Equivariant Stable Homotopy Theory and Idempotents of Burnside Rings

By
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Introduction

Let $G$ be a finite group throughout the present work. We denote by $A(G)$ the Burnside ring of $G$. The stable $G$-homotopy theory is a $G$-homology-cohomology theory of $A(G)$-modules and any idempotent of $A(G)$ decomposes it as a direct sum of $G$-homology-cohomology theories. Such a decomposition for $p$-localized case was partly investigated by Kosniowski [13] and tom Dieck [7].

Let $X$ and $Y$ be pointed $G$-CW complexes. We assume $X$ to be finite. The group of stable $G$-maps from $X$ to $Y$ is denoted by $\tilde{\omega}_G(X: Y)$. We put $\tilde{\omega}_G(X: Y) = \tilde{\omega}_G(\Sigma^v X: \Sigma^v Y)$ for $\alpha = U - V \in RO(G)$. We study $e \tilde{\omega}_G(X: Y)$ for each primitive idempotent $e$ of $A(G)$. Denote by $P$ the set of all conjugacy classes of perfect subgroups of $G$. Primitive idempotents of $A(G)$ correspond bijectively with members of $P$, Dress [9]. Denote by $e_H$ the primitive idempotent of $A(G)$ corresponding to $(H) \in P$, then

$$\tilde{\omega}_G(X: Y) = \bigsqcup_{(H) \in P} e_H \tilde{\omega}_G(X: Y).$$

Let $H$ be a perfect subgroup of $G$. We denote $N = N_G(H)$ and $W = N_G(H)/H$ for simplicity. The main result of the present work is the following.

**Theorem A.** There hold the isomorphisms

$$e_H \tilde{\omega}_G(X: Y) \cong \tilde{\omega}_G(X: Y) \cong \tilde{\omega}_{\{1\}}(X^H: Y^H)$$

which are $e_H A(G)$- and $\tilde{\omega}_H A(N)$-module isomorphisms respectively, where $\tilde{\omega}_H$ and $\tilde{\omega}_{\{1\}}$ denote the primitive idempotents of $A(N)$ and $A(W)$ corresponding to $(H)_N$ and the trivial perfect subgroup $\{1\}$ of $W$ respectively, $\alpha' = \text{res}_W \alpha$ and

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$\alpha'' = U^H - V^H$ for $\alpha' = U - V \in RO(N)$.

**Corollary B.** There hold the ring isomorphisms

$$\epsilon_H A(G) \cong e_H A(N) \cong \mathcal{E}_{\langle 1 \rangle} A(W).$$

Direct proof of this corollary is not difficult. T. Miyata and T. Yoshida communicated to the author relatively short direct proofs of this corollary.

Theorem A hold also for any $G$-homology-cohomology theories defined by $G$-spectra. The $p$-localized version of Theorem A is also true. In fact we prove the more generalized version of Theorem A (Theorem 3.6, (3.7) and Theorem 4.7). We obtain Theorem A by specializing $\pi = \{\text{all primes}\}$ and the $p$-localized version by $\pi = \{p\}$.

In Section 1 we observe certain relations between primitive idempotents of $A(G)_{(\pi)}$ and $A(N)_{(\pi)}$, and their behaviors in Mackey double coset formula. The explicit formula (1.2) for primitive idempotents due to Yoshida [17] is essential. In Section 2 we prove an isomorphism theorem (Theorem 2.5) for Mackey functors on the category $\tilde{G}$ of finite $G$-sets. In Section 3 we see briefly that stable $G$-homotopy theory provides Mackey functors on $\tilde{G}$, then we obtain the first isomorphism of Theorem A (Theorem 3.6 and (3.7)) by applying Theorem 2.5. In Section 4 we construct the fixed-point exact sequences for stable $G$-homotopy theory and prove the second isomorphism of Theorem A (Theorem 4.7).

§ 1. Idempotents of Burnside Rings

Let $\tilde{G}$ be the category of finite $G$-sets and $G$-maps. The set of all isomorphism classes in $G$ forms a commutative semi-ring $A^+ (G)$ with addition and multiplication defined by disjoint unions and direct products (with diagonal $G$-actions) respectively. The *Burnside ring* of $G$, denoted by $A(G)$, is the Grothendieck ring of $A^+ (G)$. A finite $G$-set $S$ represents an element of $A(G)$, denoted by $[S]$. Then every element of $A(G)$ can be expressed in the form $[S] - [T]$. Every finite $G$-set is expressed uniquely as the disjoint union of $G$-orbits, which implies that $A(G)$ is additively a free $\mathbb{Z}$-module with basis $\{ [G/L]; (L) \in C(G) \}$, where $C(G)$ denotes the set of conjugacy classes of subgroups of $G$.

As to the basic properties of $A(G)$ we refer to [8] [9] [10].

Let $\pi$ be a set of primes and $\mathbb{Z}_{(\pi)}$ the subring of $\mathbb{Q}$ consisting of all fractions $a/b$ such that $(a, b) = 1$ and $b$ is prime to every member of $\pi$. Thus, $\mathbb{Z}_{(\pi)} = \mathbb{Q}$ in case $\pi = \emptyset$; $\mathbb{Z}_{(\pi)} = \mathbb{Z}$ in case $\pi = \{\text{all primes}\}$; $\mathbb{Z}_{(\pi)} = \mathbb{Z}_{(p)}$ in case $\pi = \{p\}$, the
set consisting of a single prime $p$. We write $A_{(\pi)} = A \otimes \mathbb{Z}_{(\pi)}$ for any abelian group $A$. Let $G \geq L$, a subgroup. The assignment $\cdot S \mapsto |S^{L}|$ defines a semiring homomorphism $A^{+}(G) \rightarrow \mathbb{Z}$ and induces the ring homomorphism

$$
\phi_{L} : A(G)_{(\pi)} \longrightarrow \mathbb{Z}_{(\pi)},
$$

which is important in studying structure of $A(G)_{(\pi)}$ \cite{8} \cite{9} \cite{17}. E.g., $A(G)_{(\pi)} \ni x = 0 \iff \phi_{L}(x) = 0$ for all $L \leq G$.

Primitive idempotents of $A(G)_{(\pi)}$ are discussed in \cite{8} \cite{9} \cite{11} \cite{17}. Following \cite{17} we denote by $S^{\pi}(G)$ the minimal normal subgroup of $G$ by which the quotient is a solvable $\pi$-group. $S^{\pi}(G)$ is the uniquely determined characteristic subgroup of $G$ \cite{9}. $G$ is called to be $\pi$-perfect provided $S^{\pi}(G) = G$. When $\pi = \{\text{all primes}\}$, $\pi$-perfect groups are perfect groups.

$S^{\pi}(G)$ is always $\pi$-perfect as $S^{\pi}(S^{\pi}(G)) = S^{\pi}(G)$. Let $P_{\pi}$ denote the set of all conjugacy classes of $\pi$-perfect subgroups of $G$. Primitive idempotents of $A(G)_{(\pi)}$ correspond bijectively with members of $P_{\pi}$ \cite{9} \cite{17}.

Let $H$ be a $\pi$-perfect subgroup of $G$ and $e^{H}_{\pi}$ the primitive idempotent corresponding to the conjugacy class $(H)$. Put

$$
S_{\pi}(H, G) = \{L \leq G ; S^{\pi}(L) = H\}
$$

following \cite{17}. $e^{H}_{\pi}$ is characterized by

$$
\phi_{L}(e^{H}_{\pi}) = 1 \quad \text{if} \quad L \sim S_{\pi}(H, G) = 0 \quad \text{otherwise},
$$

where $\sim$ means “conjugate to a member of” \cite{8} \cite{9} \cite{17}.

Recently an explicit formula for the idempotent $e^{\pi}_{H}$ has been given by Yoshida \cite{17}. (The formula for the case $\pi = \emptyset$ is given also by Gluck \cite{11}.) Let $\mu$ be the Möbius function on the subgroup lattice of $G$. For $D \leq G$ he defines

$$
\lambda(D, H) = \sum_{L \in S_{\pi}(H, G)} \mu(D, L)
$$

and obtains the explicit formula for $e^{\pi}_{H}$ \cite{17}, Theorem 3.1, as follows:

$$
e^{\pi}_{H} = (1/|N_G(H)|) \cdot \sum_{D \leq N_G(H)} |D| \lambda(D, H) [G/D].
$$

Let $K \leq G$. Restricting $G$-actions to $K$ on each finite $G$-set $S$, one obtains the ring homomorphism

$$
\text{res}^{K}_{G} : A(G)_{(\pi)} \longrightarrow A(K)_{(\pi)},
$$

called the restriction homomorphism. Clearly

$$
\phi_{L}(\text{res}^{K}_{G} x) = \phi_{L}(x)
$$

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for $x \in A(G)_{(a)}$ and $L \leq K$. The assignment "$S \mapsto G \times_K S$" for each finite $K$-set $S$ induces the linear homomorphism

$$\text{tr}_K^G : A(K)_{(a)} \longrightarrow A(G)_{(a)},$$

called the transfer homomorphism. By definition

$$\text{tr}_K^G [K/L] = [G/L].$$

There holds the Frobenius formula

$$\text{tr}_K^G (x \cdot \text{res}_K^G y) = (\text{tr}_K^G x) \cdot y$$

for $x \in A(K)_{(a)}$ and $y \in A(G)_{(a)}$. $\text{res}_K^G$ maps idempotents to idempotents (which may be decomposable), whereas $\text{tr}_K^G$ does not in general. Obviously $\text{res}_G^G = \text{tr}_G^G = \text{id}$ for $K = G$.

Let $H$ be a $\pi$-perfect subgroup of $G$ and put $N = N_G(H)$, the normalizer of $H$ in $G$. Let $\bar{e}_H$ denote the primitive idempotent of $A(N)_{(a)}$ corresponding to $(H)_N$, the conjugacy class of $H$ in $N$, which we call the central idempotent of $A(N)_{(a)}$. It is characterized by

$$\phi_L(\bar{e}_H) = 1 \quad \text{if} \quad L \in S_\pi(H, N)$$

$$= 0 \quad \text{otherwise},$$

since $H \triangleleft N$. (Compare with (1.1).) Remark that $S_\pi(H, G) = S_\pi(H, N)$ and $\lambda(D, H), D \leq N$, is the same for $G$ and $N$. Since $N_G(H) = N_N(H)$, we compute by (1.2) as follows:

$$\text{tr}_N^G \bar{e}_H = (1/|N_N(H)|) \cdot \sum_{D \leq N_N(H)} |D| \lambda(D, H) \cdot \text{tr}_K^G [N/D]$$

$$= (1/|N_G(H)|) \cdot \sum_{D \leq N_G(H)} |D| \lambda(D, H) [G/D]$$

$$= \bar{e}_H,$$

i.e., we obtain

$$\text{tr}_N^G \bar{e}_H = e_H.$$

$\text{res}_N^G e_H$ is an idempotent of $A(N)_{(a)}$ and we see easily by (1.1) that it decomposes as a sum of primitive idempotents which correspond to conjugacy classes $(H')_N$ in $N$ such that $H' \sim H$ in $G$. Such conjugacy classes correspond bijectively to a part of the double cosets $N \backslash G / N$. Let $\{g_1, \ldots, g_s\}$ be a complete system of representatives of $N \backslash G / N$. Choose a numeration of this system so that $i \leq \circ \mapsto H_i = g_i H g_i^{-1} \leq N$ (which does not depend on the choice of the representative $g_i$). Then $\{(H_i)_N, 1 \leq i \leq s\}$ forms the complete set of the above mentioned conjugacy classes $(H')_N$ in $N$. We choose $g_1 = 1$ always, then $H_1 = H$. 

Let $\tilde{e}_i$ denote the primitive idempotent of $A(N)(\alpha)$ which corresponds to $(H_i)_N$. $\tilde{e}_i = \tilde{e}_H$, the central idempotent of $A(N)(\alpha)$. And we obtain

\[(1.5) \quad \text{res}^N_N e_H = \sum_{1 \leq i \leq s} \tilde{e}_i.\]

By (1.4) and (1.5) we see that

\[(1.6) \quad \text{res}^N_N \circ \text{tr}^N_N e_H = \sum_{1 \leq i \leq s} \tilde{e}_i.\]

Next we apply the Mackey decomposition to $\text{res}^N_N \circ \text{tr}^N_N$. Putting $N_i = N_0(H_i)$, $1 \leq i \leq t$, we obtain

\[(1.7) \quad \text{res}^N_{N_i} \circ \text{tr}^N_{N_i} = \sum_{1 \leq i \leq s} \text{tr}^N_{N_i} \circ \text{res}^N_{N_i} \circ c_i^* \
\text{where } c_i^* : A(N)(\alpha) \to A(N_i)(\alpha), \text{ the isomorphism induced by the conjugation isomorphism } N_i \cong N \text{ with respect to } g_1^{-1}.

We observe $\text{tr}^N_{N_i} \circ \text{res}^N_{N_i} \circ c_i^* (\tilde{e}_i)$ for each $i$, $1 \leq i \leq t$. $c_i^*$ maps primitive idempotents to primitive ones. By (1.3) we see that

\[
\phi_L(c_i^*(\tilde{e}_i)) = 1 \quad \text{if } L \in S_n(H_i, N_i) \\
= 0 \quad \text{otherwise.}
\]

Thus $c_i^*(\tilde{e}_i)$ is the central idempotent of $A(N_i)(\alpha)$. Then

\[
\phi_L(\text{res}^N_{N_i} \circ c_i^*(\tilde{e}_i)) = 1 \quad \text{if } L \in S_n(H_i, N \cap N_i) \\
= 0 \quad \text{otherwise},
\]

which shows that $\text{res}^N_{N_i} \circ c_i^*(\tilde{e}_i) = 0$ for $i > s$ and $=\text{the central idempotent of } A(N_i(H_i))(\alpha)$ for $1 \leq i \leq s$ by (1.3) as $N \cap N_i = N_k(H_i)$. Let $\tilde{e}_i$ denote the central idempotent of $A(N)(H_i)(\alpha)$. We have obtained

\[(1.8) \quad \text{res}^N_{N_i} \circ c_i^*(\tilde{e}_i) = \tilde{e}_i \quad \text{for } 1 \leq i \leq s \\
= 0 \quad \text{for } s < i \leq t.
\]

Apply (1.4) for the pair $(N, H_i)$ and obtain

\[
\text{tr}^N_{N_i} (\tilde{e}_i) = \tilde{e}_i \quad \text{for } 1 \leq i \leq s.
\]

We add two remarks. Since $\tilde{e}_i$ is the primitive idempotent of $A(N \cap N_i)(\alpha)$ corresponding to $(H_i)_{N \cap N_i}$, we have the decomposition

\[
\text{res}^N_{N \cap N_i} \tilde{e}_i = \tilde{e}_i + \cdots
\]

into primitive ones for $1 \leq i \leq s$ by (1.5). Thus

\[
(\text{res}^N_{N \cap N_i} \tilde{e}_i) \cdot \tilde{e}_i = \tilde{e}_i \quad \text{for } 1 \leq i \leq s.
\]

The second remark is that $g_1 = 1$, $H_1 = H$ and $N_1 = N$ by our choice. Thus
§ 2. Idempotents and Mackey Functors

Dress [10], Section 4, defined the Burnside functor on $\hat{G}$. Let $T$ be a finite $G$-set and $\hat{G}/T$ the category of objects over $T$. The set of all isomorphism classes of $\hat{G}/T$ forms a commutative semi-ring $A_G(T)$ with addition and multiplication defined by disjoint unions and pull-backs. Its Grothendieck ring is denoted by $A_G(T)$. The element of $A_G(T)$ represented by an object $f: S \to T$ of $\hat{G}/T$ is denoted by $[f: S \to T]$. The Burnside functor $A_G = (A_G*, A_G^*)$ on $\hat{G}$ is a pair of functors $A_G*: \hat{G} \to \text{Ab}$ and $A_G^*: \hat{G}^{op} \to \text{Ab}$ such that $A_G^*(T) = A_G(T) = A_G(T)$ on each object $T$ and, for a morphism $f: S \to T$ in $\hat{G}$, $A_G^*(f) = f_*$: $A_G(S) \to A_G(T)$ is given by $f_* [g: U \to S] = [f \circ g: U \to T]$ and $A_G^*(f) = f^* : A_G(T) \to A_G(S)$ by $f^* [h: W \to T] = [W \times_T S \to S]$.

As for the definition of a Mackey functor $M = (M_*, M^*)$ on $\hat{G}$ we refer to [7], p. 68. The Burnside functor $A_G$ is a Mackey functor on $\hat{G}$. Moreover, $f^*$ is multiplicative (i.e., $A_G^*$ is ring-valued) and there holds the Frobenius property among $f_*$, $f^*$ and multiplication, i.e., $A_G$ is a Green functor in the sense of [10].

There holds the canonical isomorphism

$$A_G(G/K) \cong A(K)$$

for $K \leq G$ such that

$$p_* = \text{tr}_K^G, \quad \text{and} \quad p^* = \text{res}_K^G$$

for $L \leq K \leq G$ and $p: G/L \to G/K$, the canonical projection.

Let $M = (M_*, M^*)$ be any Mackey functor on $\hat{G}$. We write $M_*(f) = f_*$ and $M^*(f) = f^*$ for a morphism $f: S \to T$ in $\hat{G}$. $M(T)$ becomes an $A_G(T)$-module by $[f: S \to T] \cdot x = f_* y$, $x \in M(T)$, [7][10]. By these module actions $M$ is an $A_G$-module in the sense that $M^*$ is a module-valued functor $(f^*(xy)) = (f^*x)(f^*y)$ for $f: S \to T$, $x \in A_G(T)$ and $y \in M(T)$) and there holds the Frobenius property among $f_*$, $f^*$ and module action [10], Proposition 4.2. We write $p_* = \text{tr}_K^G$, $p^* = \text{res}_K^G$ for any Mackey functor $M$, $L \leq K \leq G$ and $p: G/L \to G/K$, the canonical projection, in conformity with the above mentioned identities for $A_G$.

Let $\pi$ be a set of primes and $M$ a $\mathbb{Z}_\pi$-module-valued Mackey functor. Put $A_{G, \pi} = A_G \otimes \mathbb{Z}_\pi$. The above module action of $A_G$ on $M$ makes $M$ an $A_{G, \pi}$-module.
For each $K \leq G$, $M(G/K)$ is an $A(K)(\sigma)$-module. Hence primitive idempotents of $A(K)(\sigma)$ decomposes $M(G/K)$ as a direct sum of submodules. In particular

$$M(pt) = \bigoplus_{(H) \in \mathcal{P}_K} e^*_H M(pt).$$

We observe $e^*_H M(pt)$ as an $e^*_H A(G)(\sigma)$-module.

Let $H$ be a $\pi$-perfect subgroup of $G$ and $N = N_G(H)$. Let $e^*_H$ be the central idempotent of $A(N)(\sigma)$. We want to discuss $\text{res}^e_{N} \circ \text{tr}^e_{N} (\bar{e}^*_H x)$ for $\bar{e}^*_H x \in \bar{e}^*_H M(G/N)$. The axiom (M1) for the Mackey functor [7] applied to the pull-back diagram

$$
\begin{array}{ccc}
G/N \times G/N & \longrightarrow & G/N \\
\downarrow & & \downarrow \\
G/N & \longrightarrow & pt
\end{array}
$$

implies the Mackey decomposition

$$\text{res}^G_N \circ \text{tr}^G_N = \sum_{1 \leq i \leq t} \text{tr}^N_N \circ \text{res}^N_N \circ c^*_{i}$$

for $M^{10}$ [12] (the same formula as (1.6)), where we used the same notations as in Section 1, i.e., $\{g_1, \ldots, g_t\} (g_1 = 1)$ is a complete system of representatives of $N \backslash G/N$, $N_i = g_i N g_i^{-1}$, and $c^*_i : M(G/N) \simeq M(G/N_i)$, the isomorphism induced by the right multiplication with $g_i : G/N_i \simeq G/N$, for $1 \leq i \leq t$.

Put

$$\bar{x}_i = \text{res}^N_N \circ c^*_i (\bar{e}^*_H x) \in M(G/N \cap N), \ 1 \leq i \leq t.$$ 

As $\text{res}^N_N$ and $c^*_i$ preserve module actions we see that

$$\text{res}^N_N \circ c^*_i (\bar{e}^*_H x) = \bar{e}^*_i \bar{x}_i \quad \text{for} \quad 1 \leq i \leq s,$$

$$= 0 \quad \text{for} \quad s < i \leq t$$

by (1.7). Next we put

$$x_i = \text{tr}^N_N \circ (\bar{e}^*_i \bar{x}_i) \in M(G/N), \quad 1 \leq i \leq s.$$ 

Then

$$\text{tr}^N_N \circ (\bar{e}^*_i \bar{x}_i) = \text{tr}^N_N \circ ((\text{res}^N_N \circ \bar{e}^*_i) \bar{e}^*_i \bar{x}_i) = \bar{e}^*_i x_i, \quad 1 \leq i \leq s,$$

by (1.9). For $i = 1$, the remark (1.10) is applicable also for $M$ and we see that

$$\bar{e}^*_1 x_1 = \bar{e}^*_1 x,$$

the given element. Thus we obtain
**Proposition 2.1.** Using the notations of Section 1 we have the direct sum decomposition

\[(\text{res}^\mathbb{H}_M(G/N))e\mathbb{K}M(G/N) = \bigsqcup_{1 \leq i \leq s} \hat{e}_i M(G/N),\]

and, for any \(\hat{e}_i x \in \hat{e}_i M(G/N)\), we have the decomposition

\[\text{res}^\mathbb{H}_M(G/N) e\mathbb{K} \cdot \text{tr}^\mathbb{K}_M(\hat{e}_i x) = \sum_{1 \leq i \leq s} \hat{e}_i x_i\]

such that

\[\hat{e}_i x_i = \text{tr}^\mathbb{K}_M(\hat{e}_i x) \circ \text{res}^\mathbb{H}_M(G/N) e\mathbb{K} (\hat{e}_i x)\]

and

\[\hat{e}_i x_i = \hat{e}_i x, \text{ the given element}.

Put

(2.2) \[\text{tr}^\mathbb{H}_M(G/N) e\mathbb{K} : \hat{e}_i M(G/N) \rightarrow \hat{e}_i M(pt).

Suppose \(\hat{e}_i x \in \text{Ker} \text{tr}^\mathbb{H}_M(G/N)\). Then \(\text{res}^\mathbb{H}_M(G/N) e\mathbb{K} \cdot \text{tr}^\mathbb{K}_M(\hat{e}_i x) = \sum_{1 \leq i \leq s} \hat{e}_i x_i = 0\). Hence \(\hat{e}_i x_i = 0\) for all \(i, 1 \leq i \leq s\). In particular \(\hat{e}_i x = \hat{e}_i x_i = 0\). Thus we obtain

(2.3) \[\text{tr}^\mathbb{H}_M(G/N) e\mathbb{K} \text{ is monomorphic}.

Let

(2.4) \[\text{res}^\mathbb{H}_M(G/N) e\mathbb{K} : \hat{e}_i M(pt) \rightarrow \hat{e}_i M(G/N)\]

be the \(\hat{e}_i A(G)(\pi)\)-module map defined by

\[\text{res}^\mathbb{H}_M(G/N) e\mathbb{K} (x) = \hat{e}_i x \cdot \text{res}^\mathbb{H}_M(G/N) e\mathbb{K} x, x \in \hat{e}_i M(pt).

By Frobenius property and (1.4) we see that

\[\text{tr}^\mathbb{H}_M(G/N) e\mathbb{K} \cdot \text{res}^\mathbb{H}_M(G/N) e\mathbb{K} (x) = \text{tr}^\mathbb{K}_M(\hat{e}_i x \cdot \text{res}^\mathbb{H}_M(G/N) e\mathbb{K} x) = e\mathbb{H} n(x) = x\]

for \(x \in \hat{e}_i M(pt)\). Thus

\[\text{tr}^\mathbb{H}_M(G/N) e\mathbb{K} \cdot \text{res}^\mathbb{H}_M(G/N) e\mathbb{K} = \text{id},\]

which shows that \(\text{tr}^\mathbb{H}_M(G/N) e\mathbb{K}\) is epimorphic and hence isomorphic by (2.3). Clearly \(\text{res}^\mathbb{H}_M(G/N) e\mathbb{K}\) is the inverse to \(\text{tr}^\mathbb{H}_M(G/N) e\mathbb{K}\) and we obtain

**Theorem 2.5.** Let \(\pi\) be a set of primes, \(M\) a \(Z\)-module-valued Mackey functor on \(G\), \(H\) a \(\pi\)-perfect subgroup of \(G\) and \(N = N_G(H)\). Let \(e\mathbb{K}_H\) be the primitive idempotent of \(A(G)(\pi)\) corresponding to \((H) \in P_\pi\) and \(\hat{e}_i\) the central idempotent of \(A(N)(\pi)\). Then there holds the \(e\mathbb{K}_H A(G)(\pi)\)-module isomorphism

\[\text{res}^\mathbb{H}_M(G/N) e\mathbb{K} : e\mathbb{K}_H M(pt) \cong \hat{e}_i M(G/N)\].
§ 3. Stable $G$-Homotopy Theory

By a $G$-module $V$ we mean a finite dimensional real or complex $G$-module equipped with an invariant metric for simplicity. By $S^V$ and $B^V$ we denote the unit sphere and unit ball of $V$ respectively. We put $\Sigma^V = B^V/S^V$, which is $G$-homeomorphic to the one-point compactification of $V$.

Let $X$ and $Y$ be pointed $G$-CW complexes. We assume $X$ to be finite. By the group of stable-$G$-maps from $X$ to $Y$ we understand

$$\tilde{\omega}_G(X: Y) = \text{colim} [\Sigma^V X, \Sigma^V Y]^G$$

[8], Section 7, where $[\ , , ]^G$ denotes the set of $G$-homotopy classes of pointed $G$-maps, $\Sigma^V X = \Sigma^V \wedge X$, $V$ runs over the system of complex $G$-modules which is directed by $G$-embeddings as $G$-submodules, and the colimit is taken with respect to suspensions

$$\Sigma^V : [\Sigma^V X, \Sigma^V Y]^G \longrightarrow [\Sigma^{W \odot V} X, \Sigma^{W \odot V} Y]^G.$$ 

$\tilde{\omega}_G(X: Y)$ is a well-defined abelian group.

We use complex $G$-modules by the following two reasons: i) the directed system of complex $G$-modules may be regarded as a cofinal subsystem of that of real $G$-modules so that we loose nothing by this restriction; ii) the group of complex automorphisms of a complex $G$-module $V$ is connected so that $G$-maps $\Sigma^V \rightarrow \Sigma^V$ induced by complex automorphisms of $V$ are all $G$-homotopic to the identity, which makes several identifications among $G$-homotopy sets coming from isomorphisms of $G$-modules unique.

Let $f : S \rightarrow T$ be a map in $G$. Endowing discrete topology to $S$ and $T$ respectively, a $G$-embedding $i : S \subset T \times V$ such that $V$ is a complex $G$-module and $\text{pr}_1 \circ i = f$ is called an admissible embedding for $f$. The existence of an admissible embedding is easily shown by making use of the complex permutation representation $V_S$ of $S$. Let $i : S \subset T \times V$ be an admissible embedding for $f$. We may assume that $i(S) \subset T \times \text{Int } B^V$. Regard $S$ and $T$ as $0$-dimensional $G$-manifolds and let $v_i$ be the normal $G$-bundle of the embedding $i$. Then $v_i \simeq G S \times V$. Choose the normal disk $G$-bundle $Dv_i$ so that $Dv_i \subset T \times B^V$. Since $Dv_i \simeq G S \times B^V$, the Thom construction gives a pointed $G$-map

$$\text{tr } f : T^+ \wedge \Sigma^V \longrightarrow S^- \wedge \Sigma^V.$$ 

This construction is of course a very special case of the equivariant Becker-
Gottlieb transfer [15]. (Compare also with [8], §7, in which the case of compact Lie group actions is discussed.) The following properties of $trf$ are easily shown by standard techniques and left to readers.

(3.1) The stable class $\{trf\} \in \tilde{\omega}_G(T^+: S^+)$ is uniquely determined by $f$.

(3.2) Let $f: S_1 \rightarrow S_2$ and $g: S_2 \rightarrow S_3$ be morphisms in $G$. Then

$$\{tr(g \circ f)\} = \{tr f\} \cdot \{tr g\}$$

as stable $G$-maps.

(3.3) Let

$$S' \xrightarrow{g'} S \xrightarrow{f} S$$

be a pull-back diagram in $\mathcal{G}$. Then

$$\{g'^+\} \cdot \{trf\} = \{tr f\} \cdot \{g^+\}$$

as stable $G$-maps.

We define a bifunctor

$$\omega_G[X: Y]: \mathcal{G} \rightarrow Ab$$

as follows:

$$\omega_G[X: Y](S) = \tilde{\omega}_G(S^+ \wedge X: Y)$$

on objects; for a morphism $f: S \rightarrow T$ in $\mathcal{G}$ we put

$$f_* = (tr f \wedge 1)^*: \tilde{\omega}_G(S^+ \wedge X: Y) \rightarrow \tilde{\omega}_G(T^+ \wedge X: Y)$$

which gives a covariant functor by (3.2), and

$$f^* = (f^+ \wedge 1)^*: \tilde{\omega}_G(T^+ \wedge X: Y) \rightarrow \tilde{\omega}_G(S^+ \wedge X: Y)$$

which gives obviously a contravariant functor.

**Proposition 3.4.** $\omega_G[X: Y]$ is a Mackey functor.

**Proof.** (3.3) implies the axiom (M1) of [7], p. 68. As to the axiom (M2), let $S \sqcup T$ be a disjoint union of finite $G$-sets, then $(S \sqcup T)^+ = S^+ \vee T^+$ and

$$\tilde{\omega}_G((S \sqcup T)^+ \wedge X: Y) \simeq \tilde{\omega}_G((S^+ \wedge X) \vee (T^+ \wedge X): Y) \simeq \tilde{\omega}_G(S^+ \wedge X: Y) \oplus \tilde{\omega}_G(T^+ \wedge X: Y).$$

Let $L \leq G$. Since the directed system of $L$-modules which are obtained
from $G$-modules by restriction of actions is a cofinal subsystem of that of arbitrary $L$-modules, we get the homomorphism
\[ \psi^G_L = \text{res}^GL: \tilde{\omega}^L_0(X: Y) \to \tilde{\omega}^L_0(X: Y) \]
by restricting $G$-actions to $L$-actions. On the other hand we get the isomorphism
\[ \kappa: \tilde{\omega}^L_0((G/L)^+ \wedge X: Y) \cong \tilde{\omega}^L_0(X: Y) \]
by restricting stable $G$-maps to $(L)^+ \wedge X \simeq_L X$, which we regard as the canonical isomorphism. Let
\[ p: G/L \to pt \]
be the unique $G$-map. We can easily identify
\[ p^\ast = \text{res}^L_L \]
via the canonical isomorphism $\kappa$. We define
\[ \text{tr}^G_L: \tilde{\omega}^L_0(X: Y) \to \tilde{\omega}^L_0(X: Y). \]

With these setting we apply Theorem 2.5 to the Mackey functor $\omega_G[X: Y]$ and obtain

**Theorem 3.5.** Let $X$ and $Y$ be pointed $G$-CW complexes. Assume $X$ to be finite. Let $\pi$ be a set of primes. Using the same notations as in Theorem 2.5 there holds the $e^H_\pi A(G)$-module isomorphism
\[ \text{res}^G_\pi: e^H_\pi \tilde{\omega}^L_0(X: Y) \cong e^H_\pi \tilde{\omega}^L_0(X: Y). \]

The above theorem applies also to $G$-homology and $G$-cohomology theories. Any $G$-cohomology theory defined on the category of (finite) $G$-CW complexes satisfying suitable axioms is representable by a $G$-spectrum \[2\] \[14\]. So we discuss here only $G$-homology and $G$-cohomology theories defined by $G$-spectra \[2\] \[13\]. We use $G$-spectra indexed by complex (virtual) $G$-modules in the same reason as the definition of the group of stable $G$-maps. Practically we may restrict our $G$-spectra to those indexed by a cofinal subsystem of that of complex $G$-modules and will do so in the sequel.

Let $p = p^G_\ast$ be the complex regular representation of $G$. \( \{np: n \in \mathbb{Z} \} \) is one of such cofinal subsystems. We use this system particularly. A $G$-spectrum $E_G = \{ E_n, e_n: \Sigma^n E_n \to E_{n+1}; n \in \mathbb{Z} \}$ consists of a pointed $G$-CW complex $E_n$ and a pointed $G$-map (structure map) $e_n: \Sigma^n E_n \to E_{n+1}$ for each $n \in \mathbb{Z}$. When $E_n = \Sigma^{np}$ and $e_n = \text{id}: \Sigma^n \Sigma^{np} = \Sigma^{(n+1)p}$ for $n \geq 0$ ($E_n = pt$ for $n < 0$), the $G$-spectrum
is called the $G$-sphere spectrum and denoted by $\Sigma_G$.

Let $E_G = \{E_n, e_n; n \in \mathbb{Z}\}$ be a $G$-spectrum and $L \leq G$. As $\text{res}_L^E \rho_G = |G/L| \cdot \rho_L$, where $\rho' = \rho_L$ is the complex regular representation of $L$, putting

$$E'|_{G/L}|_{n+k} = \Sigma^k \rho' E_n \quad \text{for} \quad 0 \leq k < |G/L|$$
$$e'|_{G/L}|_{n+k} = \text{id} \quad \text{for} \quad 0 \leq k < |G/L| - 1$$
$$e_n \quad \text{for} \quad k = |G/L| - 1,$$

we get an $L$-spectrum

$$\psi_L E_G = \{E'_n, e'_n; n \in \mathbb{Z}\}$$
by restricting $G$-actions to $L$-actions. Clearly

$$\psi_L \Sigma_G = \Sigma_L.$$

The $E_G$-homology-cohomology group in degree 0 (homology with respect to $Y$ and cohomology with respect to $X$) is defined by

$$E^0(X; Y) = \text{colim} \left[ \Sigma^n X, E_n \wedge Y \right]^G,$$
where the colimit is taken with respect to the compositions $e_n \circ \Sigma^n$ as usual. $E^0(X; Y)$ is a well-defined abelian group. Obviously

$$\Sigma^0(X; Y) = \tilde{\omega}^0(X; Y).$$

Again we obtain a Mackey functor $\hat{G} \to \text{Ab}$ by the assignment: $S \mapsto E^0_S(S^X \wedge X; Y)$ and "$f: S \to T" \mapsto f^* = (\text{tr} f \wedge 1)^*$ and $f^* = (f^+ \wedge 1)^*$. Also we have the restriction homomorphism

$$\psi_L^0 = \text{res}_L^E: E^0(X; Y) \longrightarrow (\psi_L E_G)^0(X; Y)$$
and the transfer homomorphism

$$\text{tr}_L^0: (\psi_L E_G)^0(X; Y) \longrightarrow E^0_G(X; Y)$$
together with the canonical isomorphism

$$\kappa: E^0_G((G/L)^+ \wedge X; Y) \cong (\psi_L E_G)^0(X; Y)$$
in the parallel way to the case of $\tilde{\omega}^0$.

Now apply Theorem 2.5 to the above Mackey functor and obtain

**Theorem 3.6.** Under the same assumptions and notations as in Theorem 3.5 there holds the $e^*_H A(G)(\tau)$-module isomorphism

$$\text{res}^G_N: e^*_H E^0_G(X; Y)(\tau) \cong e^*_H (\psi_L E_G)^0(X; Y)(\tau)$$
for any $G$-spectrum $E_G$. 
Let $\alpha \in RO(G)$ and express $\alpha = U - V$ as a difference of real $G$-modules. The $E_G$-homology-cohomology group in degree $\alpha$ is defined by

$$E^\alpha_G(X: Y) = E^\alpha_G(\Sigma^\nu X: \Sigma^\mu Y).$$

Let $\alpha = U' - V'$ be another expression. We can certainly find an additive isomorphism

$$E^\alpha_G(\Sigma^\nu X: \Sigma^\mu Y) \cong E^\alpha_G(\Sigma^\nu' X: \Sigma^\mu' Y),$$

but it is no more canonical and there are many choices of this isomorphism. So, as far as we are interested in additive structures we may use the $RO(G)$-grading; but, when we are interested in multiplicative structure based on ring-$G$-spectra, we will meet with serious troubles in $RO(G)$-grading as to commutativity etc., and we need some other device which will be discussed in another occasion.

Anyway we get the restriction homomorphism

$$\psi^\alpha_L = \text{res}^\alpha_L: E^\alpha_G(X: Y) \longrightarrow (\psi_L E_G)^{\psi_L L}(X: Y)$$

and the transfer homomorphism

$$\text{tr}^\alpha_L: (\psi_L E_G)^{\psi_L L}(X: Y) \longrightarrow E^\alpha_G(X: Y)$$

in degree $\alpha \in RO(G)$, where $\psi_L \alpha = \text{res}^\alpha_L U - \text{res}^\alpha_L V \in RO(L)$ for $\alpha = U - V \in RO(G)$.

By the above definition we see that we may apply Theorem 3.6 to $E_G^\alpha$ and obtain the $e_H^L A(G(e))$-module isomorphism

$$\text{res}^\alpha_K: e_H^L E_G^\alpha(X: Y) \cong e_H^\alpha L(\psi_L E_G)^{\psi_L L}(X: Y)_{(n)}.$$

§ 4. Fixed-Point Exact Sequences

Let $G \triangleright K$, a normal subgroup; then $(\rho_G)^K = \rho_{G/K}$, the complex regular representation of $G/K$. Let $E_G = \{E_n, e_n; n \in \mathbb{Z}\}$ be a $G$-spectrum. Putting

$$E^n_* = E^n_*,$$
$$e_n^\rho = e_n^\rho \cdot \Sigma^\rho E_n^\rho \longrightarrow E_{n+1}^\rho, \quad \rho^\rho = \rho_{G/K},$$

for $n \in \mathbb{Z}$, we get a $G/K$-spectrum

$$\phi_K E_G = \{E_n^\rho, e_n^\rho; n \in \mathbb{Z}\}$$

which is called the $K$-fixed-point spectrum of $E_G$. Clearly

$$\phi_K \Sigma G = \Sigma_{G/K}.$$
By restriction to $K$-fixed-points we get a homomorphism
\[ \phi_K^*: E^n_{g}(X: Y) \longrightarrow (\phi_K E_G)^{\phi_K^*}(X^K: Y^K) \]
called the $K$-fixed-point homomorphism, where $\phi_K x = U^K - V^K \in RO(G)$ for $x = U - V \in RO(G)$.

We construct an exact sequence involving $\phi_K^*$ which generalizes the fixed-point exact sequence for $G = \mathbb{Z}/2$, [3], Section 1.

Decompose
\[ \rho_G = \rho_1 \oplus \rho_2, \rho_2 = \rho_{G/K} \cong \rho_K \quad \text{and} \quad \rho_K = \{0\}. \]

For each integer $n > 0$ we get a $G$-homotopy commutative diagram of pointed $G$-cofibrations
\[
\begin{array}{c}
S^{(n+1)\rho_1} \rightarrow B^{(n+1)\rho_1} \rightarrow \Sigma^{(n+1)\rho_1} \\
\downarrow \quad \downarrow \quad \quad \quad \downarrow \\
S^{(n+1)\rho_1}/S^{\rho_1} \times B^{n\rho_1} \rightarrow B^{(n+1)\rho_1}/S^{\rho_1} \times B^{n\rho_1} \rightarrow \Sigma^{(n+1)\rho_1},
\end{array}
\]
where we identify $B^{(n+1)\rho_1} = B^{\rho_1} \times B^{n\rho_1}$, $S^{(n+1)\rho_1} = \partial (B^{\rho_1} \times B^{n\rho_1}) = S^{\rho_1} \times B^{n\rho_1} \cup B^{\rho_1} \times S^{n\rho_1}$, which implies the following commutative diagram with two horizontal exact sequences:

\[
\begin{array}{c}
\cdots \longrightarrow E^{n\rho_1-1}_{\partial}(S_{\rho_1}\wedge X: Y) \xrightarrow{\delta_n} E^{n\rho_1}_{\partial}(X: Y) \xrightarrow{x_n} E^{n\rho_1}_{\partial}(X: Y) \longrightarrow \cdots \\
\downarrow \quad \downarrow \quad \quad \quad \downarrow \\
\cdots \longrightarrow E^{n+1\rho_1}(S_{(n+1)\rho_1}\wedge X: Y) \xrightarrow{\delta_{n+1}} E^{n+1\rho_1}_{\partial}(X: Y) \xrightarrow{x_{n+1}} E^{n+1\rho_1}_{\partial}(X: Y) \longrightarrow \cdots
\end{array}
\]  

for each $\alpha \in RO(G)$ by fixing the same expression $\alpha = U - V$, where the homomorphism $\chi$ is induced by the inclusion $\chi = \chi_{\rho_1}: \Sigma^0 \subset \Sigma^{\rho_1}$ and $\xi_n$ is induced by the collapsing map $S_{\rho_1}^{(n+1)\rho_1} \rightarrow \Sigma^{\rho_1}(S_{\rho_1}^{\rho_1})$. (Compare with the commutative diagram of [3], p. 5.) Take the colimit in vertical direction of this diagram and obtain an exact sequence which is an $S$-dual version of the localization exact sequence of tom Dieck [5] under a specified situation. We identify this exact sequence with our desired exact sequence.

Define
\[ (\lambda_K E_G)^{\partial}(X: Y) = \operatorname{colim}_n [E^{n+1\rho_1-1}_{\partial}(S_{\rho_1}^{\rho_1} \wedge X: Y), \xi_n], \]
and we prove the isomorphism
(4.3) \( \colim_n [E^n_{G^n} (X : Y), \chi] \cong (\phi_k E_G)^{\phi_k} (X^k : Y^k) \).

First we prove

**Lemma 4.4.** \( \colim_n [E^n_{G^n} (X/X^k : Y), \chi] = 0. \)

**Proof.** Take \( x = \{ f \} \in \colim [E^n_{G^n} (X/X^k : Y), \chi] \). \( x \) is represented by a \( G \)-map \( f : \Sigma^k \rightarrow \Sigma^n \rightarrow E_m \wedge Y \). We want to show that replacing \( f \) by another representative \( g \) of \( x \), \( g^L \approx 0 \) for all \( L \leq G \); then \( g \approx 0 \) by [4], Chapter II, Lemma 5.2, and hence \( x = 0 \). Suppose \( L \geq K \), then \( pt = (X/X^k)^K \approx (X/X^k)^L \); thus \( (X/X^k)^L = pt \), \( (\Sigma^k \rightarrow X^k)^L = pt \) and \( f^L = 0 \). Next, suppose \( L \leq K \). Since \( \rho_G \) is the complex regular representation of \( G \), there exists a non-zero \( v \in \rho_G \) such that \( G_v = L \). Let \( v = (v_1, v_2) \in \rho_1 \oplus \rho_2 = \rho_G \), then \( v_1 \neq 0 \) and \( \rho_2 \neq \{0\} \). Thus \( (\Sigma^k)^L \) is a sphere of dimension \( \geq 2k \) for any integer \( k > 0 \). In the present colimit \( f \) and \( (\chi^k \wedge 1) : \Sigma^k \rightarrow \Sigma^{k+n} \rightarrow E_m \wedge Y \) represent the same element \( x \) for any integer \( k > 0 \). Since \( X \) is finite by our assumption, we may choose \( k \) large enough so that \( \dim \Sigma^k (X/X^k) < 2(k+n)-1 \). Now, put \( g = (\chi^k \wedge 1) : f \); \( \dim (\Sigma^k \rightarrow X^k)^L < 2(k+n)-1 \) and \( (\Sigma^{k+n} \rightarrow E_m \wedge Y)^L \) is at least \( 2(k+n)-1 \)-connected for any \( L \leq K \); thus \( g^L \approx 0 \) for all \( L \leq G \) and \( g \approx 0 \).

**Proof of (4.3).** We prove the case \( x = 0 \). General case follows from this special case by replacing \( X \) by \( \Sigma V \) and \( Y \) by \( \Sigma V \) for \( x \in \Sigma V \) and \( V \in RO(G) \).

Consider the exact sequences associated with the \( G \)-cofibration \( X \rightarrow X \rightarrow X/X^k \) and take the colimit of these sequences with respect to \( \chi \). We get an exact sequence

\[
\colim_n [E^n_{G^n} (X/X^k : Y), \chi] \longrightarrow \colim_n [E^n_{G^n} (X : Y), \chi] \cong \colim_n [E^n_{G^n+1} (X/X^{k+1} : Y), \chi].
\]

By the above lemma \( \colim_n [E^n_{G^n} (X/X^k : Y), \chi] = 0 \) and also \( \colim_n [E^n_{G^n+1} (X/X^{k+1} : Y), \chi] = 0 \) replacing \( Y \) by \( \Sigma^2 \). Thus we get the isomorphism

(2) \( \colim_n [E^n_{G^n} (X : Y), \chi] \cong \colim_n [E^n_{G^n} (X/X^k : Y), \chi] \).

Consider the following sequence

\[
[\Sigma^m X^k, \Sigma^{n+1} E_m \wedge Y] \xrightarrow{\Sigma^m} [\Sigma^m X^k, \Sigma^{n+1} E_m \wedge Y] \xrightarrow{\Sigma^m} [\Sigma^{n+1} E_m \wedge Y] \xrightarrow{\Sigma^m} [\Sigma^{n+1} E_m \wedge Y] \xrightarrow{\Sigma^m} [\Sigma^{n+1} E_m \wedge Y] \xrightarrow{\Sigma^m} [\Sigma^{n+1} E_m \wedge Y] \xrightarrow{\Sigma^m} \cdots
\]

and observe that the composition \( = (\varepsilon_{\chi} \sigma_X)^n \), which proves the isomorphism.
\[ E^n_{\mathcal{G}}(X^K; Y) \cong \text{colim} \left[ [\Sigma^{mp+n_p} X^K, E_{n+m} \wedge Y]^G, e_{\mathcal{G}} \right]. \]

And we get the isomorphism
\[(\#\#) \quad \text{colim} \ E^n_{\mathcal{G}}(X^K; Y) \cong \text{colim} \left[ \Sigma^{mp+n_p} X^K, E_{n+m} \wedge Y \right]^G. \]

Observe the commutative diagram:

\[ \begin{array}{ccc}
[X, Y]^G & \xrightarrow{\Sigma^{p_1}} & [\Sigma^{p_1} X, \Sigma^{p_1} Y]^G \\
(\chi \wedge 1)^* & \downarrow & (\chi \wedge 1)^* \\
[X, \Sigma^{p_1} Y]^G & & \\
\end{array} \]

which shows that the homomorphism \( \chi \) may be used as \( \chi = (\chi \wedge 1)^* \) as well as \( \chi = (\chi \wedge 1)_\bullet \). In the right hand side of the isomorphism (\#\#) we may understand \( \chi = (\chi \wedge 1)_\bullet \). Then we see that the directed system of this double colimit contains the sequence \{\[\Sigma^{np_2} X^K, E_n \wedge Y]^G, \chi \circ \xi \circ \xi \}_k\} as a cofinal subsequence. Thus
\[ \text{colim} \ E^n_{\mathcal{G}}(X^K; Y) \cong \text{colim} \left[ [\Sigma^{np_2} X^K, E_n \wedge Y]^G, \chi \circ \xi \circ \xi \right] \]

Now, \( K \) acts trivially on \( \Sigma^{np_2} X^K \). Hence
\[ \left[ \Sigma^{np_2} X^K, E_n \wedge Y \right]^G = \left[ \Sigma^{np_2} X^K, E_n \wedge Y \right]^G = \left[ \Sigma^{np_2} X^K, E_n \wedge Y \right]^G, \]
and we get the isomorphism
\[ \text{colim} \ E^n_{\mathcal{G}}(X^K; Y) \cong (\phi_K E_G)^0(X^K; Y^K), \]
which, together with (\#), completes the proof of (4.3).

In the exact sequence obtained by taking the colimit of (4.1) in the vertical direction, identify one term with \((\phi_K E_G)^{\phi_K}(X^K; Y^K)\) by (4.3). It is easy to identify \( \chi^\alpha \) with the fixed-point homomorphism \( \phi_K \), and we obtain the desired exact sequence

\[ (4.5) \quad \cdots \rightarrow (\lambda_K E_G)^{\alpha}(X; Y) \rightarrow E^n_{\mathcal{G}}(X; Y) \xrightarrow{\phi_K} (\phi_K E_G)^{\phi_K}(X^K; Y^K) \rightarrow (\lambda_K E_G)^{\alpha+1}(X; Y) \rightarrow \cdots \]

for \( \alpha \in RO(G) \), which we call the \( K \)-fixed-point exact sequence.

Let \( \pi \) be a set of primes and \( H \) a \( \pi \)-perfect subgroup of \( G \). Denote \( N = N_G(H), \ W = N_G(H)/H, \ E_N = \psi_N E_G, \ E_W = \phi_H E_N, \ x' = \psi_N x \) and \( x'' = \phi_H x' \) for \( x \in RO(G) \). Consider the following \( H \)-fixed-point exact sequence (tensored with \( Z_{(\pi)} \)):

\[ \cdots \rightarrow (\lambda_H E_N)^{x'}(X; Y) \rightarrow E^n_{\mathcal{G}}(X; Y) \xrightarrow{\phi_H} E^n_{\mathcal{G}}(X^H; Y^H) \rightarrow \cdots \]
Since actions of \( A(N) \) on \( E^\cdot_\psi(X; Y) \) are natural with respect to \( X \) and \( Y \), the central idempotent \( \tilde{\varepsilon}_H^\cdot \) of \( A(N)(\pi) \) acts on this sequence as an idempotent. 

Remark that \( \tilde{\varepsilon}_H^\cdot \) acts on \( E^\cdot_\psi(X^H; Y^H)(\pi) \) through the homomorphism

\[
\phi^\cdot_H: A(N)(\pi) \longrightarrow A(W)(\pi)
\]
defined by \( \phi^\cdot_H[S] = [S^H] \) for finite \( N \)-sets \( S \). By (1.3) we see easily that

\[
\phi^\cdot_H \tilde{\varepsilon}_H^\cdot = \tilde{\varepsilon}_{\{1\}}^\cdot,
\]

the primitive idempotent of \( A(W)(\pi) \) corresponding to the trivial \( \pi \)-perfect subgroup \( \{1\} \) of \( W \). Thus we get the following exact sequence

\[
\cdots \longrightarrow \tilde{\varepsilon}_H^\cdot (\lambda_H E_N)^{\pi'}(X; Y)(\pi) \\
\quad \longrightarrow \tilde{\varepsilon}_H^\cdot E^\pi_\psi(X; Y)(\pi) \overset{\phi^\cdot_H}{\longrightarrow} \tilde{\varepsilon}_{\{1\}}^\cdot E^\cdot_\psi(X^H; Y^H)(\pi) \longrightarrow \cdots.
\]

We will prove

**Proposition 4.6.** \( \tilde{\varepsilon}_H^\cdot (\lambda_H E_N)^{\pi'}(X; Y) = 0. \)

As a corollary of this proposition we obtain

**Theorem 4.7.** Under the same setting as in Theorem 3.5 there holds the \( \tilde{\varepsilon}_H^\cdot A(N)(\pi) \)-module isomorphism

\[
\phi^\cdot_H: \tilde{\varepsilon}_H^\cdot E^\pi_\psi(X; Y)(\pi) \cong \tilde{\varepsilon}_{\{1\}}^\cdot E^\psi_\psi(X^H; Y^H)(\pi),
\]

where \( W = N_G(H)/H \), \( E_N \) is an \( N \)-spectrum, \( E_W = \phi_H E_N \), \( \pi' \in RO(N) \) and \( \tilde{\varepsilon}_{\{1\}}^\cdot \) is the primitive idempotent of \( A(W)(\pi) \) corresponding to the trivial \( \pi \)-perfect subgroup \( \{1\} \) of \( W \).

**Proof of Proposition 4.6.** Again it is sufficient to prove the special case \( \pi' = 0 \). Decompose \( \rho_N = \rho_1 \oplus \rho_2 \), \( \rho_2 = \rho^H_W \cong \rho_w \), \( \rho^H_1 = \{0\} \). By (4.2)

\[
\tilde{\varepsilon}_H^\cdot (\lambda_H E_N)^{\pi'}(X; Y)(\pi) = \operatorname{colim} \tilde{\varepsilon}_H^\cdot E^{\pi'}_\psi(S^{\rho_1} \wedge X; Y)(\pi).
\]

Therefore it is sufficient to prove

\[
\tilde{\varepsilon}_H^\cdot E^\pi_\psi(S^{\rho_1} \wedge \Sigma X; \Sigma^{\rho_1} Y)(\pi) = 0.
\]

Since \( \rho^H_1 = \{0\} \), we see that \( (S^{\rho_1})^H = \emptyset \). \( S^{\rho_1} \) is an \( N \)-CW complex. Let \( \sigma^k \times N/L \) be an \( N \)-cell of \( S^{\rho_1} \). Then \( (\sigma^k \times N/L)^H = \sigma^k \times (N/L)^H = \emptyset \), whence \( H \not\leq L \). The standard argument by induction on \( N \)-cell of \( S^{\rho_1} \) reduces the problem to show that

\[
\tilde{\varepsilon}_H^\cdot E^\cdot_\psi((\sigma^k \times N/L)(\partial \sigma^k \times N/L)) \wedge \Sigma X; \Sigma^{\rho_1} Y)(\pi) = 0.
\]

The left hand side
Apply (3.7) and Theorem 4.7 to the $G$-sphere spectrum. We get the isomorphisms

\[(4.8) \quad e_\ast H^\ast \tilde{\omega}(X: Y)_\langle \pi \rangle \cong \tilde{\omega}(X: Y)_\langle \pi \rangle \cong \tilde{\omega}(Y^H: X^H)_\langle \pi \rangle \]

for each $(H) \in P_\pi$, where $\alpha \in RO(G)$, $\alpha' = \psi N\alpha$ and $\alpha'' = \phi H\alpha'$. Specialize $\pi = \{\text{all primes}\}$, then we get Theorem A (Introduction).

Put

\[\omega_\ast (pt) = \tilde{\omega}(\Sigma^0: \Sigma^0).\]

Segal [16] showed the isomorphism

\[\omega_\ast (pt) \cong A(G).\]

Then, by (4.8) we obtain

**Corollary 4.9.** There hold the ring isomorphisms

\[e_\ast H^\ast A(G)_\langle \pi \rangle \cong \tilde{\omega}(N)_\langle \pi \rangle \cong \tilde{\omega}(W)_\langle \pi \rangle.\]

Specialize Corollary 4.9 to $\pi = \{\text{all primes}\}$, then we get Corollary B (Introduction).

Finally, we may apply the classifying spaces of families of subgroups due to tom Dieck [6]. Let $F_{\pi-\text{sol}}$ denote the family of all solvable $\pi$-subgroups of $W$ and $EF_{\pi-\text{sol}}$ its classifying space. There holds the isomorphism

\[(4.10) \quad \tilde{\omega}(X^H: Y^H)_\langle \pi \rangle \cong \tilde{\omega}(X^H: Y^H \wedge EF^+_{\pi-\text{sol}})_\langle \pi \rangle\]

by arguments of [6] [7].

Let $H_0, H_1, \ldots, H_k$ ($H_0 = \{1\}$) be a complete system of representatives of $P_\pi$. Then, from the direct sum decomposition

\[\tilde{\omega}(X: Y)_\langle \pi \rangle = \bigsqcup_{0 \leq i \leq k} e_\ast H_i \tilde{\omega}(X: Y)_\langle \pi \rangle\]

and (4.8) we get the direct sum decomposition

\[(4.11) \quad \tilde{\omega}(X: Y)_\langle \pi \rangle = e_\ast \tilde{\omega}(X: Y)_\langle \pi \rangle \oplus \bigsqcup_{1 \leq i \leq k} \tilde{\omega}(X^H: Y^H)_\langle \pi \rangle\]
where $N_I = N_G(H_I)$, $W_I = N_I/H_I$ and $x_i = \phi_{H_I}(\psi_{N_I} x)$ for $1 \leq i \leq k$.

**Example 1.** $G = S_5$, $\pi = \{\text{all primes}\}$. Conjugacy classes of perfect subgroups are $(A_5)$ and $(\{1\})$, and we obtain the direct sum decomposition

$$\varnothing \xi(X: Y) = \epsilon_{1,1} \varnothing \xi(X: Y) + \varnothing \xi_{1,2}(X^{A_5}: Y^{A_5}),$$

where $x' = \phi_{A_5} x$.

**Example 2.** $G = S_6$, $\pi = \{\text{all primes}\}$. There are 4 conjugacy classes of perfect subgroups: $(H_1)$, $(H_2)$, $(H_3)$ and $(H_4)$, where $H_1 = A_6$, $H_2 = A_5$, $H_3 \cong A_5$ and $H_4 = \{1\}$. $H_2$ and $H_3$ are isomorphic but not conjugate. There is an outer automorphism $a$ of $S_6$ such that $a(H_2) = H_3$. Thus: $N_G(H_1) = S_6$, $N_G(H_2) = S_5$ and $N_G(H_3) \cong S_2$. We obtain the direct sum decomposition

$$\varnothing \xi(X: Y) = \epsilon_{1,1,1} \varnothing \xi(X: Y) + \bigoplus_{1 \leq i \leq 3} \varnothing \xi_{2,2}(X^{H_i}: Y^{H_i}),$$

where $x_i = \phi_{H_i}(\psi_{N_i} x)$ for $1 \leq i \leq 3$.

**References**


