L_p-Spaces for von Neumann Algebra with Reference to a Faithful Normal Semifinite Weight

By
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Abstract

The L_p-space L_p(M, φ_o) for a von Neumann algebra M and its faithful normal semifinite weight φ_o is constructed as a linear space of closed linear operators acting on the standard representation Hilbert space H_o.

Any L_p-element has the polar and the Jordan decompositions relative to a positive part L^p(M, φ_o). Any positive element in L_p-space has an interpretation as the (1/p)th power φ^{1/p} of a φ ∈ M^+ with its L_p-norm given by φ(1)^{1/p}.

The product of an L_p-element and an L_p-element is explicitly defined as an L_p-element with r^{-1} = p^{-1} + q^{-1} provided 1 ≤ r and the Hölder inequality is proved. Also L_p(M, φ_o) are shown to be isomorphic for the same p and different φ_o.

There exists a vector subspace D_{φ_o} of the Tomita algebra associated with φ_o and a p-dependent injection T_p: D_{φ_o} ↪ L_p(M, φ_o) with dense range. The sesquilinear form on L_p(M, φ_o) × L_{(p')^{-1}}(M, φ_o), p^{-1} + (p')^{-1} = 1 is a natural extension of the inner product of H_o through the mapping T_p × T_{p'}.

The present work is an extension of our joint work with Araki to weights, and our L_p-spaces are isomorphic to those defined by Haagerup, Hilsum, Kosaki, and Terp.

§ 1. Introduction

The L_p-space L_p(M, τ) of a semifinite von Neumann algebra M with respect to a faithful normal semifinite trace τ is defined as the linear space of all τ-measurable operators T satisfying \|T\|_p = τ(|T|^p)^{1/p} < ∞ (see [22, 26]). Extension to non semifinite cases have been worked out by Haagerup [14] (see also Terp [30]), Hilsum [16], Kosaki [19, 20, 21], Terp [29] and our previous paper [8]. In the present paper, we shall define L_p-spaces with methods based on the technique of our previous L_p-theory [8] and different from the other authors.

In our previous L_p-theory, we assumed the existence of a faithful normal state, or equivalently the σ-finiteness of M. In this paper, we...
define $L_p$-spaces with reference to a faithful normal semifinite weight.

Our construction has an advantage in defining $L_p$-spaces on the standard representation Hilbert space rather than going over to the crossed product of $M$ with the modular action. Another salient feature of our construction is the use of a functional defined on the $L_1$-space which behaves like trace and allows us trace-like calculation for products of $L_p$-elements.

The rest of this paper is organized as follows. In Section 2, we state the main results of this paper. In Section 3, we introduce a multilinear form which plays an important role throughout the theory. Also a Hölder type inequality is proved. The result presented in Section 3 is a straightforward generalization of Lemma A in our previous paper [8] and includes multiple KMS condition as a special case (for multiple KMS condition, see [1] or [17]). In Section 4, we introduce the concept of “measurability” in our approach and fully discuss $L_p$-spaces for $2 \leq p \leq \infty$. Section 5 is devoted to the study of linear structure of $L_p$-spaces for $2 \leq p < \infty$ which is not so trivial in our situation and in Section 6, we discuss on the product between the elements of $L_p$-spaces for different $p$'s. In Section 7, special cases $p=1,2$ are discussed and the isomorphisms between the $L_1$-space and the predual $M_*$ and between the $L_2$-space and the standard representation Hilbert space $H_0$ are constructed. In Section 8, the uniform convexity as well as the uniform smoothness of $L_p$-spaces, $1 < p < \infty$, are proved and used for discussing the $L_p$-spaces for $1 < p < 2$. In Section 9, relations between $L_p$-spaces for different faithful normal semifinite weights are discussed and shown to be isomorphic. In Section 10, the positive part of an $L_p$-space is discussed and the linear polar decomposition is shown. In the same section, bilinear form is introduced and it is shown to be symmetric. In Section 11, a product of $L_p$-element and $L_q$-element is discussed and the Hölder inequality is shown. In Section 12, the dense subspace $D^*_\alpha$ and its $p$-dependent injection to $L_p$-space are discussed. In Section 13, the modular action on $L_p$-space is discussed. Section 14 provides a summary of proofs of Theorems in Section 2. A brief discussion on our theory and its connection with other works is presented in Section 15.

In Appendix A, for the sake of completeness, we define and show some properties of positive cones $V_{\alpha}, 0 \leq \alpha \leq 1/2$, for faithful normal
semifinite weights \( \phi_0 \), which are more or less known. In Appendix B, we discuss the polar decomposition in \( D(A_\phi^{1/2}) \) in terms of the positive cone \( V_\phi^\alpha \), \( 0 \leq \alpha \leq 1/2 \).

\section{Main Results}

In this paper, we use standard results on the Tomita-Takesaki theory (see [27]), and relative modular operators (see Appendix C of [8] or [4, 5, 6]). Let \( \phi_0 \) be a faithful normal semifinite weight on \( M \) and \( \sigma_\phi^t \) be the associated modular automorphism of \( M \). We denote \( N_\phi \) the set of all \( \gamma \in M \) such that \( \phi_0(\gamma^*\gamma) < \infty \), \( M_\sigma \) the set of all entire analytic elements of \( N_\phi^* \cap N_\phi \) with respect to \( \sigma_\phi^t \). We also denote the GNS mapping by \( \gamma \in N_\phi^* \rightarrow \gamma_\phi(\gamma) \in H_\phi \). Let \( I_{\phi_0} \) and \( A_\phi \) denote the modular conjugation operator and the modular operator on \( H_\phi \) associated with \( \phi_0 \), respectively.

**Definition 2.1.** A linear operator \( T \) acting on \( H_\phi \) is said to be \((\phi_0, \rho)\)-measurable \( (1 \leq \rho \leq \infty) \) if it is closed as an operator and satisfies

\[
T J_\phi \sigma_\phi^{\rho/\rho} (\gamma) J_\phi \supseteq J_\phi J_\phi^* T,
\]

for and \( \gamma \in M_\sigma \).

The definition of \( L_\rho \)-norm is the following.

**Definition 2.2.** For a \((\phi_0, \rho)\)-measurable operator \( T \),

\[
\|T\|_{\rho, \phi_0} := \left\{ \sup_{x \in N_\phi, 1 \leq \rho \leq \infty} \|T|^{\rho/\rho} \eta_\phi(x) \|^\frac{\rho}{\rho} \right\}^{1/\rho} \quad (1 \leq \rho \leq \infty),
\]

where \( T = u|T| \) is the polar decomposition of \( T \). For \( \rho = \infty \), \( \|T\|_{m, \phi_0} \) denotes the usual operator norm of \( T \).

**Definition 2.3.** The \( L_\rho \)-space \( L_\rho(M, \phi_0) \) is the set of all \((\phi_0, \rho)\)-measurable operators \( T \) with finite \( L_\rho \)-norm \( \|T\|_{\rho, \phi_0} \).

**Theorem 1.** (1) \( L_\rho(M, \phi_0) \) is a Banach space with the \( L_\rho \)-norm \( \|\cdot\|_{\rho, \phi_0} \), \( 1 \leq \rho \leq \infty \), and the linear structure given by that of \( M \) if \( \rho = \infty \) (see Theorem 3, (1)), strong sum (in the sense that \( D = D(T_1) \cap D(T_2) \), \( T_1, T_2 \in L_\rho(M, \phi_0) \) is dense and \( (T_1 + T_2)|_D \in L_\rho(M, \phi_0) \)) if \( 2 \leq \rho \leq \infty \),
that of $L^p(M, \phi_0)^*$ (see (2) below) of $1 < p < 2$, and that of $M_\phi$ if $p = 1$ (see Theorem 3, (2)), where $p^{-1} + (p')^{-1} = 1$.

(2) Sesquilinear dual pairing between $L^p(M, \phi_0)$ and $L^{p'}(M, \phi_0)$, $p^{-1} + (p')^{-1} = 1$, $1 \leq p \leq \infty$, is defined in terms of the inner product of $H_{\phi_0}$ by

\begin{equation}
\langle T, T' \rangle_{\phi_0} = \lim_{y \to 1} \langle T \eta_{\phi_0}(y), T' \eta_{\phi_0}(y) \rangle,
\end{equation}

where $T \in L^p(M, \phi_0)$ and $T' \in L^{p'}(M, \phi_0)$. The formula in the parenthesis of the right hand side is well defined for the pair $(T, T') \in L^p(M, \phi_0) \times L^{p'}(M, \phi_0)$ (the meaning of the inner product on the right hand side is $(T'^* T \eta_{\phi_0}(y), \eta_{\phi_0}(y))$ defined in Notation 3.3) and the limit $y \to 1$ is taken in the $\ast$-strong operator topology with the restriction $y \in M_\phi$, $\|y\| \leq 1$. Then,

\begin{equation}
\|T\|_{p, \phi_0} = \sup \{\langle T, T' \rangle_{\phi_0} : T' \in L^{p'}(M, \phi_0), \|T'\|_{p', \phi_0} \leq 1\}
\end{equation}

for $T \in L^p(M, \phi_0)$ for all $p$. Through this pairing, $L^p(M, \phi_0)$ is the dual of $L^p(M, \phi_0)$ if $1 \leq p < \infty$.

(3) For $1 < p < \infty$, the $L_p$-norm $\cdot \|_{p, \phi_0}$ of $L^p(M, \phi_0)$ is uniformly convex and uniformly smooth. For $2 \leq p < \infty$, the following Clarkson's inequality holds:

\begin{equation}
\|T_1 + T_2\|^p_{p, \phi_0} + \|T_1 - T_2\|^p_{p, \phi_0} \leq 2^{p-1} \{\|T_1\|^p_{p, \phi_0} + \|T_2\|^p_{p, \phi_0}\},
\end{equation}

where $T_1, T_2 \in L^p(M, \phi_0)$.

We define the positive part of the $L_p$-spaces as follows;

\begin{equation}
L^+_p(M, \phi_0) = \{T \in L_p(M, \phi_0) : T^* = T \geq 0\}.
\end{equation}

**Theorem 2.** (1) Any $T \in L_p(M, \phi_0)$ has a unique polar decomposition $T = u|T|$, where $u$ is a partial isometry in $M$ and $|T| \in L^+_p(M, \phi_0)$ satisfying $u^* u = s(|T|)$.

(2) $\|T\|_{p, \phi_0} = \|T\|_{p, \phi_0}, T \in L^p(M, \phi_0)$.

(3) If $T \in L^+_p(M, \phi_0)$, $1 \leq p < \infty$, there exists a unique $\phi \in M^+_\phi$ such that $T = A^p_{\phi, \phi_0}$. For such a unique $\phi$, $\|T\|_{p, \phi_0} = \phi(1)^{1/p}$, where $A_{\phi, \phi_0}$ is the relative modular operator.

(4) For $\phi \in M^+_\phi$ and a partial isometry $u \in M$ such that $u^* u$
\[ = s(\phi), \ T = uA_{p, e}^p \in L_p(M, \phi_0), \ (1 \leq p < \infty) \text{ with its } L_p\text{-norm} \|T\|_{p, e} = \phi(1)^{1/p}. \]

The case \( p = \infty \), excluded in Theorem 2 (3), and the case \( p = 1, 2 \) reduce to well-known objects.

**Theorem 3.** (1) \( L_\infty(M, \phi_0) = M \).

(2) The mapping from \( T \in L_1(M, \phi_0) \) to \( \psi \in M_\ast \) given by
\[
(2.7) \quad \psi(x) = \langle T, x^* \rangle_{\psi} , \quad x \in M
\]
is an isometric isomorphism from \( L_1(M, \phi_0) \) onto \( M_\ast \), where the right hand side of (2.7) is given in Theorem 1 (2) for \( p = 1 \). \( \psi(x) = \phi(xu) \), \( x \in M \), holds if \( T = uA_{p, e}^p \) is the polar decomposition of \( T \).

(3) The mapping from \( T \in L_2(M, \phi_0) \) to \( \xi(\phi) \in H_{\phi} \) is an isometric isomorphism from \( L_2(M, \phi_0) \) onto \( H_{\phi} \), where \( T = uA_{p, e}^p \) is the polar decomposition of \( T \) given by Theorem 2, and \( \xi(\phi) \) is the unique representative in a fixed natural positive cone associated with \( M \).

A linear polar decomposition is given by the following.

**Theorem 4.** Any \( T \in L_p(M, \phi_0) \) has a unique polar decomposition
\[
(2.8) \quad T = T_1 - T_2 + i(T_3 - T_4)
\]
where \( T_k \in L_p^\ast(M, \phi_0) \), \( k = 1, \cdots, 4 \), \( s(T_1) \perp s(T_2) \) and \( s(t_3) \perp s(T_4) \).

By Theorem 2 and the property of the relative modular operators, the adjoint operation on closed operators maps \( L_p(M, \phi_0) \) onto \( L_{p'}(M, \phi_0) \) isometrically. We introduce a new bilinear form on \( L_p(M, \phi_0) \times L_{p'}(M, \phi_0) \), \( p^{-1} + (p')^{-1} = 1 \) as follows:
\[
(2.9) \quad [T, T']_{\phi_0} = \langle T, T'^* \rangle_{\phi_0},
\]
where \( T \in L_p(M, \phi_0) \) and \( T' \in L_{p'}(M, \phi_0) \).

**Theorem 5.** Let \( p^{-1} + (p')^{-1} = 1 \).

(1) \( [T, T']_{\phi_0} = [T', T]_{\phi_0} \),
where \( T \in L_p(M, \phi_0) \) and \( T' \in L_{p'}(M, \phi_0) \).
(2) \( L^p_\omega (M, \phi_\omega) = \{ T \in L^p_\omega (M, \phi_\omega) : [T, T']_{\phi_\omega} \geq 0 \} \text{ for any } T' \in L^p_\omega (M, \phi_\omega) \).

Next, we consider the products between \( L^p_\omega \)-elements.

**Theorem 6.** Let \( T_k \in L^p_\omega (M, \phi_\omega) \), \( k = 1, \ldots, n \) and \( \sum_{k=1}^n p_k^{-1} = r^{-1} = 1 - (r')^{-1} \leq 1 \). Then the product \( T = T_1 \cdots T_n \) is well defined as an element of \( L^r_\omega (M, \phi_\omega)^* \cong L^r (M, \phi_\omega) \) for \( r \neq 1 \) and as an element of \( L^\infty_\omega (M, \phi_\omega)^* \cong L^\infty (M, \phi_\omega) \) for \( r = 1 \). If \( 2 \leq r \leq \infty \), the product \( T = T_1 \cdots T_n \) is defined as a strong product i.e. \( T_1 \cdots T_n \) is a densely defined preclosed operator, and \( T \) is defined as its closure. If \( 1 \leq r < 2 \), the product is understood as the element of \( L^r_\omega (M, \phi_\omega)^* \) through the multilinear pairing (see (4) below). This product satisfies the followings.

1. The product is associative and the mapping \((T_1, \ldots, T_n) \mapsto T_1 \cdots T_n\) is multilinear.
2. \( \|T\|_{\psi^r_{\omega}} \leq \|T_1\|_{\psi_{\omega}} \cdots \|T_n\|_{\psi_{\omega}} \).
3. Assume \( r = 1 \). Then,

\[ [T, 1]_{\psi_{\omega}} = [T_1 \cdots T_k, T_{k+1} \cdots T_n]_{\psi_{\omega}} \]

for any \( i \leq k \leq n - 1 \). Hence we denote it simply as \([T_1 \cdots T_n]_{\psi_{\omega}}\).

4. \([T_1 \cdots T_n]_{\psi_{\omega}} = [T_{k+1} \cdots T_n T_1 \cdots T_k]_{\psi_{\omega}}, 1 \leq k \leq n - 1\).
5. \(|[T_1 \cdots T_n]_{\psi_{\omega}}| \leq \|T_1\|_{\psi_{\omega}} \cdots \|T_n\|_{\psi_{\omega}} \).

As a corollary, we get the inequality which is originally obtained in [9].

**Corollary.** Let \( T_1, T_2 \in L^p_\omega (M, \phi_\omega) \). Then

\[ \|T_2 - T_1\|_{\psi_{\omega}} \leq 2^{1/2} \|T_2 - T_1\|_{\psi_{\omega}} . \]

The \( L^p_\omega \)-spaces for different reference weights are related as follows.

**Theorem 7.** Let \( \phi_\omega \) and \( \tilde{\phi}_\omega \) be two faithful normal semifinite weights. There exists isometric conjugate linear isomorphisms \( J_p (\tilde{\phi}_\omega, \phi_\omega) \) and isometric linear isomorphisms \( \tau_p (\tilde{\phi}_\omega, \phi_\omega) \) from \( L^p_\omega (M, \phi_\omega) \) onto \( L^p_\omega (M, \tilde{\phi}_\omega) \) having the following properties.
(1) For \(1 \leq p < \infty\),
\[
J_p(\phi_0, \phi_0) : u \Delta_{\phi_0}^{\frac{1}{p}} \mapsto u \Delta_{\phi_0}^{\frac{1}{p}} u^*,
\]
\[
\tau_p(\phi_0, \phi_0) : u \Delta_{\phi_0}^{\frac{1}{p}} \mapsto u \Delta_{\phi_0}^{\frac{1}{p}} u^*,
\]
and for \(p = \infty\), \(J_\infty\) is the usual \(*\)-operation, \(\tau_\infty\) is the identify, where \(T = u \Delta_{\phi_0}^{\frac{1}{p}} \in L_p(M, \phi_0)\), \(1 \leq p < \infty\) is the polar decomposition given in Theorem 2. If \(\tilde{\phi}_0 = \phi_0\), then the map \(J_p(\phi_0, \phi_0)\) coincides with the adjoint of a closed operator.

(2) Let \(T \in L_p(M, \phi_0)\) and \(T' \in L_p(M, \phi_0)\) with \(p^{-1} + (p')^{-1} = 1\). Then,
\[
\left[ J_p(\phi_0, \phi_0) (T) T' \right]_{\phi_0} = \left[ T J_p(\phi_0, \phi_0) (T') \right]_{\phi_0},
\]
\[
\left[ \tau_p(\phi_0, \phi_0) (T) T' \right]_{\phi_0} = \left[ T \tau_p(\phi_0, \phi_0) (T') \right]_{\phi_0}.
\]

(3) Let \(T_k \in L_p(M, \phi_0)\), \(k = 1, \ldots, n\) and \(\sum_{k=1}^n p_k^{-1} = r^{-1} \leq 1\). Then
\[
J_r(\phi_0, \phi_0) (T_1 \cdots T_n) = J_{p_1}(\phi_0, \phi_0) (T_n) \cdots J_{p_n}(\phi_0, \phi_0) (T_1),
\]
\[
\tau_r(\phi_0, \phi_0) (T_1 \cdots T_n) = \tau_{p_1}(\phi_0, \phi_0) (T_n) \cdots \tau_{p_n}(\phi_0, \phi_0) (T_1).
\]

The incompatibility of the measurability condition for different \(p\) makes the intersection of the \(L_p\)-spaces for different \(p\) trivial. As a substitute for such an intersection, we have a certain vector subspace \(D_{\phi_0}^w\) of a Tomita algebra \(\mathcal{A}_0 = \gamma_{\phi_0}(M_0)\) and a \(p\)-dependent injection \(T_p : D_{\phi_0}^w \to L_p(M, \phi_0)\) with dense image. The vector space \(D_{\phi_0}^w\) is defined as the linear span of the product of two elements in \(\mathcal{A}_0 = \gamma_{\phi_0}(M_0)\). For \(\zeta = \gamma_{\phi_0}(x_1 x_2)\), \(x_k \in M_0\), \(k = 1, 2\), \(T_p(\zeta)\) is defined by
\[
(2.10) \quad T_p(\zeta) = (\Delta_{\phi_0}^{(2p)} x_1^* (\Delta_{\phi_0}^{(2p)} x_2^*)),
\]
where the bar denotes the closure. For an arbitrary \(\zeta \in D_{\phi_0}^w\), \(T_p(\zeta)\) is defined as the sum of the operators of the form (2.10) in \(L_p(M, \phi_0)\).

We note that \(D_{\phi_0}^w\) can be viewed as a Tomita algebra.

Let \(V_{\phi_0}^\alpha (0 \leq \alpha \leq 1/2)\) be the positive cones for the weight \(\phi_0\) which are defined by the closure of (A.1) in Appendix A just as for the positive cones for states defined by Araki [2] (see also [10]).

**Theorem 8.** (1) The mapping,
\[
\zeta \in D_{\phi_0}^w \to T_p(\zeta) \in L_p(M, \phi_0)
\]
is injective and its image is dense with respect to norm topology for $1 \leq p < \infty$ and $\sigma(M, M_\#)$-topology for $p = \infty$.

1. $T_p(D_{\phi\#}^* \cap V_{\phi^n}^{(\phi^n)}) \subseteq L_p^*(M, \phi_\#)$.

2. Let $\zeta, \zeta' \in D_{\phi\#}^*$, $p^{-1} + (p')^{-1} = 1$. Then,
   $$(\zeta, \zeta') = \langle T_p(\zeta), T_p(\zeta') \rangle_{\phi\#}.$$

3. The following diagram is commutative.

4. The following diagram is commutative.

$$
\begin{array}{ccc}
J_p^\phi & \longrightarrow & D_{\phi\#}^* \\
\downarrow T_p & & \downarrow T_p \\
L_p^*(M, \phi_\#) & \overset{\ast}{\longrightarrow} & L_p^*(M, \phi_\#),
\end{array}
$$

where $J_p^\phi = J_\phi A_{\phi\#}^{(\phi^n)} - (1/p)$ and $\ast$ is the adjoint operation on $L_p^*(M, \phi_\#)$.

Remark. The case $p = 1$ in the above Theorem is already known to J. Phillips [24].

Group action on $L_p^*$-spaces is defined in our setting in Section 13 and is shown to be continuous in Lemma 13.4.

§ 3. A Multilinear Form

In this section, we shall introduce some notations which will be used throughout this paper, and also show some associated results. Throughout this section, we fix a faithful normal semifinite weight $\phi_\#$.

Lemma 3.1. Let $\phi_j \in M_\#, j = 1, \ldots, n, x_k \in M, k = 0, \ldots, n$. Then for $y \in N_{\phi_\#}$ with $\|y\| \leq 1$,

$$(3.1) \quad \zeta(z) = x_0 A_{\phi_\#}^{(\phi_\#)} \cdots x_{n-1} A_{\phi_\#}^{(\phi_\#)} x_n y(\phi_\#)$$
is well-defined for $z = (z_1, \ldots, z_n) \in I_{\phi_\#}^{(n)}$ (in the sense that $\gamma_{\phi_\#}(y)$ is in the domain of the product of operators in front), holomorphic in the interior of $I_{\phi_\#}^{(n)}$ and strongly continuous on $I_{\phi_\#}^{(n)}$ with the bound

$$(3.2) \quad \|\zeta(z)\| \leq \left( \prod_{k=0}^{n} \|x_k\| \right) \left( \prod_{k=1}^{n} \|\phi_k\|^{\mathrm{Re} z_k} \right) \phi_\#(y^* y)^{\frac{\mathrm{Re} z_0}{2}} \|y\|^{1 - \mathrm{Re} z_0},$$

where $\|\phi_k\| = \phi_\#(1)$, $z_0 = (1/2) - \sum_{k=1}^{n} z_k$ and $I_{\phi_\#}^{(n)}$ is defined for $a \geq 0$ as
\begin{equation}
I_\alpha^{(n)} = \{ z \in \mathbb{C}^n : \text{Re } z_k \geq 0, \sum_{k=1}^{n} \text{Re } z_k \leq a \}.
\end{equation}

**Proof.** Follows from Lemma A.1 and the corresponding proof of [8] by replacing \( x_n \) there by \( x_n y \) and replacing \( y_0 \) by \( y_0 (y) \). Q.E.D.

Just as Lemma A follows from Lemma A.1 in [8], the above lemma implies the following.

**Lemma 3.2.** Let \( \phi_j \in M_+^\times, j = 1, \ldots, n, \) \( x_k \in M, k = 0, \ldots, n, \) complex numbers \( z = (z_1, \ldots, z_n) \in I_\alpha^{(n)}, \) and \( y_l \in N_{\phi_i} \) with \( \| y_l \| \leq 1, l = 1, 2, \) the expression

\begin{equation}
F_{y_1, y_2}(z) = (d_{\phi_1, \phi_2}^1 x_1 d_{\phi_{1+1}, \phi_2}^{n=1} \ldots d_{\phi_n, \phi_2}^{n=1} x_n \eta_{\phi_1}(y_1),
\end{equation}

\begin{align*}
&d_{\phi_1, \phi_2}^{n=1} x_1 d_{\phi_{1+1}, \phi_2}^{n=1} \ldots d_{\phi_n, \phi_2}^{n=1} x_n \eta_{\phi_1}(y_1) \\
&\ldots d_{\phi, \phi_2}^{n=1} x_n \eta_{\phi_1}(y_1), \eta_{\phi_2}(y_2))
\end{align*}

is well defined and independent of the splitting \( z_k = z'_k + z''_k \) if

\begin{align*}
(3.5a) & \quad \text{Re } z_1 + \ldots + \text{Re } z_{k-1} + \text{Re } z'_k \leq 1/2, \quad \text{Re } z''_k \geq 0, \\
(3.5b) & \quad \text{Re } z_n + \ldots + \text{Re } z_{k+1} + \text{Re } z'_k \leq 1/2, \quad \text{Re } z''_k \geq 0.
\end{align*}

It defines a function of \( z = (z_1, \ldots, z_n) \) which is

1. holomorphic in the interior of \( I_\alpha^{(n)} \)
2. continuous on \( I_\alpha^{(n)} \)
3. bounded on \( I_\alpha^{(n)} \) by

\begin{align*}
(3.6) & \quad |F_{y_1, y_2}(z)| \leq \left( \prod_{k=0}^{n} \| x_k \| \right) \left( \| \eta_{\phi_1}(y_1) \| \| \eta_{\phi_2}(y_2) \| \right) \text{Re } z_0 \left( \sum_{k=1}^{n} \phi_k (1) \text{Re } z_k \right)
\end{align*}

where \( z_0 = 1 - \sum_{k=1}^{n} z_k. \)

**Notation 3.3.** We denote the function \( F_{y_1, y_2}(z) \) defined by (3.4) simply as

\begin{equation}
F_{y_1, y_2}(z) = (x_0 d_{\phi_1, \phi_2}^1 x_1 d_{\phi_{1+1}, \phi_2}^{n=1} \ldots d_{\phi_n, \phi_2}^{n=1} x_n \eta_{\phi_1}(y_1), \eta_{\phi_2}(y_2)).
\end{equation}

Next Lemma is used to define the sesquilinear form on \( L_p(M, \phi_0) \times L_{p'}(M, \phi_0), \ p^{-1} + (p')^{-1} = 1. \)
Lemma 3.4. Let \( \phi_k \in M^+_\star, \ k = 1, \ldots, n, \ x_k \in M, \ k = 0, \ldots, n \) and \( \text{Re} \ z_k \geq 0, \ k = 1, \ldots, n \) such that \( \sum_{k=1}^{n} z_k = 1 \). Then the limit

\[
(3.8) \quad \lim_{y_i \to y_i^+} \frac{F_{y_1, y_n}(z_1, \ldots, z_n) - \prod_{i=0}^{n} \phi_i}{y_i - y_i^+} = [x_0 \phi_0^* x_1 \phi_1^* \cdots x_n] 
\]

exists as \( y_1, y_n \to 1 \) \(*\)-strongly with the restriction \( y_1, y_n \in N_{\phi_i}, \|y_1\| \leq 1 \) and \( \|y_n\| \leq 1 \), where \( F_{y_1, y_n} \) is defined by (3.4).

Proof. Let \( z \in I_1^{(\alpha)} \) and \( y_k \in N_{\phi_i}, \|y_k\| \leq 1, \ k = 1, \ldots, 4 \). By Lemma 3.2, \( F(z) = F_{y_1, y_n}(z) - F_{y_1, y_n}(z) \) is holomorphic in the interior of \( I_1^{(\alpha)} \) and continuous on \( I_1^{(\alpha)} \). The tube domain \( I_1^{(\alpha)} \) has the following distinguished boundaries corresponding to extremal points of its base:

\[
(3.9) \quad \partial_1 I_1^{(\alpha)} = \{z: \text{Re} \ z_k = 0, \ k = 1, \ldots, n\},
\]

\[
(3.10) \quad \partial_2 I_1^{(\alpha)} = \{z: \text{Re} \ z_i = 0 (i \neq k), \text{Re} \ z_k = 1\}, \ k = 1, \ldots, n.
\]

The estimates of \( F(z) \) on the boundaries are

\[
(3.11) \quad |F_{y_1, y_n}(z) - F_{y_1, y_n}(z)| \leq (\prod_{k=0}^{n} \|x_k\|) (\|\eta_{\phi_i}(y_1)\| \|\eta_{\phi_n}(y)\| + \|\eta_{\phi_n}(y_2)\| \|\eta_{\phi_n}(y)\|)
\]

where \( z \in \partial_1 I_1^{(\alpha)} \), and

\[
(3.12) \quad F_{y_1, y_n}(z) - F_{y_1, y_n}(z) = (J_{\phi_k} \sigma_{x_k}^s (y_f - y^*_f) w_k(t) \xi (\phi_k), J_{\phi_k} \sigma_{x_k}^s (y_f) w_k(t) \xi (\phi_k)),
\]

where

\[
(3.13) \quad w_k(t) = (D\phi_k : D\phi_k)_{s_k} \sigma_{x_k}^s (x_k (D\phi_k)_{t_k} \cdots (D\phi_k)_{t_{k-1}}) \times \sigma_{x_k}^s (x_{k-1} (D\phi_k)_{t_k} \sigma_{x_k}^s (x_{k-1} (D\phi_k)_{t_k-1} \cdots),
\]

\[
(3.14) \quad s_k = t_k + \cdots + t_n,
\]

\[
(3.15) \quad w_k(t) = x_k^* (D\phi_k)_{t_k-1} (D\phi_k)_{t_k-1} \sigma_{x_k}^s (x_k^* (D\phi_k)_{t_k} \sigma_{x_k}^s (x_k^* (D\phi_k)_{t_k-1} \cdots) \times (D\phi_k : D\phi_k)_{t_k} \sigma_{x_k}^s (x_k^* (D\phi_k)_{t_k} \sigma_{x_k}^s (x_k^* (D\phi_k)_{t_k-1} \cdots),
\]

\[
(3.16) \quad s_k = t_k + \cdots + t_{k-1},
\]

where \( z \in \partial I_1^{(\alpha)} \) and we have used the formulas such as \( \Delta_{x_k}^s \alpha = (D\phi_k : D\phi_k)_{s_k} \sigma_{x_k}^s (\alpha) \Delta_{x_k}^s, \Delta_{x_k}^s \eta_{\phi_k}(y) = \eta_{\phi_k} (\sigma_{x_k}^s (y)), \) and \( \Delta_{x_k}^s \eta_{\phi_k}(y) = \)
Therefore the values of \( \exp(-\sum_{k=1}^{n} z_k^*) F_{y_1, \ldots, y_n}(z) \) at distinguished boundaries of \( I^{(n)} \) are Cauchy nets when \( y_i \to 1 \) if \( y_1, \ldots, y_n \in N_{\phi_i} \), \( \|y_1\| \leq 1 \), and \( \|y\| \leq 1 \) due to (3.12) and the uniform boundedness of \( \omega_k \) and \( \omega'_k \). By making use of the generalized three line theorem, we see that \( F_{y_1, \ldots, y_n}(z_1, \ldots, z_n) \) is a Cauchy net in \( y_i \) as \( y_i \to 1 \). Similarly \( F_{y_1, \ldots, y_n}(z_1, \ldots, z_n) \) is a Cauchy net as \( y_k \to 1 \) if \( y_1, y_2 \in N_{\phi_i}, \|y\| \leq 1, \|y_2\| \leq 1 \). Q.E.D.

Throughout this paper, we use the notation \( M_{\phi_i}^{(1)} = \{ y \in M_i : \|y\| \leq 1 \} \):

We shall see, the functional \( (x_0, x_1, \ldots, x_n, \phi_1, \ldots, \phi_n) \mapsto [x_0 \phi_1^* \cdots x_n] \) plays an important role throughout this paper for \( n = 2 \) and in Sections 9 and 11 for general \( n \). The conclusions in Sections 9 and 11 are then used to derive the properties of the multilinear form \( (T_1, \ldots, T_n) \mapsto [T_1 \cdots T_n]_{\phi_2} \) in Theorem 6, which coincides with a value of this functional under a proper identification of \( T_k \) in terms of \( x's \) and \( \phi's \).

§ 4. Polar Decomposition

In this section, we shall give a characterization of elements in \( L_p(M, \phi_0), 1 \leq p \leq \infty \) by making use of the relative modular operator.

**Lemma 4.1.** Let \( T \) be a \((\phi_0, p)\)-measurable operator \((1 \leq p < \infty)\), then there exists a unique normal semifinite weight \( \phi \) and a partial isometry \( u \in M \) satisfying \( u^* u = s(\phi) \), and \( T = u \mathcal{A}_{\phi}^{\phi_0} \).

**Proof.** Starting with Definition 2.1 of measurability and following the proof of Lemmas 3.2 and 3.3 of [8], where \( y \in M_0' \) should be replaced by \( J_{\phi_0} y J_{\phi_0} \) with \( y \in M_0 \), we obtain

\[
|T|^{\psi} J_{\phi} \sigma_{\phi}^T(y) J_{\phi} = J_{\phi} y J_{\phi}|T|^{\psi}
\]

for any \( y \in M \) and \( t \in \mathbb{R} \) as well as \( s(|T|) \in M \) and \( u \in M \).

By the proof of Lemma 3.4 in [8], \( |T| \) is of the form \( \mathcal{A}_{\phi}^{\phi_0} \) for some normal semifinite weight \( \phi \) on \( M \). The uniqueness of this decomposition follows from the uniqueness of polar decomposition as closed operators and the one-to-one correspondence between the normal semifinite weight and \( \sigma_{\phi}^T \) one cocycles with a specified support property.
Lemma 4.2. Let $T = uA^{p}_{s,s}^*$ be the polar decomposition, $1 \leq p < \infty$. Then, $\| T \|_{p,s} = \phi(1)^{1/p}$. 

Proof. Assume $\phi(1) < \infty$. Then there exists $\xi(\phi) \in P_{\phi}^0$ such that

\[ \phi = \omega_{\xi(\phi)}. \]

It follows that

\[ \sup_{x \in M_0^{(1)}} \| A_{s,s}^{1/2} \xi(\phi)(x) \| = \sup_{x \in M_0^{(1)}} \| x^* \xi(\phi) \| \leq \| \xi(\phi) \|. \]  

Due to the existence of a $*$-strong net $\{ x_n \} \subset M_0^{(1)}$ which converges to 1, inequality (4.2) is actually attained. Hence

\[ \| T \|_{p,s} = \left( \sup_{x \in M_0^{(1)}} \| A_{s,s}^{1/2} \xi(\phi)(x) \| \right)^{1/p} = \| \xi(\phi) \|^{1/p} = \phi(1)^{1/p}. \]

Next we assume $\phi(1) = \infty$. Because $\phi$ is normal and semifinite, there exists an increasing net $\{ \phi_\alpha \} \subset M_0^+$ with supremum $\phi$. For any $N > 0$, there exists $\alpha_N$ such that $\phi_\alpha(1) > N^p$. Due to Lemma C.3 of [8], $D(A_{s,s}^{1/2}) \subset D(A_{s,s}^{1/2})$ and $\| A_{s,s}^{1/2} \xi(\phi) \| \leq \| A_{s,s}^{1/2} \xi(\phi) \|$ for $\xi(\phi) \in D(A_{s,s}^{1/2})$. Hence

\[ \| T \|_{p,s} \geq \left( \sup_{x \in M_0^{(1)}} \| A_{s,s}^{1/2} \xi(\phi)(x) \| \right)^{1/p}. \]

By the preceding argument, the right hand side of (4.4) is $\phi_\alpha(1)^{1/p} > N$. It follows $\| T \|_{p,s} = \infty$. Q.E.D.

Remark. Conversely, if $T = uA_{u,\psi}^{p,s}$ with a partial isometry $u \in M$ and $\phi \in M_0^+$ such that $u^*u = s(\phi)$, $T$ is $(\phi, p)$-measurable and $\| T \|_{p,s} = \phi(1)^{1/p}$. It follows that if $T = uA_{u,\psi}^{p,s} \in L_p(M, \phi_0)$, $1 \leq p < \infty$, then $T^* = A_{u,\psi}^{p,s}u^*u = u^*A_{u,\psi}^{p,s} \in L_p(M, \phi_0)$ (\phi_\psi(x) = \phi(u^*xu), x \in M). It follows that the adjoint operation is a bijective isometry of $L_p(M, \phi_0)$. 

The case $p = \infty$ excluded from the above is given by the following lemma. It is an easy consequence of the definition.

Lemma 4.3. If $T$ is $(\phi_0, \infty)$-measurable, then $T$ is a closed operator affiliated with $M$. $L_\infty(M, \phi_0) = M$. 


§ 5. Linear Structure of $L_p(M, \phi_0)$ for $2 \leq p < \infty$

In this section, we examine the linear structure of $L_p(M, \phi_0)$, $2 \leq p < \infty$.

**Lemma 5.1.** $L_p(M, \phi_0)$, $2 \leq p < \infty$, is a linear space under strong sum.

**Proof.** Let $T_k \in L_p(M, \phi_0)$, $k = 1, 2$. According to the polar decomposition, $T_k$ is of the form $T_k = u_k \sigma_{\phi_0}^{\mathfrak{d}_k}$, $u_k \in M$, $\phi_k \in M_\phi^+$, $k = 1, 2$. By the property of relative modular operators, $\gamma_{\phi_k}(N_{\phi_k})$ is a subset of $D(T)$ for any $T \in L_p(M, \phi_0)$ ($p \geq 2$) and actually it is a core of $T$ due to the fact that $\gamma_{\phi_k}(N_{\phi_k})$ is a core of $A_{\phi_k}^{\mathfrak{d}_k}$ for any $\phi \in M_\phi^+$. It follows that $T_1 + T_2$ and $T_1^* + T_2^*$ are densely defined operators. By $(T_1 + T_2)^*$ $\supseteq T_1^* + T_2^*$, $T_1 + T_2$ is preclosed. For any $\xi \in D(T_1 + T_2)$ and $y \in M_\phi$,

\begin{equation}
J_{\phi,y}J_{\phi,T_1 + T_2}\xi = J_{\phi,y}J_{\phi,T_1}\xi + J_{\phi,y}J_{\phi,T_2}\xi
= T_1J_{\phi,y}\sigma_{\phi_0}^{\mathfrak{d}_1}(y)\xi + T_2J_{\phi,y}\sigma_{\phi_0}^{\mathfrak{d}_2}(y)\xi
= (T_1 + T_2)J_{\phi,y}\sigma_{\phi_0}^{\mathfrak{d}_1}(y)\xi.
\end{equation}

Passing to the closure $\overline{T_1 + T_2}$ of $T_1 + T_2$ in the formula (5.1), we have the $(\phi_0, p)$-measurability condition of $\overline{T_1 + T_2}$, the closure of $T_1 + T_2$. Next we show $\|T_1 + T_2\|_{p, \phi_0} < \infty$. We put $T = T_1 + T_2 \supseteq T_1 + T_2$. Let $T = u\sigma_{\phi_0}^\phi$ be the polar decomposition and let $S = u^*\sigma_{\phi_0}^\phi$, where $1/p + 1/q = 1/2$, and $\psi \in M_\phi^+$, $\|\psi\| \leq 1$. Let $y \in M^0$, then by Lemma 3.1, $S\gamma_{\phi_0}(y) \in D(T_1) \cap D(T_2) \subseteq D(T)$ and hence,

\begin{equation}
\|TS\gamma_{\phi_0}(y)\| \leq (\|\phi_1\|^{1/p} + \|\phi_2\|^{1/p}).
\end{equation}

Assume $\phi(1) = +\infty$. Because $\phi$ is normal and semifinite, there exists an increasing net $\{\phi_a\} \subseteq M_\phi^+$ with supremum $\phi_a = \phi(u^*u)$. For an arbitrary $N > 0$, there exists $\phi_a$ such that $\phi_a(1) \geq N^p$. Then by choosing $\psi = \phi_a/\|\phi_a\|$,

\begin{equation}
\|TS\gamma_{\phi_0}(y)\| \geq \|A_{\phi_a}^{\mathfrak{d}_1}A_{\phi_a}^{\mathfrak{d}_2}\gamma_{\phi_0}(y)\|
\geq \|A_{\phi_a}^{\mathfrak{d}_1}A_{\phi_a}^{\mathfrak{d}_2}\gamma_{\phi_0}(y)\|
= (1/\|\phi_a\|)^{1/2}\phi_a(y)^{1/2}.
\end{equation}
By taking the *-strong limit \( y^* \to 1 \) in (5.3), we get

\[
(5.4) \quad \| TS \eta_{\phi_0} (y) \| \geq \phi_{\lambda} (1)^{1/p} \geq N.
\]

This contradicts (5.2). Therefore \( \phi \in M_{\phi}^\ast \) and \( T \in L_p (M, \phi_0) \). Associativity of the sum follows from the fact that \( \eta_{\phi_0} (N_{\phi_0}) \) is a core for any \( T \in L_p (M, \phi_0) \).

Q.E.D.

**Lemma 5.2.** Let \( 2 < p < \infty \) and \( p^{-1} + q^{-1} = 1/2 \).

1. Let \( T = u \Delta_{\phi_0}^{1/p} \in L_p (M, \phi_0) \), \( S = v \Delta_{\phi_0}^{1/q} \in L_q (M, \phi_0) \). Then \( ST \) is preclosed and the closure \( \overline{ST} \in L_q (M, \phi_0) \). Furthermore,

\[
(5.5) \quad \| ST \|_{p, \phi_0} \leq \| S \|_{q, \phi_0} \| T \|_{p, \phi_0}.
\]

2. Let \( T \in L_p (M, \phi_0) \). Then

\[
(5.6) \quad \| T \eta_{\phi_0} (y) \| \leq \| T \|_{p, \phi_0} \| y \|_{p, \phi_0} \| \eta_{\phi_0} (y) \|^{-1/(q/p)},
\]

where \( y \in N_{\phi_0} \).

**Proof.** (1) By Lemma 3.1, \( \eta_{\phi_0} (N_{\phi_0}) \subset D(ST) \) and \( \eta_{\phi_0} (N_{\phi_0}) \subset D(T^*S^*) \). Hence \( ST \) is preclosed. For \( \xi \in \eta_{\phi_0} (N_{\phi_0}) \) and \( y \in M_0 \),

\[
(5.7) \quad J_{\phi_0} y J_{\phi_0} ST \xi = SJ_{\phi_0} \sigma_{\phi_0}^{1/2} (y) J_{\phi_0} T \xi = \overline{ST} J_{\phi_0} \sigma_{\phi_0}^{1/2} (y) J_{\phi_0} \xi.
\]

By taking the closure of \( ST \) in (5.7), we get the \( (\phi_0, 2) \)-measurability condition of \( \overline{ST} \), the closure of \( ST \). By Lemma 3.1, \( \| (ST) \eta_{\phi_0} (y) \| \leq \| S \|_{q, \phi_0} \| T \|_{p, \phi_0} \), where \( y \in M_0 \). Hence \( \overline{ST} \in L_q (M, \phi_0) \) and (5.5) holds by Definition 2.2.

(2) By Lemma 3.1. Q.E.D.

**Lemma 5.3.** Let \( T = u \Delta_{\phi_0}^{1/p} \) be the polar decomposition of a \( (\phi_0, p) \)-measurable operator, \( 2 \leq p < \infty \). Then

\[
(5.8) \quad \sup \{ \| TS \eta_{\phi_0} (y) \| : y \in M_0, S \in L_q (M, \phi_0), \| S \|_{q, \phi_0} \leq 1 \}
\]

\[
= \phi (1)^{1/p}
\]

where \( p^{-1} + q^{-1} = 1/2 \) for \( p > 2 \) and \( q = \infty \) for \( p = 2 \).

**Proof.** First assume \( \phi (1) < \infty \). Then by Lemma 3.1, inequality
holds. By substituting $S = A_l^Q$ we obtain equality. If $\phi(1) = \infty$, the same proof as (5.4) implies that the left hand side is $\infty$. Q.E.D.

**Lemma 5.4.** Let $T_1, T_2 \in L_p(M, \phi_0), 2 \leq p < \infty$. Then

\[
\|T_1^* + T_2^*\|_{p, \phi_0} = \|T_1 + T_2\|_{p, \phi_0}.
\]

**Proof.** Let $T$ and $\tilde{T}$ be the closures of $T_1 + T_2$ and $T_1^* + T_2^*$ respectively. Then $T^* = (T_1 + T_2)^* \supseteq \tilde{T}$. Because $\gamma_{\phi_0}(M_0)$ is the core of $A_{\phi_0}$ for any $\phi \in M_0$ and $2 \leq p < \infty$, the polar decomposition of Lemmas 4.1 and 4.2 and remark after Lemma 4.2 imply that it is in the domains of $T_1^*$ and hence of $T^*$. Furthermore, Lemma 5.1 and Remark after Lemma 4.2 imply that $T^* \in L_p(M, \phi_0)$ and hence the same reasoning implies that $\gamma_{\phi_0}(M_0)$ is the core of $T^*$. Equality (5.10) would then imply the assertion. Equality (5.10) is an easy consequence of Lemma 4.2 and its Remark. Q.E.D.

**Lemma 5.5.** $L_p(M, \phi_0), 2 \leq p < \infty$ with norm $\| \cdot \|_{p, \phi_0}$ is a Banach space under strong sum.

**Proof.** By Lemma 5.1 as well as (5.8), $L_p(M, \phi_0)$ is a normed space with $\| \cdot \|_{p, \phi_0}$. The rest is to show the completeness. Let $\{T_n\} \subset L_p(M, \phi_0)$ be a Cauchy sequence and $K = \sup_n \|T_n\|_{p, \phi_0}$.

Due to Lemma 5.4, $\|T_n^* - T_m^*\|_{p, \phi_0} = \|T_n - T_m\|_{p, \phi_0}$ and hence $\{T_n^*\}$ is also a Cauchy sequence. By Lemma 5.2 (2), $\{T_n\gamma_{\phi_0}(y)\}$ and $\{T_n^*\gamma_{\phi_0}(y)\}$ are Cauchy sequences in $H_{\phi_0}$. Hence there exist densely defined operators $T_\infty$ and $T_\infty'$ defined on $\gamma_{\phi_0}(N_{\phi_0})$ by,

\[
T_\infty' \gamma_{\phi_0}(y) = \lim_{n \to \infty} T_n \gamma_{\phi_0}(y), \quad T_\infty \gamma_{\phi_0}(y) = \lim_{n \to \infty} T_n^* \gamma_{\phi_0}(y).
\]

Since $T_\infty' \subset (T_\infty)^*, T_\infty$ and $T_\infty'$ are preclosed. Their closures, $\overline{T_\infty}$ and $\overline{T_\infty'}$, are easily seen to be $(\phi_0, p)$-measurable. The rest is to show the following two facts.

1. $\overline{T_\infty} \in L_p(M, \phi_0)$.
2. $\|T_n - \overline{T_\infty}\|_{p, \phi_0} \to 0$ as $n \to \infty$. 


First we assume $p = 2$. Then by (5.11), for $y \in M_\phi^p$,

\begin{equation}
\| T_n \eta_\phi(y) \| = \lim_{n \to \infty} \| T_n \eta_\phi(y) \| \leq K.
\end{equation}

By Definition 2.2, $\| T_n \|_{2, \phi} \leq K$. For $y \in N_\phi^q$,

\begin{equation}
\| (T_n - T_m) \eta_\phi(y) \| \leq \| T_n - T_m \|_{2, \phi}.
\end{equation}

For any $\varepsilon > 0$, there exists $N(\varepsilon) > 0$ such that $n, m \geq N(\varepsilon)$ implies $\| T_n - T_m \|_{2, \phi} < \varepsilon$. By taking the limit $m \to \infty$ in (5.13), we get

\[ \| (T_n - T_m) \eta_\phi(y) \| \leq \varepsilon \]

if $n \geq N(\varepsilon)$ and $y \in N_\phi^q$. Hence $\| T_n - T_m \|_{2, \phi} \leq \varepsilon$ if $n \geq N(\varepsilon)$ and we have (2).

Next we assume $2 < p < \infty$. Let $S \in L_q(M, \phi_0)$ with $p^{-1} + q^{-1} = 1/2$. Then, for $y_1, y_2 \in N_\phi^q$, $(T_n \eta_\phi(y_1), S \eta_\phi(y_2))$ converges to $(T_n \eta_\phi(y_1), S \eta_\phi(y_2))$. Since $T_n^* S$ is Cauchy in $L_2(M, \phi_0)$ due to $\| (T_n^* - T_m^*) S \|_{2, \phi_0} \leq \| T_n - T_m \|_{2, \phi_0} \| S \|_{2, \phi_0}$ by Lemma 5.2, $T_n^* S \eta_\phi(y_2)$ converges to some $\zeta$ as $n \to \infty$. Furthermore $\zeta$ has a bound $\| \zeta \| \leq K \| S \|_{2, \phi_0} \| y_2 \|$ due to (5.5).

By the fact that $\eta_\phi(N_\phi^q)$ is a core of $T_\phi^*$, we have $S \eta_\phi(y_2) \in D((T_\phi^*)^*)$ and,

\begin{equation}
\| (T_\phi^*) S \eta_\phi(y_2) \| = \| \zeta \| \leq K \| S \|_{2, \phi_0},
\end{equation}

if $y_2 \in N_\phi^q$. Hence by Lemma 5.3, we have $T_n^* \in L_p(M, \phi_0)$ and we have (1). For $y \in N_\phi^q$,

\begin{equation}
\| (T_n^* - T_m^*) S \eta_\phi(y) \| \leq \| T_n^* - T_m^* \|_{p, \phi_0} \| S \|_{q, \phi_0}.
\end{equation}

For any $\varepsilon > 0$, there exists $N(\varepsilon) > 0$ such that $n, m \geq N(\varepsilon)$ implies $\| T_n^* - T_m^* \|_{p, \phi_0} < \varepsilon$. By taking the limit $m \to \infty$ in (5.15), we get $\| (T_n^* - (T_m^*)) S \eta_\phi(y) \| \leq \varepsilon \| S \|_{q, \phi_0}$ if $n \geq N(\varepsilon)$ and $y \in N_\phi^q$. By Lemmas 5.3 and 5.4, $\| T_n - T_m \|_{n, \phi_0} = \| T_n^* - (T_m^*)^* \|_{n, \phi_0} < \varepsilon$ if $n \geq N(\varepsilon)$ and we have (2).

Q.E.D.

\section{Product}

**Notation 6.1.** Let $L_p^* (M, \phi_0)$ be the set of all formal expressions

\begin{equation}
T = x_0 d_{\phi_0}(x_0) x_1 \cdots d_{\phi_n}(x_n) x_n
\end{equation}

with $x_j \in M$, $j = 0, \ldots, n$, $\phi_k \in M_\phi^p$, $p_k > 0$, $k = 1, \ldots, n$ such that $\sum_{k=1}^n p_k^{-1} = 1 - p^{-1}$. The adjoint $T^*$ of $T$ in $L_p^* (M, \phi_0)$ is defined as
\[ T^* = x_n^* A_n^{1/p_n} \ldots x_1^* A_1^{1/p_1} \cdot x_n^* A_n^{1/p_n} \cdot x_1^* A_1^{1/p_1} \cdot x_n^* A_n^{1/p_n} \cdot x_1^* A_1^{1/p_1} \cdot x_n^* A_n^{1/p_n} \cdot x_1^* A_1^{1/p_1} \cdot x_n^* A_n^{1/p_n} \cdot x_1^* A_1^{1/p_1} . \]

The product \( TS \in L_r^* (M, \phi_0) \) of \( T \in L_r^* (M, \phi_0) \) and \( S \in L_q^* (M, \phi_0) \) is defined if \( r^{-1} = p^{-1} + q^{-1} - 1 \) and \( 1 \leq r, p, q \leq \infty \) as the expression obtained by writing expressions for \( T \) and \( S \) together in that order and combine the last \( x \) in \( T \) and the top \( x \) in \( S \) according to the product operation in \( M \). By Lemmas 4.1 and 4.2, \( L_p(M, \phi_0) \subset L_r^* (M, \phi_0) \) with \( p^{-1} + (p')^{-1} = 1 \).

**Lemma 6.2.** \( L_p^* (M, \phi_0) \) is embedded in the dual space \( L_p(M, \phi_0)^* \) of \( L_p(M, \phi_0) \), \( 2 \leq p < \infty \), through the mapping

\[
L_p^* (M, \phi_0) \rightarrow L_p(M, \phi_0)^*,
\]

where the form \( \langle \cdot, \cdot \rangle_{\phi_0} \) is defined by

\[
\langle S, T^* \rangle_{\phi_0} = \lim_{y \rightarrow 1} \langle TS\eta_{\phi_0}(y), \eta_{\phi_0}(y) \rangle, \quad S \in L_p(M, \phi_0).
\]

The limit in (6.4) is the special case of Lemma 3.4, \( y_1 = y_2 \). In particular, \( L_p^* (M, \phi_0) \subset L_p(M, \phi_0)^* \), \( p^{-1} + (p')^{-1} = 1 \), through the form (6.4) and

\[
\| T \|_{p', \phi_0} = \| T \|_{p', \phi_0}^*, \quad T \in L_{p'}(M, \phi_0),
\]

where \( \| \cdot \|_{p', \phi_0}^* \) is the dual norm in \( L_p(M, \phi_0)^* \) through the dual pairing (6.4).

**Proof.** Let \( T \) be of the form (6.1). By (3.6) and Lemmas 4.1, 4.2 as well as Remark after Lemma 4.2,

\[
| \langle TS\eta_{\phi_0}(y), \eta_{\phi_0}(y) \rangle | \leq \left( \prod_{k=0}^{n} \| x_k \| \right) \left( \prod_{k=1}^{n} \phi_k(1)^{1/p_k} \right) \| S \|_{p, \phi_0}.
\]

By taking the limit in (6.6) in the sense of Lemma 3.4, we get the first half of the assertion i.e. \( L_p^* (M, \phi_0) \) is embedded in \( L_p(M, \phi_0)^* \). Next, assume \( T = uA_{p'}^* \neq 0, \phi \in M_\phi^*, \ u^*u = s(\phi) \) (\( u \) is a partial isometry in \( M \)). By Lemmas 4.1, 4.2 as well as (6.6),

\[
| \langle S, T^* \rangle_{\phi_0} | \leq \| T \|_{p', \phi_0}^* \| S \|_{p, \phi_0}.
\]

Hence we have \( \| T \|_{p', \phi_0} \leq \| T \|_{p', \phi_0}^* \). Inequality in (6.7) is actually attained.
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by taking \( S^* = ud^1_{\phi, \phi_0} \). So we have (6.5). \[ \square \]

Remark. Let \( T \in L_p(M, \phi_0) \), \( 1 < p < \infty \). Then there exists \( T' \in L_{p'}(M, \phi_0) \), \( p^{-1} + (p')^{-1} = 1 \) such that \( \langle T, T' \rangle_{\phi_0} = \|T\|_{p, \phi_0} \|T'\|_{p', \phi_0} \), where \( \langle \cdot, \cdot \rangle_{\phi_0} \) is defined by (6.4).

Proof. Let \( T = ud^1_{\phi, \phi_0} \) be the polar decomposition. Then \( T' = ud^1_{\phi, \phi_0} \) satisfies the equality in question by Lemma 4. 2. \[ \square \]

Lemma 6. 3. Let \( T \in L^*_p(M, \phi_0), p^{-1} + (p')^{-1} = 1, 2 \leq p' < \infty \). Then \( T \) is preclosed and its closure belongs to \( L_{p'}(M, \phi_0) \).

Proof. By Lemma 3. 1, \( \eta_{\phi_0}(N_{\phi_0}) \subset D(T) \cap D(T^*) \) for \( T \in L^*_p(M, \phi_0) \). Then for \( y_1, y_2 \in N_{\phi_0} \), \( (T\eta_{\phi_0}(y_1), \eta_{\phi_0}(y_2)) = (\eta_{\phi_0}(y_1), T^*\eta_{\phi_0}(y_2)) \) and hence \( T \) is preclosed. It is also easily seen that \( T \) satisfies (2. 1) for \( p' \) and hence the closure \( \bar{T} \) is \( (\phi_0, p') \)-measurable. Let \( T \) be of the form (6.1). If \( 2 < p' < \infty \),

\[
\|TS\eta_{\phi_0}(y)\| \leq \left( \prod_{k=1}^n \|x_k\| \right) \left( \prod_{k=1}^n \|\phi_k\|^{1/p_k} \|S\|_{q, \phi_0} \right)
\]

for \( y \in N_{\phi_0}^* \) and \( S \in L_q(M, \phi_0) \), \( \sum_{k=1}^n \rho_k^{-1} + q^{-1} = 1/2 \), due to (3. 2). Hence by Lemma 5. 3, the closure of \( T \) belongs to \( L_{p'}(M, \phi_0) \). \[ \square \]

Notation 6. 4. For \( 1 \leq p \leq 2 \), \( T_1 \) and \( T_2 \) in \( L^*_p(M, \phi_0) \) is said to be equivalent if \( T_1\eta_{\phi_0}(y) = T_2\eta_{\phi_0}(y) \) for any \( y \in N_{\phi_0} \) (by Lemma 3.1, \( \eta_{\phi_0}(N_{\phi_0}) \subset D(T) \) for any \( T \in L^*_p(M, \phi_0) \)). If \( p = 1 \), this equivalence is the same as that of \( M \). For \( 2 \leq p \leq \infty \), \( T_1 \) and \( T_2 \) in \( L^*_p(M, \phi_0) \) is said to be equivalent if \( \langle T_1, S \rangle_{\phi_0} = \langle T_2, S \rangle_{\phi_0} \) for all \( S \in L_p(M, \phi_0) \), where \( \langle \cdot, \cdot \rangle_{\phi_0} \) for \( 2 \leq p \leq \infty \) is defined by (6. 4) and \( \langle T, x \rangle_{\phi_0} = \lim_{y \to 0} (x^*T\eta_{\phi_0}(y), \eta_{\phi_0}(y)) \) for \( p = \infty \) (in the sense of Lemma 3. 4). Note that for \( 1 \leq p \leq 2 \), \( T_1 \) and \( T_2 \in L^*_2(M, \phi_0) \) are equivalent iff \( \langle T_1, S \rangle_{\phi_0} = \langle T_2, S \rangle_{\phi_0} \) for all \( S \in L_p(M, \phi_0) \).

Lemma 6. 5. If \( T_k \in L^*_p(M, \phi_0) \), \( k = 1, \ldots, n \) and \( \sum_{k=1}^n T_k = 0 \) in \( L_p(M, \phi_0) \) for \( 1 \leq p \leq 2 \) and \( p^{-1} + (p')^{-1} = 1 \) (Lemma 6. 3), then \( \sum_{k=1}^n T_k^* = 0 \) in the same sense and \( \sum_{k=1}^n T_k S = \sum_{k=1}^n ST_k = 0 \) in \( L_r(M, \phi_0) \) for \( 1 \leq r \leq 2 \).
and \( r^{-1} + (r')^{-1} = 1 \) or in \( L_r(M, \phi_0)^* \) for \( 2 \leq r \leq \infty \) (Lemma 6.2 and Notation 6.4) where \( S \in L_q^*(M, \phi_0), r^{-1} = p^{-1} + q^{-1} - 1 \) and \( 1 \leq q, r \leq \infty \).

**Proof.** From the assumption,

\[
(6.9) \quad \left( \sum_{k=1}^{\infty} T_k^* \eta_{\phi_i}(y_1), \eta_{\phi_i}(y_2) \right) = \left( \eta_{\phi_i}(y_1), \sum_{k=1}^{\infty} T_k \eta_{\phi_i}(y_2) \right) = 0
\]

for any \( y_1, y_2 \in N_{\phi_i} \). Hence \( \sum_{k=1}^{\infty} T_k^* = 0 \) in \( L_{r'}(M, \phi_0) \) due to the fact that \( \eta_{\phi_i}(N_{\phi_i}) \) is a core for any \( T \in L_{r'}(M, \phi_0) \). If \( 1 \leq r \leq 2 \),

\[
(6.10) \quad \sum_{k=1}^{\infty} \left( T_k S \eta_{\phi_i}(y_1), \eta_{\phi_i}(y_2) \right) = \left( \sum_{k=1}^{\infty} T_k \eta_{\phi_i}(y_2), S \eta_{\phi_i}(y_1) \right) = 0
\]

\[
(6.11) \quad \sum_{k=1}^{\infty} \left( ST_k \eta_{\phi_i}(y_1), \eta_{\phi_i}(y_2) \right) = \left( \sum_{k=1}^{\infty} T_k \eta_{\phi_i}(y_1), \eta_{\phi_i}(y_2) \right) = 0
\]

for any \( y_1, y_2 \in N_{\phi_i} \). Hence \( \sum_{k=1}^{\infty} T_k S = \sum_{k=1}^{\infty} T_k = 0 \) in \( L_r(M, \phi_0) \). If \( 2 \leq r \leq \infty \), let

\[
(6.12) \quad S = x_0^{A_{T_1,b_1}} \cdots x_m^{A_{T_m,b_m}}
\]

with \( \sum_{k=1}^{m} q_k^{-1} + q^{-1} = 1, x_k \in M, k = 1, \ldots, m \) and \( R = v A_{T_1,b_1} \in L_r(M, \phi_0) \). Then,

\[
(6.13) \quad \sum_{k=1}^{n} \left( A_{T_k,b_k} v^{*} x_0^{A_{T_1,b_1}} \cdots x_m^{A_{T_m,b_m}} T_k \eta_{\phi_i}(y_1), \eta_{\phi_i}(y_2) \right)
\]

\[
= \left( \sum_{k=1}^{n} T_k \eta_{\phi_i}(y_1), v^{*} x_0^{A_{T_1,b_1}} \cdots x_m^{A_{T_m,b_m}} v^{*} \eta_{\phi_i}(y_2) \right) = 0,
\]

\[
(6.14) \quad \sum_{k=1}^{n} \left( A_{T_k,b_k} v^{*} T_k x_0^{A_{T_1,b_1}} \cdots x_m^{A_{T_m,b_m}} \eta_{\phi_i}(y_1), \eta_{\phi_i}(y_2) \right)
\]

\[
= \left( \sum_{k=1}^{n} T_k x_0^{A_{T_1,b_1}} \cdots x_m^{A_{T_m,b_m}} \eta_{\phi_i}(y_1), v^{*} \eta_{\phi_i}(y_2) \right) = 0
\]

for \( z = (z_0, \ldots, z_m) \in iR^{m+1} \) and \( y_1, y_2 \in N_{\phi_i} \) because \( \sum_{k=1}^{n} T_k = 0 \) as an operator. Hence by analytic continuation,

\[
(6.15) \quad \sum_{k=1}^{n} \langle R^* ST_k \eta_{\phi_i}(y_1), \eta_{\phi_i}(y_2) \rangle = 0,
\]

\[
(6.16) \quad \sum_{k=1}^{n} \langle R^* T_k S \eta_{\phi_i}(y_1), \eta_{\phi_i}(y_2) \rangle = 0.
\]

By taking the limit \( y_1 \to 1 \) and \( y_2 \to 1 \), we have

\[
(6.17) \quad \sum_{k=1}^{n} \langle ST_k, R \rangle_{\phi_0} = 0,
\]
Lemma 6.6. Let \( 2 \leq p < \infty \), \( u_k \in M \), \( \phi_k \in M_*^+ \) and \( T_k = u_k A_{\phi_k, \phi_k}^p \in L_p(M, \phi_0) \), \( k = 1, \ldots, n \). Then

\[
\sum_{k=1}^{n} \langle T_k S, R \rangle_{\phi_0} = 0.
\]

Q.E.D.

Lemma 6.7. Let \( T \in L_p^*(M, \phi_0) \) be of the form (6.1). Then

\[
\| T \|_{p', \phi_0} \leq \left( \prod_{k=1}^{n} \| x_k \| \right) \left( \prod_{k=1}^{n} \phi_k (1/\phi_k) \right).
\]

Proof. By the proof of Lemma 6.2 and inequality (3.6).

Q.E.D.

§ 7. Special Cases \( p = 1, 2 \)

By Lemma 4.3, \( L_\omega (M, \phi_0) \) is identical with \( M \). In this section, we shall give canonical isomorphisms of \( L_1(M, \phi_0) \) to \( M_* \) and of \( L_2(M, \phi_0) \) to \( H_{\phi_0} \).

Lemma 7.1. Let \( T = u A_{\phi, \phi} \in L_1(M, \phi_0) \). Then,

\[
\phi_T(x) = [xu\phi], \quad x \in M
\]

in the sense of Lemma 3.4 defines the bijection between \( L_1(M, \phi_0) \) and \( M_* \) satisfying \( \phi_T(x) = \phi(xu) \) and

\[
\| \phi_T \| = \| T \|_{1, \phi_*}.
\]

Through the mapping \( T \mapsto \psi_T \), \( L_1(M, \phi_0) \) is isomorphic to \( M_* \) as a Banach space.
**Proof.** Any $T \in L_1(M, \phi)$ is of the form $T = uA^\delta_{\phi, \theta}$ by Lemma 4.1. (7.1) is computed as

\[
\phi_T (x) = \lim_{y \to 1} (xuA_{\phi, \theta}, \eta_\phi (y)) = \lim_{y \to 1} (A_{\phi, \theta}, \phi, \theta, u \star \eta_\phi (y)) = \lim_{y \to 1} (\phi \star xu \xi (\phi), y \star \xi (\phi)) = \phi (xu)
\]

where $\xi (\phi) \in \mathcal{D}_{\phi, \theta}$ is the representative vector of $\phi \in M_\phi^*$ ($\phi \in M_\phi^*$ is due to Lemma 4.2). Hence $\phi_T \in M_\phi$ and $\|\phi_T\| = \|\phi\| = \|T\|_{1, \phi}$. Surjectivity of this mapping follows from the usual polar decomposition of $\phi \in M_\phi$ (Theorem 1.14.4 of [25]).

Q.E.D.

**Lemma 7.2.** Let $T \in L_2(M, \phi)$. Then the limit

\[
\zeta_T = \lim_{y \to 1} T \eta_\phi (y)
\]

exists and the mapping $T \mapsto \zeta_T$ is an isomorphism of $L_2(M, \phi)$ onto $H_{\phi}$. 

**Proof.** Let $T \in L_2(M, \phi)$ and $T = uA_{\phi, \theta}^{1/2}$ be the polar decomposition given by Lemma 4.1. Then (7.4) is computed as

\[
\zeta_T = \lim_{y \to 1} uA_{\phi, \theta}^{1/2} \eta_\phi (y) = \lim_{y \to 1} uJ_{\phi, \theta} y \star \xi (\phi) = uJ_{\phi, \theta} \xi (\phi) = u \xi (\phi)
\]

where $\xi (\phi) \in \mathcal{D}_{\phi, \theta}$ is the representative vector of $\phi \in M_\phi^*$ ($\phi \in M_\phi^*$ is due to Lemma 4.2), and we have used the $J_{\phi, \theta}$-invariance of $\xi (\phi) \in \mathcal{D}_{\phi, \theta}$ (see Theorem C.1 of [8]). The mapping $T \mapsto \zeta_T$ is linear by (7.4). By (7.5), $\zeta_T = u \xi (\phi) \in H_{\phi}$ and $\|\zeta_T\| = \|\xi (\phi)\| = \phi (1)^{1/2} = \|T\|_{2, \phi}$. Hence $T \mapsto \zeta_T$ is an isomorphism of Banach spaces. Since its image contains $\mathcal{D}_{\phi, \theta}$ which is total in $H_{\phi}$ (see [15] or Appendix A), we have the assertion.

Q.E.D.
§ 8. Dual Pair

In this section, we first show the uniform convexity of the norm of \( L_p(M, \phi_0) \). By making use of this fact, we show the polar decomposition of the element of \( L_p(M, \phi_0)^* \) for \( 2 \leq p < \infty \).

Lemma 8.1. For \( 2 \leq p < \infty \) and \( T_1, T_2 \in L_p(M, \phi_0) \), the following Clarkson's inequality holds,

\[
\|T_1 + T_2\|_p^p + \|T_1 - T_2\|_p^p \leq 2^{p-1} \{ \|T_1\|_p^p + \|T_2\|_p^p \}.
\]

Proof. The same as Lemma 8.1 of [8] by replacing \( \eta \) by \( \eta_{\phi_0}(y) \), \( y \in M_0^0 \) and taking the limit \( y \rightarrow 1 \), \( \ast \)-strongly. Q.E.D.

For the uniform convexity, the uniform smoothness as well as the uniform strong differentiability of the norm of a Banach space, see [18]. We follow the line of proof in Section 9, [8].

Lemma 8.2. The norm \( \| \cdot \|_{p, \phi_0} \), \( 2 \leq p < \infty \) is uniformly strongly differentiable.

Proof. Follow from the proof of Lemma 9.1, [8]. Q.E.D.

Lemma 8.3. Let \( 1 < p \leq 2 \) and \( p^{-1} + \left( p' \right)^{-1} = 1 \).

(1) For \( T_1, T_2 \in L_p(M, \phi_0) \), \( T_1 = T_2 \) in \( L_{p'}(M, \phi_0)^* \) iff \( T_1 = T_2 \) as operator.

(2) For any \( \Phi \in L_{p'}(M, \phi_0)^* \), there exists \( T \in L_p(M, \phi_0) \) such that \( \Phi(S) = \langle S, T \rangle_{\phi_0} \) for \( S \in L_{p'}(M, \phi_0) \), where the right hand side is the sequilinear pairing defined by (4.24). Through this pairing, \( L_{p'}(M, \phi_0) \cong L_p(M, \phi_0)^* \).

Proof. (1) For the polar decomposition \( T_k = u_k a_k^p, k = 1, 2, S_k = u_k a_k'^{p'} \in L_{p'}(M, \phi_0) \) satisfies

\[
\langle T_k, S_k \rangle_{\phi_0} = \|T_k\|_{p, \phi_0} \|S_k\|_{p', \phi_1}.
\]
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(See Remark after Lemma 6.2.) By the assumption, \( T_1 = T_2 \) in \( L_{p^*}(M, \phi_0) \)
i.e. \( \langle T_1, S_i \rangle_{\phi_i} = \langle T_2, S_i \rangle_{\phi_i} \) and by Lemma 6.2, \( \|T_1\|_{p^*, \phi_i} = \|T_2\|_{p^*, \phi_i} \). It follows that from (6.5) and the Clarkson's inequality (8.1) (see [8], the proof of Lemma 10.1) \( S_i = S_2 \) and hence \( u_i = u_2, \phi_i = \phi_2 \). Therefore we obtain \( T_1 = T_2 \).

(2) By Lemma 6.2, \( L_p(M, \phi_0) \subset L_{p^*}(M, \phi_0)^* \). Let \( \Phi \in L_{p^*}(M, \phi_0)^* \), \( \Phi \neq 0 \). There exists an \( S \in L_p(M, \phi_0) \), \( \|S\|_{p^*, \phi_0} = 1 \), such that \( \Phi(S) = \|\Phi\| \). Let \( S = uA^p_{\phi, \phi} \) be the polar decomposition and put \( T = uA^p_{\phi, \phi} \). Then \( \langle S, T \rangle_{\phi_0} = \|T\|_{p^*, \phi_0} = \|\Phi\| \). By the uniform convexity of \( L_{p^*}(M, \phi_0)^* \) which is an immediate consequence of Lemma 8.2, \( \Phi(\cdot) = \langle \cdot, T \rangle_{\phi_0} \) and we get the assertion.

Q.E.D.

Remark. By Lemma 8.3, the dual space \( L_p(M, \phi_0)^* \), \( 2 \leq p < \infty \), has a unique polar decomposition and it is realized by the set \( L_{p^*}(M, \phi_0) \), \( p^{-1} + (p')^{-1} = 1 \). Through this correspondence, \( L_{p^*}(M, \phi_0) \) is a Banach space. Hence by Lemma 6.2, we have the following.

Lemma 8.4. \( L_p(M, \phi_0), 1 \leq p \leq \infty \), is a Banach space.

§ 9. Change of Reference Weight

In this section, we discuss the change of reference weight \( \phi_0 \) and the associated isomorphism \( \tau_p(\widetilde{\phi}_0, \phi_0) \) from \( L_p(M, \phi_0) \) to \( L_p(M, \widetilde{\phi}_0) \). Let \( \phi_0 \) and \( \widetilde{\phi}_0 \) be two faithful normal semifinite weights.

Lemma 9.1. Let \( T \in L_p(M, \phi_0), 1 \leq p < \infty \). Then \( x^*T\eta_{\tilde{\phi}_0}(y) \in D(\Delta_{\tilde{\phi}_0, \phi_0}^{(1/p)-(1/p)}) \) and

\[
J_{\tilde{\phi}_0, \phi_0}^{(1/p)-(1/p)} x^*T\eta_{\tilde{\phi}_0}(y) = y^*J_{\phi_0}(\tilde{\phi}_0, \phi_0)(T)\eta_{\phi_0}(x)
\]

where we assume \( y \in N_{\phi_0}, \eta_{\tilde{\phi}_0}(y) \in D(T), x \in N_{\phi_0}, \eta_{\phi_0}(x) \in D(J_{\phi_0}(\tilde{\phi}_0, \phi_0)(T)) \), and we have defined in Theorem 7 as follows

\[
J_{\phi_0}(\tilde{\phi}_0, \phi_0)(T) = \Delta_{\phi_0, \tilde{\phi}_0}^{(1/p)}u^* = u^*\Delta_{\phi_0, \tilde{\phi}_0}^{(1/p)} \in L_p(M, \tilde{\phi}_0)
\]

for the polar decomposition \( T = u\Delta_{\phi_0, \tilde{\phi}_0}^{(1/p)} \).

Proof. If we have the following formula,
for \( t \in \mathbb{R} \), the assertion follows by analytic continuation.

Formula (9.3) is shown as follows.

\[
(9.4) \quad J_{\tilde{\phi}, \tilde{\phi}_0}^{\frac{1}{2} - t}x^*u \mathcal{D}^{t\frac{1}{2}}_{\tilde{\phi}, \tilde{\phi}_0}(y) = y^*\mathcal{D}^{-t\frac{1}{2}}_{\tilde{\phi}, \tilde{\phi}_0}u \mathcal{D}^{t\frac{1}{2}}_{\tilde{\phi}, \tilde{\phi}_0}(x).
\]

Next we discuss the property of \( J_p(\tilde{\phi}_0, \phi_0) \), which is defined (in Theorem 7) by (9.3) for \( 1 \leq p < \infty \) and by the following for \( p = \infty \):

\[
(9.5) \quad J_\infty(\tilde{\phi}_0, \phi_0) : L_\infty(M, \phi_0) \to L_\infty(M, \tilde{\phi}_0)
\]

**Lemma 9.2.** For \( 1 \leq p \leq \infty \), \( J_p(\tilde{\phi}_0, \phi_0) \) is a conjugate linear isometry from \( L_p(M, \phi_0) \) onto \( L_p(M, \tilde{\phi}_0) \) and satisfies

\[
(9.6) \quad \langle T, T' \rangle_{\tilde{\phi}_0} = \langle J_{p'}(\tilde{\phi}_0, \phi_0)(T'), J_p(\tilde{\phi}_0, \phi_0)(T) \rangle_{\tilde{\phi}_0},
\]

where \( T \in L_p(M, \phi_0) \), \( T' \in L_{p'}(M, \phi_0) \), \( p^{-1} + (p')^{-1} = 1 \).

**Proof.** The isometric and surjective property is immediate from the definition. In addition, the conjugate linear property will follow from (9.6). To prove (9.6), we first consider the case \( 1 < p < \infty \). Let \( T = u\mathcal{D}^t_{\phi, \phi_0}, \ T' = v\mathcal{D}^{t'}_{\phi_0, \phi_0} \) be the polar decomposition. By taking analytic elements \( \int \alpha_i(x)f(t) \, dt \) of the one parameter group of mappings \( \alpha_i(x) = (D\phi : D\phi_0)_i\mathcal{D}^{t\frac{1}{2}}_{\phi, \phi_0}(x) \) with \( x \in N_{\phi_0} \) and the Fourier transform of \( f \) having a compact support, we can choose a \( \sigma \)-weakly dense (and hence \( * \)-strongly dense) convex subset \( E \) of the unit ball of \( N_{\phi_0} \) such that \( \eta_{\phi_0}(E) \subset D(T') \). If \( 2 \leq p < \infty \), \( \eta_{\phi_0}(N_{\phi_0}) \subset D(T) \) and hence \( \eta_{\phi_0}(E) \subset D(T) \cap D(T') \). For the case \( 1 < p \leq 2 \), the same reasoning is applicable if we interchange the role of \( T \) and \( T' \). Hence we can choose convex subsets \( E \) and \( \tilde{E} \) of the unit balls of \( N_{\phi_0} \) and \( N_{\phi_0} \), respectively, such that \( \eta_{\phi_0}(E) \subset D(T) \cap \)
\[ D(T') \text{ and } \eta_{\phi}(E) \subset D(J_{\phi}(\phi_0, \phi_0)(T')) \cap D(J_{\phi}(\phi_0, \phi_0)(T')). \] By the three line method, we obtain for \( y \in E \) and \( \bar{y} \in \bar{E} \)

\begin{align*}
(9.7) \quad |\bar{y}^* T \eta_{\phi_0}(y), \bar{y}^* T' \eta_{\phi_0}(y)| & \leq |\phi(y y^* v^* \bar{y} y^* \bar{y}^* u^* y^*)^{1/p'}|^{1/p'} \\
& \leq \|y y^* \xi(\phi)\|^{1/p} \|\bar{y} y^* u^* \xi(\phi)\|^{1/p} \|\bar{y} y^* v^* \xi(\phi)\|^{1/p'} \|y y^* \xi(\phi)\|^{1/p'}.
\end{align*}

Furthermore, by (9.1) and the conjugate isometric property of \( J_{\phi_0} \), we obtain

\begin{align*}
(9.8) \quad (y^* J_{\phi_0}(\phi_0, \phi_0)(T') \eta_{\phi_0}(y), y^* J_{\phi_0}(\phi_0, \phi_0)(T) \eta_{\phi_0}(y)) \\
& = (\bar{y}^* T \eta_{\phi_0}(y), \bar{y}^* T' \eta_{\phi_0}(y)).
\end{align*}

Hence the limits

\begin{align*}
(9.9) \quad \lim_{y \to 1} (\bar{y}^* T \eta_{\phi_0}(y), \bar{y}^* T' \eta_{\phi_0}(y)) \\
(9.10) \quad \lim_{\bar{y} \to 1} (\bar{y}^* T \eta_{\phi_0}(y), \bar{y}^* T' \eta_{\phi_0}(y))
\end{align*}

are uniform with respect to \( \bar{y} \) and \( y \) respectively for our choice of \( y \in E \) and \( \bar{y} \in \bar{E} \). (Note that \( \|y\| \leq 1 \) and \( \|\bar{y}\| \leq 1 \). Then we have

\begin{align*}
(9.11) \quad \langle T, T' \rangle_{\phi_0} = \lim_{y \to 1} (T \eta_{\phi_0}(y), T' \eta_{\phi_0}(y)) \\
& = \lim_{y \to 1} \lim_{\bar{y} \to 1} (\bar{y}^* T \eta_{\phi_0}(y), \bar{y}^* T' \eta_{\phi_0}(y)) \\
& = \lim_{\bar{y} \to 1} \lim_{y \to 1} (y^* J_{\phi_0}(\phi_0, \phi_0)(T') \eta_{\phi_0}(y), \bar{y}^* J_{\phi_0}(\phi_0, \phi_0)(T) \eta_{\phi_0}(\bar{y})) \\
& = \lim_{\bar{y} \to 1} (J_{\phi_0}(\phi_0, \phi_0)(T') \eta_{\phi_0}(\bar{y}), J_{\phi_0}(\phi_0, \phi_0)(T) \eta_{\phi_0}(\bar{y})) \\
& = \langle J_{\phi_0}(\phi_0, \phi_0)(T'), J_{\phi_0}(\phi_0, \phi_0)(T) \rangle_{\phi_0},
\end{align*}

where the first and the last equalities are due to the definition (6.4), Notation 3.3, (3.4) and proper redistribution of \( \mathcal{A}' \)s between two members of the inner product and the third equality is due to (9.9) and the uniformity of limits. Next we assume \( p=1 \). Let \( T=u \mathcal{A}_{\phi_0} \) be the polar decomposition and \( x \in M \). Then,
(9.12) \[ \langle T, x \rangle_{\phi_{0}} = \lim_{y \to 1} \langle T \eta_{\phi_{0}}(y), x \eta_{\phi_{0}}(y) \rangle = \lim_{y \to 1} \phi(y y^{*} x^{*} u) = \phi(x^{*} u). \]

By the same computation,

(9.13) \[ \langle J_{\infty}(\bar{\phi}_{0}, \phi_{0})(x), J_{1}(\bar{\phi}_{0}, \phi_{0})(T) \rangle_{\bar{\phi}_{0}} = \phi(x^{*} u). \]

The case \( p = \infty \) is proved by exactly the same computation. This shows formula (9.6) for \( 1 \leq p \leq \infty \). To show the conjugate linearity of \( J_{p}(\bar{\phi}_{0}, \phi_{0}) \), let \( T_{1} + T_{2} = T \) in \( L_{p}(M, \phi_{0}) \). Then for any \( S \in L_{p'}(M, \bar{\phi}_{0}) \),

(9.14) \[ \langle J_{p}(\bar{\phi}_{0}, \phi_{0})(T_{1}), S \rangle_{\bar{\phi}_{0}} + \langle J_{p}(\bar{\phi}_{0}, \phi_{0})(T_{2}), S \rangle_{\bar{\phi}_{0}} \]
\[ = \langle J_{p'}(\phi_{0}, \bar{\phi}_{0})(S), T_{1} \rangle_{\phi_{0}} + \langle J_{p'}(\phi_{0}, \bar{\phi}_{0})(S), T_{2} \rangle_{\phi_{0}} \]
\[ = \langle J_{p'}(\phi_{0}, \bar{\phi}_{0})(S), T \rangle_{\phi_{0}} \]
\[ = \langle J_{p}(\bar{\phi}_{0}, \phi_{0})(T), S \rangle_{\bar{\phi}_{0}}. \]

Hence \( J_{p}(\bar{\phi}_{0}, \phi_{0})(T_{1}) + J_{p}(\bar{\phi}_{0}, \phi_{0})(T_{2}) = J_{p}(\bar{\phi}_{0}, \phi_{0})(T) \) and we have the conjugate linearity of \( J_{p}(\bar{\phi}_{0}, \phi_{0}) \). Q.E.D.

**Remark.** If \( \bar{\phi}_{0} = \phi_{0} \), then the mapping \( J_{p}(\phi_{0}, \phi_{0}) \) is actually the adjoint operation for closed operators. (Hence it must be conjugate linear.) Equality (9.6) then reduces to

(9.15) \[ \langle T, T' \rangle_{\phi_{0}} = \langle T'^{*}, T^{*} \rangle_{\phi_{0}} \]

for \( T \in L_{p}(M, \phi_{0}) \), \( T' \in L_{p'}(M, \phi_{0}) \), \( p^{-1} + (p')^{-1} = 1 \).

From Lemma 9.2, we immediately obtain the following lemma.

**Lemma 9.3.** Let \( \phi_{0}, \phi', \phi'' \) and \( \bar{\phi}_{0} \) be faithful normal semifinite weights and set \( \tau_{p}(\phi', \phi_{0}) = J_{p}(\phi', \bar{\phi}_{0}) J_{p}(\bar{\phi}_{0}, \phi_{0}) \). Then \( \tau_{p}(\phi', \phi_{0}) \) is a linear isometry from \( L_{p}(M, \phi_{0}) \) onto \( L_{p}(M, \phi') \) and explicitly given by

(9.16) \[ \tau_{p}(\phi', \phi_{0}) : L_{p}(M, \phi_{0}) \to L_{p}(M, \phi') \]
\[ \begin{array}{c}
\cup \\
\cup
\end{array} \]
\[ u A_{p, \phi_{0}}^{\phi'} \mapsto u A_{p, \phi_{0}}^{\phi''}, \quad (1 \leq p < \infty), \]

(9.17) \[ \tau_{\infty}(\phi', \phi_{0}) : L_{\infty}(M, \phi_{0}) \to L_{\infty}(M, \phi') \]
\[ \begin{array}{c}
\cup \\
\cup
\end{array} \]
\[ x \mapsto x. \]
The map $\tau_p(\phi_0, \phi_0)$ is independent of the choice of $\phi_0$ and satisfies
\begin{align}
(9.18) & \quad \langle \tau_p(\phi_0, \phi_0)(T), \tau_p(\phi_0, \phi_0)(S) \rangle_{\phi_0} = \langle T, S \rangle_{\phi_0}, \\
(9.19) & \quad \tau_p(\phi_0^*, \phi_0) \tau_p(\phi_0, \phi_0) = \tau_p(\phi_0^*, \phi_0),
\end{align}
where $T \in L_p(M, \phi_0)$, $S \in L_{p'}(M, \phi_0)$, $p^{-1} + (p')^{-1} = 1$.

\section{10. Positivity}

In this section, the positive cone $L_p^+(M, \phi_0)$ in $L_p$-space is discussed and the linear polar decomposition is shown. Also, a bilinear dual pairing between $L_p(M, \phi_0)$ and $L_{p'}(M, \phi_0)$, $p^{-1} + (p')^{-1} = 1$, is introduced in the present section.

**Lemma 10.1.** Let $T \in L_p(M, \phi_0)$ such that $T^* = T$. Then there exist positive selfadjoint operators $T_+$ and $T_-$ belonging to $L_p(M, \phi_0)$ such that
\begin{equation}
T = T_+ - T_-.
\end{equation}
This decomposition is unique under the condition,
\begin{equation}
\sigma(T_+) \perp \sigma(T_-).
\end{equation}

**Proof.** Same as the proof of Lemma 13.1 in [9]. Q.E.D.

**Corollary.** Any $T \in L_p(M, \phi_0)$, $1 \leq p \leq \infty$, has a unique decomposition $T = (T_+ - T_-) + i(T_+ - T_-)$ such that $T_\tau \in L_p^+(M, \phi_0)$ and $s(T_+) \perp s(T_-)$ where $\tau = r$, $i$ and $\sigma = +, -$.

**Proof.** Same as the proof of Corollary 13.2, [8]. Q.E.D.

**Lemma 10.2.** Let $T \in L_p^+(M, \phi_0)$, $1 \leq p \leq \infty$. If $y \in N_{\phi_0}$ satisfies $\eta_{\phi_0}(y) \in D(T)$, then $y^* T \eta_{\phi_0}(y) \in V_{\phi_0}^{1/(2p)}$.

**Proof.** The case $p = \infty$ is clear. We assume $1 \leq p < \infty$ and $T = \mathcal{A}_{\phi_0, \phi_0}$. Due to the polar relation of positive cones and the density of the positive cone (see Appendix A), it is enough to show the following:
(10.3) \((y^* A_{\phi_0}^{1/p} \eta_{\phi_0}(y), A_{\phi_0}^{(1-1/p)} x^* \eta_{\phi_0}(x)) \geq 0\)

for any \(x \in M_0\). We have

(10.4) \(A_{\phi_0}^{(1-1/p)} x^* \eta_{\phi_0}(x) = J_{\phi_0} \eta_{\phi_0}(\sigma_{\phi_0}^{1/(1+p)} (x^* x)) \)

\(= x^* J_{\phi_0} \eta_{\phi_0}(\sigma_{\phi_0}^{1/(1+p)} (x)) \)

where \(x' = j(\sigma_{\phi_0}^{1/(1+p)} (x)) \in M_0\). Since

(10.5) \(x' A_{\phi_0}^{1/p} \eta_{\phi_0}(y) = A_{\phi_0}^{1/p} (d_{\phi_0}^{-1/p} x' A_{\phi_0}^{1/p}) \eta_{\phi_0}(y) \)

\(= A_{\phi_0}^{1/p} y J_{\phi_0} \eta_{\phi_0}(\sigma_{\phi_0}^{1/(1+p)} (x)) \),

we obtain

(10.6) \((y^* A_{\phi_0}^{1/p} \eta_{\phi_0}(y), A_{\phi_0}^{(1-1/p)} x^* \eta_{\phi_0}(x)) = \| A_{\phi_0}^{1/(1+p)} y J_{\phi_0} \eta_{\phi_0}(\sigma_{\phi_0}^{1/(1+p)} (x)) \|^2 \geq 0\). Q.E.D.

**Lemma 10.3.** Let \(p^{-1} + (p')^{-1} = 1\). Then

\(L^+_p(M, \phi_0) = \{ T \in L_p(M, \phi_0) : \langle T, S \rangle_{\phi_0} \geq 0 \text{ for any } S \in L^+_p(M, \phi_0) \}\).

**Proof.** Let \(T \in L^+_p(M, \phi_0)\) and \(S \in L^+_p(M, \phi_0)\). By taking suitable analytic elements as in the proof of Lemma 9.2, we can choose \(\sigma\)-weakly dense (and hence \(*\)-strongly dense) convex subset \(E\) of the unit ball of \(N_{\phi_0}\) such that \(\gamma_{\phi_0}(y) \in D(S) \cap D(T)\) for any \(y \in E\). Again by the same uniformity argument as in the proof of Lemma 9.2,

(10.7) \(\langle T, S \rangle_{\phi_0} = \lim_{y \to 1} (T \gamma_{\phi_0}(y), S \gamma_{\phi_0}(y)) \)

\(= \lim_{y \to 1} (y^* T \gamma_{\phi_0}(y), y^* S \gamma_{\phi_0}(y))\).

By Lemma 10.2 and the polar relation of the positive cone, \((y^* T \gamma_{\phi_0}(y), y^* S \gamma_{\phi_0}(y)) \geq 0\). Hence \(\langle T, S \rangle_{\phi_0} \geq 0\).

Conversely, assume \(T \in L_p(M, \phi_0)\) satisfies \(\langle T, S \rangle_{\phi_0} \geq 0\) for any \(S \in L^+_p(M, \phi_0)\). By the remark after Lemma 9.2,

(10.8) \(\langle T^*, S \rangle_{\phi_0} = \langle S^*, T \rangle_{\phi_0} = \langle S, T \rangle_{\phi_0} = \langle T, S \rangle_{\phi_0} = \langle T, S \rangle_{\phi_0} = \langle T, S \rangle_{\phi_0}\).

Due to the corollary after Lemma 10.1, \(L^+_p(M, \phi_0)\) is total in \(L_p(M, \phi_0)\).
Together with (10.8), \( T \) is self-adjoint. Again by Lemma 10.1, \( T \) has a polar decomposition \( T = T_+ - T_- \) with orthogonal supports. We assume \( T_+ \neq 0 \). Let \( T_- = D_{\phi_0}^{1/p_0} \) be the polar decomposition. If we set \( S = D_{\phi_0}^{1/p_0} \), then \( \langle T, S \rangle_{\phi_0} = -\langle T_-, S \rangle_{\phi_0} = -\phi(1) < 0 \) and this contradicts to the assumption. Hence we have \( T \geq 0 \) and complete the proof. Q.E.D.

Next we introduce a bilinear dual pairing between \( L_p(M, \phi_0) \) and \( L_{p'}(M, \phi_0) \), \( p^{-1} + (p')^{-1} = 1 \).

**Notation 10.4.** For \( T \in L_p(M, \phi_0) \) and \( S \in L_{p'}(M, \phi_0) \), \( p^{-1} + (p')^{-1} = 1 \), we define

\[
[T, S]_{\phi_0} = \langle T, S^{*}\rangle_{\phi_0}.
\]

By the remark after Lemma 9.2,

\[
[S, T]_{\phi_0} = \langle S, T^{*}\rangle_{\phi_0} = \langle T, S^{*}\rangle_{\phi_0} = [T, S]_{\phi_0}.
\]

This shows that this bilinear dual pairing is symmetric.

**§ 11. Product and Hölder Inequality**

Let us recall Notation 6.1 for \( L_p^*(M, \phi_0) \), \( 1 \leq p \leq \infty \) as well as for the adjoint and the product of its elements. We may identify \( L_p^*(M, \phi_0) \) (modulo induced equivalence) with a subset of \( L_p(M, \phi_0) \), \( p^{-1} + (p')^{-1} = 1 \) (by Lemma 3.1 as operators after taking closure for \( 1 \leq p \leq \infty \), and through Lemma 6.2 and duality \( L^{p'}(M, \phi_0) \cong L_p(M, \phi_0)^* \) (Lemma 8.3 (2)) for \( 1 \leq p' \leq 2 \) and through \( L_1(M, \phi_0) = L_\infty(M, \phi_0) \).

**Lemma 11.1.** Let \( \phi_0 \) and \( \tilde{\phi}_0 \) be faithful normal semifinite weights and let \( T \in L_p^*(M, \phi_0) \), \( S \in L_p(M, \phi_0) \). Then \( \langle J_p(\tilde{\phi}_0, \phi_0)(T), J_p(\tilde{\phi}_0, \phi_0)(S) \rangle_{\phi_0} = \langle S, T \rangle_{\phi_0} \), where \( J_p(\tilde{\phi}_0, \phi_0) \) for \( L_p(M, \phi_0) \) is defined by (9.2) and \( J_{p'}(\tilde{\phi}_0, \phi_0) \) is defined for \( L_{p'}(M, \phi_0) \) by

\[
J_{p'}(\tilde{\phi}_0, \phi_0)(T) = x_n^* D_{\phi_0}^{1/p_n} \cdots x_1^* D_{\phi_0}^{1/p_1} \phi_0^*
\]

if \( T \) is given by (6.1). When \( L_p^*(M, \phi_0) \) is identified with a subset of \( L_p(M, \phi_0) \), the two definitions of \( J_p(\tilde{\phi}_0, \phi_0) \) coincide.
Proof. Let \( y \in N_{\phi} \) and \( \bar{y} \in N_{\bar{\phi}} \). Then for \( t = \sum_{k=1}^{n} t_k \),

(11.2) \[
J_{\bar{\phi},\varphi} A^{(n)-t \bar{y}} x_0 A^{t_{1}} \cdots A^{t_{n}} \sigma_{x_{0}}^{\bar{y}} \sigma_{x_{1}} \cdots \sigma_{x_{n}} \eta_{\phi} (y)
\]
\[
= J_{\bar{\phi},\varphi} A^{(n)-t \bar{y}} x_0 A^{t_{1}} \cdots A^{t_{n}} \sigma_{x_{0}}^{\bar{y}} \sigma_{x_{1}} \cdots \sigma_{x_{n}} \eta_{\phi} (y)
\]
\[
= J_{\bar{\phi},\varphi} A^{(n)-t \bar{y}} x_0 A^{t_{1}} \cdots A^{t_{n}} \sigma_{x_{0}}^{\bar{y}} \sigma_{x_{1}} \cdots \sigma_{x_{n}} \eta_{\phi} (y)
\]
\[
= J_{\bar{\phi},\varphi} A^{(n)-t \bar{y}} x_0 A^{t_{1}} \cdots A^{t_{n}} \sigma_{x_{0}}^{\bar{y}} \sigma_{x_{1}} \cdots \sigma_{x_{n}} \eta_{\phi} (y)
\]

Hence, applying the same argument as in Section 3,

(11.3) \[
F (z) = J_{\bar{\phi},\varphi} A^{(n)-t \bar{y}} x_0 A^{t_{1}} \cdots A^{t_{n}} \sigma_{x_{0}}^{\bar{y}} \sigma_{x_{1}} \cdots \sigma_{x_{n}} \eta_{\phi} (y)
\]
\[
= y^* x_0^* A^{t_{1}} \cdots A^{t_{n}} \sigma_{x_{0}}^{\bar{y}} \sigma_{x_{1}} \cdots \sigma_{x_{n}} \eta_{\phi} (y)
\]
is continuous for \( z = (z_1, \ldots, z_n) \in \Omega_{\rho}^{(n)} \) and holomorphic in the interior, where \( z_0 = \sum_{k=1}^{n} z_k \). Let \( 1 < \rho < \infty \) and \( S = u A_{\rho}^{l \rho_{x_0}} \) be the polar decomposition. If \( z \in \partial_{\rho} \Omega^{(n)}_{\rho} \), \( k = 1, \ldots, n \) then

(11.4) \[
(y^* x_0^* A^{t_{1}} \cdots A^{t_{n}} \sigma_{x_{0}}^{\bar{y}} \sigma_{x_{1}} \cdots \sigma_{x_{n}} \eta_{\phi} (y), y^* A^{l \rho_{x_0}} u^* \eta_{\phi} (y))
\]
\[
= (J_{\bar{\phi},\varphi} A^{(n)-t \bar{y}} x_0 A^{t_{1}} \cdots A^{t_{n}} \sigma_{x_{0}}^{\bar{y}} \sigma_{x_{1}} \cdots \sigma_{x_{n}} \eta_{\phi} (y),
\]
\[
= (J_{\bar{\phi},\varphi} A^{(n)-t \bar{y}} x_0 A^{t_{1}} \cdots A^{t_{n}} \sigma_{x_{0}}^{\bar{y}} \sigma_{x_{1}} \cdots \sigma_{x_{n}} \eta_{\phi} (y),
\]

by (11.3) for general \( n \) and for \( n=1 \). By the analytic continuation of (11.4) for \( z_k \to \rho_k^{-1} \), \( k = 1, \ldots, n \) we have

(11.5) \[
(u A_{\rho_k}^{l \rho_k} y^* x_0^* A^{l \rho_k} \cdots A^{l \rho_k} x_0^* \eta_{\phi} (y), y^* A^{l \rho_k} \eta_{\phi} (y),)
\]
\[
= (x_0^* A^{l \rho_k} \cdots A^{l \rho_k} x_0^* \eta_{\phi} (y), y^* A^{l \rho_k} \eta_{\phi} (y),)
\]

where \( \rho^{-1} = 1 - \sum_{k=1}^{n} \rho_k^{-1} \). By taking the limit \( y \to 1 \) and \( \bar{y} \to 1 \) and by using the uniformity argument as in the proof of Lemma 9.2, we get the assertion for \( 1 < \rho < \infty \).

Proof for remaining cases are as follows. The case \( p = 1 \) is already shown by Lemma 9.2. Let \( p = \infty \). Then for \( z \in \partial_{\rho} \Omega^{(n)}_{\rho} \), \( k = 1, \ldots, n \), \( x \in M \) and \( y \in N_{\phi} \),
(11.6) \[ (y^*x\eta_{\phi_0}(y), y^*x_0A_1^{p_s} \cdots A_n^{p_s}x_n\eta_{\phi_0}(y)) = (y^*x_0A_1^{p_s} \cdots A_1^{p_s}x_0 \eta_{\phi_0}(x), \eta_{\phi_0}(y)) \]

where \( z_0 = (1/2) \in i\mathbb{R} \).

By the analytic continuation of (11.6) for \( z_0 \to +1, k = 1, \ldots, n \), we have

(11.7) \[ (x^*A_1^{1/p_s} \cdots A_n^{1/p_s}y\eta_{\phi_0}(y), \eta_{\phi_0}(y)) = (xy^*x^*A_1^{1/p_s} \cdots A_1^{1/p_s}x^*y\eta_{\phi_0}(y), \eta_{\phi_0}(y)). \]

By taking the limit \( y \to 1 \) and by the uniformity argument, we get

\[ \langle J_1(\tilde{\phi}_0, \phi_0) (T), x^* \rangle_{\phi_0} = \langle x, T \rangle_{\phi_0}. \]

Lastly we show that the two definitions of \( J_p(\tilde{\phi}_0, \phi_0) \) coincide when \( L_{p'}(M, \phi_0) \) is identified with a subset of \( L_p(M, \phi_0) \). The case \( p = 1 \) (equivalently \( p' = \infty \)) is immediate. Let \( T \) be of the form of (6.1), \( \tilde{T} = vA_1^{p_s} \cdots A_n^{p_s}x_0 \eta_{\phi_0}(x) \), and assume \( T = \tilde{T} \in L_p(M, \phi_0)^* \). By replacing \( S \) by \( J_p(\tilde{\phi}_0, \phi_0)(\tilde{S}), \tilde{S} \in L_{p'}(M, \phi_0) \), in the formula of Lemma 11.1 which we have just proved, we obtain

\[ \langle J_p(\tilde{\phi}_0, \phi_0) (T), \tilde{S} \rangle_{\phi_0} = \langle J_p(\phi_0, \phi_0) (\tilde{S}), T \rangle_{\phi_0} = \langle J_p(\tilde{\phi}_0, \phi_0) (\tilde{T}), \tilde{S} \rangle_{\phi_0}. \]

So the two definitions coincide. Q.E.D.

**Lemma 11.2.** If \( T_k \in L_{p'}^*(M, \phi_0), k = 1, \ldots, n \) and \( \sum_{k=1}^n T_k = 0 \) in \( L_p(M, \phi_0)^* \) for \( 2 \leq p \leq \infty \), then \( \sum_{k=1}^n J_{p'}(\tilde{\phi}_0, \phi_0) (T_k) = 0 \) in \( L_p(M, \phi_0)^* \) and \( \sum_{k=1}^n T_k S = \sum_{k=1}^n ST_k = 0 \) in \( L_r(M, \phi_0)^* \) where \( S \in L_r^*(M, \phi_0), r^{-1} = p^{-1} + q^{-1} - 1 \) and \( 1 \leq q, r \leq \infty \).

**Proof.** If \( \sum_{k=1}^n T_k = 0 \). Then \( \sum_{k=1}^n J_{p'}(\tilde{\phi}_0, \phi_0) (T_k) = 0 \) by Lemma 11.1.

Let \( R \in L_r(M, \phi_0) \). Again by Lemma 11.1 for \( \tilde{\phi}_0 = \phi_0 \), and by definition (6,2) of the sesquilinear form \( \langle \cdot, \cdot \rangle \),

(11.8) \[ \sum_{k=1}^n \langle T_k S, R \rangle_{\phi_0} = \sum_{k=1}^n \langle R^*, S^* T_k \rangle_{\phi_0} = \sum_{k=1}^n \langle SR^*, T_k \rangle_{\phi_0} = 0, \]

(11.9) \[ \sum_{k=1}^n \langle ST_k, R \rangle_{\phi_0} = \sum_{k=1}^n \langle T_k, S^* R \rangle_{\phi_0} = 0. \]
Lemma 11.3. Let $1 \leq p, q, r \leq \infty$, $p^{-1} + (p')^{-1} = q + (q')^{-1} = r$ + (r')^{-1} = 1$, $p^{-1} + q^{-1} = r^{-1}$.

1. If $T_1$ and $T_2$ in $L^p_*$ ($M, \phi_0$) are equal as elements of $L_p (M, \phi_0)$, then $T_1^* = T_2^*$ in $L_p (M, \phi_0)$, $T_1 S = T_2 S$ and $ST_1 = ST_2$ in $L_r (M, \phi_0)$ where $S \in L^q_* (M, \phi_0)$.

2. $T^*$ is conjugate linear in $T$ and $TS$ is bilinear in $T$ and $S$.

3. The product is associative and $(TS)^* = S^* T^*$.

4. For $T \in L^p_* (M, \phi_0)$ and $S \in L^q_* (M, \phi_0)$,
   \[ \|TS\|_{r, s_{\phi_0}} \leq \|T\|_{p, \phi_0} \|S\|_{q, \phi_0}. \]

Proof. Viewing $\beta \in C$ as an element of $L^1_\phi (M, \phi_0)$, it is easy to check $(\beta T)^* = \bar{\beta} T^*$, $(\beta T) S = T (\beta S) = \beta TS$ and the equivalence of $T_1$ and $T_2$ with $T_1 + (-1) T_2 = 0$ in $L_p (M, \phi_0)$. Therefore Lemmas 6.5 and 11.2 imply (1), (2) and (3). Assertion (4) follows from (3.6).

Q.E.D.

§ 12. Dense Subspaces of $L_p$-spaces

In this section, we discuss the $p$-dependent injection $T_p$ of $D^p_\phi$ (defined after Theorem 7) into $L_p (M, \phi_0)$. We start with some preliminary lemmas.

Lemma 12.1. Let $x \in N_\phi \cap N_\phi$, $y \in N_\phi$ and $\|y\| = 1$. Then for partial isometry $u \in M$ and $\phi \in M^*_\phi$ satisfying $u^* u = s(\phi)$,

\[ \|A_{\phi_\delta}^{(-1)} x u A_{\phi_\delta}^* y \| \leq \|\eta_{\phi_\delta} (x^*)\|^{1 - 2i} \|x\|_{\phi_\delta}^2 (1) \]

for $0 \leq \delta \leq 1/2$.

Proof. Follows from

\[ \|A_{\phi_\delta}^{(-1)} x u A_{\phi_\delta}^* y \| = \|y^* A_{\phi_\delta}^* u^* \eta_{\phi_\delta} (x^*/\|x\|)\| \|x\| \]

and Lemma 3.2. Q.E.D.

Lemma 12.2. Let $x \in N^*_\phi \cap N_\phi$ and $2 \leq p \leq \infty$. Then $A_{\phi_\delta} x$ is
preclosed and the closure $S_p(x)$ belongs to $L_p(M, \phi_0)$, and  

\[(12.2) \quad \|S_p(x)\|_{p, \phi_0} \leq \|\eta_{\phi_0}(x^*)\|^{2/p} \|x\|^{1-(2/p)}.\]

\[\text{Proof.}\] The case $p=\infty$ is immediate. We assume $2 \leq p < \infty$. By the assumption, $\eta_{\phi_0}(N_{\phi_0}^* \cap N_{\phi_0}) \subseteq D(D_{\phi_0}^{1/p}x) \cap D(D_{\phi_0}^{1/p})$. Hence the operators $D_{\phi_0}^{1/p}x$ and $x^*D_{\phi_0}^{1/p}$ are densely defined. Because $D_{\phi_0}^{1/p}x$ is a product of closed operator and bounded operator, it is preclosed. For $y \in M_0$,

\[(12.3) \quad \|D_{\phi_0}^{1/p}xJ_{\phi_0}\sigma^{\phi_0}_{1/p}(y)J_{\phi_0}x\| \leq \|D_{\phi_0}^{1/p}x\| \|J_{\phi_0}\sigma^{\phi_0}_{1/p}(y)J_{\phi_0}x\| = \|D_{\phi_0}^{1/p}x\| \|J_{\phi_0}\sigma^{\phi_0}_{1/p}(y)J_{\phi_0}x\|.
\]

By taking the closure of $D_{\phi_0}^{1/p}x$ in (12.3), we have the $(\phi_0, p)$-measurability of the closure $S_p(x)$ of $D_{\phi_0}^{1/p}x$.

If $2 < p < \infty$, the assertion of Lemma follows from Lemmas 5.3 and 12.1 for $(1/2) - (1/p) = \lambda$. If $p = 2$, then for $y \in M_0$,

\[(12.4) \quad \|S_p(x)\|_{p, \phi_0} \leq \|\eta_{\phi_0}(x^*)\| \leq \|\eta_{\phi_0}(x^*)\|.
\]

In view of (2.2), $\|S_p(x)\|_{p, \phi_0} \leq \|\eta_{\phi_0}(x^*)\|$ and the assertion follows.

Q.E.D.

Now, we recall the definition of the $p$-dependent injection $T_p$ of $D_{\phi_0}$ into $L_p(M, \phi_0)$ defined by the linear combination of (2.10).

**Lemma 12.3.** Let $\zeta \in D_{\phi_0}$. Then $T_p(\zeta) \in L_p(M, \phi_0)$ and is independent of the expression of $\zeta$ by the linear combination of the products of two elements of the Tomita algebra $\eta_{\phi_0}(M_0)$. Moreover for $\zeta = \eta_{\phi_0}(x_1x_2), \quad x_1 \in M_k, \quad k = 1, 2, \quad 1 \leq p \leq \infty$,

\[(12.5) \quad \|T_p(\zeta)\|_{p, \phi_0} \leq \{\|\eta_{\phi_0}(x_1)\| \|D_{\phi_0}^{1/p-1/2}\eta_{\phi_0}(x_2)\|\}^{1/p} \times \{\|x_1\| \|\sigma^{\phi_0}_{1/p}(x_2)\|\}^{1-(1/p)}.
\]

\[\text{Proof.}\] The case $p=\infty$ is immediate. Let $1 < p < \infty$, and assume $\zeta = \eta_{\phi_0}(x_1x_2)$. By Lemma 12.2, $D_{\phi_0}^{(2p)}x_1^*$ and $D_{\phi_0}^{(2p)}\sigma^{\phi_0}_{1/p}(x_2)$ belong to $L_{2p}(M, \phi_0)$. Hence by taking adjoint and product, $T_p(\zeta) = (D_{\phi_0}^{(2p)}x_1^*) \times (D_{\phi_0}^{(2p)}\sigma^{\phi_0}_{1/p}(x_2)) \in L_p(M, \phi_0)$. The assertion for general $\zeta$ follows by linear combination. Inequality (12.5) follows from (12.2) and Lemma 11.3 (4).

Next we show that $T_p(\zeta)$ is independent of the expression of $\zeta$. Let
ζₖ ∈ D₀ⁿ, k = 1, 2, 3, and assume ζ₁ + ζ₂ = ζ₃. If ζₖ = ηₖ(ξₖ), k = 1, 2, 3, then x₁ + x₂ = x₃. Therefore by the explicit form of Tₚ(ζₖ), k = 1, 2, 3, Tₚ(ζₖ)ηₖ(ξₖ) + Tₚ(ζₖ)ηₖ(ξₖ) = Tₚ(ζₖ)ηₖ(ξₖ), y ∈ M₀. If 2 ≤ p < ∞, ηₖ(M₀) is a core for any T ∈ Lₚ(M, φ₀), Tₚ(ζₖ) + Tₚ(ζₖ) = Tₚ(ζₖ) as elements of Lₚ(M, φ₀). If 1 ≤ p < 2, ⟨Tₚ(ζₖ)ηₖ(ξₖ), Sηₖ(ξₖ)⟩ + ⟨Tₚ(ζₖ)ηₖ(ξₖ), Sηₖ(ξₖ)⟩ = ⟨Tₚ(ζₖ)ηₖ(ξₖ), Sηₖ(ξₖ)⟩, S = uₜₖₕ(ξₕ) ∈ Lₚ(M, φ₀), p⁻¹ + (p')⁻¹ = 1, and a proper distribution of the power of Δₕₕ between the two members of each inner products. By taking limit y → 1, we obtain ⟨Tₚ(ζₖ), S⟩φ₀ + ⟨Tₚ(ζₖ), S⟩φ₀ = ⟨Tₚ(ζₖ), S⟩φ₀, and hence Tₚ(ζₖ) + Tₚ(ζₖ) = Tₚ(ζₖ). Q.E.D.

Next we consider the special case p = 1, 2, ∞.

Lemma 12.4. Let ζ ∈ D₀ⁿ be of the form,

(12.6) ζ = ηₖ(∑ₙₖ=₁ⁿ aₖx₁(k) x₂(k))

where aₖ ∈ C, x₁(k) ∈ M₀, k = 1, ..., n, l = 1, 2.

(1) T₉(ζ) = ∑ₙₖ=₁ⁿ aₖx₁(k) x₂(k). Hence T₉(D₀ⁿ) is σ-weakly dense in L₉(M, φ₀) = M.

(2) Let T = Tₐ(ζ). Then ζₐ = ζ where T → ζₐ is the isomorphic mapping Lₐ(M, φ₀) → Hₐ defined by (7.4). Hence Tₐ(D₀ⁿ) is norm dense in Lₐ(M, φ₀) = Hₐ.

(3) Let T = T₁(ζ). Then for x ∈ M,

(12.7) ψₐ(x) = ∑ₙₖ=₁ⁿ aₖ(xηₖ(x₁(k)), Jₚₖₕ(ηₖ(ξₖ)))

where T → ψₐ is the isomorphic mapping Lₐ(M, φ₀) → Mₜ defined by (7.1). Hence T₁(D₀ⁿ) is norm dense in L₁(M, φ₀) = Mₜ.

Proof. (1) Clear from definition (2.10).

(2) By the density argument and Lemma 7.2,

(12.8) ζₐ = limᵣ T₉ηₖ(ξₖ)

where the limit means y ∈ M₀ and y converges to 1 *-strongly. Hence for T = Tₐ(ζ),

(12.9) ζₚ = ∑ₙₖ=₁ⁿ aₖ limᵣ (∫ₚₖₕ(ξₕ(y) ∫ₚₖₕ(ξₕ(y)) ηₖ(y))
By the density argument and Lemma 7.1,

\[(12.10) \quad \phi_T(x) = \lim_{y \to 1} \langle xT\eta_T(y), \eta_T(y) \rangle, \quad x \in M\]

where the limit means $u \in M_0^{(1)}$ and $yy^*$ converges to $1$ *-strongly. Hence for $T = T_1(\zeta)$,

\[(12.11) \quad \phi_T(x) = \sum_{k=1}^{n} a_k \lim_{y \to 1} \langle \left( A_{\phi_k}^{\gamma_\phi_k} T_\phi \eta_T(x_k^{(2)}) \right) \eta_T(y), \left( A_{\phi_k}^{\gamma_\phi_k} T_\phi \eta_T(x_k^{(2)}) \right) \eta_T(y) \rangle

= \sum_{k=1}^{n} a_k \lim_{y \to 1} \langle \eta_T(y \cdot x_k^{(2)}), \eta_T(y \cdot x_k^{(2)}) \rangle

= \sum_{k=1}^{n} a_k \lim_{y \to 1} \langle x_k^{(2)} \cdot T_\phi \eta_T(y), x_k^{(2)} \cdot T_\phi \eta_T(y) \rangle

= \sum_{k=1}^{n} a_k \langle x_k^{(2)} \cdot T_\phi \eta_T(y), x_k^{(2)} \cdot T_\phi \eta_T(y) \rangle.

Q.E.D.

Next we show that the image of $D^{\phi}_\phi$ by the mapping $T_\phi$ is norm dense in $L_p(M, \phi)$, $1 < p < \infty$. The case $p = 2$ is already shown in Lemma 12.4 (2).

**Lemma 12.5.** Let $x = x_0 x_1 x_2$ with $x_0 \in M$, $x_1, x_2 \in M_0$, a partial isometry $u \in M$ and $\phi \in M^{\ast}_\phi$. Then the limit

\[(12.12) \quad F(x) = \lim_{y \to 1} \langle xD_{\phi_k}^{\gamma_\phi_k} \eta_T(y), uD_{\phi_k}^{\gamma_\phi_k} \eta_T(y) \rangle

exists for $0 \leq \text{Re } z \leq 1$ and,
(12.13) \[ F_z(x) = (A_{\phi,\psi}^1 u, x_0 \phi_\psi(x_1), A_{\phi,\psi}^2 \phi_\psi(x_2^*) ) \]

for \( 0 \leq \text{Re} \ z \leq 1 \). Moreover for a fixed \( z \) such that \( 0 < \text{Re} \ z < 1 \), \( F_z(x) \) is \( * \)-strongly continuous in \( x_0 \).

**Proof.** Let \( y \in M_0^U \) and consider the function

(12.14) \[ F_{x,y}(z) = (x A_{\phi,\psi}^1 \phi_\psi(y), u A_{\phi,\psi}^2 \phi_\psi(y)) \]

Then \( F_{x,y}(z) \) is continuous on \( 0 \leq \text{Re} \ z \leq 1 \) and holomorphic in the interior of this strip region with the following estimates on the boundary. For \( t \in \mathbb{R} \),

(12.15) \[ F_{x,y}(it) = (x A_{\phi,\psi}^1 \phi_\psi(y), u A_{\phi,\psi}^2 \phi_\psi(y)) \]

where \( \xi(\phi) \) is the unique vector representative of \( \phi \) in \( \mathcal{D}_{\phi,\psi}^U \). Hence for \( \gamma, \gamma_1, \gamma_2 \in M_0^U \), \( x' = x_0^* x_1 x_2 \),

(12.17) \[ |F_{x,y_1}(it) - F_{x,y_2}(it)| \leq \| (y_1 y_1^* - y_2 y_2^*) \xi(\phi) \| \| x^* u \xi(\phi) \| , \]

(12.18) \[ |F_{x,y_1}(1+it) - F_{x,y_2}(1+it)| \leq 2 \| x_0 \phi_\psi(x_1) \| \| \phi_\psi(\sigma_{t-1}^* x_2^*) \| , \]

(12.19) \[ |F_{x,y}(it) - F_{x,y}(i)g(\phi)| \leq \| \xi(\phi) \| \| (x' - x) \xi(\phi) \| , \]

(12.20) \[ |F_{x,y}(1+it) - F_{x,y}(1+it)| \leq \| (x_0 - x) \phi_\psi(x_1) \| \| A_{\phi,\psi}^1 \phi_\psi(x_2) \| . \]

For any compact set \( K \) contained in the half open strip \( 0 \leq \text{Re} \ z < 1 \),
there exists a real number $\beta$ satisfying,
\[(12.21) \quad 0 \leq \sup_{z \in K} \Re z \leq \beta < 1.\]

Hence due to (12.17) and (12.18),
\[(12.22) \quad |F_{z, \psi}(x) - F_{z, \psi}(z)| \leq \{\| (y_1, y_2, y_3) \| \| x \| \| \xi(\phi) \| \}^{1-r} \times \{2 \| x_0 \eta_{\psi}(x) \| \| \eta_{\psi}(\sigma_{\psi}(x)) \| \}^r\]

for $0 \leq \Re z = r \leq \beta$. It follows that the convergence of limit in (12.12) is uniform on $K$. The compact convergence theorem implies that the limit is holomorphic in $z$ for $0 < \Re z < 1$ and continuous in $z$ for $0 \leq \Re z \leq 1$. For the proof of equality (12.13), we denote the right hand side of (12.13) by $G(z)$. Then $G(z)$ is continuous on $0 \leq \Re z \leq 1$ and holomorphic in the interior. By (12.15), the boundary values coincide:
\[(12.23) \quad G(it) = (A_{\psi, \xi}^* \eta_{\psi}(u^* x_0 x), A_{\psi, \xi}^* \eta_{\psi}(D\phi; D\phi; x_0 x)) = (x_1 (D\phi; D\phi; \xi(\phi), x_1 x^* u \xi(\phi)) = (A_{\psi, \xi}^* \xi(\phi), x^* u \xi(\phi)) = F_z(it).\]

Hence $G(z) = F_z(z)$ for $0 \leq \Re z < 1$. For $\Re z = 1$, the following direct calculation by using (12.16) proves the convergence (12.12) and equality (12.13):
\[(12.24) \quad F_z(1 + it) = (A_{\psi, \xi}^* u^* x_0 \eta_{\psi}(x), A_{\psi, \xi}^* \eta_{\psi}(x_0 x)) = G(1 + it).\]

Due to (12.19) and (12.20),
\[(12.25) \quad |F_{z, \psi}(z) - F_{z, \psi}(z)| \leq \{\| \xi(\phi) \| \| (x' - x) \| u \xi(\phi) \| \}^{1-\Re z} \times \{\| (x' - x_0) \eta_{\psi}(x_0) \| \| A_{\psi, \xi}^{-1/2} \eta_{\psi}(x_0) \| \}^{\Re z}.\]

By taking the limit $y \to 1$ in (12.25), $|F_{x, \psi}(z) - F_{z}(z)|$ is dominated by the right hand side of (12.25). Hence we have the second assertion.

Q.E.D.

**Lemma 12.6.** The image of $D_{\psi}^* \phi$ by the mapping $T_\rho$ is norm dense in $L_\rho(M, \phi_0)$ for $1 < \rho < \infty$.

**Proof.** By making use of the Hahn-Banach theorem, it is enough to
show that for $S \subseteq L_p(M, \phi_0)$, $p \rightarrow 1 + (p')^{-1} = 1$, $\langle T_p(\zeta), S \rangle_{\phi_0} = 0$ for any $\zeta \in D_{\phi_0}^\infty$ implies $S = 0$.

Let $S = uD_{\phi_0}^{1/p}$ be the polar decomposition. We assume

\[(12.26) \quad \langle T_p(\eta_{\phi_0}(x)), S \rangle_{\phi_0} = \lim_{y \rightarrow 1} (xD_{\phi_0}^{1/p} \eta_{\phi_0}(y), uD_{\phi_0}^{1/p} \eta_{\phi_0}(y)) = 0, \quad x = x_1x_2\]

for any $x_1, x_2 \in M_0$, where $y \in M_0^{(0)}$ and the limit $y \rightarrow 1$ is in the sense of Lemma 3.4. By Lemma 12.4 and the strong density of $M_0$ in $M$, (12.26) also holds for $x = x_0x_1x_2$ with $x_0 \in M$, $x_1, x_2 \in M_0$. We put $x_0 = u$. By (12.13), $(AD_{\phi_0}^{1/p} \eta_{\phi_0}(x_1), AD_{\phi_0}^{1/p} \eta_{\phi_0}(x_2)) = 0$ for any $x_1, x_2 \in M_0$. By putting $x_2 = \sigma_{\phi_0}(x_1)$, $\|AD_{\phi_0}^{1/p} \eta_{\phi_0}(x_1)\| = 0$ for any $x_1 \in M_0$. Since $\eta_{\phi_0}(M_0)$ is a core for $D_{\phi_0}^\infty$, $0 < \alpha \leq 1/2$, we conclude $\phi = 0$ and hence, $S = 0$.

Q.E.D.

Lemma 12.7.

(1) Let $\zeta, \zeta' \in D_{\phi_0}^\infty$, $p^{-1} + \frac{1}{(p')^{-1}} = 1$. Then

\[(12.27) \quad \langle T_p(\zeta), T_p'(\zeta') \rangle_{\phi_0} = (\zeta, \zeta').\]

(2) The following diagram is commutative.

\[
\begin{array}{ccc}
D_{\phi_0}^\infty & \xrightarrow{J_p^{\phi_0}} & D_{\phi_0}^\infty \\
\downarrow T_p & & \downarrow T_p' \\
L_p(M, \phi_0) & \ast & L_p(M, \phi_0), \quad 1 \leq p \leq \infty
\end{array}
\]

where $J_p^{\phi_0} = J_pD_{\phi_0}^{1/(p(0))} - (1/p)$ and $*$ is the adjoint operation in $L_p(M, \phi_0)$.

Proof. (1) Let

\[(12.29) \quad \zeta = \sum_{j=1}^n a_j \eta_{\phi_0}(x_j^{(0)}x_j^{(0)}), \quad \zeta' = \sum_{k=1}^m b_k \eta_{\phi_0}(y_k^{(0)}y_k^{(0)})\]

where $a_j, b_k \in C$, $x_j^{(0)}, y_k^{(0)} \in M_0$, $j = 1, \ldots, n$, $k = 1, \ldots, m$. By (2.10) and (2.3), we obtain (with the convention $y \in M_0^{(0)}$)

\[(12.30) \quad \langle T_p(\zeta), T_p'(\zeta') \rangle_{\phi_0} = \sum_{j=1}^n \sum_{k=1}^m a_j b_k \lim_{y \rightarrow 1} ((AD_{\phi_0}^{1/(p(0))}x_j^{(0)}) (AD_{\phi_0}^{1/(p(0))} \sigma_{\phi_0}(x_j^{(0)})) \eta_{\phi_0}(y), \quad (AD_{\phi_0}^{1/(p(0))}y_k^{(0)}) (AD_{\phi_0}^{1/(p(0))} \sigma_{\phi_0}(y_k^{(0)})) \eta_{\phi_0}(y))\]
\begin{align*}
&= \sum_{j=1}^{n} \sum_{k=1}^{m} a_j b_k \lim_{y \to 1} \left( D_{\phi_k}^{\alpha} \sigma_{j_{2p}}^{(t)} (x_j^{(1)} x_j^{(2)}) \eta_\phi (y) \right), \\
&\quad \quad D_{\phi_k}^{\alpha} \sigma_{j_{2p}}^{(t)} (y_k^{(1)} y_k^{(2)}) \eta_\phi (y) \\
&= \sum_{j=1}^{n} \sum_{k=1}^{m} a_j b_k \lim_{y \to 1} \left( y^* \eta_\phi \left( \sigma_{j_{2p}}^{(t)} (y_k^{(1)} y_k^{(2)}) \right) * \right), \\
&\quad \quad y^* \eta_\phi \left( \sigma_{j_{2p}}^{(t)} (x_j^{(1)} x_j^{(2)}) \right) \\
&= \sum_{j=1}^{n} \sum_{k=1}^{m} a_j b_k \left( \eta_\phi \left( \sigma_{j_{2p}}^{(t)} (y_k^{(1)} y_k^{(2)}) \right) * \right), \\
&\quad \quad \eta_\phi \left( \sigma_{j_{2p}}^{(t)} (x_j^{(1)} x_j^{(2)}) \right) \\
&= \sum_{j=1}^{n} \sum_{k=1}^{m} a_j b_k \left( \eta_\phi \left( \sigma_{j_{2p}}^{(t)} (y_k^{(1)} y_k^{(2)}) \right) * \right), \\
&\quad \quad \eta_\phi \left( \sigma_{j_{2p}}^{(t)} (x_j^{(1)} x_j^{(2)}) \right) \\
&= (\zeta, \zeta').
\end{align*}

(2) The case \( p = \infty \) is clear from Lemma 12.4 (1). So we assume \( 1 \leq p < \infty \). Let \( \zeta \) be of the form (12.29). Then

\begin{align*}
(12.31) \quad T_p (\zeta) &= \sum_{k=1}^{n} a_k \left( D_{\phi_k}^{(p_1)} (x_k^{(1)}) * \right) \left( D_{\phi_k}^{(p_2)} (x_k^{(2)}) \right), \\
(12.32) \quad J_p^h \zeta &= \sum_{k=1}^{n} a_k J_{\phi_k} \left( \sigma_{j_{2p}}^{(t)} (x_k^{(1)}) \sigma_{j_{2p}}^{(t)} (x_k^{(2)}) \right) \\
&= \sum_{k=1}^{n} a_k \left( \sigma_{j_{2p}}^{(t)} (x_k^{(1)}) * \right) \left( \sigma_{j_{2p}}^{(t)} (x_k^{(2)}) \right) \\
(12.33) \quad T_p (J_p^h \zeta) &= \sum_{k=1}^{n} a_k \left( D_{\phi_k}^{(p_1 p_2)} (x_k^{(1)}) * \right) \left( D_{\phi_k}^{(p_1 p_2)} (x_k^{(2)}) \right).
\end{align*}

It follows \( T_p (\zeta) * \eta_\phi (y) = T_p (J_p^h \zeta) \eta_\phi (y) \) for any \( y \in \mathbb{M} \) and hence, \( \langle T_p (\zeta)^*, S \rangle_{\phi} = \langle T_p (J_p^h \zeta)^*, S \rangle_{\phi} \) for any \( S \in L_{p'} (\mathbb{M}, \phi_0), \ p^{-1} + (p')^{-1} = 1 \). This shows (12.28). Q.E.D.

Next, we discuss the relation between \( D_{\phi_k}^{(p)} \cap V_{\phi_k}^{(p)} \) and \( L_p^+ (\mathbb{M}, \phi_0) \) via mapping \( T_p \).

**Lemma 12.8.** \( T_p (D_{\phi_k}^{(p)} \cap V_{\phi_k}^{(p)}) \subseteq L_p^+ (\mathbb{M}, \phi_0), \ 1 \leq p \leq \infty \).
Proof. We start with

\[(12.34) \quad \zeta = \sum_{k=1}^{n} a_k \eta_{\phi_k}(x_{(k)}^{(1)}x_{(k)}^{(2)}) \in D_{\phi}^* \cap V_{\phi}^{(1/2)} .\]

If,

\[(12.35) \quad y^* T_{\phi}(\zeta) \eta_{\phi_k}(y) \in V_{\phi}^{(1/2)} , \quad y \in M_0 \]

hold, then by Lemma 10.2 and the polar relation of the positive cones (see Appendix A), \((y^* T_{\phi}(\zeta) \eta_{\phi_k}(y), y^* S\eta_{\phi_k}(y)) \geq 0 \) for \(S \in L_\phi^+(M, \phi_0) , \quad \rho^{-1} + (\rho')^{-1} = 1, \quad \text{and} \quad y \in N_{\phi_k} \), such that \(\eta_{\phi_k}(y) \in D(S)\). Actually we can choose a *-strongly dense convex subset \(E\) of the unit ball of \(N_{\phi_k}\) such that \(\eta_{\phi_k}(E) \subset D(S)\) (see the proof of Lemma 9.2). By taking limits \(y \to 1\) and \(y \to 1\), we obtain \(\langle T_{\phi}(\zeta), S \rangle_{\phi} \geq 0, \quad S \in L_\phi^+(M, \phi_0)\) and by Lemma 10.3, we obtain the assertion. So we have only to prove (12.35). By the polar relation of positive cones and the density of the set of vectors \(D_{\phi}^{(1/2)-(1/2p)} \beta_{\phi_k}(\beta)\), \(\beta \in M_0\) in \(V_{\phi}^{(1/2)-(1/2p)}\), we compute as follows.

\[(12.36) \quad (y^* T_{\phi}(\zeta) \eta_{\phi_k}(y), D_{\phi}^{(1/2)-(1/2p)} \beta_{\phi_k}(\beta)) \]

\[= (\sum_{k=1}^{n} a_k \chi_{(k)}^{(1)}x_{(k)}^{(2)} D_{\phi}^{(1/2)} \eta_{\phi_k}(y), D_{\phi}^{(1/2)-(1/2p)} \beta_{\phi_k}(\beta)) \]

\[= (J_{\phi_k} D_{\phi}^{(1/2)-(1/2p)}(y^* \eta_{\phi_k}(\beta)), D_{\phi}^{(1/2)-(1/2p)} \beta_{\phi_k}(\beta)) \]

\[= y J_{\phi_k} D_{\phi}^{(1/2)-(1/2p)} \beta_{\phi_k}(\beta) \]

\[(12.37) \quad J_{\phi_k} D_{\phi}^{(1/2)-(1/2p)} \beta_{\phi_k}(\beta) = S_{\phi, \phi}^{(1/2)}(\beta) \eta_{\phi_k}(y) \]

\[= y^* \eta_{\phi_k}(S_{\phi, \phi}^{(1/2)}(\beta)) \]

\[= y^* J_{\phi_k} D_{\phi}^{(1/2)-(1/2p)} \eta_{\phi_k}(\beta) \]

\[= y^* J_{\phi_k} D_{\phi}^{(1/2)-(1/2p)} \zeta \]

\[(12.38) \quad y J_{\phi_k} D_{\phi}^{(1/2)-(1/2p)} \beta_{\phi_k}(\beta) \]

\[= y J_{\phi_k} D_{\phi}^{(1/2)}(y^* \beta_{\phi_k}(\beta)) \]

\[= y J_{\phi_k} D_{\phi}^{(1/2)}(y^* \beta_{\phi_k}(\beta)) \]
where we used the $J^\phi_t$-invariance of $\zeta$ in the last equality of (12.37). By applying (12.37) and (12.38) to (12.36), we get (12.35).

Q.E.D.

§ 13. Group Action on $L_p$-Spaces

On the standard representation of a given von Neumann algebra $M$ by a faithful normal semifinite weight $\phi_0$, the associated modular action $\sigma^t_\phi$ on $M$ plays an important role in the structure theory of the von Neumann algebra $M$ ([7], [11], [28]). This one parameter action can be extended to $L_p$-spaces. In this section, we discuss the group action on $L_p$-spaces. Throughout this section, we fix a $W^*$-dynamical system $(M, G, \alpha)$. The modular action on $L_p$-spaces can be considered as its special case of this discussion.

Lemma 13.1. Let $T \in L_p(M, \phi_0)$. Then

\begin{align*}
\alpha_\phi^{(p)}(T) &= \alpha_\phi(u)\Delta^t_{\phi_0}(y)^* \\
\alpha_\phi^{(\infty)}(x) &= \alpha_\phi(x), \quad x \in L_\infty(M, \phi_0) = M,
\end{align*}

defines an isometric $G$-action on $L_p(M, \phi_0)$, where $T = u\Delta^t_{\phi_0}$ is the polar decomposition and $\phi_0(x) = \phi_0(\phi_0(x), x \in M$. Furthermore, this action $\alpha_\phi^{(p)}$ $(1 \leq p < \infty)$ preserves the adjoint operation and the positive part $L_p(M, \phi_0)$.

Proof. The case $p = \infty$ is the $\alpha$-action on $M$. For the case $1 \leq p < \infty$, the group property is checked by direct computation. It is also immediately seen that this action preserves the $L_p$-norm, the adjoint operation and the positive part $L_p(M, \phi_0)$.

Q.E.D.

Now, recall that the continuous group action is implemented by the strongly continuous unitary representation on the standard form of von
Neumann algebra (see [15]). Furthermore, this unitary representation preserves the natural positive cone. The unitary representation is actually defined by the following family of unitaries and the identification of two GNS representations $H_{\phi}$, and $H_{\phi_{t}}$ ($\phi_{t} = \phi \circ \alpha_{t}^{-1}$) through $\xi(\phi) = \xi_{t}(\phi)$

$$U(g) : H_{\phi_{t}} \rightarrow H_{\phi}$$

$$U(g) \xi(\phi) = \xi(\phi_{g}), \quad \phi_{g} = \phi \circ \alpha_{g}^{-1},$$

where $\xi(\phi)$, $\xi(\phi_{g}) \in \mathfrak{P}_{\phi_{t}}$ are the (unique) representative vectors of $\phi$ and $\phi_{g}$, respectively.

**Lemma 13.2.** $U(g) A_{\phi_{t}}^{z} U(g)^{*} = A_{\phi}^{z}$, $z \in C$, $g \in G$.

**Proof.** Let $y \in N_{\phi_{t}}$. Then

$$U(g) A_{\phi_{t}}^{z} U(g)^{*} = U(g) A_{\phi}^{z} \gamma_{\phi_{t}}(y)$$

$$= U(g) J_{\phi} \alpha_{g}^{-1}(y^{*}) \xi(\phi)$$

$$= J_{\phi} U(g) \alpha_{g}^{-1}(y^{*}) \xi(\phi)$$

$$= J_{\phi_{g}} y^{*} \xi(\phi_{g})$$

$$= A_{\phi_{g}}^{z}$$

where we used $U(g) J_{\phi_{t}} = J_{\phi_{t}} U(g)$ for the third equality and (13.3) for the fourth equality. By the fact that the set $\gamma_{\phi_{t}}(N_{\phi_{t}}) = U(g) \gamma_{\phi_{t}}(N_{\phi_{t}})$ is a core for the both of operators, we obtain the assertion for $z = 1/2$. The general case follows from functional calculus of self-adjoint operators.

Q.E.D.

**Lemma 13.3.**

1. $\alpha_{g}^{(p)}(T) = \tau_{p}(\phi_{g}, \phi_{0}, T_{1}) (U(g) TU(g)^{*})$, $g \in G$, $T \in L_{\phi}(M, \phi_{0})$.

2. Let $T_{k} \in L_{\phi_{0}}(M, \phi_{0})$, $k = 1, \ldots, n$ and $\sum_{k=1}^{n} p_{k} = p^{-1} \leq 1$. Then $\alpha_{g}^{(p)}(T) = \alpha_{g}^{(p)}(T_{1}) \cdots \alpha_{g}^{(p)}(T_{n})$ if $T = T_{1} \cdots T_{n}$.

3. The mapping $T \in L_{1}(M, \phi_{0}) \rightarrow [T]_{\phi_{0}}$ is $G$-invariant.

**Proof.**

1. The case $p = \infty$ follows from the definition. The case $1 \leq p < \infty$ follows from Lemma 13.2 for $z = 1/p$ and Lemma 9.3.
(2) By the definition of the product, $U(g)TU(g)^* = U(g)T_1U(g)^*$
$\cdots U(g)T_nU(g)^*$ in $L_p(M, \phi_{0, g})$. Hence by (1) and Lemmas 9.3 and
11.1, we obtain,

$$(13.5) \quad \alpha_{g^p}(T_1 \cdots T_n) = \tau_p(\phi_{0, g}) (U(g)T_1 \cdots T_nU(g)^*)$$

$$= \tau_{p_1}(\phi_{0, g_1}) (U(g)T_1U(g)^*) \cdots$$

$$\times \tau_{p_n}(\phi_{0, g_n}) (U(g)T_nU(g)^*)$$

$$= \alpha_{g^p}(T_1) \cdots \alpha_{g^p}(T_n).$$

(3) Let $T \in L_1(M, \phi_0), T = uA_{\phi_0, g}$. Then by (9.12) and (10.9),
$[T]_{\phi_0} = \phi(u)$. Hence $[\alpha_{g^p}(T)]_{\phi_0} = \phi \circ \alpha_{g^p}(u) = \phi(u) = [T]_{\phi_0}$.

Q.E.D.

**Corollary.** Let $T \in L_p(M, \phi_0), S \in L_{p'}(M, \phi_0), p^{-1} + (p')^{-1} = 1$.

Then $[\alpha_{g^p}(T)S]_{\phi_0} = [T\alpha_{g^p}(S)]_{\phi_0}$.

**Lemma 13.4.** For a fixed $T \in L_p(M, \phi_0)$, the mapping

$$(13.6) \quad \mathbb{G} \rightarrow L_p(M, \phi_0)$$

$$\cup \quad \cup$$

$g \mapsto \alpha_{g^p}(T), \quad 1 \leq p < \infty$

is norm continuous.

**Proof.** By the isomorphism $L_2(M, \phi_0) \cong H_{\phi_0}$ and the commutativity
of the diagram

$$(13.7) \quad \begin{array}{ccc}
L_2(M, \phi_0) & \xrightarrow{\alpha_{g^p}} & L_2(M, \phi_0) \\
\| & & \| \\
\phi_{\phi_0} & U(g) & \phi_{\phi_0},
\end{array}$$

the assertion holds for $p = 2$. Let $T = uA_{\phi_0, g} \in L_1(M, \phi_0)$ and $T_1 = uA_{\phi_0, g}^2,$
$T_2 = A_{\phi_0, g}^2 \in L_2(M, \phi_0)$. Then

$$(13.8) \quad \| \alpha_{g^p}(T) - T \|_{1, \phi_0} = \| \alpha_{g^p}(T_1)\alpha_{g^p}(T_1) - T_1T_{1}\|_{1, \phi_0}$$

$$\leq \| \alpha_{g^p}(T_1) - T_{1}\|_{2, \phi_0} \| T_{1}\|_{2, \phi_0}$$

$$+ \| \alpha_{g^p}(T_1)\|_{2, \phi_0} \| \alpha_{g^p}(T_2) - T_{2}\|_{2, \phi_0} \rightarrow 0 \quad \text{as} \quad g \rightarrow e.$$
Let \( T = u A_{\phi, \phi}^{1/2} \in L_p (M, \phi) \), \( S = v A_{\phi, \phi}^{1/2} \in L_{p'} (M, \phi) \), \( p^{-1} + (p')^{-1} = 1 \). We define the function \( F(z) \) by
\[
F(z) = e^{z^2} (v A_{\phi, \phi}^{1/2} [\alpha_g (u) A_{\phi, \phi}^{1/2} - u A_{\phi, \phi}^{1/2}] \eta_{\phi} (y), \eta_{\phi} (y)),
\]
where \( g \in G, y \in M^{(1)}, z \in C \) satisfying \( 0 \leq \Re z < 1 \) and the inner product in the right hand side is in the sense of the linear combination of Notation 3.3. Then the function \( F(z) \) is holomorphic on the open strip \( \{ z : 0 < \Re z < 1 \} \) and continuous on its closure. The estimates on one of the boundaries is the following:
\[
|F(it)| = e^{-|t|} \left| (v A_{\phi, \phi}^{1/2} [\alpha_g (u) A_{\phi, \phi}^{1/2} - u A_{\phi, \phi}^{1/2}] \eta_{\phi} (y), \eta_{\phi} (y)) \right| \leq 2e^{-|t|} \phi(1) \leq 2\phi(1) \quad (0 \leq \| y \| \leq 1).
\]
Hence, if we have the next formula:
\[
\lim_{g \to e} \sup_{\phi(1) = 1} \sup_{t \in R} |F(1 + it)| = 0,
\]
then due to
\[
\| \alpha_g^{(p)} (T') - T \|_{p, \phi} = \sup_{\| S \|_{p'}, \phi = 1} |[S (\alpha_g^{(p)} (T') - T)]_{\phi} | = \sup_{\phi(1) = 1} e^{-\gamma\| p \|} \lim_{\gamma \to 1} F(1/\gamma) \leq e^{-\gamma\| p \|} 2\| p \|^{1/p} \left( \sup_{\phi(1) = 1} \sup_{t \in R} |F(1 + it)| \right)^{1/p},
\]
where inequality is due to (13.10) and the three line theorem. Hence we obtain the assertion for \( 1 < \gamma < \infty \). Now, we show (13.11). By the repeated use of Lemma 13.2 and the formula \( U(g)^* \eta_{\phi, \phi} (y) = \eta_{\phi} (\alpha_g (\phi)) \),
\[
(\nu A_{\phi, \phi}^{1/2} \alpha_g (u) A_{\phi, \phi}^{1/2}, \eta_{\phi} (y)) = (\alpha_g^{(p)} (T'), \eta_{\phi} (y))
\]

It follows that
The first term of the right hand side of (13.14) tends to zero uniformly in $t$ if $g$ tends to unit $e$ due to the strong continuity of $U$. Due to the existence of converging factor $e^{-it}$, we may consider the convergence of the second term for $t$ in compact set. In that case, due to the continuity of $t^* = u A^t u^* t(0)$, the strong continuity of $U$, and the compactness of the set in which $t$ varies, the second term also tends to zero uniformly in $t$ if $g$ tends to $e$. This shows (13.11) and completes the proof of the assertion.

Q.E.D.

**Remark 13.5.** If we assume that the reference weight $\phi_0$ is relatively $G$-invariant i.e. there exists a continuous positive character $\chi: G \to \mathbb{R}_+$ such that

\begin{equation}
\phi_0 \circ \alpha_g(x) = \chi(g) \phi_0(x), \quad x \in M_\phi, \quad g \in G
\end{equation}

(see [28]). Then the modular action $\sigma^\phi_t$ and $\alpha_g$ commute and the Tomita algebra $\mathcal{H}_t$ is $\alpha$-invariant. In this situation the unitary representation $U$ of $G$ on $H_{\phi_0}$ by (13.3) satisfies

\begin{equation}
U(g) \eta_{\phi_0}(y) = \chi(g)^{-1/2} \eta_{\phi_0}(\alpha_g(y)), \quad y \in N_{\phi_0}, \quad g \in G.
\end{equation}

Furthermore, $D^\phi_{\phi_0}$ is $U$-invariant and the followings hold.

1. $U(g) D^\phi_{\phi_0} U(g)^* = \chi(g)^{-1/2} D^\phi_{\phi_0} \alpha_g, \quad x \in \mathcal{C}, \quad g \in G.$

2. $\alpha_g^\phi(T) = \chi(g)^{-1/2} U(g) T U(g)^*, \quad g \in G, \quad T \in L_p(M, \phi_0).$

3. The following diagram is commutative.

\begin{equation}
\begin{array}{ccc}
D^\phi_{\phi_0} & \xrightarrow{U_p(g)} & D^\phi_{\phi_0} \\
\downarrow T_p & & \downarrow T_p \\
L_p(M, \phi_0) & \xrightarrow{\alpha_g^\phi} & L_p(M, \phi_0)
\end{array}
\end{equation}
where \( U_p(g) = \chi(g)^{(1/p)-(1/p')} U(g) \).

**Proof.** (1) follows from Lemma 13.2, (13.15) and the homogeneity of relative modular operators.

(2) follows from (1) and Lemma 13.3 (1).

(3) Let \( \zeta \in D^\infty_{\phi_0} \) be of the form (12.29) and \( S \in L_p(M, \phi_0), \ p^{-1} + (p')^{-1} = 1 \). Then by Corollary after Lemma 13.3,

\[
S \alpha^{(p)}(T_p(\zeta)) \left[ \varphi_0 \right] = \left[ \alpha^{(p)}(S) T_p(\zeta) \right] \varphi_0.
\]

Hence we obtain (13.17). Q.E.D.

§ 14. Proof of Theorems

Theorem 1 (1) is shown in Lemma 4.3 (also see Definition 2.2) for \( p = \infty \), Lemma 5.5 for \( 2 \leq p < \infty \), Lemma 7.1 for \( p = 1 \) and Lemma 8.4 for \( 1 < p < 2 \). (2) for \( 1 \leq p \leq 2 \) is shown in Lemmas 6.2 and 8.3 (2).

(2) for \( 2 < p < \infty \) follows from Lemma 5.5, (2) for \( 1 < p < 2 \), and the reflexivity which is implied by the uniform convexity of (3). (3) is shown in Lemmas 8.1 and 8.2.

Theorem 2 (1), (2) for \( 1 \leq p < \infty \) follows from Lemmas 4.1, 4.2 and (2.6). The case \( p = \infty \) for (1), (2) follows from the usual polar decomposition and Theorem 3 (1). (3) follows from Lemma 4.1, (2.6) and the uniqueness of the polar decomposition of closed operators. (4) is shown in the Remark after Lemma 4.2.

Theorem 3 (1) is shown in Lemma 4.3 and (2) is shown in Lemma 7.1. (3) is shown in Lemma 7.2 and (7.5) in the proof of Lemma
7.2.

Theorem 4 is shown in Corollary of Lemma 10.1.

Theorem 5 (1) is (10.10). (2) is shown in Lemma 10.3, (2.9) and Theorem 2 (3).

Theorem 6 (1), (2) are shown in Lemma 11.3. (3) is shown as follows:

\[ (14.1) \quad [T_1 \cdots T_{k-1} T_k \cdots T_n]_{\phi_i} = \langle T_1 \cdots T_{k-1}, (T_k \cdots T_n)^* \rangle_{\phi_i} = \langle T_{k+1} \cdots T_n, (T_1 \cdots T_k)^* \rangle_{\phi_i} = \langle T_1 \cdots T_k T_{k-1} \cdots T_n, 1 \rangle_{\phi_i} = [T_1 \cdots T_n, 1]_{\phi_i} \]

where we used Remark after Lemma 9.2 for the second equality. (4) is a consequence of Notation 10.4 and Lemma 11.3. (5) follows from (2.4), (2.9), Lemma 11.3 (4) and \( \|1\|_{\infty, \phi_i} = 1 \).

Proof of Corollary. Follows directly from the method of [9].

Theorem 7 (1) is obtained in Lemmas 9.2 and 9.3. And two equations of (2) can be identified with (9.6) and (9.18) if polar decompositions of \( T \) and \( T' \) are submitted into explicit definitions (9.2), (9.16) of \( \mathcal{J}_p(\mathfrak{f}_\varphi, \phi_0) \) and \( \tau_{p}(\mathfrak{f}_\varphi, \phi_0) \), and by the fact that \( [\cdot, \cdot] \) is transformed into \( \langle \cdot, \cdot \rangle \) by (2.9).

(3) for \( \mathcal{J} \) is (11.1). (3) for \( \tau \) follows from (3) for \( \mathcal{J} \) and Lemma 9.3.

Theorem 8 (1) is shown in Lemma 12.3 for \( p=1, 2, \infty \) and in Lemma 12.5 for \( 1<p<\infty \). (2) is shown in Lemma 12.8. (3) and (4) are shown in Lemma 12.6.
§ 15. Discussions

The $L_p$-space $L_p(M, \phi_0)$ we constructed in this paper is isomorphic to that of $L_p$-spaces developed by Connes-Hilsum [12, 16], Kosaki [19, 20] and Terp [29]. Actually, our $L_p$-space is identical to $L_p(M, \phi_0(J^*J))$ in the notation of Hilsum [16] due to the equality $\frac{d\phi}{d\phi_0(J^*J)} = d_{\phi_0}$ where $\phi \in M^*_\phi$ and $\phi_0$ is the fixed faithful normal semifinite weight.

As we have already mentioned, the result stated in this paper is in some sense a straightforward generalization of the results obtained in our previous paper [8]. So, the fundamental idea and the main tools are analogous to [8]. The only non trivial part is the density argument and the method of taking limit (for example, Lemma 3.4). If $\phi_0$ is bounded then $1 \in N_{\phi_0}$ and $\eta_{\phi_0}(y) = y^2(\phi_0)$, $y \in N_{\phi_0}$. Hence the density argument as well as the process of taking limit is not necessary. So we have the previous $L_p$-theory through the mapping $T = u\mathcal{A}^\phi_{\phi_0} \mapsto T^\phi(\phi_0) = u\mathcal{A}^{\phi_0}_{\phi_0}$. On the other hand, if $M$ is not $\sigma$-finite, a cyclic and separating vector is not available and the method of [8] is not directly applicable. In the present approach, we first realize the $L_p$-space as the linear space of closed linear operators acting on the standard representation Hilbert space. We also find out that the $L_p$-space can be obtained as the completion of a certain vector subspace $D^\phi_{\phi_0}$ of a Tomita algebra associated with $\phi_0$ with respect to the $L_p$-norm.

Appendix A

In this section, we extend the notion of positive cones for a faithful normal semifinite weight and show their properties which are more or less known.

Let $M$ be a von Neumann algebra with a faithful normal semifinite weight $\phi_0$. For $0 \leq \alpha \leq 1/2$, we define the set,

\begin{equation}
V^\alpha_{\phi_0} = \{ d^\alpha_{\phi_0} x^* \eta_{\phi_0}(x) : x \in M_0 \},
\end{equation}

where $M_0$ is the set of all entire analytic elements of $N^*_\phi_0 \cap N_{\phi_0}$ with respect to the modular action $\sigma^\phi_0$. We define the positive cone $V^\alpha_{\phi_0}$, $0 \leq \alpha \leq 1/2$, to be the closure of $V^\alpha_{\phi_0}$ in $H_\phi$. Note that $\mathfrak{A} = \eta_{\phi_0}(N^*_\phi_0 \cap N_{\phi_0})$ is an achieved Hilbert algebra and $\mathfrak{A}_0 = \eta_{\phi_0}(M_0)$ is a maximal Tomita
algebra equivalent to $\mathcal{A}$. By the argument of Perdrizet and Haagerup ([23], [15]), $\mathcal{P}^\mathcal{N}_{\mathcal{A}} = V^0_{\mathcal{A}}$, $\mathcal{P}^\mathcal{P}_{\mathcal{A}} = V^{1/2}_{\mathcal{A}}$, and $\mathcal{P}^\mathcal{D}_{\mathcal{A}} = V^1_{\mathcal{A}}$. Furthermore, $\mathcal{P}^\mathcal{N}_{\mathcal{A}}$ and $\mathcal{P}^\mathcal{D}_{\mathcal{A}}$ are dual cones each other, and $\mathcal{P}^\mathcal{P}_{\mathcal{A}}$ is selfdual.

**Lemma A.1.** (1) $V^a_{\mathcal{A}}$ is a pointed weakly closed convex cone invariant under $\mathcal{A}^\mathcal{A}_{\mathcal{A}}$, $t \in \mathbb{R}$.

(2) $V^a_{\mathcal{A}} \subseteq D(\mathcal{A}^{(1/2)-2a}_{\mathcal{A}})$ and $J_{\mathcal{A}} \mathcal{A}^{(1/2)-2a}_{\mathcal{A}} = \xi$ for any $\xi \in V^a_{\mathcal{A}}$.

(3) $\mathcal{A}^\mathcal{A}_{\mathcal{A}}$ is a dense subset of $V^a_{\mathcal{A}}$.

(4) $J_{\mathcal{A}} V^a_{\mathcal{A}} = V^{(1/2)-a}_{\mathcal{A}}$.

(5) $V^a_{\mathcal{A}}$ and $V^{(1/2)-a}_{\mathcal{A}}$ are the dual cone of each other.

**Proof.** (2) Let $\zeta = \mathcal{A}^\mathcal{A}_{\mathcal{A}} x^* \eta_{\mathcal{A}}(x)$ with $x \in M_0$. Then,

\[
(A.2) \quad J_{\mathcal{A}} \mathcal{A}^{(1/2)-2a}_{\mathcal{A}} = J_{\mathcal{A}} \mathcal{A}^{(1/2)-2a}_{\mathcal{A}} x^* x
\]

\[
= \mathcal{A}^\mathcal{A}_{\mathcal{A}} J_{\mathcal{A}} \mathcal{A}^{(1/2)} x^* \eta_{\mathcal{A}}(x^* x)
\]

\[
= \mathcal{A}^\mathcal{A}_{\mathcal{A}} \eta_{\mathcal{A}}(x^* x) = \zeta.
\]

It follows that $V^a_{\mathcal{A}} \subseteq D(\mathcal{A}^{(1/2)-2a}_{\mathcal{A}})$ and any element in $V^a_{\mathcal{A}}$ is pointwise fixed under $J_{\mathcal{A}} \mathcal{A}^{(1/2)-2a}_{\mathcal{A}}$. Let $\xi \in V^a_{\mathcal{A}}$. There exists a sequence $\{\zeta_n\} \subseteq V^a_{\mathcal{A}}$ which converges to $\zeta$; $J_{\mathcal{A}} \zeta_n$ converges to $J_{\mathcal{A}} \zeta$ and $\mathcal{A}^{(1/2)-2a-2a}_{\mathcal{A}} = J_{\mathcal{A}} \zeta_n$ by (A.2). By the closedness of $\mathcal{A}^{(1/2)-2a}_{\mathcal{A}}$, it follows that $\zeta \in D(\mathcal{A}^{(1/2)-2a}_{\mathcal{A}})$ and $\mathcal{A}^{(1/2)-2a}_{\mathcal{A}} = J_{\mathcal{A}} \zeta$. This shows the assertion.

(3) By (2), $V^a_{\mathcal{A}} \subseteq D(\mathcal{A}^{(1/2)}_{\mathcal{A}})$ and hence, the expression $\mathcal{A}^\mathcal{A}_{\mathcal{A}} V^a_{\mathcal{A}}$ is well defined. By definition, $\mathcal{A}^\mathcal{A}_{\mathcal{A}} V^a_{\mathcal{A}} = V^a_{\mathcal{A}}$ is a dense subset of $V^a_{\mathcal{A}}$. Let $\xi \in V^a_{\mathcal{A}}$.

There exists a sequence $\{\zeta_n\} \subseteq V^a_{\mathcal{A}}$ which converges to $\zeta$. Again by (2), $\mathcal{A}^{(1/2)-2a}_{\mathcal{A}} = J_{\mathcal{A}} \zeta$ and $\mathcal{A}^{(1/2)-2a}_{\mathcal{A}} = J_{\mathcal{A}} \zeta_n$ hence $\mathcal{A}^{(1/2)-2a}_{\mathcal{A}}$ converges to $\mathcal{A}^{(1/2)-2a}_{\mathcal{A}}$. Due to the inequality $\|\mathcal{A}^{(1/2)-2a}_{\mathcal{A}} \| \leq \| \mathcal{A}^{(1/2)-2a}_{\mathcal{A}} \|$, $\mathcal{A}^{(1/2)-2a}_{\mathcal{A}} \zeta_n$ converges to $\mathcal{A}^{(1/2)-2a}_{\mathcal{A}}$. Due to $\mathcal{A}^{(1/2)-2a}_{\mathcal{A}} \zeta_n \in V^a_{\mathcal{A}}$, we conclude that $\mathcal{A}^\mathcal{A}_{\mathcal{A}} V^a_{\mathcal{A}} \subseteq V^a_{\mathcal{A}}$.

(4) Let $\zeta = \mathcal{A}^\mathcal{A}_{\mathcal{A}} x^* \eta_{\mathcal{A}}(x)$ with $x \in M_0$. Then,

\[
(A.3) \quad J_{\mathcal{A}} \zeta = J_{\mathcal{A}} \mathcal{A}^{(1/2)-2a}_{\mathcal{A}} x^* \eta_{\mathcal{A}}(x)
\]

\[
= \mathcal{A}^{(1/2)-2a}_{\mathcal{A}} \mathcal{A}^{(1/2)}_{\mathcal{A}} \eta_{\mathcal{A}}(x^* x)
\]

\[
= \mathcal{A}^{(1/2)-2a}_{\mathcal{A}} \eta_{\mathcal{A}}(x^* x) \in V^{(1/2)-a}_{\mathcal{A}}.
\]

It follows $J_{\mathcal{A}} \mathcal{A}^a_{\mathcal{A}} \subseteq V^{(1/2)-a}_{\mathcal{A}}$ and hence $J_{\mathcal{A}} V^a_{\mathcal{A}} \subseteq V^{(1/2)-a}_{\mathcal{A}}$. We replace $\alpha$ by $(1/2) - \alpha$ and multiply $J_{\mathcal{A}}$ on both sides and get $V^{(1/2)-a}_{\mathcal{A}} \subseteq J_{\mathcal{A}} V^a_{\mathcal{A}}$. Hence
we have the assertion.

(5) Let \( \zeta_1 = x^* \eta_{\phi_0}(x), \zeta_2 = x^{(q/n) - a} y^* \eta_{\phi_0}(y) \) with \( x, y \in M_0 \). Then,

\[
(A.4) \quad (\zeta_1, \zeta_2) = (x^* \eta_{\phi_0}(x), x^{(q/n) - a} y^* \eta_{\phi_0}(y))
\]

where the last equality is due to \( f J_0 x J_0 \gamma_{\phi_0}(y) = y J_0 \gamma_{\phi_0}(x) \) for \( x, y \in N_{\phi_0} \).

It follows that \( \zeta_1 \in V_{\phi_0}, \zeta_2 \in V_{\phi_0}^{(q/n) - a} \) implies \( (\zeta_1, \zeta_2) \geq 0 \), and hence,

\[
(A.5) \quad (V_{\phi_0})^\circ = \{ \xi \in H_{\phi_0} : (\xi, \zeta) \geq 0 \text{ for any } \zeta \in V_{\phi_0} \} \supset V_{\phi_0}^{(q/n) - a}.
\]

To prove the converse inclusion, we assume \( \xi \in (V_{\phi_0})^\circ \). Then \( (\xi, \zeta) \geq 0 \) for any \( \zeta \in \overline{V}_{\phi_0}^a \). By definition, \( \overline{V}_{\phi_0}^a \) is \( \mathcal{A}_{\phi_0}^n \)-invariant and hence, \( (\mathcal{A}_{\phi_0}^n \xi, \zeta) \geq 0 \) for any \( \zeta \in \overline{V}_{\phi_0}^a \). If we put,

\[
(A.6) \quad \xi_n = (n/\pi)^{1/2} \int_{\mathbb{R}} e^{-nt} \mathcal{A}_{\phi_0}^n \xi \, dt,
\]

then \( \xi_n \) is an entire analytic element of \( \mathcal{A}_{\phi_0} \) and \( (\xi_n, \zeta) \geq 0 \) for any \( \zeta \in \overline{V}_{\phi_0}^a \) and hence, \( (\mathcal{A}_{\phi_0}^n \xi_n, \zeta) \geq 0 \) for any \( \zeta \in \overline{V}_{\phi_0}^a \). By the density of \( \overline{V}_{\phi_0}^a \) in \( V_{\phi_0}^a \), the dual relation between \( \mathcal{P}_{\phi_0}^a \) and \( \mathcal{P}_{\phi_0}^a = V_{\phi_0}^{(q/n) - a}, \mathcal{A}_{\phi_0}^n \xi_n \in V_{\phi_0}^{(q/n) - a} \).

By (4), \( J_{\phi_0} \mathcal{A}_{\phi_0}^n \xi_n \in V_{\phi_0}^0 \) and hence \( J_{\phi_0} \xi_n = \mathcal{A}_{\phi_0}^n J_{\phi_0} \mathcal{A}_{\phi_0}^n \xi_n \in V_{\phi_0}^a \). Again by (4), \( \xi_n \in J_{\phi_0} V_{\phi_0}^a = V_{\phi_0}^{(q/n) - a} \). By (A.6), \( \xi_n \) converges to \( \xi \). Since \( V_{\phi_0}^{(q/n) - a} \) is closed, we conclude \( \xi \in V_{\phi_0}^{(q/n) - a} \).

1. By (5), we can see easily that \( V_{\phi_0}^a \) is weakly closed and convex. The \( \mathcal{A}_{\phi_0}^n \)-invariance of \( V_{\phi_0}^a \) is immediate. To see that \( V_{\phi_0}^a \) is pointed, assume \( \xi \in V_{\phi_0}^a \cap \{-V_{\phi_0}^a\} \). Then by (5), \( \xi \perp V_{\phi_0}^{(q/n) - a} \). The linear span of \( V_{\phi_0}^{(q/n) - a} \) contains the linear span of \( \overline{V}_{\phi_0}^{(q/n) - a} \), so it contains \( \{ \gamma_{\phi_0}(xy) : x, y \in M_0 \} \) and hence, \( V_{\phi_0}^{(q/n) - a} \) is total in \( H_{\phi_0} \). It follows \( \xi = 0 \) and \( V_{\phi_0}^a \) is pointed.

Q.E.D.

Appendix B

In this section, we discuss the polar decomposition in \( D(\mathcal{A}_{\phi_0}^{(q/n) - a}) \) in
terms of the positive cone \( V^\alpha_{\phi_0} \), \( 0 \leq \alpha \leq 1/2 \) (see Appendix A for the positive cones) for a faithful normal semifinite weight \( \phi_0 \) on a von Neumann algebra \( M \). The results are similar to but weaker than that obtained in [8] due to the fact that we don’t assume the boundedness of \( \phi_0 \).

**Theorem B.1.** Let \( \zeta \in D(\mathcal{A}_\phi^{(q/2) - 2\alpha}) \), \( 0 \leq \alpha \leq 1/2 \). Then there exist a partial isometry \( u \in M \) (resp. \( u' \in M' = J_{\phi}M\phi_0 \)) and \( |\zeta|_\alpha \in V^\alpha_{\phi_0} \) such that \( \zeta = u|\zeta|_\alpha \) and \( u^*u = s^N(|\zeta|_\alpha) \) (or equivalently \( uu^* = s^N(\zeta) \)) (resp. \( \zeta = u'|\zeta|_\alpha \) and \( u'^*u' = s^{N'}(|\zeta|_\alpha) \) (or equivalently \( u'u'^* = s^N(\zeta) \)).

The proof of the theorem is divided into several steps. We consider the involution operator,

\[
J_\alpha = J_{\phi_0}A_\phi^{(q/2) - 2\alpha}.
\]
(Note that \( J_{\phi_0}^{(q/2)} \) discussed in Theorem 8 (3) is the restriction of this \( J_\alpha \), \( \alpha = 1/(2p) \) to \( D_{\phi_0} \).) By (B.1) and \( J_{\phi_0}\mathcal{A}_\phi^{(q/2)}J_{\phi_0} = \mathcal{A}_\phi^{(q/2)} \), \( \lambda \in \mathcal{R} \), the domain \( D(\mathcal{A}_\phi^{(q/2) - 2\alpha}) \) is \( J_\alpha \)-invariant. For \( \zeta \in D(\mathcal{A}_\phi^{(q/2) - 2\alpha}) \), we define two operators \( T_0 \) and \( R_0 \) as follows;

\[
T_0 \eta_{\phi_0}(y) = J_{\phi_0}A_\phi^{(q/2) - 2\alpha}(y^*)J_{\phi_0} \zeta,
\]

\[
R_0 \eta_{\phi_0}(y) = J_{\phi_0}A_\phi^{(q/2) - 2\alpha}(y^*)J_{\phi_0}J_\alpha \zeta,
\]

where \( M_\alpha \) is the set of all entire analytic elements of \( N_{\phi_0} \cap N_{\phi_0}^* \) with respect to the modular action \( \sigma_{\phi_0}^* \). By (B.2) and (B.3), \( D(T_0) = D(R_0) = \eta_{\phi_0}(M_\alpha) \) and hence \( T_0, R_0 \) are densely defined operators.

**Lemma B.2.** \( T_0 \) and \( R_0 \) are preclosed and

\[
T^* \supseteq R, \quad R^* \supseteq T
\]

where \( T \) and \( R \) are the closures of \( T_0 \) and \( R_0 \) respectively.

**Proof.** It is sufficient to prove that for any \( y_1, y_2 \in M_\alpha, \)

\[
(T_0 \eta_{\phi_0}(y_1), \eta_{\phi_0}(y_2)) = (\eta_{\phi_0}(y_1), R_0 \eta_{\phi_0}(y_2)).
\]

By definitions of \( T_0 \) and \( R_0 \), both sides of (B.5) are computed as follows;
\begin{align*}
(B.6) \quad (T\varphi_{\psi}(y_1), \varphi_{\psi}(y_2)) &= (J_{\psi}\sigma^2_{(U_0)-z_0}(y^*_2)J_{\psi}\zeta, \varphi_{\psi}(y_2)) \\
&= (\zeta, J_{\psi}\sigma^2_{(U_0)-z_0}(y_1)J_{\psi}\varphi_{\psi}(y_2)) \\
&= (\zeta, J_{\psi}\sigma^2_{(U_0)-z_0}(y_1)\varphi_{\psi}(\sigma^2_{(U_0)-z_0}(y^*_2))) \\
&= (\zeta, J_{\psi}\Delta_{\psi}^{\psi^2_{(1-x_0)-z_0}}(y_1)y^*_2) \\
&= (\zeta, \varphi_{\psi}(y_2\sigma^2_{(1-z_0)}(y^*_2))).
\end{align*}

\begin{align*}
(B.7) \quad (\varphi_{\psi}(y_1), R\varphi_{\psi}(y_2)) &= (\varphi_{\psi}(y_1), J_{\psi}\sigma^2_{(U_0)-z_0}(y^*_2)J_{\psi}\varphi_{\psi}(y_2)) \\
&= (\sigma^2_{(U_0)-z_0}(y^*_2)\Delta_{\psi}^{\psi^2_{(1-x_0)-z_0}}(y^*_2)J_{\psi}\varphi_{\psi}(y_1)) \\
&= (\Delta_{\psi}^{\psi^2_{(1-x_0)-z_0}}(y^*_2)\sigma^2_{(U_0)-z_0}(y_2)\varphi_{\psi}(\sigma^2_{(U_0)-z_0}(y^*_2))) \\
&= (\Delta_{\psi}^{\psi^2_{(1-x_0)-z_0}}(y^*_2)\varphi_{\psi}(\sigma^2_{(U_0)-z_0}(y_2)\sigma^2_{(U_0)-z_0}(y^*_2))) \\
&= (\zeta, \Delta_{\psi}^{\psi^2_{(1-x_0)-z_0}}(y^*_2)\varphi_{\psi}(\sigma^2_{(U_0)-z_0}(y_2)\sigma^2_{(U_0)-z_0}(y^*_2))) \\
&= (\zeta, \varphi_{\psi}(y_2\sigma^2_{(1-z_0)}(y^*_2))).
\end{align*}

Q.E.D.

\textbf{Lemma B.3.} \textit{T and R are $(\psi_0, p)$-measurable operators with $p=1/(2\alpha)$.}

\textbf{Proof.} If we have

\begin{align*}
(B.8) \quad T_{0}J_{\psi}\sigma^2_{(U_0)-z_0}(y)J_{\psi}\varphi_{\psi}(\bar{y}) &= J_{\psi}yJ_{\psi}T_{0}\varphi_{\psi}(\bar{y})
\end{align*}

for any $y, \bar{y} \in M_\alpha$, we get the assertion for $T$ by taking the closure. Two
sides of (B.8) are computed as follows:

\begin{align*}
(B.9) \quad T_{0}J_{\psi}\sigma^2_{(U_0)-z_0}(y)J_{\psi}\varphi_{\psi}(\bar{y}) &= T_{0}J_{\psi}\sigma^2_{(U_0)-z_0}(y)\varphi_{\psi}(\sigma^2_{(U_0)-z_0}(y^*)) \\
&= T_{0}J_{\psi}\varphi_{\psi}(\sigma^2_{(U_0)-z_0}(y)\sigma^2_{(U_0)-z_0}(y^*)) \\
&= T_{0}J_{\psi}\varphi_{\psi}(y\sigma^2_{(U_0)-z_0}(y^*)) \\
&= J_{\psi}\sigma^2_{(U_0)-z_0}(y\sigma^2_{(U_0)-z_0}(y^*))J_{\psi}\zeta \\
&= J_{\psi}yJ_{\psi}T_{0}\varphi_{\psi}(\bar{y})J_{\psi}\zeta,
\end{align*}

\begin{align*}
(B.10) \quad J_{\psi}yJ_{\psi}T_{0}\varphi_{\psi}(\bar{y}) &= J_{\psi}y\sigma^2_{(U_0)-z_0}(y^*)J_{\psi}\zeta.
\end{align*}

If we replace $\zeta$ by $J_{\alpha}\zeta$, we obtain the proof for $R$. Q.E.D.
**Proof of Theorem B.1.** First we assume \( \alpha > 0 \). By Lemmas 4.1, B.3 and (B.2), there exist a partial isometry \( u \in M \) and a normal semifinite weight \( \phi \) satisfying \( u^* u = \delta (\phi) \) and \( T = u \mathcal{A}^p_{\phi, u} \), i.e.

\[(B.11) \quad u \mathcal{A}^p_{\phi, u} \gamma_{\phi} \left( y \right) = J_{\phi} \sigma_{[0, y]}(y^*) J_{\phi} \zeta, \quad y \in M_0.\]

Next we show that \( P = uu^* \) and \( Q = s^M(\zeta) \) coincide. Since \( Pu = u \) (B.11) in which \( \sigma_{[0, y]}(y^*) \) is replaced by \( y \), implies \( J_{\phi} y J_{\phi} P \zeta = P J_{\phi} y J_{\phi} \zeta = J_{\phi} y J_{\phi} \zeta \). Taking the strong limit \( y \to 1 \), \( P \zeta = \zeta \) and we obtain \( P \leq Q \).

Conversely, (B.11) and \( Q J_{\phi} y J_{\phi} \zeta = J_{\phi} y J_{\phi} Q \zeta = J_{\phi} y J_{\phi} \zeta \), \( y \in M_0 \) implies \( Q u \mathcal{A}^p_{\phi, u} \gamma_{\phi} \left( y \right) = u \mathcal{A}^p_{\phi, u} \gamma_{\phi} \left( y \right) \), \( y \in M_0 \). By definition (B.2), \( \gamma_{\phi} (M_0) \) is a core for \( u \mathcal{A}^p_{\phi, u} \). This implies that \( \mathcal{A}^p_{\phi, u} \gamma_{\phi} (M_0) \) is dense in \( s(\phi) H_{\phi} \) and we obtain \( Qu = u \). Hence \( Q \geq P \). Therefore we obtain \( P = Q \).

Next, we show \( u^* \zeta = \| \zeta \|_a \in V^a_{\phi} \). By (B.11),

\[(B.12) \quad \mathcal{A}^p_{\phi, u} \gamma_{\phi} \left( y \right) = J_{\phi} \sigma_{[0, y]}(y^*) J_{\phi} \zeta, \quad y \in M_0.\]

On the other hand,

\[(B.13) \quad J_{\phi} \sigma_{[0, y]}(y^*) J_{\phi} \gamma_{\phi} \left( y \right) = y J_{\phi} \gamma_{\phi} \left( \sigma_{[0, y]}(y^*) \right) = y J_{\phi} \mathcal{A}^p_{\phi, u} \gamma_{\phi} \left( \sigma_{[0, y]}(y^*) \right) = y \mathcal{A}^p_{\phi, u} \gamma_{\phi} \left( y^* \right) = \mathcal{A}^p_{\phi, u} \gamma_{\phi} \left( y^* \right) \sigma_{[0, y]}(y^*) \gamma_{\phi} \left( y^* \right), \quad y \in M_0.\]

Combining (B.12) and (B.13),

\[(B.14) \quad 0 \leq \left( \mathcal{A}^p_{\phi, u} \gamma_{\phi} \left( y \right), \gamma_{\phi} \left( y \right) \right) = \left( J_{\phi} \sigma_{[0, y]}(y^*) J_{\phi} \zeta, \gamma_{\phi} \left( y \right) \right) = \left( |\zeta|_a, J_{\phi} \sigma_{[0, y]}(y^*) J_{\phi} \gamma_{\phi} \left( y \right) \right) = \left( |\zeta|_a, \mathcal{A}^p_{\phi, u} \gamma_{\phi} \left( y^* \right) \right),\]

where \( \mathcal{A}^p_{\phi, u} \gamma_{\phi} \left( y^* \right) \), \( y \in M_0 \). By the density of \( \mathcal{A}^p_{\phi, u} \gamma_{\phi} \left( y^* \right) \) in \( V^a_{\phi} \) and Lemma A.1 (5), we obtain \( |\zeta|_a \in V^a_{\phi} \).

Now, we give the proof for \( \alpha = 0 \). By Lemmas 4.3, B.3 and (B.2), there exists a closed operator \( T \) affiliated with \( M \) satisfying

\[(B.15) \quad T \gamma_{\phi} \left( y \right) = J_{\phi} \sigma_{[0, y]}(y^*) J_{\phi} \zeta, \quad y \in M_0.\]
Let \( T = u |T| \) be the polar decomposition. Then by the same reason as the case \( \alpha > 0 \), \( uu^* = s^M (\zeta). \) Let \( |\zeta|_o = u^* \zeta. \) Then

\[
(B.16) \quad 0 \leq (|T| \eta_{\phi_o} (\psi), \eta_{\phi_o} (\psi)) \\
= (|\zeta|_o, J_{\phi_o} \sigma_{12} (\psi) J_{\phi_o} \eta_{\phi_o} (\psi)) \\
= (|\zeta|_o, \sigma_{12} \sigma_{12} (\psi) \eta_{\phi_o} (\sigma_{12} (\psi^*) )
\]

implies \( |\zeta|_o \in V_{\phi_o}^o. \)

The polar decomposition in terms of \( M' \) (\( u' \in M' \) instead of \( u \in M \)) follows from the polar decomposition in terms of \( M \) (which we have shown above) for the complementary index \( \alpha' = (1/2) - \alpha \), if we use \( J_{\phi_o} D (\Delta (\phi_{o}^{-2a}) = D (\Delta (\phi_{o}^{-2a})^*), \) and Lemma A.1 (4). Q.E.D.

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**References**


