quasi-invariance of \( \lambda \) is clear. Next, let \( \mu \) and \( \nu \) be left \( O_{0}(H) \)-ergodic, and let \( A \in \mathfrak{B}_\infty \). As \( A g \in \mathfrak{A}_\infty \) for all \( g \in O(H) \), we have \( \mu(Ag)=1 \) or 0 for all \( g \in O(H) \). Put \( E=\{g \in O(H) \mid \mu(Ag)=1\} \). Then it follows from \( E \in \mathfrak{A}_\infty \) that we have \( \nu(E)=1 \) or 0. Hence \( \lambda(A)=1 \), if \( \nu(E)=1 \) and \( \lambda(A)=0 \), if \( \nu(E)=0 \). \( \text{Q.E.D.} \)

Now we shall consider an ergodic decomposition of \( O_{0}(H) \)-quasi-invariant measures. Let \( \mu \) be a probability measure on \( \mathfrak{B} \). As \( (O(H), d) \) is a Polish space, so for any sub-\( \sigma \)-field \( \mathfrak{A} \) of \( \mathfrak{B} \), there exists a family of conditional probability measures on \( \mathfrak{B} \) relative to \( \mathfrak{A} \) \( \{\mu(g, \mathfrak{A}, \cdot)\}_{g \in O(H)} \) which satisfy (1) for each fixed \( B \in \mathfrak{B} \), \( \mu(g, \mathfrak{A}, B) \) is an \( \mathfrak{A} \)-measurable function and (2) \( \mu(A \cap B)=\int_{A} \mu(g, \mathfrak{A}, B)\,d\mu(g) \) for all \( A \in \mathfrak{A} \) and for all \( B \in \mathfrak{B} \).
Lemma 4. Under the above notation, we take an arbitrary \( B \in \mathfrak{B} \) and fix it. Then for all \( n \), \( \mu(g, \mathfrak{A}, T \cdot B \cdot S) \) is a jointly \( \mathfrak{A} \times \mathfrak{B}(O(n)) \times \mathfrak{B}(O(n)) \)-measurable function of variables \( (g, T, S) \in O(H) \times O(n) \times O(n) \), where \( \mathfrak{B}(O(n)) \) is a usual Borel field on \( O(n) \).

Proof. Let \( f \) be a continuous bounded function on \( O(H) \). Put

\[
h(g, T, S) = \int_{O(H)} f(T^{-1} \cdot s^{-1}) d\mu(g, \mathfrak{A}, dt).
\]

Then (1) for a fixed \( (T, S) \in O(n) \times O(n) \), \( h(g, T, S) \) is \( \mathfrak{A} \)-measurable of \( g \) and (2) for a fixed \( g \in O(H) \), \( h(g, T, S) \) is continuous on \( O(n) \times O(n) \). Hence \( h(g, T, S) \) is jointly-measurable. Next, if \( f \) is an indicator function of a closed set \( B \), then we see that \( h(g, T, S) \) is again measurable, taking a family of bounded continuous functions \( \{f_n\} \), \( f_n \downarrow f \). Now a family of Borel subsets satisfying the assertion of this Lemma is a monotone class, and contains an algebra generated by closed sets by the above arguments. Thus it coincides with \( \mathfrak{B} \). Q.E.D.

Let \( \mu \) be a left \( O_b(H) \)-quasi-invariant probability measure on \( \mathfrak{B} \). First we shall ask for conditional probability measures relative to \( \mathfrak{A}_n \), using the normalized Haar measure \( dT \) on \( O(n) \) for each \( n \). We put

\[
\mu_n(B) = \int_{O(n)} \mu(T \cdot B) dT
\]

for all \( B \in \mathfrak{B} \). Then we have \( \mu_n \simeq \mu, \mu_n(A) = \mu(A) \) for all \( A \in \mathfrak{A}_n \) and \( \mu_n \) is \( O(n) \)-invariant. It follows that for all \( A \in \mathfrak{A}_n \) and for all \( B \in \mathfrak{B} \),

\[
\mu(A \cap B) = \int_{A \cap B} \frac{d\mu}{d\mu_n}(g) d\mu_n(g)
\]

\[
= \int_A \int_{T \in O(n)} \chi_B(T \cdot g) \frac{d\mu}{d\mu_n}(T \cdot g) dT d\mu_n(g)
\]

\[
= \int_A \int_{T \in O(n)} \chi_B(T \cdot g) \frac{d\mu}{d\mu_n}(T \cdot g) dT d\mu(g).
\]

Since

\[
\int_{T \in O(n)} \chi_B(T \cdot g) \frac{d\mu}{d\mu_n}(T \cdot g) dT = \mu(g, \mathfrak{A}_n, B)
\]

is an \( \mathfrak{A}_n \)-measurable function of \( g \) for each fixed \( B \in \mathfrak{B} \), so we have \( \mu(g, \mathfrak{A}_n, O(H)) = 1 \) for \( \mu \)-a.e. \( g \) and \( \{\mu(g, \mathfrak{A}_n, \cdot)\}_{g \in O(H)} \) is the family of conditional probability measures relative to \( \mathfrak{A}_n \). Let \( A \in \mathfrak{A}_n \) and \( B \in \mathfrak{B} \). Then

\[
\mu_n(A \cap B) = \int_{T \in O(n)} \mu(A \cap T \cdot B) dT
\]
\[ \int_{T \in \mathcal{O}(n)} \mu(g, \mathcal{A}, T \cdot B) \, d\mu(g) \, dT \]
\[ = \int_A d\mu_n(g) \int_{T \in \mathcal{O}(n)} \mu(g, \mathcal{A}, T \cdot B) \, dT. \]

Therefore by Fubini's theorem and Lemma 4,
\[ \int_{T \in \mathcal{O}(n)} \mu(g, \mathcal{A}, T \cdot B) \, dT = \mu_n(g, \mathcal{A}, B) \]
are conditional probability measures of \( \mu_n \) relative to \( \mathcal{A}_n \). Since it holds \( \mu_n \simeq \mu \), applying general discussions for conditional probability measures, it is assured that there exists an \( \Omega_n \subseteq \mathcal{A}_n \) with \( \mu(\Omega_n) = 1 \) such that
\[ \mu_n(g, \mathcal{A}_n, \cdot) \equiv \mu^g_n \equiv \mu(g, \mathcal{A}_n, \cdot) \equiv \mu^g \]
and the Radon-Nikodim derivative \( \frac{d\mu^g}{d\mu_n} \) can be taken as \( \frac{d\mu}{d\mu_n} \) for all \( g \in \Omega_n \). As \( \mu^g_n \) is \( O(n) \)-invariant, we conclude that for all \( g \in \bigcap_{n=1}^\infty \Omega_n \equiv \Omega_0 \), \( \mu^g \) is left \( O(H) \)-quasi-invariant. Moreover, from \( (\mu^g)_n = \mu^g_n \) we have for all \( g \in \Omega_n \),
\[ \mu^g(t, \mathcal{A}_n, B) = \int_{T \in \mathcal{O}(n)} \chi_B(T \cdot t) \frac{d\mu^g}{d\mu_n}(T \cdot t) \, dT \]
\[ = \int_{T \in \mathcal{O}(n)} \chi_B(T \cdot t) \frac{d\mu}{d\mu_n}(T \cdot t) \, dT \]
\[ = \mu(t, \mathcal{A}_n, B) \]
for all \( t \in O(H) \) and for all \( B \in \mathcal{B} \). Consequently, for all \( g \in \Omega_b \), \( \mu^g(t, \mathcal{A}_n, \cdot) = \mu(t, \mathcal{A}_n, \cdot) \) holds for all \( t \in O(H) \) and for all \( n \). In virtue of inverse martingale theorem, for all \( B \in \mathcal{B} \),
\[ 0 = \lim_n \int \mu(t, \mathcal{A}_n, B) - \mu(t, \mathcal{A}_n, B) \, d\mu(t) \]
\[ = \lim_n \int \mu(t, \mathcal{A}_n, B) - \mu(t, \mathcal{A}_n, B) \, d\mu^g(t) \, d\mu(g). \]

Taking a subsequence \( \{n_j\} \) if necessary, there exists an \( \Omega_b^{1} \subseteq \mathcal{A}_\infty \) with \( \mu(\Omega_b^{1}) = 1 \) such that for all \( g \in \Omega_b \),
\[ \lim_j \int \mu(t, \mathcal{A}_n, B) - \mu(t, \mathcal{A}_n, B) \, d\mu^g(t) = 0. \]

Hence again using the inverse martingale theorem, we have for all \( g \in \Omega_b^{1} \cap \Omega_c \),
\[ \int \mu(t, \mathcal{A}_n, B) - \mu^g(t, \mathcal{A}_n, B) \, d\mu^g(t) = 0. \]
It follows that
\[
2\int \mu(g, \mathcal{A}_\infty, B) |\psi(\mathcal{A}_\infty, B) |^2 d|\psi(\mathcal{A}_\infty, B) | d\mu(g) = 0.
\]
Thus there exists an \( \Omega^*_B \subseteq \mathcal{A}_\infty \) with \( \mu(\Omega^*_B) = 1 \) such that
\[
\int |\psi(\mathcal{A}_\infty, B) |^2 d\mu(g) = 0
\]
for all \( g \in \Omega^*_B \). Finally we shall put \( \Omega = \bigcap_{B \in \mathcal{F}} \Omega^*_B \), where \( \mathcal{F} \) is a countable algebra generated by a countable open base of \( (O(H), d) \). Then for all \( g \in \Omega \), the above formula holds for every \( B \in \mathcal{B} \), so for all \( A \in \mathcal{A}_\infty \) and \( B \in \mathcal{B} \),
\[
\mu(\Omega \cap B) = \int_A \mu(t, \mathcal{A}_\infty, B) d\mu(t) = \mu(B) \mu(A).
\]
Especially, we have \( \mu(A) = 1 \) or \( 0 \) for all \( A \in \mathcal{A}_\infty \) and it implies \( \mu \) is left \( O_0(H) \)-ergodic for all \( g \in \Omega \cap \Omega \). We shall conclude these arguments with the following theorem.

**Theorem 2.** Let \( \mu \) be a left \( O_0(H) \)-quasi-invariant probability measure on \( \mathcal{B} \). Then the conditional probability measures \( \mu(g, \mathcal{A}_\infty, \cdot) \) relative to \( \mathcal{A}_\infty \) are left \( O_0(H) \)-ergodic for \( \mu \)-a.e. \( g \).

From Theorem 2, we can derive a following theorem called canonical decomposition in a quite similar way with it in pp. 372–373 in [5].

**Theorem 3.** Let \( \mu \) be a left \( O_0(H) \)-quasi-invariant probability measure. Then there exist a family of probability measures \( \{ \mu^\tau \}_{\tau \in \mathbb{R}^1} \) on \( \mathcal{B} \) and a map \( \tau \) from \( O(H) \) to \( \mathbb{R}^1 \) which satisfy
(a) \( \mu^\tau \) is left \( O_0(H) \)-ergodic for all \( \tau \in \mathbb{R}^1 \),
(b) for each fixed \( B \in \mathcal{B} \), \( \mu^\tau(B) \) is \( \mathcal{B}(\mathbb{R}^d) \)-measurable,
(c) \( \mathbb{R}^1 \circ \mathcal{A}_\infty \subseteq \mathcal{A}_\infty \),
(d) \( \mu(B \cap \mathbb{R}^1(E)) = \int_{\mathbb{R}^1} \mu(\mathbb{R}^1) d\mu(\tau) \) for all \( B \in \mathcal{B} \) and \( E \in \mathcal{B}(\mathbb{R}^d) \),
(e) there exists \( E_0 \in \mathcal{B}(\mathbb{R}^d) \), \( \mu(\mathbb{R}^1(E_0)) = 1 \) such that \( \mu^1 \) and \( \mu^2 \) are mutually singular for all \( \tau_1, \tau_2 \in E_0(\tau_1 \neq \tau_2) \).

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Measures is carried out in parallel with the left case, only changing the integrals $\int_{O(n)} \cdots dT$ into double integrals $\iint_{O(n) \times O(n)} \cdots dTdS$. And the statements of Theorem 3 remains valid, changing "left" and $\mathcal{B}_\infty$ into "left and right" and $\mathcal{B}_\infty$, respectively.

References

[7] ————, Projective limit of Haar measures on $O(n)$, ibid., 8 (1972), 141-149.