On a Holomorphic Fiber Bundle with Meromorphic Structure

By

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Introduction

Let \( f : X \to Y \) be a proper surjective morphism of compact complex manifolds. Let \( U \subseteq Y \) be a Zariski open subset over which \( f \) is smooth. Let \( X_U = f^{-1}(U) \) and let \( f_U : X_U \to U \) be the induced morphism. Assume that \( f_U \) is a holomorphic fiber bundle with typical fiber \( F \) and the structure group \( H \). Let \( G_U \to U \) be the holomorphic fiber bundle associated with \( f_U \) with typical fiber \( H \) with the adjoint action of \( H \) on itself so that \( G_U \) acts naturally on \( X_U \) over \( U \). Let \( I_U \to U \) be the principal \( H \)-bundle associated to \( f_U \). Then \( G_U \) acts naturally on \( I_U \) over \( U \) also. We say that \( f_U \) is a holomorphic fiber bundle with meromorphic structure if there exists a compact complex space \( G^* \) (resp. \( I^* \)) over \( Y \) containing \( G_U \) (resp. \( I_U \)) as a Zariski open subset such that the action of \( G_U \) on \( X_U \) (resp. \( I_U \)) extends 'meromorphically' to that of \( G^* \) on \( X \) (resp. \( I^* \)). Then in this paper we shall prove the following: Suppose that \( f_U \) is a holomorphic fiber bundle with meromorphic structure for some \( G^* \) and \( I^* \) as above. Then

1) there exists a 'generic quotients' \( X/G^* \) of \( X \) by \( G^* \) over \( Y \), and
2) \( X/G^* \) is bimeromorphic to the product space \( (F/H) \times Y \) where \( F/H \) is a generic quotient of \( F \) by \( H \).

Actually in this paper, these results are obtained in a more general setting of comparing two proper morphisms \( f_i : X_i \to Y, \ i = 1, 2 \), over \( Y \) having isomorphic general fibers (cf. Theorems 1 and 2); the above special case corresponds to the case where one of the \( f_i \) is isomorphic to the projection \( p : F \times Y \to Y \). (This generalization is in a sense parallel with Grothendieck's generalization \([7]\) of the theory of fiber bundles to the theory of general fiber spaces with structure sheaf.)

Section 1 is preliminary, and in Section 2 we prove Theorems 1 and 2 mentioned above. Then in Section 3 we shall give some general examples which appear naturally in the study of the structure of compact complex manifolds in \( \mathcal{C} \) \([5]\); indeed, the application to these examples is the principal motivation for this paper. Finally in Section 4, as a reference for \([5]\), we gather some results.

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1) The assumption on \( I_U \) is unnecessary for the assertion 1).
related to the subject of this paper.

In this paper a complex variety means a reduced and irreducible complex
space. Let \( f : X \rightarrow Y \) be a proper surjective morphism of complex varieties. Then we write \( f \in \mathcal{C}/Y \) if there exist a proper Kähler morphism \( g : Z \rightarrow Y \) (cf. [4]) and a surjective meromorphic \( Y \)-map \( \phi : Z \rightarrow X \).

\[ \text{§ 1. Preliminaries and Basic Definitions} \]

1.1. a) Let \( Y \) be a complex space. Then a relative complex Lie group over \( Y \) is a complex space \( G \) over \( Y \) with a holomorphic section \( e : Y \rightarrow G \) (the identity section) and \( Y \)-morphisms \( \mu = \mu_{G/Y} : G \times_Y G \rightarrow G \), and \( \iota = \iota_{X/Y} : G \rightarrow G \) (relative group multiplication and inversion) satisfying the usual axioms of group law (cf. [11], Def. 0.1). Then a relative complex Lie subgroup of \( G \) is a complex subspace \( H \) of \( G \) which itself is a relative complex Lie group over \( Y \) with respect to the ‘restrictions’ of \( e, \mu \) and \( \iota \) to \( H \). Let \( f : X \rightarrow Y \) be a morphism of complex spaces and \( G \) a relative complex Lie group over \( Y \). Then a relative (biholomorphic) action of \( G \) on \( X \) over \( Y \) is a \( Y \)-morphism \( \sigma : G \times_Y X \rightarrow X \) satisfying the usual axioms of operation (cf. [11], Def. 0.3).

b) Let \( f : X \rightarrow Y \) and \( f' : X' \rightarrow Y \) be proper surjective flat morphisms of complex spaces. Let \( (\text{An}/Y) \) be the category of complex spaces over \( Y \). Then we define the contravariant functor \( \text{Isom}_Y(X, X') : (\text{An}/Y) \rightarrow (\text{Sets}) \) by the following formula; \( \text{Isom}_Y(X, X')(\tilde{Y}) : = \)the set of \( \tilde{Y} \)-isomorphisms \( \phi : X \times_Y \tilde{Y} \rightarrow X' \times_Y \tilde{Y} \) where \( \tilde{Y} \in (\text{An}/Y) \). Let \( D_{X \times_Y X'/Y} \rightarrow Y \) be the relative Douady space associated to the morphism \( f \times_Y f' : X \times_Y X' \rightarrow Y \). Then \( \text{Isom}_Y(X, X') \) is represented by a Zariski open subset \( \text{Isom}_Y(X, X') \) of \( D_{X \times_Y X'/Y} \) (cf. Schuster [13]). We set \( \text{Aut}_Y X : = \text{Isom}_Y(X, X) \). Then \( \text{Aut}_Y X \) has the natural structure of a complex Lie group over \( Y \), acting naturally on \( X \) over \( Y \).

When \( Y \) is a point, we write \( \text{Isom}(X, X') \) and \( \text{Aut} X \) instead of \( \text{Isom}_Y(X, X') \) and \( \text{Aut}_Y X \) respectively. \( \text{Aut} X \) is thus the automorphism group of \( X \) as a complex Lie group in the usual sense.

For any \( \tilde{Y} \in (\text{An}/Y) \) we have the natural isomorphisms \( \text{Isom}_Y(X, X') \times_Y \tilde{Y} \equiv \text{Isom}_Y(\tilde{X}, \tilde{X}') \) and \( (\text{Aut}_Y X) \times_Y \tilde{Y} \equiv \text{Aut}_Y \tilde{X} \) where \( \tilde{X} = X \times_Y \tilde{Y} \) and \( \tilde{X}' = X' \times_Y \tilde{Y} \) (cf. [8], [13]). In particular we have for each \( y \in Y \), \( \text{Isom}_y(X, X') \equiv \text{Isom}(X_y, X'_y) \) and \( (\text{Aut}_Y X)_y \equiv \text{Aut} X_y \).

c) \( \text{Aut}_Y X \) and \( \text{Aut}_Y X' \) act naturally on \( \text{Isom}_Y(X, X') \) over \( Y \) (from the right in the case of \( \text{Aut}_Y X \)). In relation to these actions we shall define the notion of principal subspace of \( \text{Isom}_Y(X, X') \) in a rather primitive way.

i) When \( Y \) is a point, then with respect to this action \( \text{Isom}(X, X') \) becomes a principal homogeneous space under either of \( \text{Aut} X \) and \( \text{Aut} X' \), i.e., for any \( h \in \text{Isom}(X, X') \) the induced maps \( \sigma_h : \text{Aut} X \rightarrow \text{Isom}(X, X') \), \( \sigma_h(\gamma) = hg \), and \( \sigma'_h : \text{Aut} X' \rightarrow \text{Isom}(X, X') \), \( \sigma'_h(\gamma') = g'h \), are isomorphic where \( g \in \text{Aut} X \) and \( g' \in \text{Aut} X' \). We shall call any isomorphism \( \text{Aut} X \rightarrow \text{Isom}(X, X') \) obtained in
this way admissible. The composition $h_* := \sigma^{-1}_h \sigma_h : \text{Aut} \ X \to \text{Aut} \ X'$ is an isomorphism of complex Lie groups, and is given by $h_*(g) = hgh^{-1}$, $g \in \text{Aut} \ X$. Hence $h(gx) = h_*(g)h(x)$ for any $g \in \text{Aut} \ X$ and $x \in X$. Now let $I \subseteq \text{Isom} \ X$ be a subspace. Then $I$ is called principal if there exist complex Lie subgroups $G \subseteq \text{Aut} \ X$ and $G' \subseteq \text{Aut} \ X'$ such that $G \cong I$ and $G' \cong I$ under some admissible isomorphisms. In this case $G$ and $G'$ are said to be associated to $I$.

ii) In the general case let $I \subseteq \text{Isom}_Y(X, X')$ be any analytic subset. Assume that $X$ and $Y$ are varieties. Then $I$ is called principal if there exist relative complex Lie subgroups $G \subseteq \text{Aut}_Y X$ and $G' \subseteq \text{Aut}_Y X'$ such that for each $y \in Y$, $I_y$ is principal with the associated subgroups $G_y \subseteq \text{Aut}_Y X_y$ and $G'_y \subseteq \text{Aut}_Y X_y$. In this case we call $G$ and $G'$ associated to $I$.

1.2. a) We use the following terminology. Let $h : Z \to Y$ be a proper morphism of complex varieties and $V \subseteq Y$ a Zariski open subset. Let $A \subseteq h^{-1}(V)$ be an analytic subset whose closure $\overline{A}$ in $Z$ is analytic. Then the essential closure $A^*$ of $A$ in $Z$ (over $Y$) is the union of those irreducible components of $\overline{A}$ which are mapped surjectively onto $Y$. Clearly, if $V' \subseteq V$ is another Zariski open subset, then the essential closure of $A \cap h^{-1}(V')$ in $Z$ coincides with $A^*$. Moreover, if $\overline{A}$ is proper over $Y$, there exists a Zariski open subset $U \subseteq Y$ such that for any $y \in U$, $A_y \subseteq A^*_y$ and $A^*_y$ is the closure of $A_y$. In fact, since $A^*$ is the closure of $A \cap A^*$, it suffices to show the assertion with $A^*$ replaced by $\overline{A}$. In this case the proof is standard.

b) Let $f : X \to Y$ and $f' : X' \to Y$ be proper surjective morphisms of complex varieties (not necessarily flat). Let $U \subseteq Y$ be a Zariski open subset over which both $f$ and $f'$ are flat [1]. Then $\text{Isom}_U(X_U, X'_U)$ is Zariski open in $D_{X \times_Y X'}$, $D_{X_U \times_Y X'_U}$.

**Definition 1.** $\text{Isom}_U^y(X, X')$ is the essential closure of $\text{Isom}_U(X_U, X'_U)$ in $D_{X \times_Y X'}$ over $Y$. We set $\text{Aut}_U^y X := \text{Isom}_U^y(X, X)$. When $Y$ is a point, we simply write $\text{Isom}_U^y(X, X')$ and $\text{Aut}_U^y X$.

**Remark 1.**
1) $\text{Isom}_U^y(X, X')$ and $\text{Aut}_U^y X$ is independent of the choice of $U$ as above and depends only on $f$ and $f'$ (cf. a)).

2) Let $\varphi : X_U \to X'_U$ be a $Y$-isomorphism represented by a unique holomorphic section $s : U \to \text{Isom}_U(X_U, X'_U)$. Then $\varphi$ extends to a bimeromorphic $Y$-map $\varphi^* : X \to X'$ if and only if $s$ extends to a meromorphic section $s^* : Y \to \text{Isom}_U^y(X, X')$.

3) The relative group multiplication $\mu_U : \text{Aut}_U X_U \times_Y \text{Aut}_U X'_U \to \text{Aut}_U X_U$ and inversion $\iota_U : \text{Aut}_U X_U \to \text{Aut}_U X_U$ of relative complex Lie groups $\text{Aut}_U X_U$ over $U$, and the natural relative action $\sigma_U : \text{Aut}_U X_U \times_Y X_U \to X_U$ of $\text{Aut}_U X_U$ on $X_U$ over $U$ extend to meromorphic maps $\mu^* : \text{Aut}_U^y X \times_Y \text{Aut}_U^y X \to \text{Aut}_U^y X$ and $\sigma^* : \text{Aut}_U^y X \times_Y X \to \text{X}$ respectively. Moreover the identity section $e_U : U \to \text{Aut}_U X_U$ extends to a meromorphic section $e^* : Y \to \text{Aut}_U^y X$.

4) Let $\nu : Y \to Y$ be any proper surjective morphism of complex varieties. Set $\tilde{X} = X \times_Y \tilde{Y}$ and $\tilde{X}' = X' \times_Y \tilde{Y}$. Then we have the natural isomorphisms
Isom§(X, X')×Y Y ≅ Isom§(X, X') and Aut§X×Y Y ≅ Aut§X.

5) Suppose that f, f' ∈ C/Y. Then for any relatively compact open subset \( V \subseteq Y \) any irreducible component of Isom§(X, X') and Aut§X is proper over Y. This follows from [4].

1.3. a) Let f : X → Y be a proper morphism of complex varieties.

**Definition 2.** Let \( G^* \subseteq \text{Aut}§X \) be an analytic subset such that any irreducible component of \( G^* \) is mapped surjectively onto Y. Then we call \( G^* \) (by abuse of language) a **relative quasi-meromorphic (Lie) subgroup** of Aut§X if there exists a Zariski open subset \( U \subseteq Y \) such that f is flat over U and that \( G_U := G^* \cap \text{Aut}_U \) is dense in \( G^* \) and is a relative Lie subgroup of Aut_U over U. If, further, \( G^* \) is proper over Y, we call \( G^* \) a **relative meromorphic (Lie) subgroup** of Aut§X.

**Remark 2.** 1) If Y reduces to a point, \( G^* \), or more properly, \( G = G^* \cap \text{Aut} X \), is called a (quasi-)meromorphic subgroup of Aut§X (cf. [3]).
2) If \( G^* \) is a relative quasi-meromorphic subgroup and \( G_U \) is as above, then the relative group law \( G_U \times_U G_U \rightarrow G_U \) and the relative action \( \sigma_U : G_U \times_U X_U \rightarrow X_U \) extend to meromorphic Y-maps \( G^* \times Y G^* \rightarrow G^* \), \( G^* \rightarrow G^* \) and \( \sigma^* : G^* \times Y X \rightarrow X \) respectively. Moreover the identity section \( e_U : U \rightarrow G_U \) extends to a meromorphic section \( e^* : Y \rightarrow G^* \). This follows from Remark 1, 3).

b) Let f : X → Y and \( f' : X' \rightarrow Y \) be proper surjective morphisms of complex varieties.

**Definition 3.** Let \( I^* \) be any analytic subspace of Isom§(X, X'). Then we say that \( I^* \) is a **quasi-meromorphic principal subspace** if there exist relative quasi-meromorphic subgroups \( G^* \subseteq \text{Aut}§X \) and \( G'^* \subseteq \text{Aut}§X' \) and a Zariski open subset \( U \subseteq Y \) over which both f and \( f' \) are flat, such that \( I_U := I^* \cap \text{Isom}(X_U, X'_U) \) is dense in \( I^* \) and \( I_U \subseteq \text{Isom}(X_U, X'_U) \) is principal with the associated relative Lie subgroups \( G_U := G^* \cap \text{Aut}_U X_U \) and \( G'_U := G'^* \cap \text{Aut}_U X'_U \) (cf. 1.1 c)). In this case we call \( G^* \) (resp. \( G'^* \)) associated to \( I^* \). \( I^* \) is called a **meromorphic principal subspace** if further it is proper over Y. In the latter case the associated \( G^* \) and \( G'^* \) are also proper over Y and hence are relative meromorphic subgroups of Aut§X and Aut§X' respectively.

**Remark 3.** 1) Let \( G^* \) be a relative meromorphic subgroup of Aut§X. Then the following conditions are equivalent. a) \( G^* \) is associated to some meromorphic principal subspace \( I^* \). b) Let \( \tilde{I}^*(X, X') := \text{Isom}(X, X') / G^* \) be a relative generic quotient of Isom§(X, X') by \( G^* \) over Y with the natural projection \( \varepsilon : \tilde{I}^*(X, X') \rightarrow Y \) (cf. Definition 5 and Theorem 1 below). Then \( \varepsilon \) admits a meromorphic section \( s : Y \rightarrow \tilde{I}^*(X, X') \). Moreover in this case \( I^* \) is given by \( I^* = \pi^{-1}(s(Y)) \) and \( G^* \) is given by the union of those irreducible components of \( \beta_1(I^* \cap (\text{Aut}§X \times Y I^* \times Y I^*)) \) which are mapped surjectively onto Y, where \( \pi : \text{Isom}(X, X') \)}
→Φ\(^*(X, X')\) is the natural meromorphic projection, \(\Gamma \subseteq \text{Aut}_X \times \text{Isom}_X(X, X')\) \(\times \text{Isom}_X(X, X')\) is the closure of the graph of the action of \(\text{Aut}_X\) on \(\text{Isom}_X(X, X')\) and \(p_1\) is the projection to the first factor \(\text{Aut}_X\). In particular \(\Gamma^*\) and \(G^*\) determine each other uniquely. The analogous fact holds of course for a meromorphic subgroup \(G^* \subseteq \text{Aut}_X\).

c) We consider the special case of b) where \(X'=Y \times F\) for a compact complex variety \(F\) and \(f' : Y \times F \to Y\) is the natural projection. Then we have the natural isomorphisms

\[
\text{Aut}_Y X' \cong Y \times \text{Aut} F \quad \text{and} \quad \text{Aut}_F X' \cong Y \times \text{Aut}^* F.
\]

Definition 4. Let \(I^*\) be a (quasi-)meromorphic principal subspace of \(\text{Isom}_X(X, X')\), and \(G^* \subseteq \text{Aut}_X X\) and \(G^* \subseteq \text{Aut}_X X'\) the associated relative (quasi-)meromorphic subgroups. Then we call \(I^*\) admissible (with the associated meromorphic subgroup \(H^*\)) if \(G^*\) is of the form \(G^* = Y \times H^*\) for some (quasi-)meromorphic subgroup \(H^* \subseteq \text{Aut}_F\).

Suppose that \(I^*\) is admissible as above and set \(H = H^* \cap \text{Aut} F\). Take a Zariski open subset \(U \subseteq Y\) as in Definition 3. Then it is immediate to see that \(f_U : X_U \to U\) is a holomorphic fiber bundle with structure group \(H\). In this case we say that \(f\) is a holomorphic fiber bundle over \(U\) which is (quasi-)meromorphic with respect to \(f\) and with (quasi-)meromorphic structure group \(H\). We note that in this case the natural map \(I^*_U \to U\) is the principal bundle associated to \(f_U\).

§ 2. Relative Generic Quotients and Related Results

2.1. We generalize the generic quotient theorem by a meromorphic group in [3] to a relative case.

Theorem 1. Let \(f : X \to Y\) be a proper surjective morphism of complex varieties. Let \(G^* \subseteq \text{Aut}_X X\) be any relative meromorphic subgroup over \(Y\). Then there exists a unique subspace \(\bar{X} \subseteq D_{X/Y}\) having the following properties: Let \(\rho : Z \to \bar{X}\) be the universal family \(\rho_{X/Y} : Z_{X/Y} \to D_{X/Y}\) restricted to \(\bar{X}\), i.e., \(Z = Z_{X/Y} \times \rho_{X/Y}\). Then: 1) the natural \(Y\)-morphism \(\pi : Z \to X\) is bimeromorphic, and 2) there exists a Zariski open subset \(V \subseteq \bar{X}\) such that for any \(v \in V\), the corresponding subspace \(Z_v \subseteq X_v\) is a closure of an orbit \(Z_v\) of the group \(G_v\) acting on \(X_v\), where \(y = f(v)\) (\(f : \bar{X} \to Y\) being the natural map) and \(G_y = G^*_v \cap \text{Aut} X_v\).

Proof. Define a meromorphic \(Y\)-map \(\Phi : G^* \times X \to X \times \bar{X}\) by \(\Phi(g, x) = (a^*(g, x), x)\). Let \(R := \Phi(G^* \times X) \subseteq X \times \bar{X}\). Let \(p : R \to X\) be induced by the second projection \(p_\pi : X \times \bar{X} \to X\);

2) the relative Douady space associated to \(f\).
Let \( T : X \rightarrow DXIY \) be the universal meromorphic F-map associated to this diagram where \( r \) is holomorphic exactly on the Zariski open subset over which \( p \) is flat (cf. [2], Lemma 5.1). Let \( \bar{X} \) be the image of \( \tau \). Let \( \bar{\tau} : X \rightarrow \bar{X} \) be the resulting meromorphic Y-map. We claim that \( \bar{X} \) has the desired property. Take a Zariski open subset \( U \subseteq X \) such that both \( f \) and \( G* \rightarrow Y \) are flat over \( U \), and that for each \( y \in U \), \( G_y \) is dense in \( G_y^x \) (cf. 1.2 a)). Now we consider \( \Phi \) as a meromorphic map over \( X \) where \( X = X \) is over \( X \) by \( p \). Take a Zariski open subset \( W \subseteq X \) such that for each \( x \in W \), if we set \( y = f(x) \), then \( y \in U \), and with respect to the natural identification \( (G^* \times X) \) \( \rightarrow X \) and \( (X \times X) \) \( \rightarrow X \), \( \Phi \) induces a meromorphic map \( \Phi_x : G^*_x \rightarrow X_y \). Then we have for \( x \in W \), \( R_x = \Phi_x(G_x^*) = \Phi_x(G^*_x) = G^*_x \) as a subspace of \( X_y \), where \( G_x \) is the orbit of \( x \) under \( G_y \) and \( G^*_x \) its closure. In particular for any \( \bar{w} \in \tau(W) \), \( Z_{\bar{w}} \) is a closure of an orbit of \( G_y \). Since \( \tau(W) \) contains a nonempty Zariski open subset, say \( V \), 2) follows. It remains to show that \( \pi \) is bimeromorphic. Restricting \( W \) if necessary we may assume that \( p_y : R_y \rightarrow W \) is flat [1] so that for any \( x \in W \), we have \( \dim(G_{f(x)}(x)) = m \) for a fixed integer \( m \geq 0 \). \( \bar{W} \) is a union of 'regular orbits'. cf. [11]) Then just as in the proof of the absolute case (cf. Theorem 4.1 of [3]) we can show that \( \pi \) is isomorphic on \( \rho^{-1}(V) \cap \pi^{-1}(W) \). Thus \( \pi \) is bimeromorphic.

It remains to show the uniqueness of \( \bar{X} \). In fact, from 1) and 2) alone we deduce easily the following: 1) There exists a Zariski open subset \( W_1 \subseteq X \) such that a) for every \( x \in W_1 \) with \( f(x) = y \), the point \( d(x) \in D_{X \rightarrow Y} \) corresponding to the subspace \( G^*_x \) belongs to \( \bar{X} \), and b) the set \( \{d(x); x \in W_1\} \) forms a Zariski open subset of \( \bar{X} \). Uniqueness clearly follows from this.

**Definition 5.** We call the commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{q} & \bar{X} \\
\downarrow{f} & \nearrow{\bar{f}} \\
Y & & \\
\end{array}
\]

or simply, the meromorphic Y-map \( q : X \rightarrow \bar{X} \), or \( \bar{X} \) itself, the *relative generic quotient* of \( X \) by \( G^* \) over \( Y \). We often denote \( \bar{X} \) symbolically by \( X/G^* \).

**Proposition 1.** Let \( f : X \rightarrow Y \) and \( G^* \subseteq \text{Aut}_X^* \) be as in Theorem 1.

1) Let \( \tilde{Y} \rightarrow Y \) be a surjective morphism of complex varieties. Let \( \tilde{X} = X \times_Y \tilde{Y} \) and \( \tilde{G}^* = G^* \times_Y \tilde{Y} \). Then \( \tilde{G}^* \) is a relative meromorphic subgroup of \( \text{Aut}_Y \tilde{X} \cong \text{Aut}_Y^* \times_Y \tilde{Y} \) and the relative generic quotient \( \tilde{X}/\tilde{G}^* \) of \( \tilde{X} \) by \( \tilde{G}^* \) over \( \tilde{Y} \) is isomorphic over \( \tilde{Y} \) to the pull-back \( (X/G^*) \times_Y \tilde{Y} \).

2) Suppose that \( Y \) is a complex variety over another complex variety \( T \) with a surjective morphism \( h : Y \rightarrow T \). Then there exists a Zariski open subset \( U \subseteq T \)
such that for any \( t \in U \) i) \( G^*_t \) is a relative meromorphic subgroup of \( \text{Aut}^*_Y X_t \) over \( Y \), and ii) \( (X/G^*)_t = X_t/G^*_t \) as a subspace of \( (D_X/X)_t = D_{X_t/Y} \), where we consider any complex space over \( Y \) naturally as a complex space over \( T \) via \( h \). In particular if \( Y = T \) then for each \( y \in U \subseteq Y \), \( (X/G^*)_y \) is the generic quotient of \( X_y \) by \( G^*_y \). As a special case of this, if \( X/G^* \to Y \) is bimeromorphic, then there exists a Zariski open subset \( W \subseteq X \) with \( f(W) \subseteq U \) such that if \( y \in U \) then \( G_y \) acts almost homogeneously on \( X_y \) and its unique Zariski open orbit coincides with \( W_y \).

3) If \( G^*_y = G_y \) is a complex torus for \( y \in U \), and \( G_y \) acts freely on \( X_y \) then \( q_y : X_y \to X_y/G_y \) is a holomorphic fiber bundle and hence \( q_U : X_U \to (X/G^*)_U \) is holomorphic and smooth.

**Proof.** In view of the uniqueness assertion of Theorem 1 the verification of 1) is straightforward. For the first assertion of 2) it suffices to take \( U \) in such a way that for any \( t \in U \), \( V_t = X_t \) is bimeromorphic and \( V_t = \overline{X_t} \) where \( \overline{V_t} \) is the closure of \( V_t \) in \( \overline{X_t} \) in the notation of Theorem 1. Here, restricting \( U \) if necessary, we may assume further that \( G_U \) is smooth over \( U \). Then, when \( X/G^* \to Y \) is bimeromorphic, if we set \( A = \{ x \in X_U; \dim G_U(x) > t - r \} \) \((t = \dim G_U/U, r = \dim f) \) where \( G_U(x) \) is the stabilizer of \( G_{U,f(x)} \) at \( x \in X_{f(x)} \), then \( W : = X_U - A \) is easily seen to satisfy the above condition. In 3) that \( q \) is a fiber bundle is due to Holmann [10], § 5. Since \( \dim X_Z \) is constant on \( \overline{X_U} \), from this follows the last assertion.

2.2. Let \( f : X \to Y \) and \( f' : X' \to Y \) be proper surjective morphisms of complex varieties. Let \( U \subseteq Y \) be a Zariski open subset over which both \( f \) and \( f' \) are flat. Then by the universality of the relative Douady space we have the natural transformation of functors \( \phi : \text{Isom}_U(X_U, X_U') \to \text{Isom}_U(D_{X_U/Y}, D_{X'_U/Y}). \) Let \( I^* \subseteq \text{Isom}^*(X, X') \) be an analytic subset such that \( I_U = I^* \cap \text{Isom}_U(X_U, X_U') \) is dense in \( I^* \). Let \( B \subseteq D_{X/Y} \) and \( B' \subseteq D_{X'/Y} \) be analytic subspaces which are proper over \( Y \) and are flat over \( U \). Now we assume the following condition; \((*)\) the image of \( I_U \subseteq \text{Isom}_U(X_U, X_U') \) by \( \phi \) is contained in the subfunctor \( \text{Isom}^*_U(D_{X_U/Y}, B_U), (D_{X'_U/Y}, B'_U)) \) (cf. 3.1 a) below for the notation) where we identify \( \text{Isom}_U(X_U, X_U') \) with the functor it represents. Then composed with the natural projection \( \text{Isom}_U(D_{X_U/Y}, B_U), (D_{X'_U/Y}, B'_U)) \to \text{Isom}_U(B_U, B'_U) \) we get a \( U \)-map \( \phi : I_U \to \text{Isom}_U(B_U, B'_U) \). It is immediate to see that \( \phi \) is indeed a morphism of complex spaces and that \( \phi \) extends to a meromorphic \( Y \)-map \( \phi^* : I^* \to \text{Isom}^*_U(B, B') \). The condition \((*)\) is fulfilled if for each \( y \in U \) and for each \( h \in I_{U,y} \subseteq \text{Isom}(X_y, X'_y) \) we have \( D_y h(B_y) = B'_y \) where \( D_y h \in \text{Isom}(D_{X_y}, D_{X'_y}) \) is the element canonically induced by \( h \).

**Theorem 2.** Let \( I^* \subseteq \text{Isom}^*_U(X, X') \) be a meromorphic principal subspace with the associated relative meromorphic subgroups \( G^* \subseteq \text{Aut}^*_X \) and \( G'^* \subseteq \text{Aut}^*_X \). Let \( \overline{X} = X/G^* \) and \( \overline{X'} = X'/G'^* \) be the respective relative generic quotients over \( Y \). Then there exists a natural bimeromorphic \( Y \)-map \( \overline{X} \to \overline{X'} \) which is isomorphic
over some Zariski open subset of $Y$.

Proof. Take Zariski open subsets $V \subseteq X$ and $V' \subseteq X'$ as in 2) of Theorem 1. Restricting $V$ and $V'$ we may assume that the following conditions are satisfied: 1) $f(V) = f'(V')$, and if we denote this set by $U$, then $U$ is nonsingular and Zariski open in $Y$, where $f: X \to Y$ and $f': X' \to Y$ are the natural morphisms, 2) both $f$ and $f'$ are flat over $U$, and 3) for each $y \in U$, a) $G_x^y$ is a meromorphic subgroup of $\text{Aut}^*_X y$, $X_y$ is the generic quotient of $X_y$ by $G_x^y$, and the similar condition for $G_x'^y$ and $X'_y$ is true (cf. Proposition 1), b) $V_y$ is dense in $X_y$ and c) the induced map $\pi_y: Z_y \to X_y$ is bimeromorphic where $Z$ is as in Theorem 1. Let $I_U = \text{Isom}_U(X_U, X_U') \cap I^*$. Take any $y \in U$ and any $h = h_y \in I_y$ such that $h \in I_U$. We shall first show that $D_y h((X)_y) = (X'_y)$ where $D_y h$ is defined just before the theorem. Let $X'_y = D_y h(X_y) \subseteq X'_y$ and $V'_y = D_y h(V_y) \subseteq X'_y$. Then $h$ induces an isomorphism of the following diagrams

$$X_y \xrightarrow{\pi_y} \frac{Z_y}{Z'_y} \quad X'_y \xrightarrow{\pi'_y} \frac{Z'_y}{Z'^*_y} = \frac{Z_{X/y} \times_{D_X/y} X'_y}{Z'^*_y}.$$

By the uniqueness of the generic quotient in Theorem 1 it suffices to show that $X'_y$ satisfies the conditions of that theorem for $f'_y$ and $G^*_y$. Since $\pi'_y$ is bimeromorphic as well as $\pi_y$, 1) is satisfied. We set $G_y = G^*_y \cap \text{Aut} X_y$ and $G'_y = G^*_y \cap \text{Aut} X'_y$. For 2) it suffices to show that for any point $v' \in V'_y$, $Z'_y$ is a closure of an orbit of $G'_y$ when $Z'_y$ is considered as a subspace of $X'_y$ via $\pi'_y$. In fact, take $v \in V$ with $D_y h(v) = v'$. Then $Z'_y = h(Z_v)$, which is the closure of $h(Z_v)$ where $Z_v$ is a $G_y$-orbit on $X_y$. Then, since $h_y$ is $(G_y, G'_y)$-equivariant with respect to the homomorphism $h_y: G_y \to G'_y$ (cf. 1.1 c)), $h(Z'_y)$ is an orbit of $G'_y$ as was desired.

Thus $D_y h$ induces an element of $\text{Isom}((X)_y, (X')_y)$ which we shall denote by the same letter $D_y h$. Hence by the remark just before the theorem we have obtained a $U$-morphism $\phi: I_U \to \text{Isom}_U((X)_y, (X')_y)$ which extends to a meromorphic $Y$-map $\phi^*: I^* \to \text{Isom}^*_Y((X), (X'))$. Next we show that $D_y h(y) = D_y h'(y)$ for any $y \in U$. It suffices to show that $D_y h(v) = D_y h'(v)$ for any $v \in V_y$ since $V_y$ is dense in $X_y$. In fact, since $g(Z_v) = Z_v$ for any $g \in G_y$ and $h'^{-1} h \in G y$, $h(Z_v) = h'^{-1} h h(Z_v) = h'(Z_v)$, or equivalently, $D_y h(v) = D_y h'(v)$ as was desired. Since $I_U \to U$ is surjective it follows that $\phi(I_U) \subseteq \text{Isom}_U((X)_y, (X')_y)$ gives a holomorphic section to $\text{Isom}_U((X)_y, (X')_y) \to U$, $U$ being nonsingular, and hence, $\phi^*(I^*) \subseteq \text{Isom}^*_Y((X), (X'))$ a meromorphic section to $\text{Isom}^*_Y((X), (X')) \to Y$. Hence by Remark 1, 2) $X$ and $X'$ are bimeromorphic over $Y$ by a bimeromorphic map which is isomorphic over $U$.

$q.e.d.$

In Theorem 2 assume that there exists a $Y$-isomorphism $\phi: X' \to Y \times F$ for some compact complex variety $F$. Let $H^* = \text{Aut}^* F$ be a meromorphic subgroup. Then we say that $\phi$ is admissible with respect to $(I^*, H^*)$ if $\phi$ induces an
isomorphism $G^*\cong Y\times H^*$. Then, if $\phi$ and $\phi'$ are $Y$-isomorphisms $X'\cong Y\times F$ which are admissible with respect to $(I^*, H^*)$, then $\phi'^{-1}\phi$ induces a $Y$-automorphism of $\rho_1: Y\times F\to Y$, i.e., gives a holomorphic map $Y\to \text{Aut} F$, whose image is contained in $H$ where $H=H^*\cap \text{Aut} F$. This implies that the set of admissible $Y$-isomorphisms is naturally a principal homogeneous space under the group $\text{Hol}(Y, H)$, the space of holomorphic maps of $Y$ to $H$. From this observation we get the following:

**Lemma 1.** Suppose that there exists a $Y$-isomorphism $\phi: X'\to Y\times F$ which is admissible with respect to $(I^*, H^*)$, so that we have the natural isomorphism $X'/G^*\cong Y\times (F/H^*)$. Then the composite meromorphic map $X\to X/G^*\cong X'/G^*\cong Y\times (F/H^*)\to F/H^*$ is independent of the choice of the admissible isomorphism $\phi$.

**Definition 6.** We call the meromorphic map $X\to \tilde{F}=F/H^*$ defined in the lemma, or any meromorphic map which is bimeromorphic to it, a canonical meromorphic map associated to $f$ and to $H^*$.

Clearly we have $\dim \tilde{F}=\dim \rho$ where $\rho: X/G^*\to Y$ is the natural map.

§ 3. Examples of Relative Quasi-Meromorphic Subgroups

**3.1.** $\text{Isom}_Y((X, A), (X', A'))$ and $\text{Aut}_Y(X, A)$. Let $f: X\to Y$ and $f': X'\to Y$ be proper morphisms of complex varieties. Let $A=(A_1, \ldots, A_m)$ and $A'=(A'_1, \ldots, A'_m)$ be sequences of analytic subspaces of $X$ and $X'$ respectively.

a) Suppose first that $f$ and $f'$ are flat and that $A_a$ are all flat over $Y$ with respect to $f'$. Then we define a subfunctor $\text{Isom}_f((X, A), (X', A')): (\text{An}/Y)\to (\text{Sets})$ of $\text{Isom}_Y(X, X')$ as follows; $\text{Isom}_f((X, A), (X', A'))(Y)=\{\phi\in \text{Isom}_Y(X, X')(Y); \phi$ induces isomorphisms of $A_a\times_Y \tilde{Y}$ and $A'_a\times_Y \tilde{Y}$ for all $a}\}$.

**Lemma 2.** $\text{Isom}_f((X, A), (X', A'))$ is represented by a unique analytic subspace $\text{Isom}_Y((X, A), (X', A'))$ of $\text{Isom}_Y(X, X')$.

**Proof.** Let $I=\text{Isom}_f(X, X')$ and $\xi: X\times_Y I\to X'\times_Y I$ the universal $I$-isomorphism. Let $\overline{A}_a, I:=\xi(A_a\times_Y I)$. Then by [12] Prop. 1, there exists a unique analytic subspace $T\subseteq I$ such that for any morphism $u: T'\to I$ of complex spaces $\overline{A}_a, I\times_Y T'=A'_a, I\times_Y T'$, where $A'_a, I:=A'_a\times_Y I$, as a subspace of $X'\times_Y I$ if and only if $u$ factors through $T$. (In fact, apply [12] Prop. 1 to the morphism $X'\times_Y I\to I$ and to the coherent analytic sheaves $\mathcal{E}:=\mathcal{O}_{X', I}$ and $\mathcal{F}:=\mathcal{O}_{X', I}$.) Then it is easy to see that $T$ represents the functor $\text{Isom}_f((X, A), (X', A'))$.

We then set $\text{Aut}_f(X, A)=\text{Isom}_f((X, A), (X, A))$. $\text{Aut}_f(X, A)$ is a relative complex Lie subgroup of $\text{Aut}_f X$ over $Y$.

b) In the general case, let $U\subseteq Y$ be a Zariski open subset such that $X$, $X'$, and $A'_a$ are all flat over $U$ [1].
Lemma 3. The closure $I^-$ of $\text{Isom}_0((X_U, A_U), (X_U', A_U'))$ in $D_{x \times y}X'$ is analytic, where $A_U = (A_{1,U}, \ldots, A_{m,U})$ and $A_{U'} = (A_{1,U'}, \ldots, A_{m,U'})$.

Proof. Take a proper modification $\sigma : \tilde{Y} \to Y$ such that $\sigma$ gives an isomorphism of $\sigma^{-1}(U)$ and $U$ and that the strict transforms $\tilde{X}$ and $\tilde{A}_a$ (resp. $\tilde{X}'$ and $\tilde{A}_a'$) of $X$ and $A_a$ in $X \times \tilde{Y}$ (resp. of $X'$ and $A_a'$ in $X' \times \tilde{Y}$) respectively are all flat over $\tilde{Y}$ [9]. Then by Lemma 2 $I = \text{Isom}_Y((\tilde{X}, \tilde{A}), (\tilde{X}', \tilde{A}'))$, $\tilde{A} = (\tilde{A}_1, \ldots, \tilde{A}_m)$, $\tilde{A}' = (\tilde{A}_1', \ldots, \tilde{A}_m')$, is realized as an analytic subspace of $\text{Isom}_Y(X, X')$. Let $I$ be the union of those irreducible components of $I$ whose images in $\tilde{Y}$ intersect with $\sigma^{-1}(U)$. Then the image of $I$ in $D_{x \times y}X' \times Y$ by the natural proper morphism $\text{Isom}_Y((X, A), (X', A'))$ is represented by a unique analytic subspace $I^-$ of $I$ which is a union of connected components.

Definition 7. $\text{Isom}_Y((X, A), (X', A'))$ is the essential closure of $\text{Isom}_X((X, A), (X', A'))$ in $D_{x \times y}X' \times Y$. We set $\text{Aut}_Y(X, A) = \text{Isom}_Y((X, A), (X, A))$. When $Y$ is a point, we write $\text{Aut}_X(X, A)$ for $\text{Aut}_Y(X, A)$.

Remark 4. $\text{Aut}_Y(X, A)$ is a relative quasi-meromorphic subgroup of $\text{Aut}_X(X, A)$ and $I^* = \text{Isom}_Y((X, A), (X', A'))$ is a quasi-meromorphic principal subspace with the associated quasi-meromorphic subgroups $\text{Aut}_Y(X, A)$ and $\text{Aut}_Y(X', A')$. This follows immediately from the definitions.

c) In b) assume further that $X'$ is of form $X' = Y \times F$ for some compact complex variety $F$ and $f' : X' \to Y$ is the natural projection as in 1.3 c). Suppose that there exists a sequence $B = (B_1, \ldots, B_\eta)$ of subspaces of $F$ such that $A'_a = Y \times B_a \subseteq X'$. Then $I^* = \text{Isom}_Y(F, (X, A), (X', A'))$ is admissible, if it is not empty (Definition 4). In general, let $I^* \subseteq \text{Isom}_Y(X, X')$ be a meromorphic principal subspace. Suppose that $I^*$ is admissible with the associated meromorphic subgroup $H^* \subseteq \text{Aut}_X(F)$ and that $I^* \subseteq \text{Isom}_Y((X, A), (X', A'))$. Then $f_{X'A} : (X, A) \to Y$ is a holomorphic fiber bundle over $U$ in the sense that for each $y \in U$ there exist a neighborhood $y \in V$ and a trivialization $X_y \cong V \times F$ which sends $A_a$ onto $V \times B_a$ isomorphically. In this case we say that $f_{X'A}$ is a holomorphic fiber bundle over $U$ which is meromorphic with respect to $f$ (and with meromorphic structure group $H$).

3.2. $\text{Isom}_Y(X, X')_{o, w}$ and $\text{Aut}_Y(X)$. 

a) Let $f : X \to Y$ and $f' : X' \to Y$ be proper smooth morphisms of complex varieties. Let $\omega \in \Gamma(Y, R^nf_*R)$ and $\omega' \in \Gamma(Y, R^nf'_*R)$ be fixed elements. Then we define a subfunctor $\text{Isom}_Y((X, X')_{o, w})$ of $\text{Isom}_Y((X, X')_{o, w})$ as follows, $\text{Isom}_Y(X, X')_{o, w}(\tilde{Y}) = \{ \varphi \in \text{Isom}_Y(X, X')(\tilde{Y}) \mid \varphi^*\omega = \varphi^*\omega' \}$ where $\omega$ (resp. $\omega'$) is the pull-back of $\omega$ (resp. $\omega'$) to $X \times Y$ (resp. $X' \times Y$).

Lemma 4. $\text{Isom}_Y(X, X')_{o, w}$ is represented by a unique analytic subspace $\text{Isom}_Y(X, X')_{o, w}$ of $I = \text{Isom}_Y(X, X')$ which is a union of connected components.

Proof. Let $\xi : X \times Y \to X' \times Y$ be the universal $I$-isomorphism. Let $y \in Y$ be any point and $I_{y, \tau}$ be any connected component of $I_y$. For $t \in I_{y, \tau}$ let $\xi_t : X_t \to X'_t$. 


be the isomorphism induced by \( \xi \). Then \( \xi^* \omega_{t,q} = \omega_{t,q} \) for some \( t \in I_{y,T} \), then \( \xi^* \omega_{t,q} = \omega_{t,q} \) for all \( t \in I_{y,T} \). From this the assertion follows readily.

We set \( \text{Aut}_Y X_m = \text{Isom}_Y (X, X)_{m.m} \).

b) In general let \( g : Z \rightarrow Y \) be any proper smooth morphism of complex varieties. Then any real closed \( C^\infty \) 2-form \( \alpha \) on \( Z \) determines a unique section \( \bar{\alpha} \in \Gamma(Y, R^3 g_* R) \) such that the class of \( \alpha \) equals \( \bar{\alpha} \) in \( H^3(Z_y, R) \).

**Proposition 2.** Let \( f, f', \omega, \omega' \) be as in a). Suppose that there exists a real closed \( C^\infty \) 2-form \( \beta \) (resp. \( \beta' \)) on \( X \) (resp. \( X' \)) with \( \beta = \omega \) (resp. \( \beta' = \omega' \)), which restricts to a Kähler form on each fiber of \( f \) (resp. \( f' \)). Then the closure \( \bar{I} \) of \( \text{Isom}_R (X, X')_{m,m} \) in \( D_{X \times Y} \) is proper over \( Y \).

For the proof we need a general result. Let \( f : X \rightarrow Y \) be a smooth morphism of complex varieties and \( \beta \) a \( C^\infty \) 2-form on \( X \) which restricts to a positive \((1,1)\)-form on each fiber of \( f \). Let \( D_{X/Y} \) be the relative Douady space of \( X \) over \( Y \) and \( A \subseteq D_{X/Y} \) an analytic subset. Let \( \delta : A \rightarrow Y \) be the natural morphism. Then we say that \( A \) is bounded with respect to \( \beta \) if there exist a dense Zariski open subset \( V \subseteq A \), a positive constant \( R \) and an integer \( q \geq 0 \) such that for any \( d \in V \) the corresponding subspace \( Z_d \subseteq X_{\delta(d)} \) is reduced and is of pure dimension \( q \) and that if \( \text{vol} (Z_d) := \int_{Z_d} \beta_y \) is the volume of \( Z_d \) with respect to \( \beta_{\delta(d)} \) (the restriction of \( \beta \) to \( X_{\delta(d)} \)), then \( \text{vol} (Z_d) \leq R \).

**Proposition 3.** Let \( A \subseteq D_{X/Y} \) be as above. Suppose that for any relatively compact open subset \( U \subseteq Y \), the restriction \( A_U = A \cap D_{X/U} \) of \( A \) over \( U \) is bounded with respect to \( \beta_U \). Then \( A \) is proper over \( Y \).

**Proof.** Follows immediately from Propositions 4.1 and 3.4 of [2]. (The proof there clearly applies also to \( \beta \) as above.)

**Proof of Proposition 2.** In view of a) it is clear that \( \bar{I} \) is a union of irreducible components of \( D_{X \times Y} \). To show the properness we shall apply Proposition 3 to \( A = \bar{I} \), by considering \( f \times f' : X \times X' \rightarrow Y \) and \( C^\infty \) 2-form \( \beta_\alpha : = \bar{\beta} + \beta' \) on \( X \times X' \) instead of \( f \) and \( \beta \) in the proposition respectively. Here \( \bar{\beta} \) and \( \beta' \) are the natural pull-backs to \( X \times X' \) of \( \beta \) and \( \beta' \) respectively. Then we have to show that on any relatively compact open subset \( Y \), \( \bar{I} \) is bounded with respect to \( \beta_\alpha \). Let \( V := \text{Isom} (X, X')_{\omega, \omega'} \subseteq \bar{I} \). Then for any \( d \in V \) the associated subspace \( Z_d \subseteq X_{\delta(d)} \), \( y = \delta(d) \), equals the graph \( \Gamma_h \) of the isomorphism \( h = h_d : X_y \rightarrow X'_\beta \) corresponding to \( d \), where \( \delta : I \rightarrow Y \) is the natural morphism. Hence \( Z_d \approx X_y \). Moreover, since \( h_d \omega_y = \omega_y \), we calculate easily that

\[
\text{vol} (Z_d) = \int_{Z_d} \beta_y = (q+1) \int_X \beta_y
\]

where \( q = \dim X_y \) (cf. the proof of Theorem 4.8 in [3]). Thus \( \text{vol} (Z_d) \) depends only on \( y = \delta(d) \) and is a continuous function of \( y \). Hence it is bounded on any
relatively compact open subset of $Y$ as was desired. q.e.d.

c) In general let $g : Z \to Y$ be a proper morphism of complex varieties. Then we call $\alpha \in \Gamma(Y, R^f_*R)$ a relative Kähler class if the restriction $\alpha_y \in H^\cdot(X_y, R)$ of $\alpha_y$ to each $X_y$ is a Kähler class, i.e., represented by a Kähler form. Using Proposition 2 we have shown in [6] the following:

**Proposition 4.** Let $f, f', \omega, \omega'$ be as in a). Suppose that $\omega$ and $\omega'$ are relative Kähler classes. Then $\tilde{I}$ is proper over $Y$.

**Proof.** See [6], Proposition 4.

d) Let $f : X \to Y$ and $f' : X' \to Y$ be generically smooth proper morphisms of complex varieties. Let $U \subseteq Y$ be a Zariski open subset over which both $f$ and $f'$ are smooth. Let $\omega \in \Gamma(Y, R^f_*R) \otimes \omega$ and $\omega' \in \Gamma(Y, R^{f'}_*R) \otimes \omega'$ be fixed elements.

**Definition 8.** $\text{Isom}^\#(X, X', \omega, \omega')$ is the essential closure of $\text{Isom}_{U}(X_U, X'_U)_{\omega_U, \omega'_U}$ in $\text{Isom}^\#(X, X')$. We set $\text{Aut}^\#X_{\omega} = \text{Isom}^\#(X, X)_{\omega, \omega}$.

**Remark 5.** $\text{Isom}^\#(X, X')_{\omega, \omega'}$ and $\text{Aut}^\#X_{\omega}$ are unions of irreducible components of $\text{Isom}^\#(X, X')$ and $\text{Aut}^\#X$ respectively (cf. Lemma 4).

**Proposition 5.** Suppose that $\omega_U, \omega'_U$ are relative Kähler classes, and that $f, f' \subset C/Y$. Then $\text{Isom}_{U}(X, X')_{\omega, \omega'}$ is proper over $Y$. Thus $\text{Aut}^\#X_{\omega}$ and $\text{Aut}^\#X'_{\omega'}$ are meromorphic subgroups of $\text{Aut}^\#X$ and $\text{Aut}^\#X'$ respectively and $\text{Isom}^\#(X, X')_{\omega, \omega'}$ is a meromorphic principal subspace with the associated meromorphic subgroups $\text{Aut}^\#X_{\omega}$ and $\text{Aut}^\#X'_{\omega'}$.

**Proof.** By Proposition 4 $\text{Isom}_{U}(X_U, X'_U)_{\omega_U, \omega'_U}$ has only finitely many irreducible components, say $I_{1, U}, \ldots, I_{k, U}$, which are mapped surjectively onto $U$. Then $\text{Isom}^\#(X, X')_{\omega, \omega'}$ is the union of the closures $I_j$ of $I_{j, U}$. Since $f, f' \subset C/Y$, $f \times f' \subset C/Y$, and hence each $I_j$ are proper over $Y$ by [4]. Thus the first assertion follows. The second assertion then follows readily from the definition of these spaces.

3.3. a) Let $f : X \to Y$, $f' : X' \to Y$, $U \subseteq Y$, $\omega$ and $\omega'$ be as in Proposition 5. Let $A = (A_1, \ldots, A_m)$, $A' = (A'_1, \ldots, A'_m)$ be as in 3.1.

**Definition 9.** We set

$$\text{Isom}^\#((X, A), (X', A'))_{\omega, \omega'} := \text{Isom}^\#(X, X')_{\omega, \omega'} \cap \text{Isom}^\#((X, A), (X', A'))$$

and

$$\text{Aut}^\#(X, A)_{\omega} := \text{Aut}^\#X_{\omega} \cap \text{Aut}^\#(X, A).$$

**Remark 6.** 1) $\text{Isom}^\#((X, A), (X', A'))_{\omega, \omega'}$ is a meromorphic principal subspace with the associated meromorphic subgroups $\text{Aut}^\#(X, A)_{\omega}$ and $\text{Aut}^\#(X', A')_{\omega'}$.

2) There exists a Zariski open subset $U \subseteq Y$ such that

$$(\text{Isom}^\#((X, A), (X', A'))_{\omega, \omega'})_{\omega} = \text{Isom}^\#((X_y, A_y), (X'_y, A'_y))_{\omega, \omega'}$$
for any \( y \in U \).

3) Let \( \nu: \tilde{Y} \to Y \) be a surjective morphism of complex varieties. Let \( \tilde{X} = X \times_Y \tilde{Y} \) and \( \tilde{A} = (A_1 \times_Y \tilde{Y}, \ldots, A_m \times_Y \tilde{Y}) \). Let \( \omega \) be the pull-back of \( \omega \) to \( \tilde{X} \). Then \( \text{Aut}^*_Y(X, A)_{\omega} \times_Y \tilde{Y} \cong \text{Aut}^*_Y(\tilde{X}, \tilde{A})_{\omega} \) with respect to the natural isomorphism \( \text{Aut}^*_Y(X, A)_{\omega} \times_Y \tilde{Y} \cong \text{Aut}^*_Y(\tilde{X}, \tilde{A})_{\omega} \).

In fact, since \( \text{Isom}^*_Y(X, X')_{\omega, \omega} \) is a union of irreducible components (Remark 5) it follows that \( \text{Isom}^*_Y(X, X')_{\omega, \omega} \) is the essential closure of \( \text{Isom}_U(X_U, X'_U)_{\omega_U, \omega'_U} \). From this together with Remark 4 and Proposition 5, 1) follows. For 3) it suffices to see that \( \text{Aut}^*_Y(X, A)_{\omega} \times_Y \tilde{Y} \cong \text{Aut}^*_Y(\tilde{X}, \tilde{A}) \) and \( \text{Aut}^*_Y(X, A)_{\omega} \times_Y \tilde{Y} \cong \text{Aut}^*_Y(\tilde{X}, \tilde{A})_{\omega} \). Since \( \nu \) is surjective, this follows from the isomorphisms \( \text{Aut}^*_U(X_U, A_U)_{\omega_U} \times_U \tilde{U} \cong \text{Aut}^*_U(\tilde{X}_U, \tilde{A}_U)_{\omega_U} \) and \( \text{Aut}^*_U(X_U, A_U)_{\omega_U} \times_U \tilde{U} \cong \text{Aut}^*_U(\tilde{X}_U, \tilde{A}_U)_{\omega_U} \) where \( \tilde{U} = \nu^{-1}(U) \).

b) Consider the special case where \( X' = Y \times F \) for some compact complex variety \( F \) and \( f: X' \to Y \) is the natural projection. Let \( B = (B_1, \ldots, B_m) \) be a sequence of subspaces of \( F \) as in 3.1 c). Suppose that \( CD' \) is of the form \( CD' = p \circ \sigma \) for some Kahler class \( \sigma \) on \( F \) where \( p: X' \to F \) is the natural projection. Then:

**Proposition 6.** If \( \text{Isom}^*_Y((X, A), (X', A'))_{\omega, \omega} \neq \emptyset \), then \( f \circ A \) is a holomorphic fiber bundle over \( U \) which is meromorphic with respect to \( f \) and with meromorphic structure group \( \text{Aut}(F, B)_{\omega} \) in the sense of 3.1 c).

**Proof.** We have \( \text{Aut}^*_Y(X', A')_{\omega} \cong Y \times \text{Aut}^*_Y(F, B)_{\omega} \) and hence \( \text{Isom}^*_Y((X, A), (X', A'))_{\omega, \omega} \) is admissible. Thus the proposition follows from 3.1 c).

3.4. Let \( f: X \to Y \) be a proper flat morphism of complex varieties. Let \( \text{Aut}_{X, \omega} X \) be the unique irreducible component of \( \text{Aut}_X X \) which contains the identity section \( e(Y) \). Then it is easy to see that \( \text{Aut}_{X, \omega} X \) is a relative complex Lie subgroup of \( \text{Aut}_X X \).

**Lemma 5.** Suppose that \( f \in C/Y \). Then there exists a Zariski open subset \( U \subseteq Y \) such that \( (\text{Aut}_{X, \omega} X)_y = \text{Aut}_{X_y} X_y \) for each \( y \in U \) where \( \text{Aut}_{X_y} X_y \) is the identity component of \( \text{Aut}_X X_y \).

**Proof.** Let \( \mu: \text{Aut}_{X, \omega} X \to Y \) be the natural morphism. Let \( r = \dim \mu \), and \( V = \{ y \in Y ; \dim_{\text{ker} \mu}(y) = r \} \) and \( Y \) is smooth at \( y \). Then \( V \) is Zariski open in \( Y \). Moreover \( \mu \) is smooth at every point of \( e(V) \) and hence \( \text{Aut}_{X, \omega} X \) is smooth along \( e(V) \). Let \( A = \text{Aut}_{X, \omega} X \) and \( n: A \to A \) the normalization. Since \( n \) is isomorphic along \( e(V) \), \( e \) lifts to a meromorphic section \( \bar{e} \) to \( \bar{p}: \bar{A} \to Y \). On the other hand, since \( f \in C/Y, \bar{p} \) is proper [4]. Let \( b: \bar{A} \to \bar{Y}, c: \bar{Y} \to Y \) be the Stein factorization of \( \bar{p} \). Then \( b \bar{e} \) gives a meromorphic section to \( c \). Hence the fiber of \( \bar{p} \) is connected. Since \( A \) is normal, this implies that the general fiber of \( \bar{p} \), and hence of \( \mu \), is irreducible. Thus for general \( y \in Y, A_y \) is the closure of \( \text{Aut}_{X, \omega} X_y \). Hence the assertion follows.

Let \( f: X \to Y \) be a proper surjective morphism of complex varieties. Let
$U \subseteq Y$ be a Zariski open subset over which $f$ is smooth. Then we denote by $\text{Aut}_{\mathcal{O}}^* X$ the closure of $\text{Aut}_{\mathcal{O}}^* X_U$ in $\text{Aut}_{\mathcal{O}}^* X$. This is independent of the choice of $U$ as above. $\text{Aut}_{\mathcal{O}}^* X$ is a relative meromorphic subgroup if $f \in \mathcal{C}/Y$.

**Proposition 7.** Let $f : X \to Y$ be a proper morphism of complex spaces. Let $U \subseteq Y$ be a Zariski open subset. 1) Suppose that $f$ is smooth over $U$ with each fiber a complex torus and that $f$ admits a holomorphic section $e_U : U \to X_U$ on $U$. Then $f_U : X_U \to U$ has the unique structure of a complex Lie group over $U$ with $e_U$ the identity section. 2) Suppose further that $X, Y$ are varieties, $f \in \mathcal{C}/Y$ and that $e_U$ extends to a meromorphic section $e^* : Y \to X$. Then the group law of $X_U$ over $U$ extends meromorphically over $Y$.

**Proof.** 1) Restricting the natural relative action $\sigma_U : (\text{Aut}_{\mathcal{O}}^* X_U) \times_U X_U \to X_U$ to $(\text{Aut}_{\mathcal{O}}^* X_U) \times_U \mathcal{O}_U(U) \cong \text{Aut}_{\mathcal{O}}^* X_U$ we get an isomorphism $\eta_U : \text{Aut}_{\mathcal{O}}^* X_U \cong X_U$ (cf. Appendix). Hence 1) follows. (For the uniqueness see [11], Cor. 6.6.)

2) Similarly, restricting $\sigma^* : (\text{Aut}_{\mathcal{O}}^* X_U) \times_Y X_U \to X_U$ to $(\text{Aut}_{\mathcal{O}}^* X_U) \times_Y \mathcal{O}(Y)$, which is bimeromorphic to $\text{Aut}_{\mathcal{O}}^* X$ we get a natural bimeromorphic map $\text{Aut}_{\mathcal{O}}^* X \to X$ extending $\eta_U$. Then 2) follows from Remark 1, 3). q. e. d.

### 3.5

In concluding this section, as an application of Theorem 2 combined with the consideration of this section, we shall prove a proposition which is used in [5].

Let $g : X \to Y$, $h : Y \to T$ be fiber spaces$^3$ of complex varieties. Let $A=(A_1, \ldots, A_m)$ be a sequence of analytic subspaces of $X$. Suppose that 1) there exist Zariski open subsets $U \subseteq T$, $V \subseteq Y$ with $h(V) \subseteq U$ such that for any $u \in U$, $g_u=g_{u, X_U, A_u} : (X_u, A_u) \to Y_u$ is a holomorphic fiber bundle over $V_u \subseteq Y_u$ which is meromorphic with respect to $g_u$ (cf. 3.1, c)) and 2) there exists a holomorphic section $s : T \to Y$ with $s(T) \cap V \neq \emptyset$. Suppose further that $g$ is Kähler (cf. [4]) so that in particular we can find a relative Kähler class $\omega \in \mathcal{I}^r(Y, R^2 g_* R)$ over $Y$. Then by Proposition 6 if $s(u) \in V$ we can take $G^*(u) := \text{Aut}^*(X_{s(u)}, A_{s(u)})|_{X_{s(u)}}$ as a meromorphic structure group of $g_u$ (considering $(X_{s(u)}, A_{s(u)})$ as a typical fiber of the bundle). Then we shall prove the following:

**Proposition 8.** Under the above situation there exists a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{a} & Z \\
\downarrow g & & \downarrow b \\
Y & \xrightarrow{h} & T
\end{array}
$$

where $a$ is a surjective meromorphic map and $b$ is a fiber space of complex varieties, such that if we restrict $U$ smaller, then for each $u \in U$, $Z_u$ is a generic quotient $X_u/G^*(u)$ of $X_u$ by $G^*(u)$ and $a$ induces a canonical meromorphic map $a_u : X_u \to Z_u$ associated to $g_u$ and $G^*(u)$ (cf. Def. 6).

---

$^3$ A fiber space is a proper surjective morphism with general fiber irreducible.
Proof. Let $\hat{X} := X \times_T T$ where $T$ is over $Y$ via $s$. Let $X' := X \times_T Y$ and $g' : X' \to Y$ the natural map. Let $\hat{\omega}$ (resp. $\omega'$) be the pull-back of $\omega$ (resp. $\partial$) to $\hat{X}$ (resp. $X'$). Then $g'$ is a Kähler morphism with a relative Kähler class $\omega' \in \Gamma(Y, R^s g_* R)$. Let $A_i := A_i \times_T T \subseteq \hat{X}$ and $A'_i := A_i \times T Y \subseteq X'$. Let $I^* := \text{Isom}_F((X, A), (X', A'))$ where $A := (A_1, \ldots, A_m)$ and $A' := (A'_1, \ldots, A'_m)$. Then by Remark 6, 1) $I^*$ is a principal meromorphic subspace to which $G^*$ := Aut$_F(X, A)$ and $G^*: = \text{Aut}_F(X', A')$ are associated. Let $\hat{X} := X \times G^*$ (resp. $\hat{X}' := X' \times G^*$) be the relative generic quotient of $X$ by $G^*$ (resp. $X'$ by $G^*$) over $Y$. Then by Theorem 2 there exists a canonical bimeromorphic map $\eta : \hat{X} \to \hat{X}'$ over $Y$. On the other hand, by Remark 6, 3) $G^* := \hat{G}' \times_T Y$ where $\hat{G}' : = \text{Aut}_F(\hat{X}, \hat{A})$, $\hat{A} := (\hat{A}_1, \ldots, \hat{A}_m)$. Further we have the natural meromorphic map $\pi : \hat{X}' \to \hat{X} / G^*$ over $T$ (cf. Proposition 1, 1)). Let $Z := \hat{X} / G^*$ and define $a : X \to Z$ by the composite meromorphic map $\pi \eta q : X \to Z$ where $q : X \to \hat{X}$ is the quotient meromorphic map. Let $\beta : Z \to T$ be the natural surjective morphism. Then we have $\eta f = ba$. We claim that the resulting diagram meets the requirement of the proposition. In fact, restricting $U$ smaller, we have that for each $u \in U$, $G^*_u$ is a relative meromorphic subgroup of Aut$_F(\hat{X}_u, A_u)$ over $Y_u$ and $\hat{X}_u = X_u / G^*_u$ (cf. Proposition 1, 2)), where $X_u / G^*_u$ is a relative generic quotient of $X_u$ by $G^*_u$ over $Y_u$. Further we have $G^*_u \subseteq G^*(u)$ and $Z_u = (\hat{X} / G^*)_u \subseteq X_u / G^*(u)$. Combining these facts we see readily from our construction that for sufficiently small $U$, the induced meromorphic map $a_u : X_u \to Z_u$ is a canonical meromorphic map associated to $G^*(u)$. q.e.d.

§ 4. BHol$_F(\hat{X}, X')$

a) Let $f : X \to Y$ and $f' : X' \to Y$ be proper flat morphisms of complex varieties. Let BHol$_F(X, X')$ be the contravariant functor $(A_0/Y) \to (\text{Sets})$ defined by

\[
\text{BHol}_F(X, X')(Y) := \text{the set of } \tilde{Y}\text{-morphisms } \phi : X \times_Y \tilde{Y} \to X' \times_Y \tilde{Y}.
\]

Then \text{BHol}_F(X, X') is represented by a unique Zariski open subset Hol$_F(X, X')$ of the relative Douady space $D_{X \times_Y \tilde{Y}}$ with Isom$_F(X, X') \subseteq \text{BHol}_F(X, X')$ (cf. [13]).

Suppose for simplicity that both $f$ and $f'$ are smooth with connected fibers. Let BHol$_F(X, X') := \bigcup_{\tilde{Y}} \text{BHol}_F(X, X')_\tilde{Y}$ where BHol$_F(X, X')_\tilde{Y} := \{h \in \text{BHol}_F(X, X')_\tilde{Y} : h(\gamma) \text{ is bimeromorphic}, \text{ where } h(\gamma) : X_\gamma \to X'_\gamma \text{ is a morphism corresponding to } h\}$.

Then BHol$_F(X, X')$ is Zariski open in $D_{X \times_Y X' / \tilde{Y}}$ (cf. [2], Lemma 5.5). We see that for any open subset $W \subseteq Y$ there is a natural bijective correspondence between the set of holomorphic sections of BHol$_F(X, X') \to Y$ on $W$ and the set of bimeromorphic morphisms $X \to X'$ over $W$.

Let $A \subseteq X$ and $A' \subseteq X'$ be any analytic subspaces. Suppose that $A'$ is flat over $Y$. Then the subfunctor \text{BHol}_F((X, A), (X', A')) of BHol$_F(X, X')$ defined by \text{BHol}_F((X, A), (X', A')) := \{\phi \in \text{BHol}_F((X, A), (X', A')) : \phi(A) = A'\}$ is represented by a unique analytic subspace $\text{Hol}_F((X, A), (X', A'))$ of Hol$_F(X, X')$. This can be shown just in the same way as for Lemma 2. We set BHol$_F((X, A), (X', A'))$:
b) Let \( f : X \to Y \) and \( f' : X' \to Y \) be generically smooth proper surjective morphisms of complex varieties with connected fibers. Let \( U \subseteq Y \) be a Zariski open subset over which both \( f \) and \( f' \) are smooth. Then \( \text{BHol}^F((X, A), (X', A')) \) is Zariski open in \( D_{X \times Y} \). Let \( \text{BHol}^F((X, X')) \) be the essential closure \((1.2 \ a)) \) of \( \text{BHol}^F((X, X')) \) in \( D_{X \times Y} \) which is independent of the choice of \( U \). Let \( A \subseteq X \) and \( A' \subseteq X' \) be analytic subspaces. Restrict \( U \) smaller so that \( A' \) is flat over \( U \). Then the closure of \( \text{BHol}^F((X, A), (X', A')) \) is analytic in \( D_{X \times Y} \) (cf. the proof of Lemma 3). We shall denote the essential closure of \( \text{BHol}^F((X, A), (X', A')) \) by \( \text{BHol}^F((X, A), (X', A')) \).

Remark 7. 1) A bimeromorphic morphism \( \phi : X_U \to X'_U \) defined on \( U \) extends to a bimeromorphic map \( \phi^* : X \to X' \) over \( Y \) if and only if the corresponding holomorphic section \( U \to \text{BHol}^F((X, A), (X', A')) \) extends to a meromorphic section \( Y \to \text{BHol}^F((X, X')) \).

2) If \( \tilde{Y} \to Y \) is a surjective morphism of complex varieties, then it is immediate to see that \( \text{BHol}^F((X, X') \times_Y \tilde{Y}) \equiv \text{BHol}^F((X \times_Y \tilde{Y}, X \times_Y \tilde{Y})) \).

3) If \( f \in C(Y) \), then after replacing \( Y \) by any relatively compact open subset of \( Y \) any irreducible component of \( \text{BHol}^F((X, X')) \) (resp. \( \text{BHol}^F((X, X')) \)) is proper over \( Y \). In particular if \( X \) is compact, we need no restriction to a relatively compact subset.

c) We shall include a standard application of Remark 7, 3) as a reference to [5].

Let \( f : X \to Y \) and \( f' : X' \to Y \) be surjective morphisms of compact complex varieties in \( C \). Let \( U \subseteq Y \) be a Zariski open subset over which both \( f \) and \( f' \) are smooth.

**Proposition 9.** 1) Suppose that \( f \) and \( f' \) admit meromorphic sections \( s : Y \to X \) and \( s' : Y \to X' \) respectively. Suppose further that there exists a \( U \)-isomorphism \( \gamma : X_U \to X'_U \) with \( \gamma|_{U=U} = s'|_{U} \). Then if \( \text{Aut}(X_u, s(u)) = \{e\} \) for all \( u \in U \), then \( \gamma \) extends to a bimeromorphic \( Y \)-map \( \gamma^* : X' \to X' \). 2) Suppose that \( \text{BHol}((X, X')) \) (resp. \( \text{Isom}(X, X') \)) are nonempty and discrete for all \( u \in U \). Then there exists a finite covering \( \mu : \tilde{Y} \to Y \) such that \( X \times_Y \tilde{Y} \) and \( X' \times_Y \tilde{Y} \) is bimeromorphic over \( \tilde{Y} \) by a bimeromorphic \( \tilde{Y} \)-map which is holomorphic (resp. isomorphic) over \( \tilde{U} \).
exists an irreducible component \( Y \) of \( \text{BHolf}(X, X') \) such that \( Y \cap \text{BHolf}(X_U, X'_U) \) is dense in \( Y \) and the natural morphism \( \mu: Y \to Y \) is generically finite and surjective. Let \( \tilde{X} = X_U \) and \( \tilde{X}' = X'_U \). Since \( \tilde{Y} \times_Y \tilde{Y} \subseteq \text{BHolf}(X, X') \times_Y \tilde{Y} \equiv \text{BHolf}(\tilde{X}, \tilde{X}') \), \( \text{BHolf}(\tilde{X}, \tilde{X}') \to \tilde{Y} \) admits a natural holomorphic section whose image over \( \tilde{U} \) is in \( \text{BHolf}(\tilde{X}_U, \tilde{X}'_U) \). Hence \( f_\tilde{Y} \) and \( f'_\tilde{Y} \) are bimeromorphic. Let \( \tilde{Y} \to \tilde{Y}_1 \to Y \) be the Stein factorization of \( \mu \). Then replacing \( \tilde{Y} \) by \( \tilde{Y}_1 \) which is bimeromorphic to \( \tilde{Y} \) we obtain 2). For Isom the proof is similar. q.e.d.

**Remark 8.** As is clear from the above proof the conclusion of 2) is true if there exists an analytic subset \( \tilde{Y}' \subseteq \text{BHolf}(X, X') \) (resp. \( \text{Isom}(X, X') \)) such that \( \tilde{Y}' \cap \text{BHolf}(X_U, X'_U) \) (resp. \( \tilde{Y}' \cap \text{Isom}(X_U, X'_U) \)) is dense in \( \tilde{Y}' \) and that \( \tilde{Y}_y, y \in U \), is discrete. Moreover these results (Proposition 9 and this remark) are true even if the assumption is weakened to: \( f, f' \in \mathcal{C}/Y \) (\( Y \) may not be compact), except that for 2) we have to replace \( Y \) by an arbitrary relatively compact open subset in the conclusion.

**Appendix**

In this appendix we shall summarize some well-known results on the automorphism group of a complex torus and its relative form.

a) Let \( T \) be a complex torus and \( o \in T \) a fixed point. Then \( T \) has a unique structure of a complex Lie group with identify \( o \). Then we can identify \( T \) with \( \text{Aut}_o T \) naturally. Let \( T = H_1(T, \mathbb{Z}) \) and \( H(T) \subseteq \text{Aut} T \) the Lie subgroup of isomorphisms of \( T \) as a complex Lie group. We note that \( H(T) = \text{Aut} (T, \{0\}) \). Then we have the exact sequence

\[
0 \longrightarrow T \longrightarrow \text{Aut} T \xrightarrow{\alpha} \text{Aut} \Gamma
\]

and if \( H \) is the image of \( \alpha \), then \( \alpha \) induces an isomorphism \( H(T) \equiv H \). Hence we have the natural semi-direct product decomposition \( \text{Aut} T = T \cdot H(T) \).

b) Let \( f: X \to Y \) be a proper smooth morphism of complex spaces (not necessarily reduced). Suppose that each fiber of \( f \) is a complex torus and \( f \) admits a holomorphic section \( s: Y \to X \). Then \( X \) has a unique structure of a relative complex Lie group over \( Y \). In fact we can identify \( X \) with \( \text{Aut}_s X \) in the notation of 3.4 (cf. Proposition 7). Let \( H_f X \) be the relative complex Lie subgroup of \( \text{Aut}_s X \) defined by \( H_f X = \text{Aut}_s(X, s(Y)) \). Then we have \( (H_f X)_y = H(X_y) \) for each \( y \in Y \). Let \( \Gamma_y \) be the local system of abelian groups on \( Y \) defined by the presheaf \( U \to H_1(X_U, \mathbb{Z}) \) with \( U \) open subsets of \( Y \). Let \( \tau: \text{Aut}_s \Gamma \to Y \) be the relative automorphism group of \( \Gamma \to Y \); \( \tau \) represents the functor \( K: (\text{An}/Y) \to (\text{Sets}) \) with \( K(Y) \) the set of \( \tilde{Y} \)-automorphisms of \( \Gamma \times_Y \tilde{Y} \). \( \text{Aut}_s \Gamma_y \) is a relative complex Lie group over \( Y \) with \( \tau \) locally biholomorphic. Then as in the absolute case we have the exact sequence

\[
0 \longrightarrow X \longrightarrow \text{Aut}_s X \xrightarrow{\alpha_y} \text{Aut}_s \Gamma_y
\]
of relative complex Lie groups in the sense that each map is a morphism of
complex spaces over \( Y \) and induces an exact sequence of complex Lie groups on
each fiber. Hence \( \alpha_r \) induces an isomorphism of \( H_r X \) with a relative subgroup
of \( \text{Aut}_Y(Y) \), and we have the semi-direct product decomposition

\[
\text{Aut}_Y X = X \cdot H_r X
\]

over \( Y \).

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