On the Existence of Solutions to Time-Dependent Hartree-Fock Equations

By

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§ 1. Introduction and Summary

The approximate methods in the quantum mechanical many body problems lead us to interesting non-linear equations. Consider an $N$-body Schrödinger equation $(N \geq 2)$

$$i \frac{\partial}{\partial t} \Psi(t) = H_N \Psi(t),$$

where

$$H_N = \sum_{j=1}^{N} (-\Delta_j + Q(x_j)) + \sum_{i<j} V(x_i-x_j),$$

where $x_j=(x_{j1}, x_{j2}, x_{j3}) \in \mathbb{R}^3$, $\Delta_j = \sum_{i=1}^{3} (\partial/\partial x_{ji})^2$ and $Q(x)$, $V(x)$ are real functions such that $V(x)=V(-x)$. If the system obeys the Fermi statistics, it is natural to treat (1.1) in the anti-symmetric subspace of $L^2(\mathbb{R}^{3N})$. Taking note of this anti-symmetry and using the variational principle, Dirac ([3], [4]) has derived the following time-dependent version of Hartree-Fock equation in order to obtain an approximate solution of (1.1):

$$i \frac{\partial}{\partial t} u(t) = Hu(t) + K(u(t)),$$

where the unknown $u(t)=u_1(x_1, t), \ldots, u_N(x_N, t)$ is a $C^N$-valued function of $x=(x_1, x_2, x_N) \in \mathbb{R}^3$ and $t \geq 0$,

$$H = -\Delta + Q(x),$$

$$K(u(t))(x) = \int_{R^3} V(x-y)u(x, y, t)\overline{u(y, t)}dy,$$

$$U(x, y, t) = (U_{jk}(x, y, t)) \ (the \ N \times N \ matrix),$$

$$U_{jk}(x, y, t) = u_j(x, t)u_k(y, t) - u_k(x, t)u_j(y, t).$$

Chadam and Glassey [2] have proved the existence of global solutions to (1.2), when $Q(x)$, $V(x)$ are Coulomb potentials: $Q(x) = -Z/|x|$, $V(x) = 1/|x|$, which is practically most important. In this paper, we show that their results can be extended to the more general class of potentials.

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Let $L^p = L^p(\mathbb{R}^3)$ and $\mathcal{H}^m = \mathcal{H}^m(\mathbb{R}^3)$ denote the usual Lebesgue space and the Sobolev space of order $m$, respectively. Their norms are written as $\| \cdot \|_{L^p}$ and $\| \cdot \|_{\mathcal{H}^m}$. For Banach spaces $X$ and $Y$, $B(X; Y)$ denotes the totality of bounded linear operators from $X$ to $Y$. Now, we shall state the assumptions imposed on $Q(x)$ and $V(x)$:

(A-1) $Q(x)$ is a real function and is split into two terms $Q_1(x)$ and $Q_2(x)$: $Q(x) = Q_1(x) + Q_2(x)$, where $Q_1 \in L^2$, $Q_2 \in L^\infty$.

(A-2) $V(x)$ is a real function such that $V(x) = V(-x)$, and is split into two parts: $V(x) = V_1(x) + V_2(x)$, where $V_1 \in L^2$, $V_2 \in L^\infty$.

(A-3) As the multiplication operator, $V$ belongs to $\mathcal{B}(\mathcal{H}^1; L^2)$.

Here we should note that any $f \in L^2$ can be split into two parts $f = f_1 + f_2$, where $f_1 \in L^p \cap L^p$ for any $p$ such that $1 \leq p \leq 2$, $f_2 \in L^\infty$. Indeed, we have only to take $f_1(x) = f(x)$ ($|f(x)| \geq 1$), $f_1(x) = 0$ ($|f(x)| < 1$) and $f_2(x) = f(x) - f_1(x)$. This fact can be written formally as

$$L^p + L^\infty = L^p \cap L^p + L^\infty \quad (1 \leq p \leq 2).$$

Let $p$ ($1 \leq p \leq 2$) be arbitrarily fixed, and $f \in L^p + L^\infty$. Then, as above, one can easily see that for any $\varepsilon > 0$, $f$ can be split into two parts $f_1$ and $f_2$, where

$$f = f_1 + f_2,$$

$$\|f_1\|_{L^p} + \|f_2\|_{L^\infty} < \varepsilon, \quad f_2 \in L^\infty.$$

We shall frequently use this relation in the later arguments.

Under the assumption (A-1), the differential operator $H$ restricted to $C_c^\infty(\mathbb{R}^3)$ (the smooth functions of compact support in $\mathbb{R}^3$) is essentially self-adjoint and the domain of its self-adjoint realization, which we also denote by $H$, is equal to $\mathcal{H}^2$. Then the equation (1.2) can be transformed into the integral equation

$$u(t) = e^{-itH}u(0) - \int_0^t e^{-i(s-t)H} K(u(s)) ds.$$

By the solution of (1.2), we mean an $\mathcal{H}^2$-valued continuous function of $t \geq 0$ verifying the integral equation (1.7).

The result of this paper is summarized in the following

**Theorem.** (1) Under the assumptions (A-1) and (A-2), for any Cauchy data $u(0) \in \mathcal{H}^2$, there exists a unique local solution of (1.2).

(2) Under the assumptions (A-1), (A-2) and (A-3), for any Cauchy data $u(0) \in \mathcal{H}^2$, there exists a unique global solution to (1.2).

The proof of the above theorem is carried out along the line of Chadam and Glassey [2]. For the local existence, it suffices to show that the non-linear term $K(u)$ is locally Lipshitz continuous in $\mathcal{H}^2$. As for the global existence, we have only to obtain some a-priori estimate of the solution $u(t)$, which can be proved by using the energy conservation law.
We shall end this section by giving an example of \( V \) satisfying \((A-3)\).

**Example.** Let \( V(x) \) be split into three terms: \( V(x) = V_1(x) + V_2(x) + V_3(x) \), where \( |V_i(x)| \leq C/|x| \) for a constant \( C > 0 \), \( V_2 \in L^3 \) and \( V_3 \in L^\infty \). Then \( V \in B(\mathcal{H}; L^3) \) as the multiplication operator.

Indeed, by the well-known inequality, we have
\[
\|V_1f\|_{L^3} \leq C\|f(x)/|x|\|_{L^1} \lesssim \text{Const.}\|\nabla f\|_{L^1} \lesssim \text{Const.}\|f\|_{\mathcal{H}^1}.
\]

One can also see that
\[
\|V f\|_{L^3} \leq \|V_2\|_{L^3}\|f\|_{L^6} \lesssim \text{Const.}\|V_3\|_{L^1}\|f\|_{\mathcal{H}^1},
\]
where we have used the well-known Sobolev inequality
\[
\|f\|_{L^6} \lesssim \text{Const.}\|f\|_{\mathcal{H}^1},
\]
(see e. g. [6] p. 12). These observations show that \( V \) verifies the assumption \((A-3)\).

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**§ 2. Existence of Local Solutions**

Let \( A(W; f, g, h) \) be the operator defined by

\[
(2.1) \quad A(W; f, g, h)(x) = f(x) \int_{\mathbb{R}^3} W(x-y) g(y) h(y) dy.
\]

**Lemma 2.1.** We have the following estimates:

1. \( \|A(W; f, g, h)\|_{L^2} \leq \|W\|_{L^\infty}\|f\|_{L^3}\|g\|_{L^3}\|h\|_{L^2}, \)
2. \( \|A(W; f, g, h)\|_{L^2} \leq \text{Const.}\|W\|_{L^3}\|f\|_{L^3}\|g\|_{L^3}\|h\|_{L^2}, \)
3. \( \|A(W; f, g, h)\|_{L^2} \leq \text{Const.}\|W\|_{L^2}\|f\|_{L^3}\|g\|_{L^3}\|h\|_{L^1}, \)
4. \( \|A(W; f, g, h)\|_{L^2} \leq \|W\|_{B(\mathcal{H}; L^3)}\|f\|_{L^3}\|g\|_{L^3}\|h\|_{L^2}, \)

where \( \|\cdot\|_{B(\mathcal{H}; L^3)} \) denotes the operator norm of \( W \) as the multiplication operator from \( \mathcal{H} \) to \( L^3 \).

**Proof.** Let
\[
B(W; g, h)(x) = \int_{\mathbb{R}^3} W(x-y) g(y) h(y) dy.
\]

Then we have only to estimate \( \|B(W; g, h)\|_{L^\infty} \).

(1) easily follows from the Schwarz inequality.

To show (2), we note the Sobolev inequality:
\[
\|g\|_{L^\infty} \lesssim \text{Const.}\|g\|_{L^3}.
\]

Then we have
\[
|B(W; g, h)(x)| \leq \|W\|_{L^3}\|g\|_{L^\infty}\|h\|_{L^2} \lesssim \text{Const.}\|W\|_{L^3}\|g\|_{L^3}\|h\|_{L^2}.
\]

(3) follows from the Hölder and Sobolev inequalities:
\[ |B(W; g, h)(x)| \leq \|W\|_{L^{2/2}} g \|g\|_{L^4} h \|h\|_{L^6} \leq \text{Cont.}\, \|W\|_{L^{2/2}} g \|g\|_{S^1} h \|h\|_{S^1} . \]

(4) can be proved as follows:
\[
|B(W; g, h)(x)| \leq \|W(x - \cdot) g(\cdot)\|_{L^2} h \|h\|_{L^2} \leq \|W\|_{B(S^1; L^2)} g \|g\|_{S^1} h \|h\|_{L^2} ,
\]
where we have used the fact that \(\|W(x - \cdot) g(\cdot)\|_{L^2} = \|W(\cdot) g(x - \cdot)\|_{L^2} \leq \|W\|_{B(S^1; L^2)}\).

We introduce the following notations
\[
\begin{align*}
\mathcal{P}_1(f, g, h) &= \|f\|_{S^2} \|g\|_{S^2} \|h\|_{S^2} , \\
\mathcal{P}_2(f, g, h) &= \|f\|_{S^2} \|g\|_{S^1} \|h\|_{S^1} + \|f\|_{S^2} \|g\|_{S^2} \|h\|_{S^2} .
\end{align*}
\]

**Lemma 2.2.** We have:

1. \(\|A(V; f, g, h)\|_{S^2} \leq \text{Const.} (\|V_1\|_{L^2} + \|V_2\|_{L^{2/2}} + \|V_3\|_{L^\infty}) \mathcal{P}_1(f, g, h).\)
2. \(\|A(V; f, g, h)\|_{S^2} \leq \text{Const.} (\|V_1\|_{L^{2/2}} + \|V_2\|_{L^\infty} + \|V_3\|_{B(S^1; L^2)}) \mathcal{P}_2(f, g, h).\)

**Proof.** Let \(I_j(V) (j=1, 2, \ldots, 6)\) be defined as follows:

\[
\begin{align*}
I_1(V) &= A(V; f, g, h) , \\
I_2(V) &= 2 \sum_{i=1}^3 A(V; f, g, h) , \\
I_3(V) &= 2 \sum_{i=1}^3 A(V; f, g, h), \\
I_4(V) &= A(V; f, g, h) , \\
I_5(V) &= 2 \sum_{i=1}^3 A(V; f, g, h), \\
I_6(V) &= A(V; f, g, h). 
\end{align*}
\]

Then we have
\[
\Delta A(V; f, g, h) = \sum_{j=1}^6 I_j(V). 
\]

Lemma 2.1 (2) implies that
\[
\begin{align*}
\|I_1(V_1)\|_{L^2} &\leq C \|V_1\|_{L^2} \|f\|_{S^2} \|g\|_{S^2} \|h\|_{L^4} , \\
\|I_2(V_1)\|_{L^2} &\leq C \|V_1\|_{L^2} \|f\|_{S^2} \|g\|_{S^2} \|h\|_{S^1} , \\
\|I_3(V_1)\|_{L^2} &\leq C \|V_1\|_{L^2} \|f\|_{L^2} \|g\|_{S^1} \|h\|_{S^2} .
\end{align*}
\]

Also Lemma 2.1 (3) shows that
\[
\begin{align*}
\|I_4(V_1)\|_{L^2} &\leq C \|V_1\|_{L^{2/2}} \|f\|_{S^1} \|g\|_{S^1} h \|h\|_{S^1} , \\
\|I_5(V_1)\|_{L^2} &\leq C \|V_1\|_{L^{2/2}} \|f\|_{S^1} \|g\|_{S^1} h \|h\|_{S^2} , \\
\|I_6(V_1)\|_{L^2} &\leq C \|V_1\|_{L^{2/2}} \|f\|_{L^2} \|g\|_{S^1} h \|h\|_{S^2} .
\end{align*}
\]
We have, therefore,
\begin{equation}
\|A(V; f, g, h)\|_{L^2} \leq C \|1-\Delta\|_{L^2} A(V; f, g, h) \leq C \|V_1\|_{L^2} + \|V_1\|_{L^2} p(f, g, h).
\end{equation}

Using Lemma 2.1 (1), we can show as above that
\begin{equation}
\|A(V_2; f, g, h)\|_{L^2} \leq C \|V_2\|_{L^2} p(f, g, h).
\end{equation}

(2.10) together with (2.11) proves the assertion (1).

In view of Lemma 2.1 (4), we have
\[\|I_1(V)\|_{L^2} + \sum_{j=4, 6} \|I_j(V)\|_{L^2} \leq C \|V\|_{H^1; L^2} p(f, g, h),\]
which together with (2.7) and (2.8) shows the assertion (2).

**Lemma 2.3.** Under the assumption (A-2), the non-linear term $K$ is locally Lipshitz continuous in $\mathcal{S}^3$. That is, for any bounded set $B$ in $\mathcal{S}^3$, there exists a constant $C$ such that
\[\|K(u) - K(v)\|_{\mathcal{S}^3} \leq C \|u - v\|_{\mathcal{S}^3}\]
if $u, v \in B$.

**Proof.** Let $K_j(u)$ be the $j$-th component of $K(u)$. Then $K_j(u)$ can be written as
\[K_j(u) = \sum_{k=1}^{N} \{A(V; u_j, u_k, \bar{u}_k) - A(V; u_j, \bar{u}_k)\} \quad (\text{see (1.4))}.\]
Thus to prove the Lipshitz continuity of $K_j(u)$, we have only to show that of $A(V; u_j, u_k, \bar{u}_k)$ and $A(V; u_j, \bar{u}_k)$, which can be proved by using the multi-linearity of $A(V; \cdot, \cdot, \cdot)$. For example,
\[A(V; u_j, u_k, \bar{u}_k) - A(V; v_j, v_k, \bar{v}_k) = A(V; u_j - v_j, u_k, \bar{u}_k) + A(V; v_j, u_k - v_k, \bar{u}_k) + A(V; v_j, v_k, u_k - v_k).\]

The Lipshitz continuity then follows from Lemma 2.2 (1).

The assumption (A-1) shows that $Q$ is infinitesimally small with respect to $H_0 = -\Delta$. That is, for any $\epsilon > 0$ there exists a constant $C_\epsilon > 0$ such that
\[\|Qf\|_{L^2} \leq \epsilon \|H_0f\|_{L^2} + C_\epsilon \|f\|_{L^2} \quad (f \in \mathcal{S}^3)\]
(see e.g. [7]). Therefore, for sufficiently large $\lambda > 0$, we can find a constant $C > 0$ such that
\begin{equation}
C \|f\|_{\mathcal{S}^3} \leq \|(H + \lambda)f\|_{L^2} \leq C^{-1} \|f\|_{\mathcal{S}^3}.
\end{equation}

In view of (2.12), one can easily see that $e^{-itH}$ is uniformly bounded and strongly continuous in $\mathcal{S}^3$ for $t \in \mathbb{R}$. Thus using the Lipshitz continuity of $K(u)$, one can solve the integral equation (1.7) locally.

**Theorem 2.4** (Local Existence). Assume (A-1) and (A-2). Then for any bounded set $B$ in $\mathcal{S}^3$, there exists a constant $T = T(B) > 0$ such that the solution
of (1.2) exists uniquely for $t \in [0, T]$ if $u(0) \in B$.

§ 3. Existence of Global Solutions

In this section, we shall assume that the solution $u(t) = \{u_i(t), \ldots, u_N(t)\}$ of (1.2) exists for $t \in [0, T)$ and derive its properties. In the followings, $(,)$ denotes the inner product of $L^2$ and also $(L^2)^N$, and $C$'s denote various constants independent of $T$.

The first important property has already been obtained by Dirac [4].

**Theorem 3.1.** \[ \frac{d}{dt} (u_j(t), u_k(t)) = 0 \quad \text{for any } j, k. \]

**Proof.** Using the equation (1.2), we have

\[
\int \frac{d}{dt} (u_j(t), u_k(t)) dt = \{(Hu_j(t), u_k(t)) - (u_j(t), Hu_k(t))\} + \{(K_j(u(t)), u_k(t)) - (u_j(t), K_k(u(t)))\}. 
\]

The first term of the right-hand side vanishes because of the self-adjointness of $H$. In view of (1.4), we have

\[
(K_j(u(t)), u_k(t)) = \int \int V(x-y)u_j(x, t)\overline{u_k(x, t)}|u(y, t)|^2 dx dy - \sum_{n=1}^{N} \int \int V(x-y)\overline{u_n(x, t)}u_j(x, t)\overline{u_k(x, t)}u_k(y, t)\overline{u_n(y, t)} dx dy .
\]

Therefore we have

\[
(u_j(t), K_k(u(t))) = \langle K_k(u(t)), u_j(t) \rangle \\
= \int \int V(x-y)\overline{u_k(x, t)}u_j(x, t)|u(y, t)|^2 dx dy \\
- \sum_{n=1}^{N} \int \int V(x-y)\overline{u_n(x, t)}u_j(x, t)\overline{u_k(x, t)}u_k(y, t)\overline{u_n(y, t)} dx dy .
\]

If we interchange the variables $x$ and $y$, and take into account of the property $V(x) = V(-x)$, we can see that $(u_j(t), K_k(u(t))) = \langle K_j(u(t)), u_k(t) \rangle$, which shows that $\frac{d}{dt} (u_j(t), u_k(t)) = 0$. \[ \square \]

**Corollary 3.2.** $\| u_j(t) \|_{L^2} = \| u_j(0) \|_{L^2} \quad (j = 1, \ldots, N).$

We also prepare the following lemma.

**Lemma 3.3.** \[ \text{Re}(K(u(t)), \frac{\partial}{\partial t} u(t)) = \frac{1}{4} \frac{d}{dt} \langle K(u(t)), u(t) \rangle, \]

where Re means the real part.

**Proof.** $(K(u(t)), \frac{\partial}{\partial t} u(t))$ is split into two parts $I_1$ and $I_2$, where
\[ I_1 = \sum_{j=1}^{N} \iint V(x-y)u_j(x, t) \left( \frac{\partial}{\partial t} u_j(x, t) \right) |u(y, t)|^2 \, dx \, dy, \]

\[ I_2 = -\sum_{j=1}^{N} \iint V(x-y)u_j(x, t)u_j(y, t)u_j(y, t) \frac{\partial}{\partial t} u_j(x, t) \, dx \, dy. \]

Therefore, we can see that

\[ \text{Re} \, I_1 = \frac{1}{2} \iint V(x-y) \left( \frac{\partial}{\partial t} |u(x, t)|^2 \right) |u(y, t)|^2 \, dx \, dy. \]

Exchanging the variables \( x \) and \( y \) suitably, we can rewrite this as

\[ \text{Re} \, I_1 = \frac{1}{4} \frac{d}{dt} \iint V(x-y) |u(x, t)|^2 |u(y, t)|^2 \, dx \, dy. \]

Similarly,

\[ \text{Re} \, I_2 = -\frac{1}{4} \frac{d}{dt} \iint V(x-y)u_j(x, t)u_j(y, t)u_j(y, t) \frac{\partial}{\partial t} u_j(x, t) \, dx \, dy \]

\[ = -\frac{1}{4} \frac{d}{dt} \sum_{j=1}^{N} \iint V(x-y)u_j(x, t)u_j(y, t)u_j(y, t) \frac{\partial}{\partial t} u_j(x, t) \, dx \, dy. \]

Thus we have

\[ \text{Re} \left( \frac{d}{dt} u(t) \right) = \frac{1}{4} \frac{d}{dt} \iint V(x-y)u(x, t)U(x, y, t)u(y, t) \, dx \, dy \]

\[ = \frac{1}{4} \frac{d}{dt} \langle K(u(t)), u(t) \rangle. \]

We can now prove an important theorem concerning the conservation of energy.

**Theorem 3.4** (The Energy Conservation Law). Let \( E(t) \) be defined by

\[ E(t) = \langle Hu(t), u(t) \rangle + \frac{1}{2} \langle K(u(t)), u(t) \rangle. \]

Then, \( E(t) = E(0) \).

**Proof.** By the equation (1.2), we have

\[ i \left( \frac{\partial}{\partial t} u(t), \frac{\partial}{\partial t} u(t) \right) = \langle Hu(t), \frac{\partial}{\partial t} u(t) \rangle + \langle K(u(t)), \frac{\partial}{\partial t} u(t) \rangle. \]

Taking the real part, we have

\[ \text{Re} \left( Hu(t), \frac{\partial}{\partial t} u(t) \right) + \text{Re} \left( K(u(t)), \frac{\partial}{\partial t} u(t) \right) = 0. \]

Since \( \text{Re} \left( Hu(t), \frac{\partial}{\partial t} u(t) \right) = \frac{1}{2} \frac{d}{dt} \langle Hu(t), u(t) \rangle \), in view of Lemma 3.3, we have

\[ \frac{d}{dt} \left\{ \langle Hu(t), u(t) \rangle + \frac{1}{2} \langle K(u(t)), u(t) \rangle \right\} = 0. \]

**Lemma 3.5.** Under the assumption (A-1), for sufficiently large \( \lambda > 0 \), there
exists a constant $C > 0$ such that
\[ C\|f\|_{s_1}^2 \leq (H + \lambda) f, f \leq C^{-1}\|f\|_{s_1}^2. \]

Proof. We have only to show that for any $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that
\[ (Qf, f) \leq \varepsilon (H_\varepsilon f, f) + C_\varepsilon\|f\|_{s_1}^2. \]

$\langle H_\varepsilon = -\Delta \rangle$. By the Hölder and Sobolev inequalities
\[ |(Q_i f, f)| \leq \|Q_i\|_{L^{3/2}}\|f\|_{s_1}^2 \leq C\|Q_i\|_{L^{3/2}}\|f\|_{s_1}^2.
\]
We also have
\[ |(Q_\varepsilon f, f)| \leq \|Q_\varepsilon\|_{L^{3/2}}\|f\|_{s_1}^2. \]
Since $\|Q_\varepsilon\|_{L^{3/2}}$ can be made arbitrarily small, we see (3.1). \qed

Lemma 3.6 (A-priori $s^1$ Bound). If we assume (A-1) and (A-2), we have $\|u(t)\|_{s^1} \leq C$ for a suitable constant $C > 0$.

Proof. Theorem 3.4 and Corollary 3.2 show that
\[ ((H + \lambda)u(t), u(t)) + \frac{1}{2}(K(u(t)), u(t)) = E(0) + \lambda \|u(0)\|_{s_1}^2. \]
Choosing $\lambda$ large enough and taking note of Lemma 3.5, we have
\[ \|u(t)\|_{s_1}^2 \leq C(1 + \|K(u(t))\|_{L^2}), \]
where we have again used Corollary 3.2. Now, $K(u(t))$ can be divided into two parts $K^{(1)}(u(t))$ and $K^{(2)}(u(t))$, where
\[ K^{(2)}(u(t)) = \int V_j(x - y)U(x, y, t)u(y, t)dy. \]
Lemma 2.1 (1) shows that
\[ \|K^{(1)}(u(t))\|_{L^2} \leq C\|V_1\|_{L^\infty}\|u(t)\|_{s_1}^2 \leq C\|V_1\|_{L^\infty}. \]
Lemma 2.1 (3) implies that
\[ \|K^{(1)}(u(t))\|_{L^2} \leq C\|V_1\|_{L^{3/2}}\|u(t)\|_{L^2} \leq C\|V_1\|_{L^{3/2}} \]
\[ \leq C\|V_1\|_{L^{3/2}}\|u(t)\|_{s_1}^2. \]
Since $\|V_1\|_{L^{3/2}}$ can be made arbitrarily small, we have for small $\varepsilon > 0$
\[ \|u(t)\|_{s_1}^2 \leq \varepsilon \|u(t)\|_{s_1}^2 + C_\varepsilon, \]
proving the present lemma. \qed

We can now obtain an a-priori bound of $\|u(t)\|_{s^2}$.

Lemma 3.7 (A-priori $s^2$ Estimate). The assumptions (A-1), (A-2) and (A-3) imply that
\[ \|u(t)\|_{s^2} \leq M \exp(Mt), \]
for a suitable constant $M > 0$.\]
Proof. Since $e^{-tH}$ is uniformly bounded in $\mathcal{S}$, we have by the integral equation (1.7),
\[
\|u(t)\|_{\mathcal{S}} \leq C \left(1 + \int_0^t \|K(u(s))\|_{\mathcal{S}} \, ds \right).
\]
In view of Lemma 2.2 (2) and Lemma 3.5, we have
\[
\|K(u(t))\|_{\mathcal{S}} \leq C \|u(t)\|_{\mathcal{S}} \|u(t)\|_{\mathcal{S}} \leq C \|u(t)\|_{\mathcal{S}}.
\]
We have thus obtained the integral inequality
\[
\|u(t)\|_{\mathcal{S}} \leq M \left(1 + \int_0^t \|u(s)\|_{\mathcal{S}} \, ds \right).
\]
The assertion of the lemma now follows from the well-known Gronwall’s inequality.

Since we have obtained the apriori estimate of $u(t)$, we can easily prove the global existence of solutions to (1.2) by the standard arguments.

Theorem 3.8 (Global Existence). Assume (A–1), (A–2) and (A–3). Then for any Cauchy data $u(0) \in \mathcal{S}$, there exists a unique global solution to (1.2).

References
