Pursell-Shanks Type Theorem for Orbit Spaces of G-Manifolds

By

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§ 0. Introduction

Pursell and Shanks [8] proved that a Lie algebra isomorphism between Lie algebras of all $C^\infty$ vector fields with compact support on paracompact connected $C^\infty$ manifolds $M$ and $N$ yields a diffeomorphism between the manifolds $M$ and $N$. Similar results hold for some other structures on manifolds. Indeed, Omori [6] proved the corresponding results in the case of volume structures, symplectic structures, contact structures and fibering structures with compact fibers. The case of complex structures was studied by Amemiya [1]. Koriyama [5] proved that in the case of Lie algebras of vector fields with invariant submanifolds.

Recently, Fukui [4] studies the case of Lie algebras of $G$-invariant $C^\infty$ vector fields with compact support on paracompact free smooth $G$-manifolds when $G$ is a compact connected semi-simple Lie group. The corresponding result is no longer true when $G$ is not semi-simple or $G$ does not act freely.

In this paper, we consider Pursell-Shanks type theorem for orbit spaces of smooth $G$-manifolds in the case of $G$ a compact Lie group. For a smooth $G$-manifold $M$, the orbit space $M/G$ inherits a smooth structure by defining a function on $M/G$ to be smooth if it pulls back to a smooth function on $M$, and the Zariski tangent space of $M/G$ can be defined. This smooth structure of the orbit space was studied by Schwarz [9], [11], Bierstone [2], Poénaru [7] and Davis [3]. Schwarz [10] defined a Lie algebra $\mathfrak{X}(M/G)$ of smooth vector fields on the orbit

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space $M/G$, and proved $\pi_* (\mathfrak{g}_G(M)) = \mathfrak{k}(M/G)$, where $\mathfrak{g}_G(M)$ is the Lie algebra of all $G$-invariant $C^\infty$ vector fields with compact support on $M$ and $\pi: M \to M/G$ is a natural projection.

The purpose of this paper is to prove the following:

**Theorem.** Let $G$ and $G'$ be compact Lie groups. Let $M$ and $N$ be connected paracompact smooth $G$-manifold and $G'$-manifold without boundary, respectively. There exists a Lie algebra isomorphism $\Phi: \mathfrak{k}(M/G) \to \mathfrak{k}(N/G')$ if and only if there exists a strata preserving diffeomorphism $\sigma: M/G \to N/G'$ such that $\Phi = \sigma_*$. 

Main part of the proof of our theorem is to find maximal ideals of $\mathfrak{k}(M/G)$. By the theorem of Schwarz, maximal ideals of $\mathfrak{k}(M/G)$ are induced from those of $\mathfrak{g}_G(M)$. To determine the maximal ideals of $\mathfrak{g}_G(M)$, we use the parallel method to those of Pursell-Shanks [8] and Koriyama [5].

§ 1. The Tangent Space of an Orbit Space

In this paper, we consider $C^\infty$ smooth category. Let $G$ and $G'$ be compact Lie groups. Let $M$ and $N$ be connected paracompact smooth $G$-manifold and $G'$-manifold without boundary, respectively. Put $\overline{M} = M/G$, $\overline{N} = N/G'$. The orbit space $\overline{M}$ has an induced smooth structure such that a function $f: \overline{M} \to \mathbb{R}$ is smooth if the composition $\overline{M} \to \overline{M} \to \mathbb{R}$ is smooth, where $\pi$ is the natural projection. Let $C^\infty(\overline{M})$ denote the set of all smooth functions on $\overline{M}$. A map $h: \overline{M} \to \overline{N}$ is smooth if, $f \circ h \in C^\infty(\overline{M})$ for any $f \in C^\infty(\overline{N})$, and we say that $h$ is diffeomorphic if $h^{-1}$ is also smooth.

We can define a tangent space of the orbit space as usual. A tangent vector $v$ of $\overline{M}$ at $p$ is a correspondense assigning to any smooth functions $f$, $g$ around $p$ real numbers $v(f)$, $v(g)$ with the following conditions:

1. $v(\lambda f + \mu g) = \lambda v(f) + \mu v(g)$ for $\lambda, \mu \in \mathbb{R}$,
2. $v(fg) = v(f)g(p) + f(p)v(g)$.

Put $\tau(\overline{M}) = \bigcup_{p \in \overline{M}} \tau_p(\overline{M})$. 

Given \( a \in M \), let \( G_a \) denote the isotropy group at \( a \) and \( V_a \) be a linear slice at \( a \). Then \( V_a \) is a \( G_a \)-module. Put \( \rho = \pi (a) \) and \( \overline{V}_\rho = V_a / G_a \). Then \( \overline{V}_\rho \) is an open neighborhood of \( \rho \) in \( \overline{M} \).

**Proposition 1.1** (cf. Davis [3], Proposition 2.3).

1. \( \tau_\rho (\overline{M}) \cong \tau_\rho (\overline{V}_\rho) \).
2. Let \( \overline{\mathcal{H}}_\rho \) denote the germs of smooth functions on \( \overline{V}_\rho \), which vanish at \( \rho \). Then \( \tau_\rho (\overline{V}_\rho) \cong \text{Hom} (\overline{\mathcal{H}}_\rho / \overline{\mathcal{H}}_\rho ^*, R) \).

Let \( H \) be a compact Lie group and let \( V \) be an \( H \)-module. By a theorem of Hilbert ([13], p. 275), the algebra of \( H \)-invariant polynomials \( R[V]^H \) is finitely generated.

**Theorem 1.2** (Schwarz [9]). Let \( \{ \theta_1, \ldots, \theta_i \} \) be a set of generators for \( R[V]^H \), and let \( \theta = (\theta_1, \ldots, \theta_i) : V \to R^i \). Then

1. \( \theta^* C^\infty (R^i) = C^\infty _H (V) \).
2. The orbit map \( \overline{\theta} : V/H \to R^i \) of \( \theta \) is a topological embedding.

**Proposition 1.3** (cf. Davis [3] Lemma 2.1). Let \( R[V]^H \) denote the algebra of \( H \)-invariant polynomials which vanish at \( 0 \). Then

1. \( \overline{\mathcal{H}}_\rho / \overline{\mathcal{H}}_\rho ^* \cong R[V]^H / (R[V]^H)^* \).
2. If \( \{ \theta_1, \ldots, \theta_i \} \) is a minimal set of generators for \( R[V]^H \), then the dimension of \( \tau_\rho (V/H) \) is \( s \).

§ 2. Smooth Vector Fields on an Orbit Space

Let \( X : \overline{M} \to \tau (\overline{M}) \) be a section. For any \( f \in C^\infty (\overline{M}) \), we can define a function \( X(f) : \overline{M} \to R \) by \( X(f) (\rho) = X_\rho (f) \). If \( X(f) \in C^\infty (\overline{M}) \) for any \( f \in C^\infty (\overline{M}) \), then we say \( X \) is a smooth vector field on \( \overline{M} \). Let \( \mathfrak{X}(\overline{M}) \) denote the Lie algebra of all smooth vector fields on \( \overline{M} \). Let \( DC^\infty (\overline{M}) \) denote the set of all derivations of \( C^\infty (\overline{M}) \). Using Theorem 1.2 (1) we have:

**Proposition 2.1.** \( \mathfrak{X}(\overline{M}) \) is isomorphic to \( DC^\infty (\overline{M}) \) as a Lie algebra.
The orbit space $\overline{M}$ is stratified by its orbit type.

**Definition 2.2** (Schwarz [11]). A smooth vector field $X$ on $\overline{M}$ is said to be strata preserving if $X_p \in \tau_p(\sigma_p)$ for any $p \in \overline{M}$, where $\sigma_p$ denotes the stratum of $\overline{M}$ containing $p$. Let $\mathfrak{x}(\overline{M})$ denote the set of all strata preserving smooth vector fields with compact support on $\overline{M}$. $\mathfrak{x}(\overline{M})$ is a Lie subalgebra of $\mathcal{D}C^\infty(\overline{M})$. Let $\mathfrak{x}_0(M)$ denote the set of all $G$-invariant smooth vector fields with compact support on $M$. There is a Lie algebra homomorphism $\pi_* : \mathfrak{x}_0(M) \rightarrow \mathcal{D}C^\infty(M)$ defined by $\pi_*(X)(\tilde{f}) = X(f)$, where $f \in C_0^\infty(M)$ and $\tilde{f}$ is the orbit map of $f$.

**Theorem 2.3** (Schwarz [11]). The image of the homomorphism $\pi_* : \mathfrak{x}_0(M) \rightarrow \mathcal{D}C^\infty(\overline{M})$ is $\mathfrak{x}(\overline{M})$.

§ 3. Maximal Ideals of $\mathfrak{x}(\overline{M})$

Let $a \in M$ and put $p = \pi(a) \in \overline{M}$. Let $V_a$ be a linear slice at $a$. Then $N_a = G \times_{\sigma_a} V_a$ is equivalent to a linear tubular neighborhood of the orbit $G(a)$ of $a$. Let $\tau(N_a)$ be the tangent bundle of the $G$-manifold $N_a$, and let $\Gamma_0(\tau(N_a))$ denote the set of all $G$-invariant smooth sections of $\tau(N_a)$. Let $\tau(V_a)$ be the tangent bundle of the $G_a$-manifold $V_a$, and let $\Gamma_{0_a}(\tau(V_a))$ denote the set of all $G_a$-invariant smooth sections of $\tau(V_a)$. Then we have canonical isomorphisms $\Gamma_0(\tau(N_a)) \cong \Gamma_{0_a}(\tau(N_a)|V_a)$ and $C_0^\infty(N_a) \cong C_0^\infty_a(V_a)$. It is easy to see the following:

**Lemma 3.1.** (1) For any $X \in \Gamma_{0_a}(\tau(V_a))$, there exists $Y \in \mathfrak{x}_0(M)$ such that $Y = X$ on a $G_a$-invariant neighborhood $U_a$ of $a$ in $V_a$.

(2) For any $f \in C_0^\infty_a(V_a)$, there exists $F \in C_0^\infty(M)$ such that $F = f$ on a $G_a$-invariant neighborhood $U_a$ of $a$ in $V_a$.

Put $\overline{M}_0 = \{ q \in \overline{M} ; X_q = 0 \text{ for any } X \in \mathfrak{x}(\overline{M}) \}$, and put $\overline{M}_1 = \overline{M} - \overline{M}_0$.

**Proposition 3.2.** $\overline{M}_0$ is discrete.

**Proof.** For any $a \in M$, let $\{ x_1, \ldots, x_n \}$ be a canonical coordinate of a
linear slice $V_a$ of $a$. We can assume $G_a$ acts orthogonally on $V_a$. Then the radial vector field $X = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}$ is a $G_a$-invariant smooth vector field on $V_a$. Let $f: V_a \to R$ be a $G_a$-invariant smooth function defined by $f(x_1, \ldots, x_n) = x_1^2 + \cdots + x_n^b$. By Lemma 3.1, there exist $Y \in \mathfrak{f}_a(M)$ and $F \in C^\infty_0(M)$ such that $Y = X$ and $F = f$ on a $G_a$-invariant neighborhood $U_a$ of $a$ in $V_a$, respectively. Put $\overline{U}_p = U_a/G_a$, $\overline{Y} = \pi_*(Y)$ and let $\overline{F}$ be the orbit map of $F$. Then $\overline{F}(\overline{F}) \neq 0$ on $\overline{U}_p - \{p\}$, and Proposition 3.2 follows.

Note that $\pi_a: N_a \to G(a)$ is a $G$-vector bundle. The tangent bundle $r(N_a)$ of $N_a$ is isomorphic to $\pi_a^*(r(G(a))) \oplus \xi_a$ as a $G$-vector bundle, where $\xi_a$ is a bundle along the fibres of $N_a$. Let $r_a$ be the composition

\[
\mathfrak{f}_a(M) \xrightarrow{\text{restriction}} \Gamma_a(r(N_a)) \cong \Gamma_a(r(N_a)|V_a) \\
\xrightarrow{\text{projection}} \Gamma_a(\xi_a|V_a) \cong \Gamma_a(r(V_a)).
\]

It is easy to see that $r_a$ is a Lie algebra homomorphism. Put $\Gamma_a(\tau(V_a))_v = \{X \in \Gamma_a(\tau(V_a))_v; X_a = 0\}$. For $X \in \Gamma_a(\tau(V_a))_v$, we denote $j^r_a(X)$ the $r$-jet of $X$ at $a$ ($r = 1, 2, \ldots$). Put $\Gamma_a(\tau(V_a))_v^k = \{X \in \Gamma_a(\tau(V_a))_v; j^r_a(X) = 0$ for $1 \leq r \leq k\}$ ($1 \leq k \leq \infty$).

For $q \in \overline{M}_b$, choose a point $b \in \pi^{-1}(q)$. Let $\mathfrak{gl}_a(V_b)$ denote the set of $G_b$-invariant endomorphisms of $V_b$. Note that, for $X \in \Gamma(\tau(V_b))$, $j^1_b(X)$ defines an element of $\mathfrak{gl}(V_b)$ as usual. It is easy to see that $j^1_b(X) \in \mathfrak{gl}_a(V_b)$, for $X \in \Gamma_a(\tau(V_b))$.

**Lemma 3.3.** $j^1_b: \Gamma_a(\tau(V_b)) \to \mathfrak{gl}_a(V_b)$ is an onto Lie algebra homomorphism.

**Proof.** Since $\pi(b) = q \in \overline{M}_b$, $\Gamma_a(\tau(V_b)) = \Gamma_a(\tau(V_b))_v$. Then $\Gamma_a(\tau(V_b))/\Gamma_a(\tau(V_b))_v \cong \mathfrak{gl}_a(V_b)$ and Lemma 3.3 follows.

By Proposition 3.2, $\overline{M}_b$ is discrete. Then $\overline{M}_b$ is a countable set $\{q_i; i \in I\}$. Choose a point $b_i \in \pi^{-1}(q_i)$ for each $q_i \in \overline{M}_b$. Put $J^1_i(M) = \mathfrak{gl}_{a_i}(V_{b_i})$ and put $J^1(M) = \prod_{i \in I} J^1_i(M)$ which consists of those elements having only finite number of non-zero factors. Then we have:
Corollary 3.4. The composition
\[ J^1: \mathfrak{g}_0(M) \xrightarrow{\prod t_i} \prod_{i \in I} \Gamma_{\theta_i}(\tau(V_{t_i})) \xrightarrow{\prod j_{t_i}^1} J^1(M) \]
is an onto Lie algebra homomorphism.

\( G_{t_i}\)-module \( V_{t_i} \) is isomorphic to \( \bigoplus d_{t_i} W_{t_i} \). Here \( d_{t_i} \) is a non-negative integer and \( W_{t_i} \) runs over the inequivalent irreducible \( G_{t_i}\)-modules. Let \( K_{t_j} \) be the real numbers \( R \), complex numbers \( C \) or quaternionic numbers \( H \) if \( \dim_R \mathfrak{g}_l(\theta_{t_i}(W_{t_j})) = 1, 2 \) or 4, respectively. Then \( \mathfrak{g}_l(\theta_{t_i}(V_{t_i})) \cong \bigoplus \mathfrak{g}_l(d_{t_i}, K_{t_j}) \).

Proposition 3.5. \( \mathfrak{gl}(d, R) \cong R \oplus \mathfrak{gl}(d, R) \),
\( \mathfrak{gl}(d, C) \cong C \oplus \mathfrak{gl}(d, C) \) and
\( \mathfrak{gl}(d, H) \cong R \oplus \mathfrak{gl}(d, H) \),
where \( \mathfrak{gl}(d, K) = [\mathfrak{gl}(d, K), \mathfrak{gl}(d, K)] \) for \( K = R, C \) or \( H \).

Proof. Note that \( \mathfrak{sl}(d, H) = \{ X \in \mathfrak{gl}(n, H) ; \text{Re } \text{Tr}(X) = 0 \} \) and \( \mathfrak{sl}(d, H) \) is a simple Lie algebra. Other cases are similar to this.

Next we consider maximal ideals of \( \Gamma_{\theta_a}(\tau(V_a)) \) for \( a \in M \) such that \( \pi(a) = p \in M_1 \). First we need the following:

Lemma 3.6. Let \( H \) be a compact Lie group and let \( V \) be an \( H \)-module. For \( Y \in \Gamma(\tau(V)) \), we define \( \tilde{Y} \in \Gamma_H(\tau(V)) \) by \( \tilde{Y}_p = \int_H (h_* Y)_p \) dh for \( p \in V \). Then \( [X, \tilde{Y}] = \int_H (h_* [X, Y]) \) dh for \( X \in \Gamma_H(\tau(V)) \). Here \( (h_* Y)_p = (dh)_{h^{-1}p} Y_{h^{-1}p} \).

Proof. Let \( \{ x_1, \ldots, x_n \} \) be a canonical coordinate of \( V \). For \( p \in V \), \( f \in C^\infty(V) \), we have:
\[
\tilde{Y}_p(f) = \left( \int_H \left( \sum_{i=1}^{n} (h_* Y)_p(x_i) \left( \frac{\partial}{\partial x_i} \right)_p \right) dh \right)(f)
= \sum_{i=1}^{n} \left( \int_H (h_* Y)_p(x_i) dh \right) \left( \frac{\partial}{\partial x_i} \right)_p (f)
\]
\[= h \left( \sum_{i=1}^{n} (h \ast Y)_{\varphi}(x_i) \left( \frac{\partial f}{\partial x_i} \right) \right) dh \]

\[= \int_{H} (h \ast Y)_{\varphi} (f) dh. \]

Then

\[ [X, \bar{Y}]_{\varphi}(f) = X_{\varphi} \left( \int_{H} (h \ast Y) (f) dh \right) - \int_{H} (h \ast Y)_{\varphi}(Xf) dh \]

\[= \int_{H} (X_{\varphi}(h \ast Y)(f) - (h \ast Y)_{\varphi}(Xf)) dh \]

\[= \int_{H} [X, h \ast Y]_{\varphi}(f) dh \]

\[= \left( \int_{H} [X, h \ast Y]_{\varphi} dh \right)(f). \]

**Lemma 3.7.** Suppose that \( \mathcal{R} \) is a proper ideal of \( \Gamma_{\sigma}(\tau(V_a)) \) which contains \( \Gamma_{\sigma}(\tau(V_a))_{0^\circ} \) for \( a \in M \) such that \( \pi(a) = \rho \in \overline{M} \). Then \( \mathcal{R} \) is contained in \( \Gamma_{\sigma}(\tau(V_a))_0 \).

**Proof.** Suppose there exists \( X \in \mathcal{R} \) with \( X \neq 0 \). By Koriyama [5] Lemma 2.1, for any \( Z \in \Gamma_{\sigma}(\tau(V_a)) \) there exist a \( G_a \)-invariant neighborhood \( U \) of \( a \) in \( V_a \) and \( Y \in \Gamma(\tau(V_a)) \) such that \([X, Y] = Z\) on \( U \). Put \( \bar{Y} = \int_{\sigma_a} g \ast Y \, dg \in \Gamma_{\sigma}(\tau(V_a)) \). By Lemma 3.6, we have

\[ [X, \bar{Y}] = \int_{\sigma_a} g \ast [X, Y] \, dg = \int_{\sigma_a} g \ast Z \, dg = Z \text{ on } U. \]

Put \( Z_a = Z - [X, \bar{Y}] \). Then \( Z_a \in \Gamma_{\sigma}(\tau(V_a))_{0^\circ} \) which is contained in \( \mathcal{R} \). Since \( \mathcal{R} \) is an ideal, \( Z \in \mathcal{R} \). Thus \( \mathcal{R} = \Gamma_{\sigma}(\tau(V_a))_0 \) which is a contradiction to \( \mathcal{R} \) a proper ideal.

By Lemma 3.7, there exists a unique maximal ideal \( \mathcal{L}_a \) of \( \Gamma_{\sigma}(\tau(V_a)) \) satisfying \( \Gamma_{\sigma}(\tau(V_a))_0 \subset \mathcal{L}_a \subset \Gamma_{\sigma}(\tau(V_a))_0 \). Put \( \mathcal{Z}_a = \{X \in \mathcal{L}_a(M) ; r_a(X) \in \mathcal{L}_a\} \).

**Proposition 3.8.** \( \mathcal{Z}_a \) is a maximal ideal of \( \mathcal{L}_a(M) \).

**Proof.** Put \( V_a(\rho) = \{v \in V_a ; \|v\| < \rho\} \) for a positive number \( \rho \). \( \mathcal{R}_a(\rho) = \{Y \in \Gamma_{\sigma}(\tau(V_a)) ; \text{supp } Y \subset V_a - V_a(\rho)\} \) is an ideal of \( \Gamma_{\sigma}(\tau(V_a)) \) which is contained in \( \Gamma_{\sigma}(\tau(V_a))_0 \). Then \( \mathcal{R}_a(\rho) \) is contained in \( \mathcal{L}_a \). It is clear that \( \mathcal{Z}_a \) is an ideal of \( \mathcal{L}_a(M) \).
Let $\mathcal{M}$ be a maximal ideal of $\mathfrak{k}_0(M)$ which contains $\mathfrak{F}_a$. Suppose that there exists $X \in \mathcal{M}$ with $r_\alpha(X) \neq 0$. Similarly as in the proof of Lemma 3.7, we can prove $\mathcal{M} = \mathfrak{k}_0(M)$, which is a contradiction. Then $r_\alpha(\mathcal{M})$ is contained in $\mathcal{R}_a(\tau(V_\alpha))$. Combining $\mathcal{R}_a(\rho) \subset \mathcal{R}_a$ and Lemma 3.1, we see $r_\alpha(\mathcal{M}) + \mathcal{R}_a$ is an ideal of $\mathcal{R}_a(\tau(V_\alpha))$. Therefore $r_\alpha(\mathcal{M})$ is contained in $\mathcal{R}_a$, and $\mathcal{M} = \mathfrak{F}_a$. Thus Proposition 3.8 follows.

Put $\overline{\mathfrak{F}}_p = \pi_\alpha(\mathfrak{F}_a)$ and $\mathfrak{k}(\overline{M})_p = \{X \in \mathfrak{k}(\overline{M}) ; X_p = 0\}$. Then $\overline{\mathfrak{F}}_p$ is contained in $\mathfrak{k}(\overline{M})_p$, and $\overline{\mathfrak{F}}_p$ is a maximal ideal.

**Lemma 3.9.**

1. $\overline{\mathfrak{F}}_p$ is an infinite codimensional maximal ideal of $\mathfrak{k}(\overline{M})$ for $p \in \overline{M}_1$.
2. For a maximal ideal $\mathfrak{L}$ of $J^1(M)$, put $\mathcal{M} = (J^1)^{-1}(\mathfrak{L})$. Then $\mathcal{M}$ is a finite codimensional maximal ideal of $\mathfrak{k}_0(M)$.

**Proof.**

1. For $a \in \pi^{-1}(p)$, there exists $X \in \Gamma_a(\tau(V_a))$ with $X_a \neq 0$. Then there exists a $G_a$-invariant local one parameter group of transformations $\phi; (-\varepsilon, \varepsilon) \times U \rightarrow V_a$ defined on a $G_a$-invariant neighborhood $U$ such that $\frac{\partial \phi}{\partial t}(t, u) = X_{\phi(t, u)}$. Let $\theta: (-\varepsilon, \varepsilon) \rightarrow V_a$ be a map defined by $\theta(t) = \phi(t, a)$. Since $X_a \neq 0$, $\theta$ is an embedding for a sufficiently small number $\varepsilon$. Let $W$ be a $G_a$-invariant normal space of $\theta((\varepsilon, \varepsilon))$ at $a$ in $V_a$. Then we may assume that $\phi: (-\varepsilon, \varepsilon) \times W \rightarrow V_a$ is a $G_a$-invariant embedding. Let $\{w_1, \ldots, w_{n-1}\}$ be a canonical coordinate of $W$. We have a local coordinate $\{x_1, \ldots, x_n\}$ of $V_a$ around a neighborhood $U_i = \phi((\varepsilon, \varepsilon) \times W)$ of $a$ given by $x_1(\phi(t, w_1, \ldots, w_{n-1})) = t$, $x_1(\phi(t, w_1, \ldots, w_{n-1})) = w_{i-1}$ for $i = 2, \ldots, n$. Note that $X = \frac{\partial}{\partial x_1}$ on $U_i$.

By Lemma 3.1, there are $X_i \in \mathfrak{k}_0(M)$ and $f \in C^\infty(M)$ such that $X_i = X$ and $f = x_1$ on a neighborhood $U_i \subset U$ of $a$ in $V_a$, respectively. Let $Y \in \mathfrak{F}_a$ and $r_\alpha(Y) = \sum_{i=1}^n \xi_i \frac{\partial}{\partial x_i}$ on $U_i$. Then $r_\alpha[X_i, Y] = \left[\frac{\partial}{\partial x_1}, \sum_{i=1}^n \xi_i \frac{\partial}{\partial x_i}\right] = \sum_{i=1}^n \xi_i \frac{\partial}{\partial x_1} \frac{\partial}{\partial x_i}$ on $U_i$. Since $r_\alpha[X_i, Y] \in \mathcal{R}_a$, we have $\frac{\partial^k \xi_i}{\partial x_1^k} (a) = 0$ for $i = 1, \ldots, n$. Inductively we have $\frac{\partial^k \xi_i}{\partial x_1^k} (a) = 0$ for $i = 1, \ldots, n$ and $k = 1, 2, \ldots$. Let $\alpha: \mathfrak{k}_a(M) \rightarrow R[[x_1]]$ be an $R$-module homomorphism defined by $\alpha(Z) = \sum_{k=1}^n \frac{\partial^k \xi_i}{\partial x_1^k} (a) x_1^k$ if $r_\alpha(Z) = \sum_{i=1}^n \xi_i \frac{\partial}{\partial x_i}$ on $U_i$. Since $\alpha(\mathfrak{F}_a) = 0$. 


the above map \( \alpha \) induces an \( R \)-module homomorphism \( \beta: \mathfrak{I}_o(M)/\mathfrak{F} \to R[[x_1]] \). Note that \( \alpha(f^jX_i) = j! \cdot x_i^j \) for \( j = 1, 2, \ldots \), and \( \dim(\text{Image } \beta) = \infty \). Since \( \mathfrak{F} \supset \ker \pi_* \), we have \( \dim \mathfrak{I}(M)/\mathfrak{F} = \infty \).

(2) There is an index \( i \in I \) such that \( \mathfrak{J} \) does not contain \( \mathfrak{J}_i(M) \). Since \( \mathfrak{J} \) is a maximal ideal, \( \mathfrak{J} + \mathfrak{J}_i(M) = \mathfrak{J}_1(M) \). Then \( \mathfrak{I}_o(M)/\mathfrak{M} \cong \mathfrak{J}_1(M)/\mathfrak{J} \cong \mathfrak{J}_1(M)/(\mathfrak{J} \cap \mathfrak{J}_1(M)) \). Since \( \mathfrak{J}_1(M) \) is finite dimensional, \( \mathfrak{M} \) is finite codimensional. This completes the proof of Lemma 3.9.

**Proposition 3.10.** Let \( \mathfrak{M} \) be a maximal ideal of \( \mathfrak{I}(M) \). Then \( \mathfrak{M} = \mathfrak{M}_p \) for \( p \in M_1 \) or \( \pi_*(\mathfrak{M}) = (\mathfrak{J}_1)^{-1}(\mathfrak{J}) \) for some maximal ideal \( \mathfrak{J} \) of \( \mathfrak{J}_1(M) \).

Proposition 3.10 plays a key role to prove our theorem. We shall prove Proposition 3.10 in Section 6.

**§ 4. Stone Topology of Maximal Ideals of \( \mathfrak{I}(M) \)**

Let \( M^* \) be the set of all maximal ideals of \( \mathfrak{I}(M) \). \( M^* \) is determined by Proposition 3.10.

**Definition 4.1** (Stone topology of \( M^* \), cf. Pursell-Shanks [8]).

The Stone topology on \( M^* \) is defined by closure operator \( \text{CL} \) as follows:

1. \( \text{CL}(\phi) = \phi \).
2. If \( B \neq \phi \) is a subset of \( M^* \), then \( \text{CL}(B) = \{ \mathfrak{M} \in M^*; \mathfrak{M} \supset \cap B \} \).

Let \( S(M_0) \) be the set of all subsets of \( M_0 \). Let \( \tau_\pi \) (or simply \( \tau \)) be a map from \( M^* \) to \( M_1 \cup S(M_0) \) defined as follows:

1. \( \tau(\mathfrak{M}_p) = p \) for \( p \in M_1 \).
2. If \( \mathfrak{M} \) is a maximal ideal of \( \mathfrak{I}(M) \) such that \( \mathfrak{J}_1(\pi^{-1}_*(\mathfrak{M})) \) is a maximal ideal of \( \mathfrak{J}_1(M) \), then \( \tau(\mathfrak{M}) = \{ q_t \in M_0; \mathfrak{J}_1(\pi^{-1}_*(\mathfrak{M})) \) does not contain \( \mathfrak{J}_1(M) \} \). (Making use of Proposition 3.5, we see that \( \tau(\mathfrak{M}) \) does not consist of a single point.)

For a subset \( A \) of \( M \), we denote the closure of \( A \) in \( M \) by \( \text{cl}(A) \).

**Lemma 4.2.** If \( \text{cl}(A) \) is contained in \( M_1 \), then \( \text{CL}(\tau^{-1}(A)) \) ...
Proof. First we shall prove "\( \subset \)". Let \( \mathcal{M} \in \text{CL}(\tau^{-1}(A)) \). Assume that \( \mathcal{M} \) is not contained in \( \text{cl}(A) \).

In the case \( \mathcal{M} = \overline{\mathcal{N}}_p \) for \( p \in \mathcal{M} \) : We can find \( X \in \mathcal{F}(\mathcal{M}) \) such that \( X \neq 0 \) and \( \text{supp} X \cap \text{cl}(A) = \emptyset \). Then \( X \in \bigcap_{p' \in A} \overline{\mathcal{N}}_{p'} \subset \mathcal{M} \), which is a contradiction to \( X \neq 0 \).

In the case \( \pi_*^{-1}(\mathcal{M}) = (J')^{-1}(\mathcal{L}) \) for a maximal ideal \( \mathcal{L} \) of \( J'(\mathcal{M}) \) :

There exists \( Y \in \mathcal{F}(\mathcal{M}) \) such that \( J'(Y) \notin \mathcal{L} \). Let \( \{i_1, \ldots, i_k\} \) be a set \( \{i \in I; \beta_{i_j}(r_{i_j}(Y)) \neq 0\} \). There exists \( \psi \in C_0(M) \) such that \( \psi = 1 \) on a neighborhood of \( b_{i_j} \) \( (j = 1, \ldots, k) \) and \( \psi = 0 \) on \( \pi^{-1}(\text{cl}(A)) \). Put \( X = \psi Y \). Then \( J'(X) = J'(Y) \), and \( \pi_*(X) \in \mathcal{M} \). Moreover \( \pi_*(X) \in \bigcap_{p' \in A} \overline{\mathcal{N}}_{p'} \subset \mathcal{M} \), which is a contradiction.

Next we shall prove "\( \supset \)". Note that an ideal \( \bigcap_{\tau(\mathcal{N}) \in A} \overline{\mathcal{N}}_p \) is contained in \( \mathcal{F}(\mathcal{M}) \) for any \( p \in \text{cl}(A) \). Then \( \bigcap_{\tau(\mathcal{N}) \in A} \mathcal{R} \) is contained in a maximal ideal \( \overline{\mathcal{N}}_p \) and \( \overline{\mathcal{N}}_p \subset \text{CL}(\tau^{-1}(A)) \) for any \( p \in \text{cl}(A) \). This completes the proof of Lemma 4.2.

If \( \Phi: \mathcal{F}(\mathcal{M}) \rightarrow \mathcal{F}(\mathcal{N}) \) is a Lie algebra isomorphism, then \( \Phi* : \overline{\mathcal{M}}* \rightarrow \overline{\mathcal{N}}* \) is homeomorphic. Combining Lemma 3.9 and Proposition 3.10, \( \Phi^*(\tau_N^{-1}(\mathcal{M})) = \tau_N^{-1}(\mathcal{N}) \) and \( \Phi^*(\tau_N^{-1}(\mathcal{M}_0)) = \tau_N^{-1}(\mathcal{N}_0) \). By Lemma 4.2, we have

**Corollary 4.3.** If \( \Phi: \mathcal{F}(\mathcal{M}) \rightarrow \mathcal{F}(\mathcal{N}) \) is a Lie algebra isomorphism, there exists a homeomorphism \( \sigma : \overline{\mathcal{M}}_1 \rightarrow \overline{\mathcal{N}}_1 \) defined by \( \sigma(p) = \tau_N(\Phi^*(\overline{\mathcal{N}}_p)) \).

We shall extend the homeomorphism \( \sigma : \overline{\mathcal{M}}_1 \rightarrow \overline{\mathcal{N}}_1 \) to a homeomorphism from \( \overline{\mathcal{M}} \) to \( \overline{\mathcal{N}} \).

**Lemma 4.4.** Let \( U \) be a neighborhood of \( q \in \mathcal{M}_0 \) such that \( \text{cl}(U) \cap \mathcal{M}_0 = \{q\} \). Then \( \text{CL}(\tau^{-1}(U)) = \tau^{-1}(\text{cl}(U)) \).

The proof of Lemma 4.4 is similar to that of Lemma 4.2.
Proposition 4.5. The map $\sigma: \overline{M} \to \overline{N}$ is extended to a homeomorphism from $\overline{M}$ to $\overline{N}$.

Proof. For $q \in \overline{M}$, let $b \in M$ such that $\pi(b) = q$. Let $V_b$ be a linear slice at $b$. Put $U_q = V_b/G_b$, $U_q = U_q \setminus \{q\}$. Since $q \in \overline{M}$, it is easy to see that $U_q$ is connected. Note that $\cap_{r(\mathfrak{m}) \in \overline{U}_q} \mathfrak{m} = \cap_{r(\mathfrak{m}) \in U_q} \mathfrak{m}$. By Lemma 4.4,

$$CL(\tau^{-1}_N(U_q)) = CL(\tau^{-1}_M(U_q))$$

$$= \tau^{-1}_N(cl(U_q)) = \tau^{-1}_M(cl(U_q)).$$

Since $\theta^*: \overline{M}^* \to \overline{N}^*$ is homeomorphic and since $\sigma \circ \tau_M = \tau_N \circ \theta^*$,

$$CL(\tau^{-1}_N(\sigma(U_q))) = CL(\theta^*(\tau^{-1}_M(U_q)))$$

$$= \theta^*(CL(\tau^{-1}_M(U_q))) = \theta^*(\tau^{-1}(cl(U_q))).$$

There exists a maximal ideal $\mathfrak{m} \in \overline{M}^*$ such that $\tau_M(\mathfrak{m}) = q$. Then $\tau_N(CL(\tau^{-1}_N(\sigma(U_q)))) \cap \overline{N}_0$ contains $\tau_N(\theta(\mathfrak{m}))$, and, by Lemma 4.2, $cl(\sigma(U_q)) \cap \overline{N}_0 \neq \emptyset$. Since $\sigma(U_q)$ is connected, $cl(\sigma(U_q)) \cap \overline{N}_0 = \{q'\}$ for some $q' \in \overline{N}_0$. Let $\sigma(q) = q'$ for $q \in \overline{M}$. Then it is clear that $\sigma: \overline{M} \to \overline{N}$ is homeomorphic.

§ 5. Proof of Theorem

In this section we shall prove our theorem. Let $\Phi: \mathfrak{g}(\overline{M}) \to \mathfrak{g}(\overline{N})$ be a Lie algebra isomorphism. By Proposition 4.5, we have a homeomorphism $\sigma: \overline{M} \to \overline{N}$ such that $\sigma(\rho) = \tau_N(\Phi(\mathfrak{g}_\rho))$ for $\rho \in \overline{M}$.

Proposition 5.1. $\sigma: \overline{M}_1 \to \overline{N}_1$ is diffeomorphic.

In order to prove Proposition 5.1, we need the following lemma.

Lemma 5.2. For $X \in \mathfrak{g}(\overline{M})$ and $\rho \in \overline{M}$, $X_\rho \neq 0$ if and only if $[X, \mathfrak{g}(\overline{M})] + \overline{X}_\rho = \mathfrak{g}(\overline{M})$.

Proof. Let $a \in M$ such that $\pi(a) = \rho$. Assume that $X_\rho \neq 0$. There exists $\mathfrak{X} \in \mathfrak{g}_a(\overline{M})$ such that $\mathfrak{X}_a \neq 0$. By the similar argument as the
proof of Lemma 3.7, we can prove that $[\tilde{X}, \mathfrak{h}(M)] + \mathfrak{z}_p = \mathfrak{h}(M)$. Conversely, suppose that $X_p = 0$. Moreover we shall assume that $[X, \mathfrak{h}(M)] + \mathfrak{z}_p = \mathfrak{h}(M)$. There exists $\tilde{X} \in \mathfrak{h}(M)$ such that $\pi_*(\tilde{X}) = X$. Put $\tilde{X}' = r_{a}(\tilde{X}) \in \Gamma_{\alpha}(\tau(V_a))$. Then $[\tilde{X}', \Gamma_{\alpha}(\tau(V_a))] + \mathfrak{N} = \Gamma_{\alpha}(\tau(V_a))$ and $\tilde{X}' = 0$. Let $V_a(0)$ and $V_a(1)$ be the trivial and non-trivial direct summand of the $G_a$-module $V_a$, respectively. Then $j_a'(\tilde{X}')$ can be expressed as $A \alpha B$, where $A \in \mathfrak{g}(V_a(0))$ and $B \in \mathfrak{gl}_{\alpha}(V_a(1))$. Since $[\tilde{X}', \Gamma_{\alpha}(\tau(V_a))] + \mathfrak{N} = \Gamma_{\alpha}(\tau(V_a))$, we can prove that $A$ is invertible. Then we have $[\tilde{X}', \Gamma_{\alpha}(\tau(V_a))] + \mathfrak{N} = \Gamma_{\alpha}(\tau(V_a))$, which implies that a linear mapping

$$
\beta: \mathfrak{gl}_{\alpha}(V_a)/j_a'(\mathfrak{N}) \rightarrow \mathfrak{gl}_{\alpha}(V_a)/j_a'(\mathfrak{N}),
$$

defined by $\beta(C + j_a'(\mathfrak{N})) = [j_a'(\tilde{X}'), C] + j_a'(\mathfrak{N})$ for $C \in \mathfrak{gl}_{\alpha}(V_a)$, is isomorphic. But this is impossible because $j_a'(\tilde{X}') \in j_a'(\mathfrak{N})$. This completes the proof of Lemma 5.2.

Proof of Proposition 5.1. Let $f$ be any smooth function on $N$. Put $g = f \circ \sigma$. We have $fY - f(\sigma(p)) = Y \in \mathfrak{X}(\tilde{N})_{\sigma(p)}$ for any $Y \in \mathfrak{X}(\tilde{N})$, $p \in M_1$, and hence, using Lemma 5.2, $\Phi^{-1}((fY) - g(p) \Phi^{-1}(Y)) \in \mathfrak{X}(\tilde{M})$, for any $p \in M_1$. Thus we have $\Phi^{-1}((fY)) = g \Phi^{-1}(Y)$ for any $Y \in \mathfrak{X}(\tilde{N})$. For any $p \in M_1$, there exist $Y \in \mathfrak{X}(\tilde{N})$ and $h \in C^\omega(\tilde{M})$ such that $\Phi^{-1}(Y)(h) \neq 0$ on a neighborhood $U$ of $p$ in $\tilde{M}$. Then $g = \Phi^{-1}(Y)(h)$ $\Phi^{-1}(Y)(h)^{-1}$ on $U$, and $g$ is smooth on $U$. Thus $f \circ \sigma$ is smooth on $M_1$ for any $f \in C^\omega(\tilde{N})$, and $\sigma$ is smooth on $M_1$. Similarly $\sigma^{-1}$ is smooth on $\tilde{N}$, and Proposition 5.1 follows.

Now we shall prove that $\sigma: M \rightarrow \tilde{N}$ is diffeomorphic. By Proposition 5.1, it is sufficient that $\sigma$ is smooth at $q \in \tilde{M}$. Let $f$ be any smooth function on $\tilde{N}$, and put $g = f \circ \sigma$. As in the proof of Proposition 5.1, $g \Phi^{-1}(Y) = \Phi^{-1}(fY)$ for any $Y \in \mathfrak{X}(\tilde{N})$. Since $\Phi$ is a Lie algebra isomorphism, $gX(h) \in C^\omega(\tilde{M})$ for any $X \in \mathfrak{X}(\tilde{M})$, $h \in C^\omega(\tilde{M})$. Let $b$ be a point of $M$ such that $\pi(b) = q$, and let $V$ be a linear slice at $b$. Let $H$ be the isotropy subgroup at $b$, and put $\overline{V} = V/H$. It suffices to prove the following:

Proposition 5.3. Let $g$ be a continuous function on $\overline{V}$ such that
\[ gX(h) \in C^\infty(\mathbb{V}) \] for any \( X \in \Gamma(\tau(\mathbb{V})) \), \( h \in C^\infty(\mathbb{V}) \). Then \( g \) is a smooth function.

Let \( \{\theta_i, \ldots, \theta_s\} \) be a minimal set of homogeneous generators of \( R[V]_\theta \) (see Davis [3], Lemma 4.6). Let \( \{x_1, \ldots, x_n\} \) be a canonical coordinate of \( V \) such that \( H \) acts orthogonally on this coordinate. Since \( q \in M_\theta \), \( \Gamma(\tau(\mathbb{V})) = \Gamma(\tau(\mathbb{V}))_0 \). It is easy to see that \( \deg \theta_i > 1 \) for \( i = 1, \ldots, s \). Then we can assume \( \theta_i = x_1 + \cdots + x_n \). Let \( X \) be a radial vector field \( \sum_{i=1}^n x_i \frac{\partial}{\partial x_i} \). Then \( X(\theta_i) = (\deg \theta_i) \theta_i \) for \( i = 1, \ldots, s \), and Proposition 5.3 follows from the following:

**Proposition 5.4.** Let \( g \) be an \( H \)-invariant continuous function on \( V \) such that \( \theta_i g \in C^\infty_H(V) \) for \( i = 1, \ldots, s \). Then \( g \) is an \( H \)-invariant smooth function.

**Proof.** Since \( \theta_i g \in C^\infty_H(V) \), it is sufficient to prove that \( g \) is smooth at 0. Put \( g_1 = \theta_i g \in C^\infty_H(V) \). First we consider the case \( s = 1 \). In this case \( V \) is a half line \( R_+ \), it follows from Theorem 1.2 that \( \bar{g}_1 = \bar{x}_1 \bar{g} \) is a smooth function on \( R_+ \), where \( \bar{g}_1 \) and \( \bar{g} \) are functions on \( R_+ \) such that \( g_1 = \bar{g}_1 \circ \bar{\theta}_i \) and \( g = \bar{g} \circ \bar{\theta}_i \), respectively. By Koriyama [5] Lemma 6.2, \( \bar{g} \in C^\infty(R_+) \), and hence \( g \in C^\infty_H(V) \).

Now we consider the case \( s \geq 2 \). Put \( R[V]_f = \{ h \in R[V]^H ; \deg h = 1 \} \). From Taylor's formula, for an integer \( m \geq 2 \), there exist \( P_m(x) \in \sum_{I \subseteq \{1\}} R[V]^H \) and \( R_m(x) \in \sum_{|I| = m + 1} x^I C^\infty(V) \) such that \( g_1(x) = P_m(x) + R_m(x) \), where \( x^I = x_1^{i_1} \cdots x_n^{i_n} \) and \( |I| = i_1 + \cdots + i_n \). Put \( g_2 = \theta_i \cdot g_1 \), \( k = \deg \theta_i + m \). Then \( \theta_i P_m \in \sum_{I \subseteq \{1\}} R[V]^H \) and \( \theta_i R_m \in \sum_{|I| = k + 1} x^I C^\infty(V) \). Let \( \Delta = \sum_{i=1}^m \frac{\partial^2}{\partial x_i} \) be the Laplacian, and put \( H^k(V) = \{ f \in R[V]_k ; \Delta f = 0 \} \). Here we need the following result:

**Theorem 5.5** (cf. [13] § 14).

\[ R[V]_t = \theta^t R[V]_{t-2} \oplus H^t(V) \] for an integer \( t \geq 2 \).

From Theorem 5.5, there exist \( Q_m \in \sum_{k \leq t-2} R[V]_t \) and \( T_m \in \sum_{k \leq t-2} H^t(V) \) such that \( \theta_i P_m = \theta_i Q_m + T_m \).
Proposition 5.6. \( T_m = 0 \) \((m=2,3,\cdots)\).

Proof. It is easy to see the following:

1. \( f \in H(V), \Delta^q(f/\theta_i) = N(p, l)f/\theta_i^{q+1}, \) where \( N(p, l) = -[4pl + 2ln - 4(p - 1)^2 + 12(p - 1) + 8] \).

2. \( T \subseteq C^\infty(V) \) with \( |I| = k + 1 \), put \( F(x) = \sum_{k=1}^{|I|-k+1} F_i(x) x^I. \) Then we have \( \Delta^q(F/\theta_i) = \sum_{k=1}^{|I|-k+1} R_{q^k} x^I \) for some \( R_{q^k} \).

Now we assume that \( T_m = \sum_{i=1}^k T_i^i \) such that \( T_i^i \subseteq H(V) \) and \( T_i^d \neq 0. \) Let \( d_i \) be an integer such that \( d = 2d_i + 1 \) or \( 2d_i + 2. \) To see Proposition 5.5, we can assume that \( k \) is an even integer, and hence \( k \geq 2d_i + 2. \) By the definitions of \( g_1 \) and \( g_2, \theta g_1 = g_2/\theta_i = Q_m + T_m/\theta_i + \theta_1 R_m/\theta_i. \)

Since \( \theta g \) is a smooth map, \( T_m/\theta_1 + \theta_2 R_m/\theta_i \) is also a smooth map.

Put \( F = \theta_2 R_m. \) Applying (2), we have \( \Delta^q(F/\theta_i) = \sum_{k=1}^{|I|-k+1} R_{q^k} x^I \) for some \( R_{q^k}(x) = \sum_{k=1}^{|I|-k+1} x^I C^\infty(V). \) Let \( a = (a_1, \cdots, a_m) \in V \) with \( a \neq 0 \) and \( \xi \) be a positive real number. Since \( k - 2d_i + 1 > 0, \) it is easy to see that

\[
\lim_{\xi \to 0} \Delta^q(F/\theta_i)|_{x = e^a} = 0.
\]

It follows from (1) that

\[
\lim_{\xi \to 0} \Delta^q(T_m^i /\theta_i)|_{x = e^a} = 0 \quad \text{for } i \geq 2d_i + 2.
\]

We can write

\[
T_m^{2d_i + 1}(x) = \sum_{|I|=2d_i + 1} \lambda_I x^I \quad \text{for } \lambda_I \in R,
\]

\[
T_m^{2d_i + 2}(x) = \sum_{|I|=2d_i + 2} \mu_I x^I \quad \text{for } \mu_I \in R.
\]

Then \( \Delta^q(T_m^{2d_i + 1}(x)/\theta_i + T_m^{2d_i + 2}(x)/\theta_i)|_{x = e^a} = (N(d_i, 2d_i + 1) \sum \lambda_I a_I) / (\xi \left| a \right|^q) + (N(d_i, 2d_i + 2) \sum \mu_I a_I) / \left| a \right|^q. \) Note that \( N(d_i, 2d_i + 1) \neq 0, \)

\( N(d_i, 2d_i + 2) \neq 0. \) Since \( T_m/\theta_1 + F/\theta_i \) is smooth, it follows from (3) that the limit \( \lim_{\xi \to 0} \Delta^q(T_m/\theta_i)|_{x = e^a} \) exists. From (4) we have \( \lambda_I = \mu_I = 0 \) for any \( I, J. \) Then \( T_m^d = 0, \) which is a contradiction. Therefore \( T_m = 0. \)

Proof of Proposition 5.4 continued. From Proposition 5.6, we have \( \theta_1 P_m = \theta_1 Q_m. \) Since \( \{\theta_1, \cdots, \theta_i\} \) is a minimal set of generators, there exists an \( H \)-invariant polynomial \( P_m' \) such that \( P_m = \theta_1 P_m'. \) Then \( g = g_1/\theta_i = P_m' + R_m/\theta_i \) for \( R_m(x) \sum_{|I|=m+1} x^I C^\infty(V) \) \((m=2,3,\cdots), \) and \( g \) is a smooth
map. This completes the proof of Proposition 5.4.

To complete the proof of our theorem, we shall prove that \( \theta = \sigma \).

Similar way as in the proof of Proposition 5.1, for any \( f \in C^\infty(M) \), \( X \in \mathfrak{X}(M) \), we have \( \theta(fX) = (f \circ \sigma^{-1}) \theta(X) \). Then \( \theta(X)(f \circ \sigma^{-1}) \theta(X) = \theta([X,fX]) = [\theta(X), \theta(fX)] = [\theta(X), f \circ \sigma^{-1} \theta(X)] = \theta(X)(f \circ \sigma^{-1}) \theta(X) \). Hence \( \theta(X)(f \circ \sigma^{-1}) \theta(X) = \theta(X)(f \circ \sigma^{-1}) \theta(X) \), and we see \( \theta(X)(f \circ \sigma^{-1}) = X(f) \circ \sigma^{-1} \). Then, for any \( g \in C^\infty(\overline{M}) \), \( X \in \mathfrak{X}(\overline{M}) \), \( \theta(X)(g) = X(g \circ \sigma) \circ \sigma^{-1} = \sigma(X)(g) \), and hence \( \theta = \sigma \). This completes the proof of our theorem.

**Remark.** We can prove that \( \sigma \) is strata preserving.

§ 6. Proof of Proposition 3.10

In this section we shall prove Proposition 3.10. The proof is parallel to those of Pursell-Shanks [8] and Koriyama [5]. We start with some lemmas.

**Lemma 6.1** (Sternberg [12]). Let \( X \) be a radial vector field on \( \mathbb{R}^n \) defined by \( X = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} \). Let \( Y \) be a smooth vector field on \( \mathbb{R}^n \) such that \( j_k(X) = j_k(Y) \). Then there exists a local coordinate system \( (y_1, \ldots, y_n) \) defined on a neighborhood \( U \) of 0 such that \( Y = \sum_{i=1}^{n} y_i \frac{\partial}{\partial y_i} \) on \( U \).

**Lemma 6.2.** Let \( \mathfrak{M} \) be an ideal of \( \mathfrak{X}(\overline{M}) \) such that for any \( p \in \overline{M} \) there exists \( Y \in \mathfrak{M} \) such that \( Y \neq 0 \). Then \( \mathfrak{M} \) contains an ideal \( \mathfrak{N}_1 = \{ X \in \mathfrak{X}(\overline{M}); \text{supp } X \subseteq \overline{M} \} \).

**Proof.** Put \( \mathfrak{M} = \pi^{-1}(\mathfrak{M}) \), \( \mathfrak{N}_1 = \pi^{-1}(\mathfrak{N}_1) \). We shall prove that \( \mathfrak{M} \) contains \( \mathfrak{N}_1 \). Similarly as in the proof of Lemma 3.9, for any point \( a \in \pi^{-1}(\overline{M}) \) there exist a local coordinate system \( (x_1, \ldots, x_n) \) around a \( G_a \)-invariant neighborhood \( U_a \) of \( a \) in \( V_a \) and \( Y \in \mathfrak{M} \) such that \( \tau_a(Y) = \frac{\partial}{\partial x_1} \) on \( U_a \). \( x_1 \) is extended to a \( G \)-invariant smooth function \( f \) on \( M \).
Put $Y_1 = Y - r_a(Y)$ on $V_a$. Then $\pi_\ast(Y_1) = 0$ on $\overline{V_a}$. Using a partition of unity, we can find $Y_2 \in \text{Ker } \pi_\ast \subset \mathfrak{M}$ such that $Y_2 = Y_1$ on $U_a$. Put $Y_3 = Y - Y_2$. Then $\frac{\partial Y_3}{\partial x_i} = \frac{\partial Y_2}{\partial x_i}$ on $U_a$ and $Y_3 \in \mathfrak{M}$.

Let $X \in \mathfrak{Z}_1$. To prove $X \in \mathfrak{M}$ we can assume that supp $X$ is contained in $G \times a_a V_a$ by using arguments of invariant partition of unity. Moreover, we can assume that $r_a(X) = X$ on $U_a$ since Ker $\pi_\ast$ is contained in $\mathfrak{M}$. Put $X = \sum_{i=1}^n \xi_i \frac{\partial}{\partial x_i}$ on $U_a$. Since $x_i$ is a $G_a$-invariant function, we see that $\xi_i$ is a $G_a$-invariant function. Similarly as in the proof of Koriyama [5] Lemma 2.13, we can prove that $X$ is an element of $\mathfrak{M}$. This completes the proof of Lemma 6.2.

Put $X_6(M)_{\circ \omega} = \bigcap_{i \in I} r_{a_i}^{-1}(\Gamma_{a_i}(\tau(V_{a_i}))_{\circ \omega})$.

**Lemma 6.3.** Let $\overline{\mathfrak{M}}$ be an ideal of $\mathfrak{X}(M)$ such that $J^1(\pi_\ast^{-1}(\overline{\mathfrak{M}})) = J^1(M)$ and for any $p \in \mathfrak{M}$, there exists an element $\overline{Y} \in \overline{\mathfrak{M}}$ such that $\overline{Y}_p \neq 0$. Then an ideal $\mathfrak{M} = \pi_\ast^{-1}(\overline{\mathfrak{M}})$ of $\mathfrak{X}_6(M)$ contains an ideal $X_6(M)_{\circ \omega}$.

**Proof:** Since $J^1(M) = J^1(M)$, there exists an element $X \in \mathfrak{M}$ such that $\mathfrak{R}_1(r_{a_i}(X)) = \mathfrak{R}_1(\mathfrak{R})$. Here $\mathfrak{R}_1$ is a radial vector field on $V_{a_i}$ defined by $\mathfrak{R}_1 = \sum_{j=1}^n y_j \frac{\partial}{\partial y_j}$ on $V_{a_i}$, where $\{y_1, \ldots, y_n\}$ is a canonical coordinate of $V_{a_i}$. Since $\mathfrak{M}$ contains Ker $\pi_\ast$, we can assume that $X = \sum_{j=1}^n y_j \frac{\partial}{\partial y_j}$ on $U_i$.

As in the proof of Koriyama [5] Lemma 2.13, for any element $Z \in X_6(M)_{\circ \omega}$, there exist vector fields $Y_i, i \in I$, such that $[X, Y_i] = r_{a_i}(Z)$ on a $G_b$-invariant neighborhood $W_i \subset U_i$ of $b_i$ and supp $Y_i \subset U_i$. Put $\overline{Y}_i = \int g_s Y_i dg \in \Gamma_{a_i}(\tau(V_{a_i}))$. By Lemma 3.6, we have $[X, \overline{Y}_i] = r_{a_i}(Z)$ on $W_i$. By Lemma 3.1 (1), we can assume $\overline{Y}_i \in X_6(M)$ and supp $\overline{Y}_i$ is contained in $G \times a_{b_i} U_i$. Since supp $Z$ is compact, there is a finite index set $\{i_1, \ldots, i_k\} \subset I$ such that supp $Z \cap G \times a_{b_j} V_{b_j} \neq \emptyset$. Put $\overline{Y} = \overline{Y}_{i_1} + \cdots + \overline{Y}_{i_k}$. Since Ker $\pi_\ast$ is contained in $\mathfrak{M}$, there exists an element $Z_0 \in \mathfrak{M}$ such that $Z_0 = Z - r_{a_i}(Z)$ on $W_j$ for $j = 1, \ldots, k$. Then $Z_i = Z - Z_0 - [X, \overline{Y}]$ is an element of $\mathfrak{Z}_1$ in Lemma 6.2, and $Z_i \in \mathfrak{M}$. Thus we have $Z \in \mathfrak{M}$,
and this completes the proof of Lemma 6.3.

**Lemma 6.4.** Let $\mathfrak{M}$ be a maximal ideal of $\mathfrak{k}(\mathcal{M})$ such that for any $p \in \bar{\mathcal{M}}$, there exists an element $Y \in \mathfrak{M}$ such that $Y_p \neq 0$. Then $J^1(\mathfrak{M})$ is a maximal ideal of $J^1(\mathcal{M})$, where $\mathfrak{M} = \pi_*^{-1}(\mathfrak{M})$.

**Proof.** Suppose $J^1(\mathfrak{M}) = J^1(\mathcal{M})$. We shall prove that $\ker J^1$ is contained in $\mathfrak{M}$. By Lemma 6.2, it is enough to prove that an element $Z \in \ker J^1$ satisfying $\text{supp } Z \subset G \times \mathcal{O}_t \mathcal{V}_t$ is an element of $\mathfrak{M}$. Since $\ker \pi_*$ is contained in $\mathfrak{M}$, we can assume $Z = r_{\mathcal{O}_t}(Z)$ on $\mathcal{V}_t$. As in the proof of Lemma 6.3, there exist $X \in \mathfrak{M}$ and a local coordinate system $(x_1, \ldots, x_n)$ defined on a $G_{\mathcal{O}_t}$-invariant neighborhood $U_t$ of $b_t$ in $\mathcal{V}_t$ such that $X = \sum_{i=1}^n x_t \frac{\partial}{\partial x_i}$ on $U_t$.

From the proof of Koriyama [5] Lemma 2.10, there exists a smooth vector field $Y$ on $\mathcal{V}_t$ such that $Z = Z - [X, Y] \in \Gamma(\tau(\mathcal{V}_t))_0$. Put $\tilde{Y} = \int_{\mathcal{O}_t} g_* Y dg$. Then $Z - [X, \tilde{Y}] = \int_{\mathcal{O}_t} g_* Z dg \in \Gamma(\tau(\mathcal{V}_t))_0$. By Lemma 3.1 (1), we can assume $\tilde{Y} \in \mathfrak{k}_0(\mathcal{M})$, and $Z - [X, \tilde{Y}] \in \mathfrak{k}_0(\mathcal{M})_0$. Then it follows from Lemma 6.3 that $Z \in \mathfrak{M}$. Thus $\ker J^1$ is contained in $\mathfrak{M}$. Since $J^1(\mathfrak{M}) = J^1(\mathcal{M})$, $\mathfrak{M} = \mathfrak{k}_0(\mathcal{M})$. This is a contradiction, and this completes the proof of Lemma 6.4.

**Proof of Proposition 3.10.** Let $\mathfrak{M}$ be a maximal ideal of $\mathfrak{k}(\mathcal{M})$. If there exists a point $p \in \bar{\mathcal{M}}$ such that $\mathfrak{M}$ is contained in $\mathfrak{k}(\mathcal{M})_p$, then $\mathfrak{M} = \mathfrak{k}_p$. Suppose for any point $p \in \bar{\mathcal{M}}$ there exists an element $X \in \mathfrak{M}$ such that $X_p \neq 0$. By Lemma 6.4, $J^1(\pi_*^{-1}(\mathfrak{M}))$ is a maximal ideal of $J^1(\mathcal{M})$. This completes the proof of Proposition 3.10.

**References**


