Hukuhara's problem for equations of evolution

By

Setuzo Yosida*

§ Introduction.

In the former work [12] we have investigated boundary value problem formulated with Hukuhara's data for parabolic system. Now in this paper, we shall treat initial value problem formulated with Hukuhara's data, or shortly, Hukuhara's problem for equations of evolution.

In §1 we shall consider a system of equations of evolution

\[ \partial_t u(t, x) = \sum_{|k| \leq n} A_k(t) D^k u(t, x) + b(t, x), \]

and find a solution \( u(t, x) = [u_1(t, x), u_2(t, x), \ldots, u_N(t, x)] \) of it defined for \( 0 \leq t \leq T_0 \)

and satisfying

\[ u_j(t, x) = \varphi_j(x), \quad 0 \leq t \leq T_0, \quad j = 1, 2, \ldots, N, \]

for a properly chosen constant \( B_0 \).

We shall apply Fourier transformation \( \mathcal{F}_x \) with respect to the variables \( x = (x_1, x_2, \ldots, x_m) \) to the system (0.1) and transform the problem into a Hukuhara's problem for ordinary differential equations, which can be solved under a suitable assumption.

In §2 we shall investigate a system of special type, which can be treated under a weaker assumption than the former one.

In §3 we shall treat Hukuhara's problem for convolution equations, which can be solved by the same method as in the preceding sections.

* Univ. of Tokyo and Research Inst. Math. Sci., Kyoto Univ., Received July 29, 1966.
In the last section we shall consider Hukuhara's problem for a semi-separate symmetric hyperbolic system. This is a system of differential equations of which the principal part consists of separate symmetric hyperbolic systems. Partial differential equations of "semi-separate type" have been investigated by several authors. See for example T. Kusano [4], [5] and references in them. K. Akô [1] called them "semi-decomposable type".

We shall prove existence, uniqueness and stability of solution of such a system.

The author wishes to express his hearty thanks to Prof. M. Hukuhara and Prof. K. Yosida for their helpful advice and incessant encouragement.

§1. Hukuhara's problem for general equations of evolution.

1.1. We use following notations; bold letters stand for vectors;

\[ u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix}, \quad b(t, x) = \begin{bmatrix} b_1(t, x) \\ b_2(t, x) \\ \vdots \\ b_N(t, x) \end{bmatrix}, \quad 0 = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}, \quad x = (x_1, x_2, \ldots, x_m), \]

\[ \sigma = (\sigma_1, \sigma_2, \ldots, \sigma_N), \quad \sigma x = \sigma_1 x_1 + \sigma_2 x_2 + \cdots + \sigma_m x_m, \quad |\sigma| = |\sigma_1| + |\sigma_2| + \cdots + |\sigma_m|, \]

\[ k = (k_1, k_2, \ldots, k_m), \quad k_i's \text{ are nonnegative integers.} \quad |k| = k_1 + k_2 + \cdots + k_m, \]

\[ \partial_t = \partial/\partial t, \quad \partial_j = \partial/\partial x_j, \quad D_j = -i\partial_j, \quad D^k = D_1^k D_2^k \cdots D_m^k. \quad t, x_j, \sigma_j \text{ are real variables.} \]

\[ u(t, x), \quad b(t, x), \quad \varphi_j(x) \text{ etc. are in general complex valued functions.} \]

Consider the following equations;

\[ \partial_t u = \sum_{|k| \leq k} A_k(t) D^k u + b(t, x), \tag{1.1} \]

in which \( A_k(t)'s \) are \( N \times N \) matrices. \( b(t, x) \) is an \( N \)-dimensional vector. We shall solve Hukuhara's problem for this system and find as solution \( u(t, x) \), defined for \( 0 \leq t \leq T_0 \) and satisfying the initial condition with Hukuhara's data

\[ u_j(t_j, x) = \varphi_j(x), \quad 0 \leq t_j \leq T_0, j = 1, 2, \ldots, N, \tag{1.2} \]
Hukuhara's problem for equations of evolution

or in vector form

\( u(t, x) |_{y} = \varphi(x) \).

We apply to \( u(t, x) \) Fourier transformation \( \mathcal{F}_x \) with respect to the variable \( x \) formally and obtain

\[
\begin{align*}
(1.3) & \quad v(t, \sigma) = \mathcal{F}_x u(t, x) = \int \exp(-i\sigma x) u(t, x) dx, \\
(1.4) & \quad \partial_t v(t, \sigma) = \sum_{|k| \leq \rho} A_k(t) \sigma^k v(t, \sigma) + b(t, \sigma), \\
(1.5) & \quad v_j(t_j, \sigma) = \tilde{\varphi}_j(\sigma), \quad j = 1, 2, \ldots, N,
\end{align*}
\]

or

\[
(1.5') \quad v(t, \sigma) |_{x} = \tilde{\varphi}(\sigma),
\]

in which we used the notations \( \tilde{b}(t, \sigma) = \mathcal{F}_x b(t, x), \tilde{\varphi}(\sigma) = \mathcal{F}_x \varphi(x) \).

We assume:

\( \Lambda \) i) The elements \( a_{ki}(t) \)'s of the matrices \( A_k(t) \)'s are defined and continuous for \( 0 \leq t \leq T_0 \), satisfy \( |a_{ki}(t)| \leq \alpha_0 \) with some constant \( \alpha_0 \).

ii) The components \( b_j(t, x) \)'s of the vector \( b(t, x) \) are defined and integrable in \( x \) for fixed \( t \), which we denote by the notation \( b_j(t, x) \in L_x^1 \). The Fourier transforms \( \tilde{b}_j(t, \sigma) \) are continuous with respect to \( t \) and satisfy the inequality \( |\tilde{b}_j(t, \sigma)| = \beta_j(\sigma) \) with a function \( \beta_j(\sigma) \in L_\sigma^1 \) with compact support; \( \beta_0(\sigma) \equiv 0 \) for \( |\sigma| > \chi_0 \) with a constant \( \chi_0 \).

iii) The components \( \varphi_j(\sigma) \)'s of the vector \( \varphi(x) \) belong to \( L_\sigma^1 \). \( \hat{\varphi}_j(\sigma) \)'s have compact supports; \( \hat{\varphi}_j(\sigma) \equiv 0 \) for \( |\sigma| > \chi_0 \).

1.2. Since (1.4) is a system of ordinary differential equations including the parameters \( \sigma \), we consider the reduced equation of it:

\[
(1.6) \quad \partial_t v(t, \sigma) = \sum_{|k| \leq \rho} A_k(t) \sigma^k v(t, \sigma).
\]

Let \( v^{(j)}(t, \sigma) \), \( j = 1, 2, \ldots, N \), be its fundamental system of solutions satisfying

\[
(1.7) \quad v^{(j)}(0, \sigma) = d^{(j)} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} <_j.
\]
or in matrix form

\[(1.7') \quad V(0, \sigma) = E, \quad \text{where} \]

\[V(t, \sigma) = (v^{(1)}(t, \sigma), v^{(2)}(t, \sigma), \cdots, v^{(N)}(t, \sigma)).\]

One particular solution \(v^{(0)}(t, \sigma)\) of the original system (1.4) is given by the formula

\[(1.8) \quad v^{(0)}(t, \sigma) = V(t, \sigma) \int_0^t (V(s, \sigma))^{-1} \tilde{b}(s, \sigma) \, ds.\]

We notice that from the assumption A) ii) we have \(v^{(0)}(t, \sigma) \equiv 0\) for \(|\sigma| > X_0\).

We try to find a solution \(v(t, \sigma)\) of (1.4) (1.5) in the form

\[
(1.9) \quad \begin{cases} 
    v(t, \sigma) = v^{(0)}(t, \sigma) + \sum_{i=1}^N v^{(i)}(t, \sigma) c_i(\sigma) = v^{(0)}(t, \sigma) + V(t, \sigma) c(\sigma) \\
    v(t, \sigma) \equiv 0 \quad \text{for} \quad |\sigma| \leq X_0, \\
    v(t, \sigma) \equiv 0 \quad \text{for} \quad |\sigma| > X_0.
\end{cases}
\]

The condition (1.5) for this \(v(t, \sigma)\) becomes

\[(1.10) \quad (V(t, \sigma)|_H) c(\sigma) = \tilde{\phi}(\sigma) - v^{(0)}(t, \sigma) |_H \equiv \psi(\sigma),\]

in which we used the notation

\[(1.11) \quad V(t, \sigma)|_H = \begin{bmatrix}
    v_1^{(1)}(t_1, \sigma) & v_1^{(2)}(t_1, \sigma) & \cdots & v_1^{(N)}(t_1, \sigma) \\
    v_2^{(1)}(t_2, \sigma) & v_2^{(2)}(t_2, \sigma) & \cdots & v_2^{(N)}(t_2, \sigma) \\
    \vdots & \vdots & \ddots & \vdots \\
    v_N^{(1)}(t_N, \sigma) & v_N^{(2)}(t_N, \sigma) & \cdots & v_N^{(N)}(t_N, \sigma)
\end{bmatrix}.
\]

1.3. We shall give a lower bound of the absolute value of the Hukuhara’s determinant \(\det(V|_H)\).

Consider any matrix \(C = (c_{jk})_{j,k=1,2,\ldots,N}\) whose elements \(c_{jk}\) satisfy

\[|c_{jk} - \delta_{jk}| \leq \varepsilon, \quad j, k = 1, 2, \ldots, N, \quad \delta_{jk}\text{ are Kronecker’s symbols, } \varepsilon \text{ is a small positive constant.}\]

We can easily give a lower bound of \(|\det C|\) as follows;

\[(1.12) \quad |\det C| \geq (1 - \varepsilon)^N - \varepsilon^2 (1 + \varepsilon)^{N-2} (N! - 1) \equiv f(\varepsilon).\]

An elementary calculation gives

\[(1.13) \quad f(\varepsilon) \geq f(\varepsilon_0) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \quad \text{for} \quad 0 \leq \varepsilon \leq \varepsilon_0,\]
Hukuhara’s problem for equations of evolution

where $e_0$ can be taken as

$$e_0 = \min\left(1 - \left(\frac{1}{2}\right)^{N^N}, (1/2^N(N! - 1))^{1/2}\right).$$

To estimate the elements $v_k^{(j)}(t_*, \sigma)$ of the matrix $V(t, \sigma)$, we define for any fixed $j$,

$$w_k(t, \sigma) = v_k^{(j)}(t, \sigma) - \delta_{jk}, \quad k = 1, 2, \ldots, N,$

or in vector form

$$w(t, \sigma) = v(t, \sigma) - d^{(j)}.$$

From (1.6) and (1.7) we have

$$\partial_t w(t, \sigma) = \sum_{|k| \leq \rho} A_k(t) e^k(w(t, \sigma) + d^{(j)}),$$

$$w(0, \sigma) = 0.$$

Since $w(t, \sigma)$ is an $N$-dimensional complex valued vector, it is equivalent to a $2N$-dimensional real valued vector, and we can apply the comparison theorem of ordinary differential equations to it. We compare $w_k(t, \sigma)$ with the solution $z(t)$ of

$$\frac{d}{dt} z(t) = 2N\alpha_0 \sum_{|k| \leq \rho} X_0^{(|k|} z(t) + \alpha_0,$$

$$z(0) = 0$$

and have

$$|v_k^{(j)} - \delta_{jk}| = |w_k| \leq z(t)$$

$$= \frac{1}{2N \sum_{|k| \leq \rho} X_0^{(|k|}} \left\{ \exp(2N\alpha_0 \sum_{|k| \leq \rho} X_0^{(|k|} t) - 1 \right\}$$

for $0 \leq t \leq T_0$.

Let $B_0$ be a constant determined by the equation

$$\frac{1}{2N \sum_{|k| \leq \rho} X_0^{(|k|}} \left\{ \exp(2N\alpha_0 \sum_{|k| \leq \rho} X_0^{(|k|} B_0) - 1 \right\} = e_0.$$

From (1.13) and (1.20) we have

$$|\det(V(t, \sigma)|_u)| \geq 1/4 \quad \text{for } 0 \leq t \leq B_0,$$
which assures us the existence of the inverse matrix $(V(t, \sigma)|_n)^{-1}$.

1.4. Now (1.10) can be solved with respect to $c(\sigma)$ and we have

\[(1.23)\quad c(\sigma) = \begin{cases} (V(t, \sigma)|_n)^{-1}\psi(\sigma) & \text{for } |\sigma| \leq X_0, \\ 0 & \text{for } |\sigma| > X_0. \end{cases}\]

This completes the construction of $v(t, \sigma)$ defined by (1.9), and we have a solution of (1.4) and (1.5). We apply inverse Fourier transformation $\mathcal{F}_\sigma^{-1}$ to $v(t, \sigma)$ and have a solution of our Hukuhara’s problem (1.1) and (1.2);

\[(1.24)\quad u(t, x) = \mathcal{F}_\sigma^{-1} v(t, \sigma) = \frac{1}{(2\pi)^n} \int \exp(i\sigma x)v(t, \sigma) d\sigma.\]

Indeed, we have assumed in A) ii) that the absolute values of $b_j(t, \sigma)$‘s are bounded by some function of $\sigma$ independently of $t$. As is seen easily from the very way of construction of $v(t, \sigma)$ the same property is bequethed to $\partial_v(t, \sigma)$’s so that we can change the order of derivation and integration in differentiating the expression (1.24) with respect to $t$.

To sum up;

**Theorem 1.1.** Under the assumption A), (1.1) has a solution $u(t, x)$ defined for $0 \leq t \leq T_0$, which satisfies (1.2) for $0 \leq t_j \leq B_0$, $j = 1, 2, \ldots, N$ with a properly chosen constant $B_0 > 0$.

1.5. We shall state some remarks on stability and uniqueness of the solution constructed above.

i) Under the assumption A), $v(t, \sigma)$ is the unique solution of (1.4) and (1.5), which means the uniqueness of $u(t, x)$ in the class of solution to be obtained by the above procedure. That is to say, the solution $u(t, x)$ obtained above is the unique solution of our problem (1.1) and (1.2) in the class of functions satisfying

\[u_j(t, x) \in L^1_x, \quad |\partial_t u_j(t, x)| < u_0(x) \quad \text{with some } u_0(x) \in L^1_x, \quad \mathcal{F}_\sigma u_j(t, x) = v_j(t, \sigma) \in L^1_x,\]
Hukuhara's problem for equations of evolution

367

|∂ν(t, a)|<v0(σ) with some v0(σ)∈L1.

ii) Let t^n_i, j=1, 2, ..., N be converge to t_i as n→∞. Corresponding to each crew t^n_i, j=1, 2, ..., N we have V| demean, c^(o), v^(o), u^(o) etc. It can be easily seen that we have successively V(t, σ)| demean, c^(o)(σ)→c(σ), v^(o)(t, σ)→v(t, σ) u^(o)(t, x)→u(t, x) uniformly as n→∞. This means the stability of the solution u(t, x) given by (1.24) with respect to the variation of t_i.

iii) Let φ^(o)(x), j=1, 2, ..., N be such functions that the Fourier transforms of them converge to φ_j(σ) in L_x:

Then we have successively

v^(o)(t, σ)→v(t, σ) in L_x, u^(o)(t, x)→u(t, x) in L_x,

which means the stability of u(t, x) with respect to the variation of φ(x) expressed in terms of its Fourier transform φ.(σ).

§2. A special case.

2.1. The assumption A) ii) iii) can be weakened to some extent for differential equations of special type. Consider equations of the following type

\[
\begin{align*}
\partial_t u_1(t, x) &= u_2(t, x) + b_1(t, x), \\
\partial_t u_2(t, x) &= \sum_{|k|\leq p} a_k(t) D^k u_1(t, x) + b_2(t, x).
\end{align*}
\]

We shall solve Hukuhara's problem for this system, and find a solution of it, defined for 0≤t≤T, and satisfying the initial condition with Hukuhara's data

\[
(2.2) \quad u_j(t_j, x) = \varphi_j(x), \quad j=1, 2.
\]

We assume;

B) i) a_k(t)'s are defined for 0≤t≤T, are continuous, bounded, and satisfy

\[
(2.3) \quad a_k(t) \geq 0 \quad \text{for} \quad |k| \leq p,
\]

\[
|σ| \geq X_0,
\]
with some constant $X_0$ and some continuous function $\alpha_0(\sigma)(\geq 0)$. Without any restriction of generality, we can assume that $\alpha_0(\sigma) = O(|\sigma|^{|\sigma|/2})$ as $|\sigma| \to \infty$.

ii) $b_j(t, x)$'s are defined for $0 \leq t \leq T_\sigma$, belong to $L^1_x$. $\tilde{b}_j(t, \sigma)$'s are continuous with respect to the variable $t$, and satisfy $|\tilde{b}_j(t, \sigma)| \leq \beta_0(\sigma)$ with a function $\beta_0(\sigma)$ possessing the properties $\beta_0(\sigma)(|\sigma|^{3/2} + 1) \in L^1_d$ and

$$\beta_0(\sigma) = O(\exp(-T_0 + \varepsilon_0(\sigma))) \quad \text{with} \quad \varepsilon > 0$$

for $|\sigma| \to \infty$.

iii) $\Phi_j(x)$'s belong to $L^1_x$. $\tilde{\Phi}_j(\sigma)$'s satisfy $|\tilde{\Phi}_j(\sigma)| \leq \varphi_0(\sigma)$ with a function $\varphi_0(\sigma)$ possessing the properties $\varphi_0(\sigma)(|\sigma|^{p + 1}) \in L^1_d$ and

$$\varphi_0(\sigma) = O(\exp(-T_0 + \varepsilon_0(\sigma))) \quad \text{with} \quad \varepsilon_0 > 0$$

for $|\sigma| \to \infty$.

2.2. Fourier transformation applied to (2.1) and (2.2) gives

$$\begin{align*}
\partial_t v_1(t, \sigma) &= v_2(t, \sigma) + \tilde{b}_1(t, \sigma), \\
\partial_t v_2(t, \sigma) &= \hat{a}(t, \sigma) v_1(t, \sigma) + \tilde{b}_2(t, \sigma), \\
v_j(t, \sigma) &= \varphi_j(\sigma), \quad j = 1, 2.
\end{align*}$$

As we have seen in §1, there is a solution $v(t, \sigma)$ of (2.6) defined for $0 \leq t \leq T_\varepsilon$ and $|\sigma| \leq X_0$, satisfying (2.7) for $0 \leq t \leq B_\sigma$ with a properly chosen $B_\sigma$. We shall investigate (2.6) and (2.7) when $|\sigma| > X_0$.

Let $v^{(j)}(t, \sigma), \quad j = 1, 2$ be a fundamental system of solution of the reduced equations

$$\begin{align*}
\partial_t v_1(t, \sigma) &= v_2(t, \sigma), \\
\partial_\sigma v_2(t, \sigma) &= \hat{a}(t, \sigma) v_1(t, \sigma),
\end{align*}$$

satisfying the initial conditions

$$v^{(1)}(0, \sigma) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad v^{(2)}(0, \sigma) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

It is easily seen that the components $v_j^{(j)}(t, \sigma), \quad j = 1, 2$ of the vector $v^{(j)}(t, \sigma)$ satisfy $v_j^{(j)}(t, \sigma) \geq 1$ and are monotone increasing as
Hakuhara’s problem for equations of evolution

$t$ increases. Exact form of the solution $v^{(0)}(t, \sigma)$ expressed by $v^{(0)}(t, \sigma)$ can be obtained by the method of depression of the order;

\[
\begin{align*}
    v_1^{(0)}(t, \sigma) &= v_1^{(1)}(t, \sigma) \int_0^t \frac{ds}{(v_1^{(1)}(s, \sigma))^2}, \\
    v_2^{(0)}(t, \sigma) &= v_2^{(1)}(t, \sigma) \int_0^t \frac{ds}{(v_1^{(1)}(s, \sigma))^2} + \frac{1}{v_1^{(1)}(t, \sigma)}.
\end{align*}
\]

The Wronskian of these solutions $V(t, \sigma) = (v^{(1)}(t, \sigma), v^{(0)}(t, \sigma))$ can be calculated easily;

$$\det V(t, \sigma) = \text{const} = \det V(0, \sigma) = 1.$$ 

As for the Hukuhara’s determinant, we have

\begin{align*}
    \det (V(t, \sigma)) &= \begin{vmatrix} v_1^{(1)}(t_1, \sigma) & v_1^{(2)}(t_1, \sigma) \\ v_2^{(1)}(t_2, \sigma) & v_2^{(2)}(t_2, \sigma) \end{vmatrix} \\
    &= v_1^{(1)}(t_1, \sigma) \left( v_2^{(2)}(t_2, \sigma) \int_0^{t_1} \frac{ds}{(v_1^{(1)}(s, \sigma))^2} + \frac{1}{v_1^{(1)}(t_2, \sigma)} \right) \\
    &= v_1^{(1)}(t_1, \sigma) \left( \int_0^{t_1} \frac{-1}{(v_1^{(1)}(s, \sigma))^2} ds + \frac{1}{v_1^{(1)}(t_2, \sigma)} \right) \\
    &= v_1^{(1)}(t_1, \sigma) \left( [v_1^{(1)}(s, \sigma)]_{s=0}^{t_1} + \frac{1}{v_1^{(1)}(t_2, \sigma)} \right) = 1.
\end{align*}

2.3. To obtain an estimation of $v^{(1)}(t, \sigma)$, we consider equations

\[
\begin{align*}
    \frac{\partial}{\partial t} w_1^{(0)}(t, \sigma) &= w_1^{(2)}(t, \sigma), \\
    \frac{\partial}{\partial t} w_2^{(0)}(t, \sigma) &= (a_0(\sigma) + \delta)^2 w_1^{(3)}(t, \sigma),
\end{align*}
\]

with a parameter $\delta > 0$, together with initial conditions

\[
\begin{align*}
    w_1^{(0)}(0, \sigma) &= 1 + 1/(a_0(\sigma) + \delta), \\
    w_2^{(0)}(0, \sigma) &= 1 + a_0(\sigma) + \delta.
\end{align*}
\]

(2.11) and (2.12) have the unique solution

\[
\begin{align*}
    w_1^{(0)}(t, \sigma) &= (1 + 1/(a_0(\sigma) + \delta)) \exp((a_0(\sigma) + \delta)t), \\
    w_2^{(0)}(t, \sigma) &= (1 + a_0(\sigma) + \delta) \exp((a_0(\sigma) + \delta)t),
\end{align*}
\]
for which we have
(2.14) \( v_i^{(1)}(0, \sigma) < w_i^{(1)}(0, \sigma), \ j = 1, 2, \)
(2.15) \( \partial_i v_i^{(j)}(t, \sigma) < \partial_i w_i^{(j)}(t, \sigma) \) as long as \( v_i^{(j)}(t, \sigma) < w_i^{(j)}(t, \sigma) \)
\( l \neq j; \ j = 1, 2. \)

From these relations we can easily deduce
(2.16) \( v_i^{(j)}(t, \sigma) < w_i^{(j)}(t, \sigma) \) for \( t \geq 0, \ j = 1, 2, \)
and as the limit of \( \delta \to 0 \) we have
(2.17) \( \begin{cases} 1 \leq v_i^{(1)}(t, \sigma) \leq (1 + 1/\alpha_0(\sigma)) \exp(\alpha_0(\sigma)t), \\ 1 \leq v_i^{(2)}(t, \sigma) \leq (1 + \alpha_0(\sigma)) \exp(\alpha_0(\sigma)t), \end{cases} \)
which, together with (2.9), gives
(2.18) \( \begin{cases} 0 \leq v_i^{(1)}(t, \sigma) = v_i^{(1)}(t, \sigma) \int_0^t \frac{ds}{(v_i^{(1)}(s, \sigma))^2} \leq tv_i^{(1)}(t, \sigma) \\ \leq t(1 + 1/\alpha_0(\sigma)) \exp(\alpha_0(\sigma)t), \\ 1 \leq v_i^{(2)}(t, \sigma) = v_i^{(2)}(t, \sigma) \int_0^t \frac{ds}{(v_i^{(1)}(s, \sigma))^2} + \frac{1}{v_i^{(1)}(t, \sigma)} \\ < tv_i^{(1)}(t, \sigma) + 1 < t(1 + \alpha_0(\sigma)) \exp(\alpha_0(\sigma)t) + 1. \end{cases} \)

2.4. We take up a particular solution \( v^{(r)}(t, \sigma) \) of the inhomogeneous equations (2.6) given by the formula
(2.19) \( v^{(r)}(t, \sigma) = V(t, \sigma) \int_0^t (V(s, \sigma))^{-1} b(s, \sigma) ds, \)
in which we used the notations
\[
V(t, \sigma) = \begin{pmatrix} v_1^{(1)}(t, \sigma) & v_1^{(2)}(t, \sigma) \\ v_2^{(1)}(t, \sigma) & v_2^{(2)}(t, \sigma) \end{pmatrix},
\]
\[
(V(s, \sigma))^{-1} = \begin{pmatrix} v_2^{(2)}(s, \sigma) & -v_2^{(1)}(s, \sigma) \\ -v_1^{(2)}(s, \sigma) & v_1^{(1)}(s, \sigma) \end{pmatrix}.
\]

Since each column vector \( w^{(j)}(t, s, \sigma), j = 1, 2 \) of the matrix \( W(t, s, \sigma) = V(t, \sigma)(V(s, \sigma))^{-1}, \) \( 0 \leq s \leq t \leq T, \) satisfies the equations (2.8) together with the initial condition \( w^{(j)}(s, s, \sigma) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) or \( w^{(2)}(s, s, \sigma) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \), it can be expressed as a linear combination of fundamental solutions.
From (2.17) and (2.18) we know that the absolute values of the elements of $V(t, \sigma) (V(s, \sigma))^{-1}, 0 \leq s \leq t \leq T_0$ do not exceed $\tau(\exp(\alpha_0(\sigma)t))$. The coefficient $\gamma = \gamma(t, \alpha_0(\sigma))$ can be taken so as to be monotone increasing with respect to $t$ and satisfy

$$\gamma(t, \alpha_0) = O(\alpha_0), \text{ as } \alpha_0 \to \infty.$$  

From (2.19) and (2.20) we have

$$|v^{(0)}(t, \sigma)| \leq t\tau\beta_0(\sigma) \exp(\alpha_0(\sigma)t), \text{ for } 0 \leq t \leq T_0.$$  

Especially we have

$$|v^{(0)}(t, \sigma)| \leq B_0 \tau'\beta_0(\sigma) \exp(\alpha_0(\sigma)\beta_0), \text{ for } 0 \leq t \leq B_0 + T_0$$  

with $\tau' = \tau(\beta_0, \alpha_0(\sigma))$.

2.5. We shall find a solution $v(t, \sigma)$ of (2.6) and (2.7) for $|\sigma| > X_0$ in the form of

$$v(t, \sigma) = v^{(0)}(t, \sigma) + V(t, \sigma) c(\sigma), \quad c(\sigma) = \left( \begin{array}{c} c_1(\sigma) \\ c_2(\sigma) \end{array} \right).$$

From (2.7) we have

$$v(t, \sigma) |_H = v^{(0)}(t, \sigma) |_H + V(t, \sigma) |_H c(\sigma) = \tilde{\phi}(\sigma).$$

As we have seen in 2.2, det $v(t, \sigma) |_H$ $= 0$ and we can solve (2.24) with respect to $c(\sigma)$;

$$c(\sigma) = (V(t, \sigma) |_H)^{-1}(\tilde{\phi}(\sigma) - v^{(0)}(t, \sigma) |_H).$$

The elements of the matrix $(V(t, \sigma) |_H)^{-1}$ can be estimated by help of (2.10), (2.17) and (2.18), and we have from (2.25)

$$|c_1(\sigma)| \leq r''\varphi_0(\sigma) \exp(\alpha_0(\sigma)B_0) + r''\beta_0(\sigma) \exp(2\alpha_0(\sigma)B_0),$$

in which $r''$ and $r''\beta_0(\sigma)$ depend on both $T_0$ and $\alpha_0(\sigma)$, and $r'' = O(\alpha_0)$, $r''\beta_0(\sigma) = O(\alpha_0^2)$ as $\alpha_0 \to \infty$. Since we have estimated all the quantities appearing in the second member of (2.23), we have at last

$$|v_1(t, \sigma)| \leq r^{IV'}\beta_0(\sigma) \{\exp(\alpha_0(\sigma)T_0) + \exp(\alpha_0(\sigma)(2B_0 + T_0)) \}$$

$$+ r^{IV}\varphi_0(\sigma) \exp(\alpha_0(\sigma)(B_0 + T_0)),$$

in which $r^{IV}$ and $r^{IV'}$ depend on both $T_0$ and $\alpha_0(\sigma)$, $r^{IV} = O(\alpha_0^3), r^{IV'} = O(\alpha_0^2)$.
For any value of the constant \( B_0(\leq T_0) \) taken so small as to satisfy
\[
\begin{cases}
2B_0 < T_0 + \varepsilon, \\
B_0 < \varepsilon',
\end{cases}
\]
we can find a positive constant \( \varepsilon_0 \) for which we have
\[
|v_j(t, \sigma)| \leq \gamma_0 \exp(-\varepsilon_0 \alpha_0(\sigma)), \text{ for } |\sigma| > X_0,
\]
with some constant \( \gamma_0 \) depending on \( T_0 \) and \( \alpha_0(\sigma) \) and \( \gamma_0 = O(\alpha_0) \) as \( \alpha_0 \to \infty \). We have seen already in 2.2 that a solution \( v(t, \sigma) \) of (2.6) and (2.7) exists for \( |\sigma| \leq X_0 \). We have thus accomplished the construction of \( u(t, \sigma) \) for any value of the parameter.

From (2.27) (2.29) and the assumptions \( \mathcal{B}) \) ii) iii), we know that \( v_j(t, \sigma) \)'s belong to \( L^1_\sigma \). We have a solution \( u(t, x) \) of (2.1) and (2.2) by inverse Fourier transformation;
\[
(2.30) \quad u(t, x) = \mathcal{F}^{-1}_\sigma \{ v(t, \sigma) \} = \frac{1}{(2\pi)^n} \int \exp(i\sigma x) v(t, \sigma) d\sigma.
\]

To sum up;

**Theorem 2.1.** Under the assumption \( \mathcal{B}) \), (2.1) has a solution \( u(t, x) \) defined for \( 0 \leq t \leq T_0 \), which satisfies (2.2) for \( 0 \leq t \leq B_0 \), \( j = 1, 25 \) ze;\( \mathcal{B} \) a proper chosen constant \( B_0 \).

2.6. The remarks on uniqueness and stability of solution stated in 1.5, remain still valid with an evident alteration that the assumption \( \mathcal{A}) \) should be replaced by the assumption \( \mathcal{B}) \) in this case. Also the condition \(" \phi^{(n)}(\sigma) \to \phi_j(\sigma) \) in \( L^2_\sigma \) as \( n \to \infty \)" stated there, becomes \(" (|\sigma|^p + 1) \phi^{(n)}(\sigma) \to (|\sigma|^p + 1) \phi_j(\sigma) \) in \( L^2_\sigma \) as \( n \to \infty \)" in this case.

\[ 3. \] **Convolution equations.**

3.1. Hukuhara’s problem for convolution equations can be treated in the same manner as in the preceding sections. Consider the following convolution equations
\[
(3.1) \quad \partial_t u(t, x) = A(t, x) \ast u(t, x) + b(t, x),
\]
Hukuhara's problem for equations of evolution

in which \( A(t, x) \) means an \( N \times N \) matrix and the symbol \(*\) means convolution with respect to the variables \( x = (x_1, x_2, \ldots, x_n) \). We shall solve Hukuhara's problem for (3.1), and find a solution of it defined for \( 0 \leq t \leq T_0 \) and satisfying the initial condition with Hukuhara's data

\[
(3.2) \quad u_j(t, x) = \varphi_j(x), \ j = 1, 2, \ldots, N,
\]
or in vector form

\[
(3.2') \quad u(t, x) \mid _n = \varphi(x).
\]

Fourier transformation with respect to \( x \), applied to (3.1) and (3.2') gives

\[
(3.3) \quad \partial_t v(t, \sigma) = A(t, \sigma)v(t, \sigma) + \hat{b}(t, \sigma),
\]

\[
(3.4) \quad v(t, \sigma) \mid _n = \hat{\varphi}(\sigma).
\]

We assume

C) i) The elements \( a_{ij}(t, x) \)'s of the matrix \( A(t, x) \) are defined for \( 0 \leq t \leq T_0 \), belong to \( L^\infty \). Their Fourier transforms \( \hat{a}_{ij}(t, \sigma) \)'s are continuous with respect to \( t \) and belong to \( L^1 \), and satisfy \( |\hat{a}_{ij}(t, \sigma)| \leq a_0 \) for \( 0 \leq t \leq T_0 \) with a constant \( a_0 > 0 \).

ii) The components \( b_j(t, x) \)'s of the vector \( b(t, x) \) are defined for \( 0 \leq t \leq T_0 \), belong to \( L^\infty \). Their Fourier transforms \( \hat{b}_j(t, \sigma) \)'s are continuous with respect to \( t \) and satisfy \( |\hat{b}_j(t, \sigma)| \leq b_0(\sigma) \) with a function \( b_0(\sigma) \) belonging to \( L^1 \).

iii) The components \( \phi_j(x) \)'s of the vector \( \phi(x) \) belong to \( L^\infty \). Their Fourier transforms \( \hat{\phi}_j(\sigma) \)'s belong to \( L^1 \).

3.2. Consider the reduced equations of (3.3);

\[
(3.5) \quad \partial_t v(t, \sigma) = \hat{A}(t, \sigma)v(t, \sigma).
\]

From the assumption C) i) we know that we can treat (3.5) in the same manner as (1.6) in §1, but without any restriction on the value of \( \sigma \). There is a fundamental system of solutions \( V(t, \sigma) = (v^{(1)}(t, \sigma), v^{(2)}(t, \sigma), \ldots, v^{(\infty)}(t, \sigma)) \) which satisfies

\[
(3.6) \quad \text{det}(V(t, \sigma) \mid _n) \geq 1/2 \quad \text{for} \quad 0 \leq t \leq B_0(\leq T_0),
\]
with a sufficiently small constant $B_0$.

A particular solution $v^{(0)}(t, \sigma)$ of the equations (3.3) is given in the form

\[(3.7)\quad v^{(0)}(t, \sigma) = V(t, \sigma) \int_0^t (v(s, \sigma))^{-1} \hat{b}(s, \sigma) \, ds,\]

which is bounded and belongs to $L^1_\sigma$.

We shall find a solution $v(t, \sigma)$ of (3.3) and (3.4) expressed in the form

\[(3.8)\quad v(t, \sigma) = v^{(0)}(t, \sigma) + V(t, \sigma) c(\sigma), \quad c(\sigma) = \begin{pmatrix} c_1(\sigma) \\ \vdots \\ c_n(\sigma) \end{pmatrix}.\]

From (3.4) we have

\[(3.9)\quad c(\sigma) = (V(t, \sigma)|_n)^{-1} (\hat{\varphi}(\sigma) - v^{(0)}(t, \sigma)|_n),\]

which, substituted in (3.8), gives the desired solution $v(t, \sigma)$ belonging to $L^1_\sigma$. Inverse Fourier transformation applied to $v(t, \sigma)$ gives a solution $u(t, x)$ of our Hukuhara's problem (3.1) and (3.2);

\[(3.10)\quad u(t, x) = \mathcal{F}_\sigma^{-1} v(t, \sigma) = \frac{1}{(2\pi)^n} \int \exp(i\sigma x) v(t, \sigma) \, d\sigma.\]

To sum up;

**Theorem 3.1.** Under the assumption C), (3.1) has a solution $u(t, x)$ defined for $0 \leq t \leq T_0$, which satisfies (3.2) for $0 \leq t \leq B_0$ with a properly chosen constant $B_0 > 0$.

The remarks on uniqueness and stability of solution stated in the preceding sections, remain still valid in this case.

3.3. The assumption C) i) can be weakened to some extent for convolution equations of special type. Analogous circumstances have occurred for differential equations (1.1) and (2.1) in the preceding sections.

Consider convolution equations of the following type;

\[(3.11)\quad \begin{cases} \partial_t u_1(t, x) = u_2(t, x) + b_1(t, x) \\ \partial_t u_2(t, x) = a(t, x) * u_1(t, x) + b_2(t, x). \end{cases}\]
We shall find a solution \( u(t, x) \) of (3.11) defined for \( 0 \leq t \leq T_0 \), satisfying the condition

\[
(3.12) \quad u_j(t_j, x) = \phi_j(x), \quad j = 1, 2.
\]

Fourier transformation with respect to \( x \) applied to (3.11) and (3.12) gives

\[
(3.13) \quad \begin{cases} \partial_t \psi_1(t, \sigma) = \psi_2(t, \sigma) + \hat{b}_1(t, \sigma), \\ \partial_t \psi_2(t, \sigma) = \hat{a}(t, \sigma) \psi_1(t, \sigma) + \hat{b}_2(t, \sigma), \end{cases}
\]

or in vector form

\[
(3.14') \quad v(t, \sigma) |_R = \hat{\phi}(\sigma).
\]

We assume:

D) i) The function \( a(t, x) \) is defined for \( 0 \leq t \leq T_0 \), belongs to \( L^1_x \). Its Fourier transform \( \hat{a}(t, \sigma) \) is continuous with respect to \( t \), belongs to \( L^1_\sigma \), and satisfies

\[
(3.15) \quad \alpha_0(\sigma)^* \hat{a}(t, \sigma) \geq 0, \quad \text{for } |\sigma| > X_0,
\]

with a constant \( X_0 > 0 \) and a continuous function \( \alpha_0(\sigma) \), and

\[
(3.16) \quad \alpha_1 \geq |\hat{a}(t, \sigma)|, \quad \text{for } |\sigma| \leq X_0,
\]

with a constant \( \alpha_1 \).

ii) The components \( b_j(t, x) \)'s of the vector \( \mathbf{b}(t, x) \) are defined for \( 0 \leq t \leq T_0 \), belong to \( L^1_x \). Their Fourier transforms \( \hat{b}_j(t, \sigma) \)'s are continuous with respect to \( t \) and satisfy \( |\hat{b}_j(t, \sigma)| \leq \beta_0(\sigma) \) with a function \( \beta_0(\sigma) \) possessing the properties \( (\alpha_0(\sigma)^* + 1) \beta_0(\sigma) \in L^1_\sigma \) and \( \beta_0(\sigma) = O(\exp(-((T_0 + \varepsilon)\alpha_0(\sigma)))), \quad \varepsilon > 0, \) for \( |\sigma| \to \infty \).

iii) The components \( \phi_j(x) \)'s of the vector \( \mathbf{\phi}(x) \) belong to \( L^1_x \). The Fourier transforms \( \hat{\phi}_j(\sigma) \)'s satisfy \( |\hat{\phi}_j(\sigma)| \leq \phi_0(\sigma) \) with a function \( \phi_0(\sigma) \) possessing the properties \( (\alpha_0(\sigma)^* + 1) \phi_0(\sigma) \in L^1_\sigma \) and \( \phi_0(\sigma) = O(\exp(-(T_0 + \varepsilon')\alpha_0(\sigma)))) \) with \( \varepsilon' > 0 \) for \( |\sigma| \to \infty \).

3.4. Under the assumption D), (3.13) and (3.14) can be treated in the same manner as in §2. There is a solution \( \mathbf{v}(t, \sigma) \) of (3.13) de-
Setuzô Yosida

fined for $0 \leq t \leq T_0$ and satisfying (3.14) for $0 \leq t \leq B_0$ with a properly chosen constant $B_0 > 0$. Inverse Fourier transformation $\mathcal{F}^{-1}$ applied to $v(t, \sigma)$ gives a solution $u(t, x)$ of (3.11) satisfying (3.12).

The remarks on uniqueness and stability of solution remain still valid in this case with some obvious changes similar to the one given in 2.6 of §2.


First we quote from Nagumo [8] well known results concerning Cauchy problem for symmetric hyperbolic equations. Consider symmetric hyperbolic system

$$\begin{align*}
\Phi u(t, x) &= \partial_t u(t, x) + \sum_{i=1}^{n} A_i(t, x) \partial_t u(t, x) + B(t, x) u(t, x) \\
&= f(t, x),
\end{align*}$$

(4.1)

together with a usual initial condition

$$u(0, x) = \varphi(x).$$

(4.2)

The matrices $A_i(t, x)$'s are symmetric; $A_i(t, x) = (a_{ij}(t, x))_{j, k = 1, 2, \ldots, n}$, $a_{ij}(t, x) = a_{kj}(t, x)$, $B(t, x)$ is not necessarily so;

$$B(t, x) = (b_{jk}(t, x))_{j, k = 1, 2, \ldots, n}.$$

(4.1) is taken in generalized sense, that is to say, a solution $u(t, x)$ is such a vector function that it belongs to $L^1(t, x)$ on a domain $\mathcal{D}$ in $(t, x)$ space and satisfies

$$(u, \Phi^* w) = (f, w)
\left(\int_{t=1}^{N} \sum_{j=1}^{n} f_j(t, x) w_j(t, x) \, dt \, dx\right)
$$

for all $w(t, x) = \begin{pmatrix}
w_1(t, x) \\
w_2(t, x) \\
\vdots \\
w_n(t, x)
\end{pmatrix}$, $w_j(t, x) \in C^0_0(\mathcal{D})$,

in which $\Phi^*$ means the adjoint operator of $\Phi$.

We shall make use of the following theorems (Nagumo loc cit. pp. 70–71).

Theorem I. Let the elements of the matrices $A_i(t, x)$, $\partial_t A_i(t, x)$, $A_i(t, x)$, \ldots, $A_n(t, x)$, $\partial_t A_1(t, x)$, $\partial_t A_2(t, x)$, \ldots, $\partial_t A_n(t, x)$ be $C^0_0(\mathcal{D})$ functions.
(i = 0, 1, ..., m; \partial_0 = \partial/\partial t) and \(B(t, x)\) be bounded continuous functions defined on \((t, x)\)-space \(E^{m+1}\). Let the components \(f_i(t, x)\)'s of the vector \(f(t, x)\) belong to \(L_1 \cap L_2^{x}\) on \(E^{m+1}\), and let the components \(\varphi_j(x)\) of the vector \(\varphi(x)\) belong to \(L_2^{x}\) on \(x\)-space \(E^{n}\). (Hereafter we prefer to say shortly "\(f(t, x)\) belongs to \(L_1^{x}\), "\(\varphi(x)\) belongs to \(L_2^{x}\)" etc.)

For any fixed value \(T_0 > 0\), there is a generalized solution \(u(t, x)\) of (4.1) defined for \(0 \leq t \leq T_0\), satisfying (4.2) as an element of \(L_2^{x}\).

For fixed \(t\) the components of \(u(t, x)\) and \(\partial u(t, x)\) belong to \(L_2^{x}\).

Under this additional condition, solution of (4.1) and (4.2) is determined uniquely for \(0 \leq t \leq T_0\), \(x \in E^{n}\).

**Theorem II.** (Energy integral inequality) The solution \(u(t, x)\) of (4.1) and (4.2) given in Theorem I satisfies

\[(4.3) \quad \|u(t)\|^2 + \|u\|^2_{\infty, T_0} \leq c_{T_0}(\|u(t_0)\|^2 + \|f\|^2_{\infty, T_0}),\]

in which the constant \(c_{T_0}\) depends on \(T_0\), and \(c_{T_0} \to 1\) as \(T_0 \to 0\).

In the above formulation we used the notations

\[\|u(t)\|^2 = \int_{E^{n}} u(t, x)^2 dx, \quad \|u\|^2_{\infty, T_0} = \int_0^{T_0} \|u(t)\|^2 dt.\]

4.1. Consider the following equations

\[(4.4) \quad \partial_t u_j(t, x) + \sum_{i=1}^{m} A_{ij}(t, x) \partial_j u_i(t, x) + B_{ij}(t, x) u_j(t, x) + \sum_{\kappa \neq j} C_{\kappa j}(t, x) u_\kappa(t, x) = f_j(t, x), \quad j = 1, 2, ..., N,\]

or in concise form

\[(4.4') \quad \partial_t \tilde{u} + \tilde{C} \tilde{u} = \tilde{f},\]

in which we used the notations

\[\tilde{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix}, \quad u_j = \begin{pmatrix} u_{i1} \\ u_{i2} \\ \vdots \\ u_{iN_j} \end{pmatrix}, \quad f = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_N \end{pmatrix}, \quad f_j = \begin{pmatrix} f_{j1} \\ f_{j2} \\ \vdots \\ f_{jN_j} \end{pmatrix},\]

\(A_{ij}(t, x)\)'s are symmetric \(N_j \times N_j\) matrices, \(B_{ij}(t, x)\)'s are \(N_j \times N_j\) matrices, \(C_{\kappa j}(t, x)\)'s are \(N_j \times N_\kappa\) matrices.

We shall set up Hukuhara's problem for the equations (4.4) and
find out a solution in generalized sense, defined for $0 \leq t \leq T_0$, satisfying the condition

\[(4.5) \quad u_j(t, x) = \varphi_j(x) \text{ in } L^2_x, \quad j = 1, 2, \cdots, N,\]

that is, every component of the vector $u_j$ belong to $L^2_x$ and equal to the corresponding component of the vector $\varphi_j$ as an element of $L^2_x$. \((4.5)\) is written in the concise form

\[(4.5') \quad \tilde{u} |_{\tilde{\xi}} = \tilde{\varphi} \text{ in } L^2_x,\]

in which we used the notations

\[
\tilde{\varphi}_{(j)} = \begin{cases} 
\varphi_1(x) \\
\varphi_2(x) \\
\vdots \\
\varphi_N(x)
\end{cases}, \quad \varphi_j(x) = \begin{cases} 
\varphi_{1j}(x) \\
\varphi_{2j}(x) \\
\vdots \\
\varphi_{Nj}(x)
\end{cases}.
\]

We assume:

E) i) The elements of the matrices $A_{jk}(t, x)$'s, $\partial_t A_{jk}(t, x)$'s, $l = 0, 1, 2, \cdots, m$, $B_j(t, x)$'s and $C_{jk}(t, x)$'s are bounded continuous functions of $(t, x)$. Moreover, the elements $c_{jk}(t, x)$'s of the matrices $C_{jk}(t, x)$'s satisfy $|c_{jk}(t, x)| \leq c_0$ with such a constant $c_0$ that it satisfies

\[(4.6) \quad (N-1)c_0^2(\max_j N^j) < 1.\]

ii) $f_j(t, x)$'s belong to $L^2_x$ in $E^{r+1}$. $\varphi_j(x)$'s belong to $L^2_x$ in $E^*$.  

4.2. We put $\tilde{u}^{(0)}(t, x) = 0 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, and define $\{\tilde{u}^{(n)}(t, x)\}_{n=1}^{\infty}$ inductively as solutions of the following Hukuhara's problem

\[(4.7) \quad \tilde{\varphi} \tilde{u}^{(n+2)} = \tilde{f} - \tilde{C}\tilde{u}^{(n)},\]

\[(4.8) \quad \tilde{u}^{(n+1)} |_{\tilde{\xi}} = \tilde{\varphi}, \quad n = 1, 2, \cdots.\]

\[(4.7)\] and \[(4.8)\] can be separated into $N$ Cauchy problems with respect to $u_j^{(n+1)}$, $j = 1, 2, \cdots, N$. From theorem I we know that $\tilde{u}^{(n)}(t, x)$ exists and is unique for $0 \leq t \leq T_0$ with any $T_0 > 0$, $n = 1, 2, \cdots$.

From theorem II we have

\[(4.9) \quad \|u_j^{(1)}(t)\|_2^2 + \|u_j^{(n)}\|_{2, T_0}^2 \leq c_{rt_0} \|\varphi_j\|_2^2 + \|f_j\|_{2, T_0},\]
Using the notations \( \|\mathbf{u}(t)\|^2 = \sum_{j=1}^{N} \|\mathbf{u}_{j}(t)\|^2 \), \( \|\mathbf{\Phi}\|^2 = \sum_{j=1}^{N} \|\mathbf{\Phi}_{j}\|^2 \) etc., we have

\[
(4.10) \quad \|\mathbf{\hat{u}}(t)\|^2 + \|\mathbf{\hat{u}}^{(3)}(0,\tau_0)\| \leq c_{\tau_0}^2 (\|\mathbf{\hat{\Phi}}\|^2 + \|\mathbf{\hat{f}}\|^2_{(0,\tau_0)}),
\]

with \( c_{\tau_0}^2 = \max_{j} c_{\tau_0}^2 \).

For general value of \( n \) we have

\[
(4.11) \quad \hat{\theta}(\mathbf{\hat{u}}^{(n+1)} - \mathbf{\hat{u}}^{(n)}) = -\mathbf{C}(\mathbf{\hat{u}}^{(n)} - \mathbf{\hat{u}}^{(n-1)}),
\]

\[
(4.12) \quad \langle \mathbf{\hat{u}}^{(n+1)} - \mathbf{\hat{u}}^{(n)} \rangle_{\mathbf{y}} = \mathbf{0}.
\]

Application of Theorem II to \( \mathbf{u}^{(n)} + 1 \), \( \mathbf{u}^{(n)} \) gives

\[
(4.13) \quad \|\mathbf{u}^{(n+1)}(t) - \mathbf{u}^{(n)}(t)\|^2 + \|\mathbf{u}^{(n+1)}(0,\tau_0) - \mathbf{u}^{(n)}(0,\tau_0)\|^2
\]

\[
\leq c_{\tau_0} \sum_{k=1}^{n} N_k \|\mathbf{u}^{(n)} - \mathbf{u}^{(n-1)}\|^2_{(0,\tau_0)},
\]

and

\[
(4.14) \quad \|\mathbf{\hat{u}}^{(n+1)}(t) - \mathbf{\hat{u}}^{(n)}(t)\|^2 + \|\mathbf{\hat{u}}^{(n+1)}(0,\tau_0) - \mathbf{\hat{u}}^{(n)}(0,\tau_0)\|^2
\]

\[
\leq c_{\tau_0} \sum_{k=1}^{n} c_k^2 N_k^2 \|\mathbf{u}^{(n)} - \mathbf{u}^{(n-1)}\|^2_{(0,\tau_0)}.
\]

Since we can let \( c_{\tau_0} \to 1 \) as \( \tau_0 \to 0 \), we can take

\[
(4.15) \quad c_{\tau_0}^2 = c_{\tau_0}^2 (N-1) \left( \max_{j} N_j^2 \right) < 1,
\]

for sufficiently small \( \tau_0 \), and we have inductively

\[
(4.16) \quad \|\mathbf{u}^{(n+1)} - \mathbf{u}^{(n)}\|^2_{(0,\tau_0)} \leq c_{\tau_0}^2 N_k \|\mathbf{u}^{(n)} - \mathbf{u}^{(n-1)}\|^2_{(0,\tau_0)}.
\]

\[
(4.17) \quad \|\mathbf{\hat{u}}^{(n+1)} - \mathbf{\hat{u}}^{(n)}\|^2_{(0,\tau_0)} \leq c_{\tau_0}^2 N_k \|\mathbf{\hat{u}}^{(n)} - \mathbf{\hat{u}}^{(n-1)}\|^2_{(0,\tau_0)}.
\]

From (4.16) and (4.17) we have convergence in \( L^2_{(t,\tau)} \) of \( \mathbf{u}^{(n,\tau)} \), and convergence in \( L^2_{\tau} \) for fixed \( t \) of \( \mathbf{u}^{(n,\tau)} \), and from (4.7) also convergence in \( L^2_{\tau} \) for fixed \( t \) of \( \hat{\theta} \mathbf{\hat{u}}^{(n)}(t,\tau) \). Hence we have a solution \( \mathbf{\hat{u}}(t,\tau) \) of (4.7) and (4.8) defined as the limit of \( \mathbf{\hat{u}}^{(n)}(t,\tau) \) as \( n \to \infty \) for \( 0 \leq \tau \leq \tau_0 \) with sufficiently small \( \tau_0 > 0 \).

To sum up,
Theorem 4.1. Under the assumption E) there is a generalized solution $\tilde{u}(t, x)$ of (4.4) defined for $0 \leq t \leq T_0$ with a properly chosen $T_0 > 0$ and satisfying (4.5) for $0 \leq t \leq T_0$.

4.3. The solution $\tilde{u}(t, x)$ obtained above is unique and stable with respect to the variation of $\phi_j'$'s.

Indeed, let $\tilde{u}(t, x)$ and $\tilde{v}(t, x)$ be two solutions of (4.4) and (4.5) given for $0 \leq t \leq T_0$ with a constant $T_0$ satisfying (4.15) under the assumption E). By the same argument as the one used to derive (4.14), we have

\[
\|\tilde{u}(t) - \tilde{v}(t)\|^2 + \|\tilde{u} - \tilde{v}\|^2_{(0, t_0)} < c\gamma' \|\tilde{u} - \tilde{v}\|^2_{(0, t_0)} ,
\]

which gives $\tilde{u}(t, x) = \tilde{v}(t, x)$ in $L^2(t, x)$ or uniqueness of solution for $0 \leq t \leq T_0$, since we have $c\gamma' < 1$ by (4.15).

Stability of solution with respect to the variation of $\phi(x)$ is obtained similarly. Let $\tilde{u}(t, x)$ and $\tilde{v}(t, x)$ be the solutions of (4.4) and (4.5) corresponding to the data $\tilde{\phi}(x)$ and $\tilde{\psi}(x)$ respectively, defined for $0 \leq t \leq T_0$ with a constant $T_0$ satisfying (4.15) under the assumption E). After the manner of (4.18) we have

\[
\|\tilde{u}(t) - \tilde{v}(t)\|^2 + (1 - c\gamma') \|\tilde{u} - \tilde{v}\|^2_{(0, t_0)} < c\gamma' \|\tilde{\phi} - \tilde{\psi}\|^2 ,
\]

which gives the convergence of $\tilde{v}(t, x)$ to $\tilde{u}(t, x)$ in both $L^2_x$ and $L^2(t, x)$ as $\tilde{\psi}(x)$ converges to $\tilde{\phi}(x)$ in $L^2_x$.

To investigate stability of solution with respect to the variation of $\phi_j$'s, let $\tilde{u}(t, x)$ and $\tilde{v}(t, x)$ be the solutions of (4.4) defined for $0 \leq t \leq T_0$ and satisfying

\[
\phi_j(t_j, x) = \phi_j(x)
\]

or

\[
\phi_j(s_j, x) = \phi_j(x) ,
\]

with $|t_j - s_j| \leq \delta$, $j = 1, 2, \ldots, N$, respectively, under the assumption E). We assume in addition, E') The components $\varphi_{\phi_j}(x)$ of the vector $\tilde{\phi}(x)$ have derivatives in generalized sense belonging to $L^2_x$; $\partial_\phi \varphi_{\phi_j}(x) \in L^2_x$.

From ((4.20') and E') we have
Hukuhara's problem for equations of evolution

(4.21) \[ \Phi_j(v_j - \varphi_j) + \sum_{k \neq j} C_{jk} v_k = f_j(t, x) - \Phi_j \varphi_j , \]
(4.22) \[ v_j(s_j, x) - \varphi_j(x) = 0. \]

Since \( \| \tilde{v} \|_{(3, R^g)} \) is finite, we can apply Theorem II to \( v_j(t, x) - \varphi_j(x) \) for \( t_j - \delta \leq t \leq t_j + \delta \) and obtain

(4.23) \[ \| v_j(t) - \varphi_j \|^2 \leq c_\delta \| f_j - \Phi_j \varphi_j - \sum_{k \neq j} v_k \|^2_{(t_j - \delta, t_j + \delta)} , \]

with a constant \( c_\delta \), which remains bounded as \( \delta \to 0 \). From (4.23) we have \( v_j(t_j, x) - \varphi_j(x) = u_j(t_j, x) \) in \( L^2_{(t, x)} \) as \( s_j \to t_j \), \( j = 1, 2, \cdots, N \), from which it follows \( \tilde{v}(t, x) \to \tilde{u}(t, x) \) in \( L^2_{(t, x)} \) as we have seen already.

It is well known that starting from the energy integral inequality (4.3), one can deal with genuine solution of Cauchy problem and Cauchy problem for semilinear system. But it is immediately seen that an energy integral inequality for the system (4.4) can be given in the same form as (4.3), using the simple method by which we have derived (4.14) from (4.13). Therefore it seems to be unnecessary to state here the same results for Hukuhara's problem as the well known ones for Cauchy problem concerning semilinear system or gunune solution.

REFERENCES

[10] Z. Szymydt, Sur un nouveau type de problèmes pour un système d'équations dif-
