On Unbounded Derivations Commuting with a Compact Group of *-Automorphisms

By

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Abstract

Let $\mathcal{A}$ be a C*-algebra with identity, $\alpha$ a continuous action of a compact abelian group $G$ as *-automorphisms of $\mathcal{A}$, $\mathcal{A}(\gamma)$ the spectral subspace of $\alpha$ corresponding to $\gamma$ in the dual $\hat{G}$ of $G$ and $\mathcal{A}^*(\gamma) = \mathcal{A}(\gamma)(0)$ the fixed point algebra of $\alpha$. Let $\delta$ be a closed symmetric derivation of $\mathcal{A}$ which commutes with $\alpha$ and generates a one-parameter group of *-automorphisms of $\mathcal{A}$. We assume that the linear span of $\mathcal{A}(\gamma)^* \mathcal{A}(\gamma)$ is dense in $\mathcal{A}$ for each $\gamma \in \hat{G}$ and then deduce that $\delta$ is a generator on $\mathcal{A}$. Some relevant material on covariant representations is also given.

§ 1. Introduction

Let $\delta$ be a closed (symmetric) derivation of C*-algebra $\mathcal{A}$ which commutes with a continuous action $\alpha$ of a topological group $G$ as *-automorphisms of $\mathcal{A}$. Several authors [1] [2] [3] [4] [5] recently derived conditions on $\mathcal{A}$, $G$, and $\delta$, which ensure that $\delta$ is a generator, i.e., the generator of a strongly continuous one-parameter group of *-automorphisms of $\mathcal{A}$. For example, if $G$ is compact abelian, and $\delta$ vanishes on the fixed point algebra $\mathcal{A}^*$ of $\alpha$, then this result is established in [4]. If, alternatively, $\delta$ is an inner derivation of $\mathcal{A}^*$ it follows from this result, and perturbation theory, that $\delta$ is a generator. But bounded derivations are generators of uniformly continuous groups and hence this can be viewed as an extension result; if $G$ is compact abelian, $\delta$ commutes with $\alpha$, and $\delta$ generates a uniformly continuous one-parameter group $\tau^0$ of inner automorphisms of the fixed point algebra $\mathcal{A}^*$ then $\tau^0$ extends to a strongly continuous group $\tau$, with generator $\delta$, on $\mathcal{A}$. Example 6.1 of [4] also establishes that this result does not necessarily extend to the case that $\delta$ generates a strongly continuous
group of \(*\)-automorphisms of \(\mathcal{A}^a\). Nevertheless in this note we demonstrate
that strong continuity of \(\tau^0\) suffices if, in addition, \(\mathcal{A}^a = \mathcal{A}^a(\gamma) \star \mathcal{A}^a(\gamma)\) for each
\(\gamma \in \hat{G}\), where the bar denotes the closed linear span. (Here, and throughout the
sequel, we adopt the notation of [4]. In particular \(\mathcal{A}^a(\gamma)\) denotes the spectral
subspace of \(\alpha\) corresponding to \(\gamma\) in the dual group \(\hat{G}\).) Thus we aim to establish
the following;

**Theorem 1.** Let \(\mathcal{A}\) be a C*-algebra with identity, \(G\) a compact abelian
group, and \(\alpha\) a continuous action of \(G\) as \(*\)-automorphisms of \(\mathcal{A}\). Furthermore
let \(\delta\) be a closed symmetric derivation satisfying;

1. \(\alpha_g \circ \delta = \delta \circ \alpha_g,\ g \in G,\)
2. \(\delta |_{\mathcal{A}^a}\) is a generator on \(\mathcal{A}^a\).

Finally assume that the closed linear span of \(\mathcal{A}^a(\gamma) \star \mathcal{A}^a(\gamma)\) equals \(\mathcal{A}^a\) for each
\(\gamma \in \hat{G}\).

It follows that \(\delta\) is a generator.

In this theorem we do not know whether the assumption on \(\mathcal{A}^a(\gamma) \star \mathcal{A}^a(\gamma)\)
can be weakened, e.g., to the assumption that \(\mathcal{A}^a(\gamma) \star \mathcal{A}^a(\gamma)\), an ideal of \(\mathcal{A}^a\), is in-
variant under the automorphism group generated by \(\delta_0\), for each \(\gamma \in \hat{G}\), which
is apparently necessary for \(\delta\) to be a generator. (In the example in [4] we referred
to above, this weaker assumption is violated.) We want to point out two
typical cases where the assumption on \(\mathcal{A}^a(\gamma) \star \mathcal{A}^a(\gamma)\) is satisfied. One is the case
where each \(\mathcal{A}^a(\gamma)\) contains a unitary. For example, for a C*-algebra \(B\) with
identity with action \(\beta\) of a discrete abelian group \(\Gamma\), let \(\mathcal{A}\) be the crossed product
\(B \times_\beta \Gamma\) and \(\alpha\) the dual action \(\hat{\beta}\) of \(G = \hat{\Gamma}\). Then for the system \((\mathcal{A}, \Gamma, \alpha)\), \(\mathcal{A}^a(\gamma)\)
contains a unitary. The other is the case where \(\mathcal{A}^a\) is simple, e.g., the Cuntz
algebras \(O_n\) with the gauge action of \(T\).

The general lines of proof of this theorem are very similar to those of [4].
There are two basic arguments. First one proves that \(\delta\) is a generator of a
group of bounded operators on each \(\mathcal{A}^a(\gamma)\) and second one argues that this is
sufficient for \(\delta\) to be a generator on \(\mathcal{A}\). This second step is independent of the
assumption on \(\mathcal{A}^a(\gamma) \star \mathcal{A}^a(\gamma)\) and is based upon the construction and exploitation
of appropriate covariant representations. Hence we begin with the discussion
of this latter lifting procedure in Section 2 and then return to the proof of
Theorem 1, and discussion of the action of \(\delta\) on the spectral subspaces \(\mathcal{A}^a(\gamma)\),
in Section 3. Relevant information about covariant representations is collected
in an appendix.
§ 2. Generators and Spectral Subspaces

In this section we examine the generator problem under the assumption that \( \delta_y \), the restriction of \( \delta \) to the spectral subspace \( \mathcal{A}^*(y) \), is a generator for each \( y \in \hat{G} \). In fact we need information on \( \delta \) under slightly weaker assumptions on the \( \delta_y \), but we will state this as a corollary of the proof of the following general result.

**Proposition 2.** Let \( \mathcal{A} \) be a C*-algebra, \( G \) a compact group, and \( \alpha \) a continuous action of \( G \) as \(*\)-automorphisms of \( \mathcal{A} \). Furthermore let \( \delta \) be a closed symmetric derivation \( \mathcal{A} \) satisfying,

1. \( \alpha_g \delta = \delta \alpha_g, \ g \in G \),
2. \( \delta_y = \delta |_{\mathcal{A}^*(y)} \) is the generator of a strongly continuous one-parameter group of bounded operators on the Banach space \( \mathcal{A}^*(y) \) for each \( y \in \hat{G} \).

It follows that \( \delta \) is the generator of a strongly continuous one-parameter group of \(*\)-automorphisms of \( \mathcal{A} \).

**Remarks.** 1. This result is valid for non-abelian \( G \) too.
2. A weaker version of this proposition is given in [4] Lemma 4.2, where it is further assumed that \( \mathcal{A}^* \subset D(\delta) \), but this domain requirement is in fact irrelevant. The following proof via covariant representations is an ‘integrated’ version of the ‘infinitesimal’ proof of Lemma 4.2 in [4]. It is the elimination of infinitesimal methods which avoids the domain requirements.

**Proof.** Let \( 0 \) denote the trivial representation of \( G \). Since \( \mathcal{A}^* (= \mathcal{A}^*(0)) \) is a C*-subalgebra it follows that \( \delta_0 \) generates a \(*\)-automorphism group \( \tau^0 \). Next define a projection \( P \) from \( \mathcal{A} \) onto the fixed point algebra \( \mathcal{A}^* \) by

\[
P(x) = \int_G d\gamma \gamma(x),
\]

where \( d\gamma \) is the normalized Haar measure on \( G \). Now for any state \( \omega_0 \) of \( \mathcal{A}^* \) define a state \( \omega \) of \( \mathcal{A} \) by

\[
\omega(x) = \frac{1}{2} \int_{-\infty}^{\infty} dte^{-|t|} \omega_0(\tau_t^0(x)),
\]

Thus for \( x \in \mathcal{A}^* \)

\[
\omega(\tau_t^0(x^*x)) = \frac{1}{2} \int_{-\infty}^{\infty} dte^{-|t|} \omega_0(\tau_t^0(x^*x))
\]

and hence
\[ \omega(\tau_0^0(x^*x)) \leq e^{s|s|}\omega(x^*x). \]

Consequently if \((\mathcal{H}, \pi, \Omega)\) denotes the cyclic representation associated with \(\omega\) and in fact one can define a strongly continuous one-parameter group of bounded linear operators acting on the subspace \(\mathcal{H}_0 = [\pi(\mathfrak{F})]\Omega\) by

\[ T_s\pi(x)\Omega = \pi(\tau^0_s(x))\Omega, \]

and in fact one has \(\|T_s\| \leq \exp\{|s|/2\}\). Moreover

\[ T_s\pi(x)T_s^{-1} = \pi(\tau^0_s(x)) \]

for all \(x \in \mathfrak{F}\). Next we argue that by multiplication with an element of \((\pi(\mathfrak{F})|_{\mathcal{H}_0})'\) the group \(T\) may be arranged to be unitary without affecting this covariant implementation law.

Since \(\tau^0\) is a group of *-automorphisms

\[ T^*_x\pi(x) = \pi(\tau^0_s(x)) \]

for all \(x \in \mathfrak{F}\) and hence \(T^*_s T_s \in (\pi(\mathfrak{F})|_{\mathcal{H}_0})'\). Next let \(L_0\) be the generator of \(T\) and remark that

\[ |s|^{-1}(\|T_s\psi\|^2 - \|\psi\|^2) \leq |s|^{-1}(e^{s|s|} - 1)\|\psi\|^2 \]

for all \(\psi \in \mathcal{H}_0\) by the above estimate on \(\omega(\tau^0_s P)\). Hence

\[ |(L_0\psi, \psi) + (\psi, L_0\psi)| \leq \|\psi\|^2 \]

for all \(\psi \in D(L_0)\). It follows that

\[ |(L_0\psi, \phi) + (\psi, L_0\phi)| \leq \|\phi\| \|\psi\| \]

for all \(\phi, \psi \in D(L_0)\). Hence \(D(L_0) \subseteq D(L_0^*\cdot)\) and

\[ |(\psi, (L_0^* + L_0)\phi)| \leq \|\phi\| \|\psi\|. \]

Therefore \((L_0^* + L_0)/2\) has a bounded self-adjoint extension \(h_0\) with \(\|h_0\| \leq \frac{1}{2}\). But \(L_0\) generates the strongly continuous one-parameter group \(T\) on \(\mathcal{H}_0\) and hence by perturbation theory \(iH_0 = L_0 - h_0\) generates a similar group. Since \(H_0\) is symmetric on \(D(L_0)\), it is automatically self-adjoint. Now if \(U_s = \exp\{iH_0s\}\) the Trotter product formula implies that

\[ U_s = \lim_{n \to \infty} (T_{s/n}e^{-h_0s/n})^n. \]

Finally since \(T^*_s T_s \in (\pi(\mathfrak{F})|_{\mathcal{H}_0})'\) it follows that \(h_0 \in ((\pi(\mathfrak{F})|_{\mathcal{H}_0})'\) and hence

\[ U_s\pi(x)U_s^* = \pi(\tau^0_s(x)) \]
for all \( x \in \mathfrak{H}^* \).

Thus \( \tau^0 \) is covariantly implemented on \( \mathfrak{H}^* \) by either \( T \) or \( U \) and consequently \( \delta \) is spatially implemented on \( \mathfrak{H}^* \) either by \( L_0 \) or \( iH_0 \). Specifically

\[
i[H_0, \pi(x)] = \pi(\delta(x))
\]

for all \( x \in \mathfrak{H}^* \cap D(\delta) \). Our next aim is to derive a similar spatial implementation law for \( \delta \) on \( \mathfrak{H} \) and for this we begin by extending \( h_0 \) and \( H_0 \) to \( \mathfrak{H} \).

Define \( h \) on \( \pi(\mathfrak{H})\Omega \) by

\[
h\pi(x)\Omega = \pi(x)h_0\Omega.
\]

Since \( \omega \) is \( \alpha \)-invariant one has

\[
\|h\pi(x)\Omega\|^2 = (h\pi(x)\Omega, \pi(x^*x)h_0\Omega)
= (h\pi(x)\Omega, \pi(P(x^*x))h_0\Omega)
\leq \|h\|_2\|\pi(x)\Omega\|^2.
\]

Hence \( h \) is well-defined and extends by continuity to a bounded operator with \( \|h\| \leq \|h_0\| \leq \frac{1}{2} \). A number of properties of \( h \) follow straightforwardly, e.g.,

\[
h = h^* \in \pi(\mathfrak{H})', \quad \|h\| = \|h_0\|,
E_\gamma h = hE_\gamma, \quad hE_\gamma = h_0,
\]

where

\[
E_\gamma = [\pi(\mathfrak{H}(\gamma))\Omega].
\]

Next define \( H \) by

\[
iH\pi(x)\Omega = \pi(\delta(x))\Omega - h\pi(x)_h\Omega
= \pi(\delta(x))\Omega - \pi(x)h_0\Omega
\]

for \( x \in D(\delta) \). If \( \pi(x)\Omega = \Omega \) and \( y \in D(\delta) \) one calculates that

\[
(\pi(y)\Omega, \pi(\delta(x))\Omega) = \omega(y^*\delta(x))
= \omega(y^*(\delta^*)x) - \omega(y^*(\delta^*))x
= \omega(\delta(y^*x)).
\]

But for \( z \in D(\delta) \) one has

\[
\omega(\delta(z)) = \omega(P(\delta(z)))
= \omega(\delta(P(z)))
= (\Omega, L_0\pi(P(z))\Omega)
= (L_0^*\Omega, \pi(P(z))\Omega)
= 2(h_0\Omega, \pi(P(z))\Omega)
\]
where we have used $L_0^* = -L_0 + 2h_0$ and $L_0\Omega = 0$. Combining these two observations one concludes that
\[(\pi(y)\Omega, \pi(\delta(x))\Omega) = 2(h_0\Omega, \pi(y^*x)\Omega) = 0\]
and hence $H$ is well-defined. But for $x, y \in D(\delta)$
\[(\pi(y)\Omega, iH\pi(x)\Omega) = \omega(y^*\delta(x)) - (\pi(y)\Omega, h\pi(x)\Omega)
= \omega(\delta(y^*x)) - \omega(\delta(y^*)x) - (h_0\Omega, \pi(y^*x)\Omega)
= (h_0\Omega, \pi(y^*x)\Omega) - \omega(\delta(y^*)x)
= -(iH\pi(y)\Omega, \pi(x)\Omega),\]
i.e., $H$ is symmetric. Moreover
\[i[H, \pi(x)]\pi(y)\Omega = iH\pi(xy)\Omega - \pi(x)iH\pi(y)\Omega
= \pi(\delta(xy))\Omega - \pi(x\delta(y))\Omega
= \pi(\delta(x))\pi(y)\Omega,\]
i.e., $\delta$ is implemented by $iH$. Next we prove that $H$ is essentially self-adjoint. It is at this point we use the assumption that $\delta_y$ is the generator of a group of bounded operators.

Set $L = iH + h$ and note that if $x \in D(\delta) \cap \mathcal{D}(\gamma)$ then
\[(I + \beta L)\pi(x)\Omega = \pi((I + i\beta \delta_y)(x))\Omega\]
for all real $\beta$. This demonstrates that $I + \beta L$ leaves $E_y\mathcal{H}$ invariant and since $\delta_y$ generates a strongly continuous group of bounded operators on $\mathcal{D}(\gamma)$ it also establishes that there is a $\beta_y$ such that $R((I + \beta L)E_y)$ is dense in $E_y\mathcal{H}$ for all $|\beta| < \beta_y$. Thus in this range of $\beta$, $(I + \beta L)^{-1}E_y$ is well defined. But
\[\|(I + \beta L)\pi(x)\Omega\| \geq \Re(\pi(x)\Omega, (1 + \beta L)\pi(x)\Omega)\|\pi(x)\Omega\|
= (\pi(x)\Omega, (1 + \beta h)\pi(x)\Omega)\|\pi(x)\Omega\|
\geq (1 - |\beta| \|h\|)\|\pi(x)\Omega\|\]
and hence
\[\|(I + \beta L)^{-1}E_y\| \leq (1 - |\beta| \|h\|)^{-1}.\]
Now define $H_y$ as the restriction of $H$ to $E_y\mathcal{H}$. It follows from perturbation theory that $(I + i\beta H_y)^{-1}$ is defined as a bounded operator for all sufficiently small $\beta$. But this establishes that $H_y$ is essentially self-adjoint and hence $R(I + i\beta H_y)$ is dense for all real $\beta$. Since this is true for all $\gamma \in \hat{G}$ it follows that $H$ is essentially self-adjoint on $\mathcal{H}$.

Now if $\overline{H}$ denotes the self-adjoint closure of $H$ then
defines a \( \sigma \)-weakly continuous group of isometries of \( \mathcal{L}(\mathcal{H}) \) such that

\[
\frac{d}{dt} \tau_t(x) = \tau_t(\pi(x)), \quad x \in D(\delta) .
\]

It follows from semigroup theory that for all real \( \beta \) and all \( x \in D(\delta) \). Since by varying \( \omega_0 \) one can construct a faithful family of covariant states \( \omega \) one then concludes that

\[
\| \pi((I + \beta \delta)(x)) \| \geq \| \pi(x) \|
\]

for all real \( \beta \) and \( x \in D(\delta) \). Finally since \( \delta \), and hence \( \delta_\gamma \), is implemented by the self-adjoint operator \( \overline{H} \) the \( \delta_\gamma \) must generate groups of isometries. Therefore

\[
R(I + \beta \delta_\gamma) = \mathcal{A}(\gamma)
\]

and since this is true for all \( \gamma \in \hat{G} \)

\[
R(I + \beta \delta) = \mathcal{A}.
\]

The two properties (*) and (**) imply, however, that \( \delta \) is a generator.

In the above proof we have not used all the assumptions on \( \delta \). The first part of the proof relies upon the assumption that \( \delta_0 \) is a generator but the second part uses less information about the \( \delta_\gamma \).

**Corollary 3.** Let \( (\mathcal{A}, G, \alpha) \) be as in Proposition 2 and let \( \delta \) be a closed symmetric derivation of \( \mathcal{A} \) satisfying

1. \( \alpha_g^* \delta = \delta \alpha_g \), \( g \in G \),
2a. \( \delta_0 \) is a generator on \( \mathcal{A} \),
2b. For each non-zero \( \gamma \in \hat{G} \) there is a \( \beta_\gamma > 0 \) such that \( R(I + \beta \delta_\gamma) \) is dense in \( \mathcal{A}(\gamma) \) for all \( |\beta| < \beta_\gamma \).

It follows that \( \delta \) is the generator of a strongly continuous one-parameter group of \( * \)-automorphisms of \( \mathcal{A} \).

\[\text{§ 3.  Proof of Theorem 1}\]

The proof of Theorem 1 is based upon verification of the assumptions of Corollary 3. This relies upon algebraic arguments, similar to those employed to prove Theorem 5.1 of [4], combined with perturbation theoretic techniques.
An essential part of the perturbation argument is summarized in the next lemma.

**Lemma 4.** Let $X$ be a Banach space and $X_1, X_2, \ldots, X_n$ closed subspaces such that $X = X_1 + X_2 + \cdots + X_n$. Furthermore let $\delta$ be a closed operator and $\delta_1, \delta_2, \ldots, \delta_n$ bounded operators on $X$. Assume that for $i = 1, 2, \ldots, n$

$$(\delta + \delta_i)(X_i \cap D(\delta)) \subseteq X_i$$

and that $\delta + \delta_i$ is the generator of a semigroup of bounded operators on $X_i$.

It follows that $R(I + \beta \delta)$ is dense in $X$ for all sufficiently small $\beta$.

**Proof.** Let $\hat{X} = X_1 \oplus X_2 \oplus \cdots \oplus X_n$ with the norm of $\hat{x} = (x_1, x_2, \ldots, x_n)$ defined by

$$\|\hat{x}\| = \sum_{i=1}^{n} \|x_i\|.$$ 

Thus $\hat{X}$ is a Banach space. Next consider the linear map $\phi$ from $\hat{X}$ to $X$ with the action

$$\phi(\hat{x}) = \sum_{i=1}^{n} x_i.$$ 

This map is continuous and since $X = X_1 + \cdots + X_n$ its range is equal to $X$. Thus the quotient space $\hat{X}/\text{ker} \phi$, with the quotient norm $\| \cdot \|_\phi$, is canonically isomorphic to $X$. Hence there is a $c > 0$ such that

$$\|\hat{x}\|_\phi \leq c \|\phi(\hat{x})\|$$

for all $\hat{x} \in \hat{X}$. Thus for any $x \in X$ one may choose $x_i \in X_i$ such that

$$x = \sum_{i=1}^{n} x_i$$

and

$$\sum_{i=1}^{n} \|x_i\| < M\|x\|$$

where $M$ is a constant slightly larger than $c$.

Next for $x \in X$ choose $x_i \in X_i$ with the foregoing properties. Then by the assumption that $\delta + \delta_i$ is a generator on $X_i$ one may choose $y_i \in X_i \cap D(\delta)$ such that

$$y_i + \beta(\delta + \delta_i)(y_i) = x_i$$

for $\beta$ sufficiently small. Therefore by semigroup theory there are constants $c_i, d_i > 0$ such that

$$\|x_i\| \geq c_i \|y_i\|(I - |\beta|d_i)$$
for $|\beta|d_i < 1$. Thus setting
\[ y = \sum_{i=1}^{n} y_i \]
one has
\[ \| y + \beta \delta(y) - x \| = \| \sum_{i=1}^{n} (y_i + \beta(\delta + \delta_i)(y_i) - x_i) - \sum_{i=1}^{n} \beta \delta_i(y_i) \| \]
\[ \leq |\beta| \sum_{i=1}^{n} \| \delta_i \| \| y_i \| \]
\[ \leq \sum_{i=1}^{n} \frac{|\beta| \| \delta \|}{c_i(I-|\beta|d_i)} \| x_i \| \]
\[ \leq \| x \| \max_{1 \leq i \leq n} \frac{|\beta| \| \delta_i \| M}{c_i(I-|\beta|d_i)} \]
\[ < \| x \|/2 \]
for all sufficiently small $\beta$. But if $R(I + \beta \delta)$ is not dense in $X$ then for any $\varepsilon > 0$ there is an $x' \in X$ such that
\[ \| y + \beta \delta(y) - x' \| > \| x' \| (1 - \varepsilon) \]
for all $y \in D(\delta)$. Since this contradicts the previous estimate one concludes that $R(I + \beta \delta)$ is dense in $X$ for all sufficiently small $\beta$.

At this stage we are prepared to prove Theorem 1.

Corollary 3 establishes that it is sufficient to show that for each $y \in \mathcal{G}$ there is a $\beta > 0$ such that $R(I + \beta \delta_{y})$ is dense in $\mathcal{U}^\ast(y)$ for all $|\beta| < \beta_{y}$.

Fix $\gamma \in \mathcal{G}$. Since $D_{\gamma} = D(\delta) \cap \mathcal{U}^\ast(y)$ is dense in $\mathcal{U}^\ast(y)$ the closed linear span of $D_{\gamma}^\ast D_{\gamma}$ is dense in $\mathcal{U}^\ast$. This follows from the final assumption of Theorem 1. Moreover $\mathcal{U}^\ast$ contains the identity $1$. Hence there exists a finite number $m$ of $x_i, y_i \in D_{\gamma}$ such that
\[ \| \sum_{i=1}^{m} x_{i}^\ast y_{i} - 1 \| < \frac{1}{2} \]
and consequently
\[ \sum_{i=1}^{m} (x_{i}^\ast x_{i} + y_{i}^\ast y_{i}) \geq \sum_{i=1}^{m} (x_{i}^\ast y_{i} + y_{i}^\ast x_{i}) \geq 1 . \]
Thus we may suppose that there are $n(= 2m)$ elements $y_i \in D_{\gamma}$ with the property
\[ \sum_{i=1}^{n} y_{i}^\ast y_{i} \geq 1 . \]
Similarly there are a finite number $n'$ of $z_j \in D_{\gamma}$ such that
\[ \sum_{j=1}^{n'} z_{j} z_{j}^\ast \geq 1 . \]
because $D_yD_y^*$ is dense in $\mathfrak{U}_*$.

Now define

$$a = \sum_{i=1}^{n} y_i y_i^* + \sum_{i=1}^{n'} z_i z_i^*$$

and set $x_i = a^{-\frac{1}{2}} y_i$ and $x_{n+i} = a^{-\frac{1}{2}} z_i$. It follows that $a^{-\frac{1}{2}} \in \mathfrak{U}_* \cap D(\delta)$, $x_i \in D_y$, and

$$\sum_{i=1}^{N} x_i x_i^* = 1$$

where $N = n + n'$. Furthermore

$$\sum_{i=1}^{N} x_i^* x_i \geq \sum_{i=1}^{n} y_i^* a^{-1} y_i \geq \|a\|^{-1} \sum_{i=1}^{n} y_i^* y_i \geq \|a\|^{-1} 1.$$

Next consider the system (\(\mathfrak{U}_N = \mathfrak{U} \otimes M_N\), \(G, \tilde{a}\)) where \(M_N\) is the \(N \times N\) matrix algebra and \(\tilde{a} = \gamma_\alpha \otimes \iota\). Here \(\iota\) denotes the trivial action. Further let \(\tilde{\delta} = \delta \otimes \iota\) with \(D(\tilde{\delta}) = D(\delta) \otimes M_N\). Thus \(\tilde{a}\) and \(\tilde{\delta}\) satisfy the same properties as \(a\) and \(\delta\). Now define

$$v = \left( x_1, x_2, \ldots, x_N, 0 \right) \in \mathfrak{U}_N^\mathfrak{S}(\gamma) \cap D(\tilde{\delta}).$$

It follows from the above construction that \(v v^* = e_{11}\), where \(e_{11}\) is the matrix unit with \((e_{11})_{ij} = \delta_{11} \delta_{jk} 1\), and \(\tilde{\delta}(v v^*) = 0\). Now for \(b \in \mathfrak{U}_N^\mathfrak{S}(\gamma) v^* v \cap D(\delta)\) one has

$$\tilde{\delta}(b) = \delta(b v^* v)$$

$$= \tilde{\delta}_0(b v^* v) + v^* \tilde{\delta}(v)$$

where \(\tilde{\delta}_0 = \delta_0 \otimes \iota\) with \(D(\tilde{\delta}_0) = D(\delta_0) \otimes M_N\). Therefore

$$(\tilde{\delta} + \delta_{v^* \delta_0}) (b) = \tilde{\delta}_0(b v^* v) + v^* \tilde{\delta}(v) + b$$

$$= \{ \tilde{\delta}_0(b v^*) + v^* \tilde{\delta}(v) v^* \} v$$

where \(\delta_0\) denotes the derivation with action \(\delta_0(b) = ub - bu\).

Now the map from \(b \in \mathfrak{U}_N^\mathfrak{S}(\gamma) v^* v\) to \(b v^* \in \mathfrak{U}_N^\mathfrak{S} v v^*\) is an isomorphism from the Banach space \(\mathfrak{U}_N^\mathfrak{S}(\gamma) v^* v\) onto the Banach space \(\mathfrak{U}_N^\mathfrak{S} v v^*\). But since \(\delta_0(v v^*) = 0\) the restriction of \(\delta_0\) to \(\mathfrak{U}_N^\mathfrak{S} v v^*\) is also a generator. Moreover the operator of left multiplication by \(v^* \tilde{\delta}(v)\) is bounded and leaves \(\mathfrak{U}_N^\mathfrak{S} v v^*\) invariant. Therefore \(\delta_0 + v^* \tilde{\delta}(v)\) is the generator of a group of bounded operators on \(\mathfrak{U}_N^\mathfrak{S} v v^*\). Hence \(\tilde{\delta} + \delta_{v^* \delta_0}\) is a generator on \(\mathfrak{U}_N^\mathfrak{S}(\gamma) v^* v\).

Next we repeat this argument with matrices \(v_\iota(\sigma)\) whose elements are zero except in the \(i\)-th row which is given by

$$\sigma(i) x_i, \sigma(i+1)x_{i+1}, \ldots, \sigma(N)x_N, \sigma(1)x_1, \ldots, \sigma(i-1)x_{i-1}$$
where the $\sigma(j)$ take values $\pm 1$. Then $v_l(\sigma) \in \mathfrak{H}_K(\gamma) \cap D(\delta)$ and

$$v_l(\sigma)v_l(\sigma)^* = e_{ll}.$$  

By the above reasoning $\delta + \delta v_l(\sigma)^* \delta(v_l(\sigma))$ is a generator on $\mathfrak{H}_K(\gamma)v_l(\sigma)^*v_l(\sigma)$. But

$$2^{-N} \sum_{\sigma} v_l(\sigma)^* v_l(\sigma) = \begin{pmatrix} x_i^* x_{i+1} & \cdots & x_i^* x_{i-1} \\ 0 & \cdots & 0 \end{pmatrix}$$

and

$$2^{-N} \sum_{i=1}^{N} v_l(\sigma)^* v_l(\sigma) = \sum_{i=1}^{N} x_i^* x_i 1_N \geq \|a\|^{-1}$$

where $1_N$ is the identity of $M_N$. Therefore

$$\mathfrak{H}_K(\gamma) = \sum_{\sigma} \mathfrak{H}_K(\gamma)v_l(\sigma)^*v_l(\sigma)$$

and we can apply Lemma 4 to the family

$$X = \mathfrak{H}_K(\gamma), \quad X_i = \mathfrak{H}_K(\gamma)v_l(\sigma)^*v_l(\sigma),$$

and the bounded operators $\delta v_l(\sigma)^* \delta(v_l(\sigma))$ and conclude that $(I + \beta \delta)(\mathfrak{H}_K(\gamma) \cap D(\delta))$ is dense in $\mathfrak{H}_K(\gamma)$ for sufficiently small $\beta$. Since $\mathfrak{H}_K(\gamma) = \mathfrak{H}(\gamma) \otimes M_N$ this implies that $(I + \beta \delta)(\mathfrak{H}(\gamma) \cap D(\delta))$ is dense in $\mathfrak{H}(\gamma)$ for small $\beta$ and hence $\delta$ is a generator by Corollary 3.

**Appendix**

**Covariant Representations**

Throughout this appendix $(\mathfrak{A}, \tau, \omega)$ denotes a $C^*$-algebra $\mathfrak{A}$, a strongly continuous one-parameter group of $^*$-automorphisms $\tau$ of $\mathfrak{A}$, and a state $\omega$ over $\mathfrak{A}$. Furthermore $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ denotes the cyclic representation of $\mathfrak{A}$ associated with $\omega$. It follows from the proof of Proposition 2 that the state

$$\omega_\omega = \frac{1}{2} \int_{-\infty}^{\infty} dte^{-|t|} \omega \tau_t$$

generates a covariant representation, i.e., there exists a strongly continuous one-parameter group of unitary operators $U_{\omega_\omega}$ on $\mathcal{H}_\omega$ which implements the automorphisms $\tau$,

$$\pi_{\omega_\omega}(\tau(A)) = U_{\omega_\omega}(t)\pi_{\omega_\omega}(A)U_{\omega_\omega}(t)^{-1}.$$
The purpose of this appendix is to further analyze this phenomenon by proving the following.

**Theorem A1.** Let \( f \) be an almost everywhere positive integrable function over \( \mathbb{R} \) with total integral one.

It follows that the state

\[
\omega_f = \int \text{d}t f(t) \omega^\otimes \tau_t
\]
generates a covariant representation.

**Remark.** If the Fourier transform \( \hat{f} \) of \( f \) has compact support this result is a corollary of a spectral theorem of Arveson (Theorem 5.3. of [6]).

The proof of Theorem A1 will be divided into two pieces each of which have a separate interest. The first piece of information extends a construction used by Tomita in the decomposition theory of states (see [7] Chapter 4, in particular Lemma 4.1.21). In the following \( E_\mathcal{W} \) will denote the state space of \( \mathcal{W} \) equipped with the weak*-topology.

**Proposition A2.** Let \( \mu \) be a regular probability measure on \( E_\mathcal{W} \) with barycentre \( \omega \) and let \( f \) be a non-negative \( \mu \)-integrable function over \( E_\mathcal{W} \). Define the positive sesquilinear form \( s_f \) over \( \mathcal{H}_{\omega} \) by \( D(s_f) = \pi_{\omega}(\mathcal{W}) \Omega_{\omega} \) and

\[
s_f(\pi_{\omega}(x) \Omega_{\omega}, \pi_{\omega}(y) \Omega_{\omega}) = \int_{E_\mathcal{W}} d\mu(\omega) f(\omega)(x^*y).
\]

It follows that \( s_f \) is closable and the positive self-adjoint operator \( S_f \) associated with the closure \( \tilde{s}_f \) of \( s_f \) is affiliated with the commutant \( \pi_{\omega}(\mathcal{W})' \) of \( \pi_{\omega}(\mathcal{W}) \). Moreover if \( f \) is positive \( \mu \)-almost everywhere then \( S_f \) is invertible.

**Proof.** Define \( f_n \) by \( f_n(x) = \min (f(x), n) \). Thus the \( f_n \) form an increasing family of positive functions which converges pointwise to \( f \). Next let \( \kappa_\mu(f_n) \in \pi_{\omega}(\mathcal{W})' \) denote the bounded operators defined by

\[
(\Omega_{\omega}, \kappa_\mu(f_n) \pi_{\omega}(x) \Omega_{\omega}) = \int d\mu(\omega) f(\omega')(x)
\]

(see [7] Lemma 4.1.21). Now introduce the increasing family of bounded quadratic forms

\[
t_\mu(\psi) = (\psi, \kappa_\mu(f_n) \psi), \quad \psi \in \mathcal{H}_{\omega}
\]

and their monotone limit
where $D(t)$ is the family of $\psi \in \mathfrak{H}_{\omega}$ for which the supremum is finite. It follows from [8] Lemma 5.2.13 that $t$ is closed. But

$$t(\pi_{\omega}(x)\Omega_{\omega}) = \int d\mu(\omega') f(\omega') \omega'(x*x)$$

$$= s_f(\pi_{\omega}(x)\Omega_{\omega})$$

for all $x \in \mathfrak{H}$. Thus $t$ is a closed extension of $s_f$, i.e., $s_f$ is closable.

Now $\pi_{\omega}(\mathfrak{H})\Omega_{\omega}$ is automatically a core for $S_f^{\frac{1}{2}}$. Moreover

$$\|S_f^{\frac{1}{2}} \pi_{\omega}(x)\pi_{\omega}(y)\Omega_{\omega}\|^2 = \int d\mu(\omega') f(\omega') \omega'(y*y)$$

$$\leq \|x\|^2 \int d\mu(\omega') f(\omega') \omega'(y*y)$$

$$= \|x\|^2 \|S_f^{\frac{1}{2}} \pi_{\omega}(y)\Omega_{\omega}\|^2.$$ 

Thus it follows that $\pi_{\omega}(\mathfrak{H})D(S_f^{\frac{1}{2}}) \subseteq D(S_f^{\frac{1}{2}})$. Moreover one concludes from the identity

$$(S_f^{\frac{1}{2}} \pi_{\omega}(xy)\Omega_{\omega}, S_f^{\frac{1}{2}} \pi_{\omega}(z)\Omega_{\omega}) = \int d\mu(\omega') f(\omega') \omega'(y*x*z)$$

$$=(S_f^{\frac{1}{2}} \pi_{\omega}(y)\Omega_{\omega}, S_f^{\frac{1}{2}} \pi_{\omega}(x*z)\Omega_{\omega})$$

by a double approximation procedure that

$$(S_f^{\frac{1}{2}} \pi_{\omega}(x)\phi, S_f^{\frac{1}{2}} \psi) = (S_f^{\frac{1}{2}} \phi, S_f^{\frac{1}{2}} \pi_{\omega}(x^*)\psi)$$

for all $\phi, \psi \in D(S_f) \subseteq D(S_f^{\frac{1}{2}})$. But the left hand side is continuous in $\phi$ and the right hand side is continuous in $\psi$. Hence one deduces that $\pi_{\omega}(\mathfrak{H})D(S_f) \subseteq D(S_f)$ and

$$(S_f \pi_{\omega}(x)\phi, \psi) = (S_f \phi, \pi_{\omega}(x^*)\psi)$$

$$=(\pi_{\omega}(x)S_f \phi, \psi).$$ 

Thus $S_f$ is affiliated with $\pi_{\omega}(\mathfrak{H})'$.

Next suppose $f$ is positive $\mu$-almost everywhere. The approximants $f_n$ introduced above then have this property. Moreover since $f \geq f_n \geq 0$ it follows that

$$S_f \geq \kappa_{\mu}(f_n) \geq 0$$

where the operator ordering is in the sense of quadratic forms. Thus to prove that $S_f$ is invertible it suffices to prove that $\kappa_{\mu}(f_n)$ is invertible and this effectively
reduces the problem to the examination of bounded $f$. Therefore we now assume $f$ is bounded.

Next define
\[ \theta_n = \{ \omega' \in E_{\mathcal{H}} ; f(\omega') \geq \frac{1}{n} \} \]
and consider the bounded sesquilinear forms $t_n$ over $\mathcal{H}_\omega \times \mathcal{H}_\omega$ with the property
\[ t_n(\pi_\omega(x)\Omega_\omega, \pi_\omega(y)\Omega_\omega) = \int_{\theta_n} d\mu(\omega')\omega'(x^*y). \]
Since $nf(\omega') \geq 1$ on $\theta_n$
\[ t_n(\pi_\omega(x)\Omega_\omega, \pi_\omega(y)\Omega_\omega) \leq n \int d\mu(\omega')f(\omega')\omega'(x^*y) \]
\[ = n\|\pi_\omega(x)S^\frac{1}{2}\Omega_\omega\|^2. \]
Hence there is a sequence of positive bounded operators $S_n$ on the range $\mathcal{H}_n$ of $E_n = [\pi_\omega(\mathbb{1})S^\frac{1}{2}\Omega_\omega]$ such that
\[ t_n(\pi_\omega(x)\Omega_\omega, \pi_\omega(y)\Omega_\omega) = (\pi_\omega(x)S^\frac{1}{2}\Omega_\omega, S_n\pi_\omega(y)S^\frac{1}{2}\Omega_\omega) \]
and $S_n$ is in the commutant $\pi_\omega(\mathbb{1})'$ restricted to $\mathcal{H}_n$. But $E_n \in \pi_\omega(\mathbb{1})'$ and hence $S_n = S_nE_n = E_nS_nE_n$ may be regarded as an operator in $\pi_\omega(\mathbb{1})'$ acting on $\mathcal{H}_\omega$. Moreover the family of forms associated with $t_n$ is monotone increasing and
\[ \lim_{n \to \infty} t_n(\pi_\omega(x)\Omega_\omega, \pi_\omega(y)\Omega_\omega) = \lim_{n \to \infty} (\pi_\omega(x)\Omega_\omega, S_n^\frac{1}{2}S_n\pi_\omega(x)\Omega_\omega) \]
\[ = (\pi_\omega(x)\Omega_\omega, \pi_\omega(x)\Omega_\omega). \]
Thus $S_n^\frac{1}{2}S_nS_n^\frac{1}{2}$ converges weakly, hence strongly, to the identity.

Finally suppose $S_f \phi = 0$. Then
\[ (\phi, S_n^\frac{1}{2}S_nS_n^\frac{1}{2}\phi) = 0. \]
But this contradicts the previous convergence result unless $\phi = 0$, i.e., $S_f$ is invertible.

Next we compare the representations generated by the states obtained from two probability measures on the state space $E_{\mathcal{H}}$.

**Proposition A3.** Let $\mu_1$ and $\mu_2$ be two regular probability measures on $E_{\mathcal{H}}$ with barycentres $\omega_1$ and $\omega_2$ respectively.

If $\mu_1$ is absolutely continuous with respect to $\mu_2$ then $\pi_{\omega_1}$ is unitarily
equivalent to a subrepresentation of \( \pi_{o_2} \), and if \( \mu_1 \) and \( \mu_2 \) are mutually absolutely continuous then \( \pi_{o_1} \) and \( \pi_{o_2} \) are unitarily equivalent.

**Proof.** If \( \mu_1 \) is absolutely continuous with respect to \( \mu_2 \) there is a non-negative \( f \in L^1(\mu_2) \) such that \( d\mu_1 = fd\mu_2 \). Now define \( S_f \) on \( \mathcal{H}_{o_2} \) by the construction of Proposition A2. Thus

\[
(S_\frac{1}{2} \Omega_{o_2}, \pi_{o_2}(x)S_\frac{1}{2} \Omega_{o_2}) = \int d\mu_{2}(\omega')f(\omega')\omega'(x)
= \int d\mu_{1}(\omega')\omega'(x) = \omega_{1}(x)
\]

and \( S_f \) is affiliated to \( \pi_{o_2}(\mathcal{M})' \). Next define an operator from \( \mathcal{H}_{o_1} \) to \( R(S_\frac{1}{2}) \), the closure of the range of \( S_\frac{1}{2} \), such that

\[
U\pi_{o_1}(x)\Omega_{o_1} = \pi_{o_2}(x)S_\frac{1}{2} \Omega_{o_2}.
\]

Note that

\[
\|U\pi_{o_1}(x)\Omega_{o_1}\|^2 = \|\pi_{o_2}(x)S_\frac{1}{2} \Omega_{o_2}\|^2 = \omega_{1}(x^*x) = \|\pi_{o_1}(x)\Omega_{o_1}\|^2.
\]

Hence \( U \) extends to a well defined isometry. But then one readily calculates that

\[
U\pi_{o_1}(x)U^* = \pi_{o_2}(x) \big|_{R(S_\frac{1}{2})}
\]

i.e., \( \pi_{o_1}(\mathcal{M}) \) is unitarily equivalent to the subrepresentation of \( \pi_{o_2}(\mathcal{M}) \) acting on \( R(S_\frac{1}{2}) \).

Finally if \( \mu_1 \) and \( \mu_2 \) are mutually absolutely continuous then \( f \) is positive \( \mu_2 \)-almost everywhere and \( S_f \) is invertible by Proposition A2. Thus \( R(S_\frac{1}{2}) = \mathcal{H}_{o_2} \) and \( (\mathcal{H}_{o_1}, \pi_{o_1}) \) and \( (\mathcal{H}_{o_2}, \pi_{o_2}) \) are unitarily equivalent.

Now we are in a position to prove Theorem A1.

**Proof of Theorem A1.** Let \( E \) be an arbitrary Borel set in \( E_\phi \). Since \( \tau \) is strongly continuous one can define a unique regular Borel measure \( \mu_f \) such that

\[
\mu_f(E) = \int_{\omega^* \tau \in E} dt f(t).
\]

But \( f \) has total integral one and hence \( \mu_f \) is a probability measure. Moreover

\[
\int d\mu_f(\omega')\omega' = \int dt f(t)\omega^* \tau_t = \omega_f.
\]

Similarly for \( e; e(t) = \exp \{-|t|\}/2 \) one can introduce a measure \( \mu_e \) and a state \( \omega_e \). But since \( f \) and \( e \) are almost everywhere positive the measures \( \mu_f \) and \( \mu_e \)
are mutually absolutely continuous and $\omega_f$ and $\omega_e$ generate unitarily equivalent representations by Proposition A3. But the representation associated with $\omega_e$ is covariant, by the proof of Proposition 2, and hence the representation associated with $\omega_f$ is also covariant.

Finally we remark that the observation that $\omega_e$ generates a covariant representation can be used to reestablish a result of Borchers [9]; the representation $(\mathcal{H}_e, \pi_\omega)$ extends to a covariant representation if, and only if, $t \mapsto \omega \circ \tau_t$ is norm continuous. The necessity of the continuity condition is straightforward. The sufficiency follows by noting that $\omega$ is the norm limit of the sequence of states

$$\omega_n = \frac{1}{2} \sum dt e^{-|t|} \omega \circ \tau_{t/n}$$

and hence $\pi_\omega$ is quasi-contained in the direct sum of the covariant representations $\pi_{\omega_n}$. In fact Borchers obtains his result for general locally compact groups of automorphisms.

References