Some remarks on the orthogonality of generalized eigenfunctions for singular second-order differential equations

By

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Let us consider a differential equation of second order

\[(0, 1) \quad \frac{d^2 u}{dx^2} - q(x) u(x) + \lambda u(x) = 0, \quad (0 < x < \infty).\]

Here \(q(x)\) is a real-valued function which is locally summable in \((0, \infty)\).

In the case where \(x = 0\) is a regular point of the equation, M. Matsuda has proved that any pseudo-spectral measure in the limit point case at \(x = \infty\) is the Weyl spectral measure (Matsuda [6]).

In this paper we try to extend this result to the case where \(x = 0\) may be a singular point of the equation.

We take a linearly independent system of solutions \((\varphi_1(x, l), \varphi_2(x, l))\) of the equation \((0, 1)\) which satisfies

\[\varphi_i(1, l) = \eta_i(l), \quad \frac{\partial \varphi_i}{\partial x}(1, l) = \zeta_i(l)\]

for \(i = 1, 2\) where \(\eta_i(l)\) and \(\zeta_i(l)\) are entire functions of \(l\) which satisfy \(\eta_i(l)\zeta_j(l) - \eta_j(l)\zeta_i(l) = 1\) for every complex number \(l\).

M. H. Stone, E. C. Titchmarsh and K. Kodaira proved that there exists a spectral measure matrix \(P(\lambda) = (\rho_{ij}(\lambda))_{i, j=1, 2}\) which satisfies the following three conditions (Kodaira [3], [4]):

(A) \(P(\lambda)\) is a positive semi-definite measure matrix on \((-\infty, \infty)\).

(B) Denote by \(L_2(-\infty, \infty; dP(\lambda))\) the Hilbert space with the norm

\[||v||^2 = \int_{-\infty}^{\infty} \delta(\lambda) dP(\lambda) v(\lambda),\]

where \(v(\lambda)\) is a vector-valued function on \((-\infty, \infty)\)

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\[ v(\lambda) = \begin{pmatrix} v_1(\lambda) \\ v_2(\lambda) \end{pmatrix}, \]

\( \bar{v}(\lambda) \) is the transpose of \( v(\lambda) \), and \( \bar{\alpha} \) means the conjugate complex number of \( \alpha \). Then the generalized Fourier transformation

\[ \mathcal{F}_P: f(x) \rightarrow \int_0^\infty f(x) y(x, \lambda) dx, \quad y(x, \lambda) = \begin{pmatrix} \varphi_1(x, \lambda) \\ \varphi_2(x, \lambda) \end{pmatrix} \]

from \( L_2(0, \infty; dx) \) into \( L_2(-\infty, \infty; dP(\lambda)) \) is isometric.

(C) \( \mathcal{F}_P \) transforms \( L_2(0, \infty; dx) \) onto \( L_2(-\infty, \infty; dP(\lambda)) \).

In Theorem 1 we shall prove that if both \( \alpha=0 \) and \( \alpha=\infty \) belong to the limit point case, then the measure matrix which satisfies (A), (B) and (C) is unique.

We shall prove in Theorem 2 that if both \( \alpha=0 \) and \( \alpha=\infty \) belong to the limit point case, then any measure matrix which satisfies (A) and (B) satisfies (C).

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\section{The measure matrix in the eigenfunction expansion for singular differential equations.}

Let us consider a differential equation of the second order

\[ \frac{d^2u}{dx^2} - q(x)u(x) + \lambda u(x) = 0, \quad (0 < x < \infty), \]

where \( q(x) \) is a locally summable function in \( (0, \infty) \). We assume that \( x=0 \) is a singular point of the equation. Moreover we assume that the equation \((1.1)\) is of the limit point type both at 0 and at \( \infty \).

Let \( \varphi_1(x, \lambda), \varphi_2(x, \lambda) \) be a linearly independent system of solutions of \((1.1)\) which satisfies

\[ \varphi_i(1, \lambda) = \eta_i(\lambda), \quad \frac{\partial \varphi_i(1, \lambda)}{\partial x} = \zeta_i(\lambda) \]

for \( i = 1, 2 \), where \( \eta_i(\lambda) \) and \( \zeta_i(\lambda) \) are entire functions of \( \lambda \) such that

\[ \eta_1(\lambda)\zeta_2(\lambda) - \eta_2(\lambda)\zeta_1(\lambda) = 1 \]
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for every complex number $l$. Then for $(\varphi_i, \varphi_j)$ there exists a matrix function on $(-\infty, \infty)$

$$P(\lambda) = \begin{pmatrix} \rho_{11}(\lambda) & \rho_{12}(\lambda) \\ \rho_{21}(\lambda) & \rho_{22}(\lambda) \end{pmatrix}, \quad \rho_{12}(\lambda) \equiv \rho_{21}(\lambda),$$

which satisfies the following three conditions (A), (B) and (C) (Kodaira [3]):

(A) Each $\rho_{ij}(\lambda)$ is a function of bounded variation on every finite interval in $(-\infty, \infty)$, and $P(\lambda)$ is a positive semi-definite measure on $(-\infty, \infty)$. Namely for every finite interval $\Delta$ and for every pair of continuous functions

$$v_0(\lambda) = \begin{pmatrix} v_1^0(\lambda) \\ v_2^0(\lambda) \end{pmatrix},$$

we have the inequality

$$\int_\Delta \bar{v}_0(\lambda) dP(\lambda) v_0(\lambda) = \sum_{i,j=1,2} \int_\Delta v_0^i(\lambda) \bar{v}_0^j(\lambda) d\rho_{ij}(\lambda) \geq 0,$$

where $\bar{v}_0(\lambda)$ is the transpose of $v_0(\lambda)$.

(B) The generalized Fourier transformation from $L_2(0, \infty; dx)$ into $L_2(-\infty, \infty; dP(\lambda))$

$$\mathcal{F}_P: f(x) \rightarrow \int_0^\infty f(x) \varphi(x, \lambda) dx, \quad (\varphi_1(x, \lambda))$$

is isometric. Here the element of $L_2(-\infty, \infty; dP(\lambda))$ is a pair of measurable functions

$$v(\lambda) = \begin{pmatrix} v_1(\lambda) \\ v_2(\lambda) \end{pmatrix}$$

such that

$$||v(\lambda)||^2 \equiv \int_{-\infty}^{\infty} \bar{v}(\lambda) dP(\lambda) v(\lambda) < \infty.$$

(C) $\mathcal{F}_P$ transforms $L_2(0, \infty; dx)$ onto $L_2(-\infty, \infty; dP(\lambda))$.

We shall prove the following two theorems.

**Theorem 1.** Let the equation (1.1) be of the limit point type both at 0 and at $\infty$. Then the measure matrix which satisfies (A), (B) and (C) is uniquely determined by $y(x, l)$. 
Theorem 2. Let the equation (1.1) be of the limit point type both at 0 and at \(\infty\). Then if a measure matrix satisfies (A) and (B) with respect to \(y(x, I)\), then it satisfies (C).

To prove Theorem 1 and Theorem 2 we prepare the following lemma.

Lemma 1. Let \(P(\lambda)\) be a measure matrix which only satisfies (A) and (B) with respect to \(y(x, I)\) and put

\[
E_{P}(x, y; \Delta) = \int_{\Delta} \check{y}(x, \lambda) dP(\lambda) y(y, \lambda).
\]

Then

(i) \(E_{P}(x, y; \Delta)\) is a symmetric kernel of Carleman type such that

\[
\int_{0}^{\infty} (E_{P}(x, y; \Delta))^{2} dx \leq \int_{\Delta} \check{y}(y, \lambda) dP(\lambda) y(y, \lambda),
\]

and

\[
\int_{0}^{\infty} E_{P}(x, y; \Delta) f(y) dy = \int_{\Delta} \check{F}_{P} f(\lambda) dP(\lambda) y(x, \lambda)
\]

hold for every \(f(x)\) in \(L_{2}(0, \infty; dx)\).

(ii) Let \(E_{P}(\Delta)\) be a linear transformation defined by

\[
E_{P}(\Delta) f(x) = \int_{0}^{\infty} E_{P}(x, y; \Delta) f(y) dy
\]

for \(f(x)\) in \(L_{2}(0, \infty; dx)\). Then \(E_{P}(\Delta)\) is a bounded symmetric operator on \(L_{2}(0, \infty; dx)\) and we have

\[
\|E_{P}(\Delta)\| \leq 1, \quad \lim_{\Delta \to -\infty} E_{P}(\Delta) = \text{identity},
\]

\[
\langle E_{P}(\Delta) f, u \rangle = \int_{\Delta} \check{F}_{P} f(\lambda) dP(\lambda) \check{F}_{P} u(\lambda)
\]

for every pair of \(f(x), u(x)\) in \(L_{2}(0, \infty; dx)\).

(iii) For \(y\) fixed \(\frac{\partial E_{P}(x, y; \Delta)}{\partial y}\) belongs \(L_{2}(0, \infty; dx)\) and we have

\[
\int_{0}^{\infty} \left( \frac{\partial E_{P}(x, y; \Delta)}{\partial y} \right)^{2} dx \leq \int_{\Delta} \frac{\partial \check{y}(y, \lambda)}{\partial y} dP(\lambda) \frac{\partial \check{y}(y, \lambda)}{\partial y}.
\]

In the case where \(x=0\) is a regular point, we have a corresponding fact as follows:
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Lemma 2. Let \( \rho(\lambda) \) be a pseudo-spectral measure for the equation

\[
\frac{d^2u}{dx^2} - q(x)u(x) + \lambda u(x) = 0, \quad (0 \leq x < \infty),
\]

with respect to the solution \( \varphi(x, \lambda) \) and put

(1.8) \[ E_\rho(x, y; \Delta) = \int_{\Delta} \varphi(x, \lambda)\varphi(y, \lambda)d\rho(\lambda), \]

where \( \Delta \) is a finite interval and \( x, y \geq 0 \). Then

(i) \( E_\rho(x, y; \Delta) \) is a bounded symmetric kernel of Carleman type such that

(1.9) \[ \int_0^\infty (E_\rho(x, y; \Delta))^2 dx \leq \int_{\Delta} \varphi^2(y, \lambda)d\rho(\lambda), \]

and

(1.10) \[ \int_0^\infty E_\rho(x, y; \Delta)f(y)dy = \int_{\Delta} \varphi(x, \lambda)F_n(\lambda)d\rho(\lambda) \]

hold for \( f(x) \) in \( L^2_2(0, \infty; dx) \).

(ii) Let \( E_\rho(\Delta) \) be a linear transformation defined by

\[ E_\rho(\Delta)f(x) = \int_0^\infty E_\rho(x, y; \Delta)f(y)dy \]

for \( f(x) \) in \( L^2_2(0, \infty; dx) \). Then \( E_\rho(\Delta) \) is a bounded symmetric operator on \( L^2_2(0, \infty; dx) \) and we have

(1.11) \[ ||E_\rho(\Delta)|| \leq 1, \quad \lim_{\Delta \to (-\infty, \infty)} E_\rho(\Delta) = \text{identity}, \]

for every pair of \( f(x), u(x) \) in \( L^2_2(0, \infty; dx) \).

(iii) For \( y \) fixed, \( \frac{\partial E_\rho(x, y; \Delta)}{\partial y} \) belongs to \( L^2_2(0, \infty; dx) \) and we have

(1.12) \[ \int_0^\infty \left( \frac{\partial E_\rho(x, y; \Delta)}{\partial y} \right)^2 dx \leq \int \left( \frac{\partial \varphi(y, \lambda)}{\partial y} \right)^2 d\rho(\lambda). \]

We only prove Lemma 2, because the proof of Lemma 1 is similar.

Proof of (i). Consider a linear functional on \( L^2_2(0, \infty; dx) \)

(1) Matsuda [6].
(1.13) \[ l_{x_0, \Delta}(f) = \int_{\Delta} \varphi(x_0, \lambda) \mathcal{F}_\rho f(\lambda) d\rho(\lambda) \]

for \( x_0 \) and \( \Delta \) fixed. Then we have

(1.14) \[
|l_{x_0, \Delta}(f)| \leq \left[ \int_{\Delta} |\varphi(x_0, \lambda)| d\rho(\lambda) \right]^{1/2} \left[ \int_{\Delta} |\mathcal{F}_\rho f(\lambda)|^2 d\rho(\lambda) \right]^{1/2} \\
\leq \left[ \int_{\Delta} \varphi^2(x_0, \lambda) d\rho(\lambda) \right]^{1/2} \|\mathcal{F}_\rho f\|_\rho \\
= M_{x_0, \Delta} \|f\|_\rho,
\]

(1.15) \[ M_{x_0, \Delta} = \left[ \int_{\Delta} \varphi^2(x_0, \lambda) d\rho(\lambda) \right]^{1/2}. \]

This shows that \( l_{x_0, \Delta} \) is a bounded linear functional. And hence by Riesz theorem we can find a function \( e_{x_0, \Delta}(x) \) in \( L_2(0, \infty ; dx) \) such that

(1.15) \[ l_{x_0, \Delta}(f) = \int_{\Delta} \varphi(x_0, \lambda) \mathcal{F}_\rho f(\lambda) d\rho(\lambda) = \int_0^\infty e_{x_0, \Delta}(x)f(x) dx. \]

On the other hand we have

(1.16) \[ l_{x_0, \Delta}(f_0) = \int_0^\infty E(x_0, x; \Delta)f_0(x) dx \]

for \( f_0(x) \) in \( L_2(0, \infty ; dx) \) which has a compact carrier. It follows from (1.15) and (1.16) that

\[ E(x_0, x; \Delta) = e_{x_0, \Delta}(x), \]

and hence \( E(x, y; \Delta) \) is a kernel of Carleman type and we have (1.10) by (1.15) and (1.16). (1.9) follows from (1.14).

Proof of (ii). Putting \( f_\Delta(x) = E_\rho(\Delta) f(x) \), we have

(1.17) \[
\left| \int_0^\infty f_\Delta(x) u(x) dx \right| = \left| \int_0^\infty \left[ \int_{\Delta} \varphi(x, y) \mathcal{F}_\rho f(\lambda) d\rho(\lambda) \right] u(x) dx \right| \\
= \left| \int_{\Delta} \mathcal{F}_\rho f(\lambda) \mathcal{F}_\rho u(\lambda) d\rho(\lambda) \right| \\
\leq \left[ \int_{\Delta} |\mathcal{F}_\rho f(\lambda)|^2 d\rho \right]^{1/2} \left[ \int_{\Delta} |\mathcal{F}_\rho u(\lambda)|^2 d\rho \right]^{1/2} \\
\leq \|f\| \|u\|.
\]

for \( u(x) \) in \( L_2(0, \infty ; dx) \) which has a compact carrier. For a positive \( N \) and a finite interval \( \Delta \), let us define \( u_{N, \Delta}(x) \) by

\[ u_{N, \Delta}(x) = \begin{cases} f_\Delta(x), & 0 \leq x \leq N \\ 0, & x > N. \end{cases} \]
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Then \( u_{N,\Delta}(x) \) belongs to \( L_2(0, \infty; dx) \) and so (1.17) implies

\[
\int_0^N |f_\Delta(x)|^2 \, dx \leq \left[ \int_0^N |f_\Delta(x)|^2 \, dx \right]^{1/2} |f|,
\]

namely

(1.18) \[
\left[ \int_0^N |f_\Delta(x)|^2 \, dx \right]^{1/2} \leq |f|.
\]

Since \( N \) is arbitrary, (1.18) implies

\[
||E_\rho(\Delta)f|| \leq ||f||.
\]

Proof of (iii). Consider a linear functional on \( L_2(0, \infty; dx) \)

\[
k_{x_0,\Delta}(f) = \int_\Delta \frac{\partial \varphi(x_0, \lambda)}{\partial x} \overrightarrow{f_\rho}(\lambda) \, d\rho(\lambda)
\]

for \( x_0 \) and \( \Delta \) fixed. Then we have

(1.19) \[
|k_{x_0,\Delta}(f)| \leq \overline{M}_{x_0,\Delta} ||f||, \quad \left( \overline{M}_{x_0,\Delta} = \left[ \int_\Delta \left( \frac{\partial \varphi(x_0, \Delta)}{\partial x} \right)^2 \, d\rho(\lambda) \right]^{1/2} \right)
\]

as in the proof of (i). By the method used in the proof of (i), we can show that \( \frac{\partial E_\rho(x, y; \Delta)}{\partial y} \) belongs to \( L_2(0, \infty; dx) \) and that (1.12) holds.

Proof of Theorem 1. Let \( P_1(\lambda) \) and \( P_2(\lambda) \) satisfy (A), (B) and (C).

We shall denote by \( \mathcal{D}_\infty \) the space of all functions \( u(x) \) in \( L_2(0, \infty; dx) \) that satisfy the following conditions:

i) \( u(x) \in L_2(0, \infty; dx) \).

ii) \( u(x) \) is differentiable in the open interval \( (0, \infty) \).

iii) \( \frac{du}{dx} \) is absolutely continuous in every closed subinterval \( [a, b] \) \( (0 < a < b < \infty) \) in \( (0, \infty) \).

iv) \( u(x) \) has a compact carrier in \( (0, \infty) \).

v) \( -\frac{d^2 u}{dx^2} + q(x)u(x) \in L_2(0, \infty; dx) \).

Define an operator \( L_\infty \) which transforms \( u(x) \in \mathcal{D}_\infty \) to

\[
L_\infty u(x) = -\frac{d^2 u}{dx^2} + q(x)u(x).
\]

By the assumption of Theorem 1, if we denote the closure of \( L_\infty \) by \( L, L \) is
a self-adjoint operator. Let \( l \) be a complex number with \( Im\,l \neq 0 \) and \( L_l \) be the resolvent \((l - L)^{-1}\). We have for \( u(x) \) in \( \mathcal{D}_\omega \)

\[
\mathcal{T}_{P_k}(l - L)u(\lambda) = (l - \lambda)\mathcal{T}_{P_k}u(\lambda), \quad (k = 1, 2).
\]

Therefore we obtain

\[
\langle L_l(l - L)u, f \rangle = \langle u, f \rangle = \langle \mathcal{T}_{P_k}u, \mathcal{T}_{P_k}f \rangle_{P_k} = \langle \frac{\mathcal{T}_{P_k}(l - L)u}{l - \lambda}, \mathcal{T}_{P_k}f \rangle_{P_k}, \quad (k = 1, 2)
\]

for \( u(x) \) in \( \mathcal{D}_\omega \) and \( f(x) \) in \( L_2(0, \infty; dx) \). Since the family of functions \( \{(l - L)u(x)/u(x) \in \mathcal{D}_\omega\} \) is dense in \( L_2(0, \infty; dx) \), we have

\[
(1.20) \quad \langle L_l f, h \rangle = \int_{-\infty}^{\infty} \frac{\mathcal{T}_{P_k}f(\lambda)dP_k(\lambda)\mathcal{T}_{P_k}h(\lambda)}{\lambda - l}, \quad (k = 1, 2)^{(1)}
\]

for every pair of \( f(x) \) and \( h(x) \) in \( L_2(0, \infty; dx) \).

Let \( E_{P_1}(\Delta) \) and \( E_{P_2}(\Delta) \) be the operators in Lemma 1 with respect to \( P_1(\Delta) \) and \( P_2(\Delta) \). Then making use of the inversion formula for Stieltjes transformation\(^{(2)}\) we have from (1.20)

\[
(1.21) \quad \langle E_{P_1}(\Delta)f, h \rangle = \langle E_{P_2}(\Delta)f, h \rangle
\]

for every finite interval \( \Delta \) in \((-\infty, \infty)\). From (1.21) and (1.5) we get

\[
(1.22) \quad E_{P_1}(x, y; \Delta) = E_{P_2}(x, y; \Delta),
\]

namely

\[
(1.23) \quad \int_{\Delta} \tilde{g}(x, \lambda)dP_1(\lambda)\tilde{g}(y, \lambda) = \int_{\Delta} \tilde{g}(x, \lambda)dP_2(\lambda)\tilde{g}(y, \lambda).
\]

Let \( y_\delta(x, l) \) be a system of solutions such that

\[
y_\delta(x, l) = \begin{pmatrix} \varphi_1^{(\delta)}(x, l) \\ \varphi_2^{(\delta)}(x, l) \end{pmatrix}
\]

such that

\[
(1.24) \quad \begin{cases} \varphi_1^{(\delta)}(1, l) = 0, \quad \frac{\partial \varphi_1^{(\delta)}}{\partial x}(1, l) = 1, \\ \varphi_2^{(\delta)}(1, l) = 1, \quad \frac{\partial \varphi_2^{(\delta)}}{\partial x}(1, l) = 0. \end{cases}
\]

\(^{(1)}\) This formula is due to M. Matsuda.

\(^{(2)}\) Neumark [7], Anhang.
Then there exists a matrix
\[ A(l) = \begin{pmatrix} \alpha(l) & \gamma(l) \\ \beta(l) & \delta(l) \end{pmatrix}, \]
\[ \alpha, \beta, \gamma, \delta \text{ being entire functions of } l \]
such that
\[ y(x, l) = A(l)y_0(x, l). \]

Define two density matrices \( d\mathcal{P}^{(\lambda)}_1 \) and \( d\mathcal{P}^{(\lambda)}_2 \) by
\[ (1.26) \quad d\mathcal{P}^{(\lambda)}_k = A(\lambda)d\mathcal{P}_k(\lambda)A(\lambda), \quad k = 1, 2. \]

Then the measure matrices \( \mathcal{P}^{(\lambda)}_k \) \((k = 1, 2)\) will satisfy (A), (B) and (C) with respect to \( y_0(x, \lambda) \).

To prove \( P_1(\Delta) = P_2(\Delta) \) it is sufficient to prove \( \mathcal{P}^{(\lambda)}_1 = \mathcal{P}^{(\lambda)}_2 \).

From (1.2), (1.25) and (1.26) we obtain for \( k = 1, 2 \)
\[ (1.28) \quad E_{\mathcal{P}^{(\lambda)}_k}(x, y; \Delta) = \int_{\Delta} \tilde{y}_0(x, \lambda)d\mathcal{P}^{(\lambda)}_k(y, \lambda), \]
and hence (1.22) implies
\[ (1.27) \quad \int_{\Delta} \tilde{y}_0(x, \lambda)d\mathcal{P}^{(\lambda)}_k(y, \lambda) = \int_{\Delta} \tilde{y}_0(x, \lambda)d\mathcal{P}^{(\lambda)}_k(y, \lambda). \]

We differentiate (1.27) with respect to \( x \) or \( y \) to obtain
\[ (1.28) \quad \int_{\Delta} \frac{\partial \tilde{y}_0(x, \lambda)}{\partial x}d\mathcal{P}^{(\lambda)}_k(y, \lambda) = \int_{\Delta} \frac{\partial \tilde{y}_0(x, \lambda)}{\partial x}d\mathcal{P}^{(\lambda)}_k(y, \lambda). \]

and
\[ (1.29) \quad \int_{\Delta} \frac{\partial \tilde{y}_0(x, \lambda)}{\partial x}d\mathcal{P}^{(\lambda)}_k(y, \lambda) = \int_{\Delta} \frac{\partial \tilde{y}_0(x, \lambda)}{\partial y}d\mathcal{P}^{(\lambda)}_k(y, \lambda). \]

Set
\[ \mathcal{P}^{(\lambda)}_k(\Delta) = \begin{pmatrix} \rho_{11}^{(\lambda)}(\Delta) & \rho_{12}^{(\lambda)}(\Delta) \\ \rho_{21}^{(\lambda)}(\Delta) & \rho_{22}^{(\lambda)}(\Delta) \end{pmatrix}, \quad (k = 1, 2). \]

Then putting \( x = y = 1 \) in (1.27), (1.28) and (1.29) we have
\[ \rho_{22}^{(\lambda)}(\Delta) = \rho_{22}^{(\lambda)}(\Delta), \quad \rho_{21}^{(\lambda)}(\Delta) = \rho_{12}^{(\lambda)}(\Delta), \quad \rho_{11}^{(\lambda)}(\Delta) = \rho_{11}^{(\lambda)}(\Delta), \quad \rho_{12}^{(\lambda)}(\Delta) = \rho_{21}^{(\lambda)}(\Delta) \]
respectively, which completes the proof.

**Proof of Theorem 2.** Let \( P(\lambda) \) be a measure matrix which satisfies (A) and (B). Then we have
(1.30) \[
\langle L_1 f, h \rangle = \int_{-\infty}^{\infty} \frac{\hat{f}(\lambda) d\mathbf{P}(\lambda) \hat{f}(\lambda)}{I - \lambda},
\]

and \(E_\rho(\Delta)\) becomes a resolution of the identity\(^3\).

Let \(y_\rho(x, \lambda)\) be a system of solutions which satisfies the initial conditions (1.24). Putting

\[
d\mathbf{P}_\rho(\lambda) = \hat{A}(\lambda) d\mathbf{P}(\lambda) A(\lambda)
\]

for \(A(l)\) satisfying (1.25), we have a resolution of the identity \(E_\rho(\Delta)\).

Defining \(u_\rho(x, \Delta)\) by

\[
(1.31) \quad u_\rho(x, \Delta) = \int_{\Delta} d\mathbf{P}_\rho(\lambda) y_\rho(x, \lambda) = \begin{pmatrix} u_1^{\rho}(x, \Delta) \\ u_2^{\rho}(x, \Delta) \end{pmatrix},
\]

we shall prove

\[
(1.32) \quad P_\rho(\Delta \cap \Delta_i) = \int_{0}^{\infty} u_\rho(x, \Delta) \tilde{u}_\rho(x, \Delta_i) dx
\]

for every pair of intervals \(\Delta\) and \(\Delta_i\).

Since \(E_\rho(\Delta)\) is a resolution of the identity, we have

\[
(1.33) \quad \int_{0}^{\infty} E_\rho(s, x; \Delta) E_\rho(s, y; \Delta_i) ds = E_\rho(x, y; \Delta \cap \Delta_i).
\]

By (iii) of Lemma 1 we can differentiate (1.33) with respect to \(x\) or \(y\) to obtain

\[
(1.34) \quad \int_{0}^{\infty} \frac{\partial E_\rho(s, x; \Delta)}{\partial x} E_\rho(s, y; \Delta_i) ds = \frac{\partial E_\rho(x, y; \Delta \cap \Delta_i)}{\partial x},
\]

and

\[
(1.35) \quad \int_{0}^{\infty} \frac{\partial E_\rho(s, x; \Delta)}{\partial x} \frac{\partial E_\rho(s, y; \Delta_i)}{\partial y} ds = \frac{\partial^2 E_\rho(x, y; \Delta \cap \Delta_i)}{\partial x \partial y}.
\]

Setting \(x = y = 1\) in (1.33) and (1.34) and (1.35), we have

\[
(1.36) \quad \int_{0}^{\infty} u_i^{\rho}(s, \Delta) u_j^{\rho}(s, \Delta_i) ds = \rho_{ij}^{\rho}(\Delta \cap \Delta_i), \quad (i, j = 1, 2),
\]

where

\[
P_\rho(\Delta) = \begin{pmatrix} \rho_{11}^{\rho}(\Delta) & \rho_{12}^{\rho}(\Delta) \\ \rho_{21}^{\rho}(\Delta) & \rho_{22}^{\rho}(\Delta) \end{pmatrix}.
\]

(1) See the proof of Theorem 1 of Matsuda [6].
Thus the identity (1.32) is proved.

For $y(x, l)$ and $P(\Delta)$ putting

$$u(x, \Delta) = \int_\Delta dP(\lambda) y(x, y)$$

we have by (1.36)

$$P(\Delta \cap \Delta_1) = \int_0^\infty u(x, \Delta)\overline{u}(x, \Delta_1)\,dx.$$  

Define a transformation $\mathcal{T}_P$ from $L_2(-\infty, \infty; dP(\lambda))$ onto $L_2(0, \infty; dx)$ by

$$\mathcal{T}_P: v(\lambda) \rightarrow \int_{-\infty}^\infty y(x, y)dP(\lambda)v(\lambda).$$

Then $\mathcal{T}_P \cdot \mathcal{T}_P$ proves to be the identity operator.\(^{(1)}\)

By (1.38), we can prove that $\mathcal{T}_P$ is an isometric transformation from $L_2(-\infty, \infty; dP(\lambda))$ onto $L_2(0, \infty; dx)$ (Kodaira [3], [5]). $\mathcal{T}_P$ is therefore surjective, and the proof is completed.

**Remark.** If we assume the existence of the measure matrix $P_\ast(\lambda)$ which satisfies (A), (B) and (C), calculated by Titchmarsh-Kodaira's spectral formula, the proof of Theorem 2 will be easier (Kodaira [3], [4]).

In fact, let $P(\lambda)$ be a measure matrix satisfying (A) and (B). Then we have

$$\langle L_1f, h \rangle = \int_{-\infty}^\infty \mathcal{T}_P f(\lambda)dP_\ast(\lambda)\mathcal{T}_P h(\lambda) = \int_{-\infty}^\infty \mathcal{T}_P f(\lambda)dP(\lambda)\mathcal{T}_P h(\lambda).$$

Using Lemma 1 we have

$$E_{P_\ast}(x, y; \Delta) = E_P(x, y; \Delta).$$

We obtain $P_\ast(\Delta) = P(\Delta)$ by the method used in the proof of Theorem 1. Therefore $P(\Delta)$ satisfies (C).

If the equation (1.1) is of the unit circle type at $0$, the situation is essentially the same as in the case where $x=0$ is a regular point.

§ 2. **The spectrum in the limit circle case at infinity.**

In the case where $x=0$ is a regular point of the equation (1.1),

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(1) See Proposition 1 of Matsuda [6].
M. Matsuda has proved that the spectrum is unbounded below in the limit circle case at $x = \infty$ (M. Matsuda [6]).

In this section we assume that the equation (1.1) belongs to the limit circle case at $x = \infty$. Then by setting some boundary conditions at $\infty$ and also at 0 if necessary, we obtain a self-adjoint operator $L$ which is a symmetric extension of the $L_{\infty}$ in §1. The spectrum of this operator is simple\(^{(2)}\).

Then we shall prove the following theorem:

**Theorem 3.** Let the equation (1.1) belong to the limit circle case at $\infty$. Then the self-adjoint operator $L$ is unbounded below.

In fact, let $I_1 = [1, \infty)$ and $I_2 = (0, 1]$. Setting some boundary conditions at $x = 1$, we obtain $L_1$ and $L_2$ which are the restrictions of $L$ to $I_1$ and to $I_2$ respectively. Then $L$ is bounded below if and only if $L_1$ and $L_2$ are both bounded below\(^{(2)}\). $L_1$ is unbounded below by virtue of Theorem 2 of Matsuda [6], and hence $L$ is also unbounded below.

Using Weyl’s classification of the limit point case and the limit circle case, we can see that Theorem 3 is equivalent to the following fact:

Let $q(x)$ be locally summable in $(0, \infty)$. Then if $L_{\infty}$ in §1 is bounded below, $L_{\infty}$ is essentially self-adjoint.

Let us make a remark on this fact. In the $m$-dimensional case, the following result is known (Wienholtz [8], Kato [2]):

Let $L_0$ be a partial differential operator

$$L_0 = -\Delta + q(x),$$

where $q(x)$ has a following property: there exists a constant $\alpha (0 < \alpha < 1)$ such that

$$M(x) = \int_{|x-y| \leq \delta} |x-y|^{\mu(m, \alpha)} |q(y)|^2 dy,$$

$$\mu(m, \alpha) = \begin{cases} 0, & m \leq 3 \\ -m + 4 - \alpha, & m \geq 4 \end{cases}$$

is locally bounded. The domain of $L_0$ consists of $C^\infty$-functions of compact carrier. Then if $L_0$ is bounded below, $L_0$ is essentially self-adjoint.

By slight modification of their method we can replace the local

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(1) Kodaira [3].
(2) Dunford-Schwartz [1], p. 1455.
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boundedness of $M(x)$ with the local summability of $q(x)$ to prove the fact we obtained above. However, our method seems to be of some interest in that we derived this in the scheme of the inverse problem of Gelfand-Levitan.

BIBLIOGRAPHY
