Representation Theorems of Cohomology on Weakly 1-Complete Manifolds

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Introduction

Vanishing, finiteness and harmonic representation of cohomology on complex manifolds are one of the most interesting problems in analytic geometry (for example, see [2], [5], [9], [13], [19], [20], [38], [45]), and here we are concerned especially with cohomology on noncompact complex manifolds. Namely, in this paper, we deal with cohomology groups with coefficients in a locally free sheaf of rank one on domains with pseudoconvex boundaries and weakly 1-complete manifolds. We say

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that a complex manifold $X$ is weakly 1-complete if there exists a $C^\infty$-exhausting plurisubharmonic function on $X$. S. Nakano pointed out the importance of the concept in [31] (or [11]) by solving the inverse problem of monoidal transformations and the main point was some vanishing theorem of cohomology groups for positive line bundles (see [32], [33]). Then the theorem was generalized in various aspects (see [1], [12], [37], [40], [41], [42], [43], [44]). Recently, in [34], [35] and [37] T. Ohsawa showed the finiteness and isomorphic theorems of cohomology groups for line bundles on weakly 1-complete manifolds which are positive outside compact subsets. In these articles, the main method was the $\bar{\partial}$-operator theory without boundary condition, which was originated by Andreotti and Vesentini's works [3], [4] and [5], while in this paper, we want to make use of the $\bar{\partial}$-operator theory with boundary condition, which was studied by Kohn and Hörmander (for example, see [9], [17], [22], [23], [24], etc.). But of course we have to recreate from these authors because of weak pseudoconvexity of our domain. Hence we present, in Chapter I, some calculation i.e. a complex tensor calculus for Kähler manifolds with boundary. Although it is more or less a routine calculation, it is worth to do so because, in the opinion of the author, the complex tensor calculus is one of the important methods to treat the cohomology groups from differential geometric view point. The result is a simple generalization of a formula for compact complex Kähler manifolds due to K. Kodaira [20] and is more or less known.

The main purpose of this paper is to make Ohsawa's results more exact in the following sense. When we let $X$ be a weakly 1-complete manifold with a $C^\infty$-exhausting plurisubharmonic function $\Phi$ on $X$, we shall show that for a line bundle $B$ on $X$ which is positive outside a compact subset $K$ of $X$, the $p$-th cohomology group $H^p(X_c, \mathcal{O}(B\otimes K_x))$ ($p \geq 1$) is finite dimensional and represented by the space of harmonic forms which are obtained as the harmonic part of $B\otimes K_x$-valued differential forms of type $(0, p)$ being smooth up to the boundary $\partial X_c$ (see Theorem 3.9). Here we write $X_c = \{x \in X | \Phi(x) < c\}$ and $\partial X_c = \{x \in X | \Phi(x) = c\}$ for some non-critical value $c$ of $\Phi$ with $c > \sup_{x \in K} \Phi(x)$ and $K_x$ is the canonical line bundle of $X$. This seems to be the analogy of the case of compact or $q$-convex complex manifolds (see [9], [17], [19]).
Chapter 1. Differential Analysis on Complex Manifolds and Functional Analysis

§ 1.1. A Complex Tensor Calculus for Kähler Manifolds with Boundary

Let $M$ be an $n$-dimensional complex manifold and let $X$ be a relatively compact domain on $M$ with smooth boundary $\partial X$. In this paper, this means that there exist a neighborhood $W$ of $\partial X$ and a real valued function $\Phi$ of class $C^\infty$ on $W$ such that i) $X \cap W = \{ x \in W | \Phi(x) < 0 \}$, ii) the gradient of $\Phi$ does not vanish on $\partial X$. From now on, we assume that $M$ is provided with a kählerian metric $ds^2$. Let $\{ U_t \}$ be a coordinate cover of $M$ and let $(z_1, \cdots, z^n)$ be local coordinates on $U_t$. We use the notation $\partial_{i,a} = \frac{\partial}{\partial z_i^a}$ and $\partial_{i,a;b} = \frac{\partial}{\partial z_i^b}$. (We sometimes omit $i$ for simplicity.) We set

$$(1.1) \quad ds^2 = \sum_{a,b=1}^n g_{i,a}dz_i^a \wedge \overline{dz_i^b}.$$ 

The Kähler form is defined by

$$(\Omega = \sqrt{-1} \sum_{a,b=1}^n g_{i,a}dz_i^a \wedge \overline{dz_i^b}.$$ 

With respect to this metric, we can define a connection $\{ \omega_i \}$, $\omega_t = (\omega_{i,a})$ for the holomorphic tangent bundle $TM$ on $M$:

$$(1.2) \quad \omega_{i,a} = \sum_{r=1}^n \Gamma_{i,ra}dz_r \quad \text{where} \quad \Gamma_{i,ra} = \sum_{\sigma=1}^n g_{i,\sigma} \partial_r g_{i,\sigma}.$$ 

The Riemann curvature tensor is defined by

$$(1.3) \quad R_{i,\sigma ; k} = \partial_k \Gamma_{i,\sigma} - \partial_{\sigma} \Gamma_{i,k} + \sum_{\rho=1}^n g_{i,\rho} R_{i,\sigma ; \rho k}.$$ 

and also we set

$$(1.4) \quad R_{i,\sigma ; \rho k} = \sum_{\beta=1}^n g_{i,\rho} R_{i,\sigma ; \rho k}.$$
As for the conjugates of the above, we define
\[ (1.5) \quad \Gamma_{t, \beta}^\alpha = \Gamma_{t, \beta}^\alpha, \quad R_{t, \beta}^\alpha = \overline{R_{t, \beta}^\alpha} \quad \text{and} \quad R_{t, \alpha \beta}^\gamma = \overline{R_{t, \alpha \beta}^\gamma}. \]

The Ricci curvature is defined by
\[ (1.6) \quad R_{t, \alpha} = \sum_{\beta=1}^{n} R_{t, \beta}^\alpha. \]

Since \( ds^2 \) is a Kähler metric, in the same manner as in [21], p. 109, Theorem 5.1 and p. 117, Proposition 6.2, we have
\[ (1.7) \quad \Gamma_{t, \alpha} = \Gamma_{t, \alpha}^\gamma \]
and
\[ (1.8) \quad R_{t, \alpha \beta} = R_{t, \beta \alpha} = R_{t, \alpha \beta}^\gamma = R_{t, \alpha \beta} \].

As in [21], p. 111–112 and p. 118, Proposition 6.4, it is easily verified that
\[ (1.9) \quad \sum_{\ell=1}^{n} \Gamma_{t, \alpha}^\ell \partial_\ell \log g_\ell \]
and
\[ (1.10) \quad R_{t, \alpha} = \partial_\alpha \log g_\ell \quad \text{where} \quad g_\ell = \det(g_{\alpha \beta}). \]

Let \( C^\infty(TM) \) (resp. \( C^\infty(T\overline{M}) \)) be the sheaf of germs of the \( C^\infty \)-sections of the holomorphic tangent bundle \( TM \) (resp. conjugate tangent bundle \( T\overline{M} \)) and let \( \Gamma(M, C^\infty(TM)) \) (resp. \( \Gamma(M, C^\infty(T\overline{M})) \)) be the space of global sections of \( C^\infty(TM) \) (resp. \( C^\infty(T\overline{M}) \)).

We define covariant differentiation \( \nabla_\alpha, \nabla_\beta \) induced by (2) on \( \Gamma(M, C^\infty(TM)) \) and \( \Gamma(M, C^\infty(T\overline{M})) \) as follows:
\[ (1.11) \]
\[
\begin{align*}
\nabla_{t, \alpha} \xi^\alpha &= \partial_{t} \xi^\alpha + \sum_{\beta=1}^{n} \Gamma_{t, \alpha}^{\alpha} \xi^\beta \\
\nabla_{t, \beta} \xi^\alpha &= \partial_{t} \xi^\alpha
\end{align*}
\]
for \( \xi = \sum_{\alpha=1}^{\infty} \xi^\alpha \frac{\partial}{\partial z^\alpha} \in \Gamma(M, C^\infty(TM)) \),
\[ (1.12) \]
\[
\begin{align*}
\nabla_{t, \alpha} \eta^\alpha &= \partial_{t} \eta^\alpha \\
\nabla_{t, \beta} \eta^\alpha &= \partial_{t} \eta^\alpha + \sum_{\beta=1}^{n} \Gamma_{t, \beta}^{\alpha} \eta^\beta
\end{align*}
\]
for \( \eta = \sum_{\beta=1}^{\infty} \eta^\beta \frac{\partial}{\partial z^\beta} \in \Gamma(M, C^\infty(T\overline{M})) \).

Then we remark that
\[ (1.13) \quad \nabla_{t, \alpha} g_{\alpha \beta} = 0, \quad \nabla_{t, \alpha} g_{\beta}^\alpha = 0. \]
For any open subset $Y$ of $M$, let $C^{p,q}(Y)$ be the space of differential forms of type $(p,q)$ and of class $C^\infty$ on $Y$ and let $C^{p,q}_c(X)$ be the space of the forms in $C^{p,q}(Y)$ with compact supports. Let $C^{p,q}(X)$ be the image of the restriction mapping from $C^{p,q}(M)$ to $C^{p,q}(X)$. We denote the length of the gradient of $\Phi$ with respect to $(1,1)$ by $|\text{grad} \Phi|$, then $|\text{grad} \Phi|^2 = \sum_{\alpha, \beta=1}^n g_{\alpha \beta} \partial_\alpha \Phi \partial_\beta \Phi$ and, from the hypothesis, we may assume that $|\text{grad} \Phi| > 0$ on a neighborhood $W$ of $\partial X$. We define a function $f$ of class $C^\infty$ on $W$ by

$$f = \frac{\Phi}{|\text{grad} \Phi|} \quad \text{on} \quad W.$$ 

Then we obtain

$$\begin{align*}
\text{i)} & \; f = 0 \quad \text{on} \quad \partial X, \\
\text{ii)} & \; df = |\text{grad} \Phi|^{-1} d\Phi \quad \text{on} \quad \partial X, \\
\text{iii)} & \; |\text{grad} f| = 1 \quad \text{on} \quad W.
\end{align*}$$

We separate $z^\alpha_i$ into the real and imaginary parts: $z^\alpha_i = x^{2n-1}_i + \sqrt{-1} x^\nu_i$ ($\alpha = 1, 2, \cdots, n$) and set $\partial_{i,k} = \frac{\partial}{\partial x^k_i}$ ($k = 1, 2, \cdots, 2n$). Let $dV$ be the volume element of $M$ with respect to $(1.1)$. Then we have, by direct calculations,

$$dV = \frac{1}{n!} g_{\alpha \beta} dx^\alpha_i \wedge dx^\beta_i \wedge \cdots \wedge dx^{2n}.$$

Let $dS$ be the volume element of the real differential manifold $\partial X$ of real dimension $2n-1$. Since $(1.13)$ implies that $\text{grad} f$ is the outward unit normal on $\partial X$, we have

$$dV = df / dS \quad \text{on} \quad \partial X.$$

We consider a vector field $\sum_{\alpha=1}^n \psi^\alpha_i \partial / \partial x^\alpha_i + \sum_{\beta=1}^n \psi^\beta_i \partial / \partial x^\beta_i$ on a neighborhood of $\bar{X}$. Then $\sum_{\alpha=1}^n \nabla_i \psi^\alpha_i + \sum_{\beta=1}^n \nabla_i \psi^\beta_i$ is called the divergence of the vector field $\vec{\psi} = \{\psi^\alpha, \psi^\beta\}$.

**Lemma 1.1.** We have

$$\int_X \left( \nabla_i \psi^\alpha + \nabla_i \psi^\beta \right) dV = \int_{\partial X} \left( \sum_{\alpha=1}^n \psi^\alpha \partial_\alpha f + \sum_{\beta=1}^n \psi^\beta \partial_\beta f \right) dS,$$

where $\partial_\alpha = \frac{\partial}{\partial x^\alpha_i}$ and $\partial_\alpha = \frac{\partial}{\partial x^\alpha_i}$ on $U_i$ ($\alpha = 1, 2, \cdots, n$).
Proof. From (1.13) and (1.15), Green's formula may be written in the following form:

$$
\int_{\Omega \cap \gamma_t} \partial_{t,k} u \varphi dV = - \int_{\Omega \cap \gamma_t} u \left( \partial_{t,k} + \partial_{t,k} \log g_t \right) \varphi dV
$$

$$
+ \int_{\partial \Omega \cap \gamma_t} u \varphi d\gamma
$$

for any \( u \in C^0_0(\Omega_t) \), \( v \in C^0(\bar{\Omega}) \) and \( k = 1, 2, \ldots, 2n \).

Since \( \partial_{t,a} = \partial_{t,a-1} + \sqrt{-1} \partial_{t,a} \) and \( \partial_{t,a} = \partial_{t,a-1} - \sqrt{-1} \partial_{t,a} \), we have

\[ \text{i) } \int_{\Omega \cap \gamma_t} \partial_{t,a} u \varphi dV = - \int_{\Omega \cap \gamma_t} u \left( \partial_{t,a} + \partial_{t,a} \log g_t \right) \varphi dV \]

$$
+ \int_{\partial \Omega \cap \gamma_t} u \varphi d\gamma
$$

(1.17)

\[ \text{ii) } \int_{\Omega \cap \gamma_t} \partial_{t,b} u \varphi dV = - \int_{\Omega \cap \gamma_t} u \left( \partial_{t,b} + \partial_{t,b} \log g_t \right) \varphi dV 
$$

$$
+ \int_{\partial \Omega \cap \gamma_t} u \varphi d\gamma
$$

for any \( u \in C^0_0(\Omega_t) \), \( v \in C^0(\bar{\Omega}) \) and \( \alpha, \beta = 1, 2, \ldots, n \).

Let \( \{ \rho_i \}_{i=1}^m \) be a family of \( C^\infty \)-functions on \( M \) such that supp \( \rho_i \subseteq U_i \), \( 0 \leq \rho_i \leq 1 \) and \( \sum \rho_i = 1 \) on \( \bar{X} \). We set \( \psi_i^\alpha = \rho_i \psi_i^\alpha \) and \( \psi_i^\beta = \rho_i \psi_i^\beta \) respectively. Replacing \( u \) and \( v \) by \( \psi_i^\alpha \) and \( 1 \) in (1.17) i), we have

$$
\int_{\Omega \cap \gamma_t} \left( \partial_{t,a} \psi_i^\alpha + \partial_{t,a} \log g_i \psi_i^\alpha \right) dV = \int_{\partial \Omega \cap \gamma_t} \psi_i^\alpha \partial_{t,a} \varphi d\gamma.
$$

While,

$$
\sum_{a=1}^n \left( \partial_{t,a} \psi_i^\alpha + \partial_{t,a} \log g_i \psi_i^\alpha \right)
$$

$$
= \sum_{a=1}^n \partial_{t,a} \psi_i^\alpha + \sum_{a=1}^n \left( \sum_{t=1}^n \Gamma_{t,a} \psi_i^\alpha \right) \quad \text{(Use (1.9).)}
$$

$$
= \sum_{a=1}^n \left( \partial_{t,a} \psi_i^\alpha + \sum_{t=1}^n \Gamma_{t,a} \psi_i^\alpha \right)
$$

$$
= \sum_{a=1}^n \nabla_{t,a} \psi_i^\alpha \quad \text{(Use (1.11).)}
$$

Therefore we have

$$
\int_{\Omega \cap \gamma_t} \left( \sum_{a=1}^n \nabla_{t,a} \psi_i^\alpha \right) dV = \int_{\partial \Omega \cap \gamma_t} \left( \sum_{a=1}^n \psi_i^\alpha \partial_{t,a} \varphi \right) d\gamma.
$$
For $\psi^\alpha_{\beta}$, we have similarly

$$\int_{X \cap U_i} \left( \sum_{\beta=1}^n \nabla_{i, \beta} \psi^\alpha_{\beta} \right) dV = \int_{\partial X \cap U_i} \left( \sum_{\beta=1}^n \psi^\alpha_{\beta} \partial_{i, \beta} f \right) dS.$$  

Using $\psi^a = \sum_{\beta=1}^n \frac{\partial x^a_j}{\partial z^\beta_j} \psi^\beta_{\beta} \nabla^i_{a, \alpha} = \sum_{\beta=1}^n \frac{\partial x^a_j}{\partial z^\beta_j} \nabla^i_{a, \beta} \psi^\beta_{\beta}$ and $\sum_i \rho_i = 1$ on $\overline{X}$, we have the conclusion.

**Remark 1.1.** If $d^g$ is a hermitian metric on $M$ which is Kähler on a neighborhood $U$ of $\partial X$ and $\chi$ is a $C^\infty$-function on $M$ whose support is contained in $U$, then it is clear that the equation (1.16) holds for $\chi^\overline{\beta} = \{ \chi^\alpha_{\beta}, \chi^\beta_{\beta} \}$ i.e.

$$(1.16)' \quad \int_{X \cap U} \left( \sum_{\alpha=1}^n \nabla_\alpha (\chi^\alpha_{\beta}) + \sum_{\beta=1}^n \nabla_\beta (\chi^\beta_{\beta}) \right) dV = \int_{\partial X} \left( \sum_{\alpha=1}^n \chi^\alpha_{\beta} \partial_\alpha f + \sum_{\beta=1}^n \chi^\beta_{\beta} \partial_\beta f \right) dS.$$

Let $F$ be a holomorphic line bundle over $M$ and suppose that $F$ is defined by the system of transition functions $\{ f_{ij} \}$ with respect to the coordinate cover $\{ U_i \}_{i \in I}$. A hermitian metric on $F$ with respect to this covering is given by the system of positive $C^\infty$-functions $\{ a_i \}$, each defined on $U_i$, such that $a_i \cdot a_j^{-1} = |f_{ij}|^2$ on $U_i \cap U_j$. (In this paper, we use the notation of a system of metric along the fibres in the sense of Kodaira [20], p. 1268, (1)). From now on, we fix a hermitian metric of $F$

$$(1.18) \quad \{ c_i \}.$$  

With respect to the covering $\{ U_i \}_{i \in I}$ a hermitian inner product $\langle , \rangle$ of $F$ is expressed by the metric $\{ c_i \}$ i.e. for $C^\infty$-sections $\xi = \{ \xi_i \}$ and $\eta = \{ \eta_i \}$ of $F$ on $M$.

$$\langle \xi, \eta \rangle = c_i^{-1} \overline{\xi_i} \overline{\eta_i}.$$  

With respect to this metric, we can define a connection $\{ \theta_i \}$ of $F$ as follows:

$$(1.19) \quad \theta_i = \sum_{a=1}^n - \partial_a \log c_i dz^a_i.$$  

The curvature tensor of the above connection is defined by
The canonical line bundle $K_M$ of $M$ is defined by the system of transition functions

$$K_M = \{K_{M_{ij}}\}$$

where

$$K_{M_{ij}} = \frac{\partial (x^1_{I_j}, \ldots, x^n_{I_j})}{\partial (x^1_{I_i}, \ldots, x^n_{I_i})}$$
on $U_i \cap U_j$.

We see that

$$|K_{M_{ij}}|^2 = g_i g_j^{-1}$$non $U_i \cap U_j$.

Hence

(1.21) $\{g_i\}$
determines a metric of $K_M$.

Let $C^{p,q}(M,F)$ (resp. $C^{p,q}(X,F)$) be the space of $F$-valued differential forms of type $(p,q)$ and of class $C^\infty$ on $M$ (resp. on $X$) and let $C^{p,q}(X,F)$ be the image of the restriction map from $C^{p,q}(M,F)$ to $C^{p,q}(X,F)$. Let $C^{p,q}_o(M,F)$ (resp. $C^{p,q}_o(X,F)$) be the space of the forms in $C^{p,q}(M,F)$ (resp. $C^{p,q}(X,F)$) with compact supports. It is clear that $C^{p,q}_o(X,F)$ is a subspace of $C^{p,q}(X,F)$. We express $\varphi = \{\varphi_i\} \in C^{p,q}(M,F)$ as $\varphi_i = \frac{1}{p! q!} \sum \sum \varphi_{i_1, i_2, \ldots, i_p, j_1, j_2, \ldots, j_q} dx^{i_1} \wedge \cdots \wedge dx^{i_p} \wedge dy^{j_1} \wedge \cdots \wedge dy^{j_q}$.

For simplicity, we write

$$\varphi_{i_{1p}} \cdots \varphi_{i_{1q}} \cdots \varphi_{i_{2q}},$$

where $A_p = (\alpha_1, \ldots, \alpha_p)$, $B_q = (\beta_1, \ldots, \beta_q)$, $C_p = (c_1, \ldots, c_p)$ and $D_q = (d_1, \ldots, d_q)$.

We set

$$\langle \varphi, \psi \rangle = c_i^{-1} \sum_{A_p \leq \beta_q} \varphi_{i_{1p}} A_p \varphi_{i_{2p}} B_q$$

where $A_p = (\alpha_1, \ldots, \alpha_p)$ and $B_q = (\beta_1, \ldots, \beta_q)$ run through the sets of multi-indices with $1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_p \leq n$ and $1 \leq \beta_1 < \beta_2 < \cdots < \beta_q \leq n$ respectively. Then we have

$$c_i^{-1} \varphi_{i_{1p}} \cdots \varphi_{i_{2q}} = \langle \varphi, \psi \rangle dV,$$
where \(*\) is the star operator with respect to the metric \(ds^2\).

For \(\varphi\) and \(\psi \in C^{p,q}(\overline{X}, F)\), we define
\[
\langle \varphi, \psi \rangle = \int_X \langle \varphi, \psi \rangle dV.
\]

For any real valued \(C^\infty\)-function \(\Psi\) on \(M\), we define
\[
\langle \varphi, \psi \rangle_\Psi = \int_X \langle \varphi, \psi \rangle e^{-\Psi} dV,
\]
where \(\langle \varphi, \psi \rangle_\Psi = \langle \varphi, \psi \rangle e^{-\Psi} \).

We set
\[
\|\varphi\|^2 = \langle \varphi, \varphi \rangle
\]
(1.22)
\[
\|\varphi\|_\Psi^2 = \langle \varphi, \varphi \rangle_\Psi.
\]

We have the operator \(\overline{\partial}: C^{p,q}(\overline{X}, F) \to C^{p,q+1}(\overline{X}, F)\) defined by \(\overline{\partial} \varphi = \overline{\partial} \varphi\). Then formal adjoint operators \(\overline{\partial}\) and \(\partial\) are defined by
\[
\langle \overline{\partial} \varphi, \psi \rangle = \langle \varphi, \partial \psi \rangle
\]
(1.23)
\[
\langle \partial \varphi, \psi \rangle_\Psi = \langle \varphi, \partial \Psi \psi \rangle_\Psi
\]
for any \(\varphi \in C_{\infty}^{p,q}(X, F)\) and \(\psi \in C_{\infty}^{p,q+1}(X, F)\).

**Lemma 1.2.** If \(\varphi \in C^{p,q}(\overline{X}, F)\) and \(\psi \in C^{p,q+1}(\overline{X}, F)\), we have

i) \(\langle \overline{\partial} \varphi, \psi \rangle_\Psi = \langle \varphi, \partial \Psi \psi \rangle_\Psi\)

\[
+ \int_{\partial X} (c_i \exp(\Psi) q!^{-1} \sum \varphi_i c_{a e} g_{i e} \partial_s \varphi_i e_{a e} F e_{\partial} \frac{\partial f}{\partial z_i}) dS,
\]

ii) \(\langle \partial \psi \rangle_{i, A u} \Psi = -\sum_{\beta=1}^n (\partial_\beta + \partial_\beta \log(c_i e^{-\Psi} g_i)) \psi_i, A u, \Psi\)

for any real valued \(C^\infty\)-function \(\Psi\) on \(M\).

**Remark 1.2.** In this lemma, \(ds^2\) need not be a Kähler metric.

**Proof.** Take elements \(u\) and \(v\) of \(C_{\infty}^{0,0}(\overline{X}, F)\). Applying a family of \(C^\infty\)-functions \(\{\rho_i\}\) on \(M\) which was taken as in the proof of Lemma 1.1, to \(u\) and \(v\), we have the following formula similar to (1.17):

\[
\int_X c^{-1} \partial_a u \overline{v} e^{-\Psi} dV =
\]
for any real valued $C^\infty$-function $\Psi$ on $M$ and $\alpha=1,2,\ldots,n$. Here we omitted $i$ for simplicity and, in this proof, we do so. For an element $\varphi$ of $C^{p,q}(X,F)$, $\varphi$ and $\bar{\varphi}$ are represented as

$$
\varphi = \frac{1}{q!} \sum_{p_1,\ldots,p_q} \varphi_{p_1,\ldots,p_q} d\bar{z}^{p_1} \wedge \cdots \wedge d\bar{z}^{p_q}
$$

and

$$
\bar{\varphi} = \frac{1}{(q+1)!} \sum_{p_1,\ldots,p_q} (\bar{\varphi})_{p_1,\ldots,p_q} d\bar{z}^{p_1} \wedge \cdots \wedge d\bar{z}^{p_q}
$$

where $A_p = (\alpha_1,\ldots,\alpha_p)$ and $d\bar{z}^{p_s} = dx^{\hat{\alpha}_s} \wedge \cdots \wedge dx^{\hat{\alpha}_s}$ and so on.

Since $(\bar{\varphi})_{p_1,\ldots,p_q} = \sum_{k=0}^{n} (-1)^{q+k} \partial_{\bar{\beta}_k} \varphi_{p_1,\ldots,p_q,\bar{\beta}_1,\ldots,\bar{\beta}_k}$, we have

$$
(\bar{\varphi}, \psi) = \int_{X} \frac{1}{(q+1)!} \sum_{p_1,\ldots,p_q} \sum_{k=0}^{n} (-1)^{q+k} \partial_{\bar{\beta}_k} \varphi_{A_p,\bar{\beta}_1,\ldots,\bar{\beta}_k} d\bar{z}^{p_1} \wedge \cdots \wedge d\bar{z}^{p_q} e^{-\Psi} d\sigma_x\wedge \cdots \wedge d\sigma_{x^n}
$$

(by (1.24))

$$
= \int_{X} \frac{2^n}{q!} \sum_{p_1,\ldots,p_q} \sum_{k=0}^{n} \partial_{\beta} \varphi_{A_p,B_q} \bar{\psi}^{p_1,\ldots,p_q,\beta_1,\ldots,\beta_q} (c^{-1} e^{-\Psi} g) d\sigma_x\wedge \cdots \wedge d\sigma_{x^n}
$$

$$
- \int_{X} \frac{2^n}{q!} \sum_{p_1,\ldots,p_q} \varphi_{A_p,B_q} (\psi^{p_1,\ldots,p_q,\beta_1,\ldots,\beta_q} (c^{-1} e^{-\Psi} g)) d\sigma_x\wedge \cdots \wedge d\sigma_{x^n}
$$

$$
+ \int_{X} (ce^\Psi g)^{-1} \sum_{A_p} \varphi_{A_p,B_q} \sum_{\beta=1}^{n} g^{A_p}C_qB_q (\sum_{\beta=1}^{n} \psi^{p_1,\ldots,p_q,\beta_1,\ldots,\beta_q} d\sigma_{x^n}) d\sigma_x\wedge \cdots \wedge d\sigma_{x^n}
$$

In the last line, the equality of the first term holds by definition. Hence we have i) and ii).

Q.E.D.

From now on, we consider the following subspace of $C^{p,q}(X,F)$.

\begin{equation}
(1.25) \quad B^{p,q}(X,F) = \{ \varphi \in C^{p,q}(X,F) \mid \sum_{t=1}^{n} \varphi_t a_{c_p,B_q} \partial_{t,a} \varphi = 0 \}
\end{equation}

on $\partial X$ for every multi-index $C_p$ and $D_{q-1}$.

**Lemma 1.3.** If $\varphi \in C^{p,q}(X,F)$ and $\psi \in B^{p,q-1}(X,F)$, we have
(\bar{\varphi}, \psi)_\varphi = (\varphi, \partial_\varphi \psi)_\varphi \text{ for any real valued } C^\infty \text{-function } \varphi \text{ on } M.

Proof. Since } df \text{ and } d\varphi \text{ are proportional on } \partial X, \text{ our lemma follows immediately from Lemma 1.2. } \text{Q.E.D.}

Let } \mathcal{T}_{p,q}(M,F) \text{ (resp. } \mathcal{T}_{p,q}(M)) \text{ be the space of } F\text{-valued (resp. scalar) tensor fields of type } (p,q). \text{ The connections (1.2) and (1.19) give rise to covariant differentiations } \nabla_a, \nabla^{(0)}_a \text{ of type } (1,0) \text{ and } \nabla_\beta \text{ of type } (0,1) \text{ in } \mathcal{T}_{p,q}(M,F) \text{ and } \mathcal{T}_{p,q}(M) \text{ as follows:}

\begin{align*}
\text{i) } & \quad \nabla_{t_i} a \varphi_{i_1 \cdots i_p, j_1 \cdots j_q} = \tilde{\partial}_{t_i} \varphi_{i_1 \cdots i_p, j_1 \cdots j_q} \\
& = \sum_{i_1 = 1}^p \sum_{j_1 = 1}^q \Gamma_{t_i \alpha}^{a_1} \varphi_{a_1 i_2 \cdots i_p, j_1 \cdots j_q} \\
& \quad - \sum_{i_1 = 1}^p \sum_{j_1 = 1}^q \Gamma_{t_i \alpha}^{a_1} \varphi_{a_1 i_2 \cdots i_p, j_1 \cdots j_q} \\
& = (1, 26) \\
\text{ii) } & \quad \nabla_{t_i} \bar{\varphi}_{i_1 \cdots i_p, j_1 \cdots j_q} = \tilde{\partial}_{t_i} \bar{\varphi}_{i_1 \cdots i_p, j_1 \cdots j_q} \\
& = \sum_{i_1 = 1}^q \sum_{j_1 = 1}^p \Gamma_{t_i \alpha}^{a_1} \varphi_{a_1 i_2 \cdots i_p, j_1 \cdots j_q} \\
& \quad - \sum_{i_1 = 1}^q \sum_{j_1 = 1}^p \Gamma_{t_i \alpha}^{a_1} \varphi_{a_1 i_2 \cdots i_p, j_1 \cdots j_q}
\end{align*}

for every } \varphi \in \mathcal{T}_{p,q}(M) .

\begin{align*}
\text{i) } & \quad \nabla_{t_i} \alpha \varphi_{i_1 \cdots i_p, j_1 \cdots j_q} = \nabla_{t_i} \alpha \varphi_{i_1 \cdots i_p, j_1 \cdots j_q} \\
& = \tilde{\partial}_{t_i} \alpha \varphi_{i_1 \cdots i_p, j_1 \cdots j_q} \\
& = \sum_{i_1 = 1}^q \sum_{j_1 = 1}^p \Gamma_{t_i \alpha}^{a_1} \varphi_{a_1 i_2 \cdots i_p, j_1 \cdots j_q} \\
& \quad - \sum_{i_1 = 1}^q \sum_{j_1 = 1}^p \Gamma_{t_i \alpha}^{a_1} \varphi_{a_1 i_2 \cdots i_p, j_1 \cdots j_q}
\end{align*}

for every } \varphi \in \mathcal{T}_{p,q}(M,F).

Then, by using the term of covariant differentiations, we can describe the operators } \tilde{\partial}, \tilde{\partial} \text{ as follows (see [21], p. 110, Proposition 5.2 and p. 122, Proposition 6.7) :}

\begin{align*}
\tilde{\partial} \varphi_{i_1 \cdots i_p, j_1 \cdots j_q} = & \sum_{\beta = 0}^q (-1)^{p+\beta} \nabla_{t_i} \bar{\varphi}_{i_1 \cdots i_p, j_1 \cdots j_q} \\
& = (1, 28) \\
\tilde{\partial} \varphi_{i_1 \cdots i_p, j_1 \cdots j_q} = & \sum_{\beta = 1}^q g_{\beta}^{\alpha} \nabla_a \varphi_{i_1 \cdots i_p, j_1 \cdots j_q} \tilde{\partial}_{t_i} \varphi_{i_1 \cdots i_p, j_1 \cdots j_q}
\end{align*}

for any } \varphi \in \mathcal{T}_{p,q}(M,F).
Lemma 1.4. For an element $\varphi = \{\varphi_{i,\beta}\}$ $(D_\varphi = (d_1, \ldots, d_\beta) \in \mathcal{T}_{a,b}(M, F)$, we set

$$\xi^\beta = \sum_{\beta=1}^n c_i^{-1} \nabla_{i,\beta} \varphi_{i,\beta} \cdot \varphi_i^\beta \quad (\beta = 1, 2, \ldots, n)$$

$$\eta^\gamma = \sum_{\gamma=1}^n c_i^{-1} \nabla_{i,\gamma} \varphi_{i,\gamma} \cdot \varphi_i^\gamma \quad (\gamma = 1, 2, \ldots, n)$$

then $\xi = \{\xi^\beta\}$ and $\eta = \{\eta^\gamma\}$ are vector fields on $M$.

Proof. We recall the following relations:

i) $G_i = (\frac{\partial z_j^i}{\partial z_i^1}) G_j = (\frac{\partial z_j^i}{\partial z_i^1}) G_i = (\frac{\partial z_j^i}{\partial z_i^1}) G_i (\frac{\partial z_j^i}{\partial z_i^1})$

(1.29) where $G_i = (g_{i,\alpha})$ and $G_j = (g_{j,\alpha})$

ii) $\nabla_{i,\beta} = \sum_{\beta=1}^n \frac{\partial z_j^i}{\partial z_i^1} \nabla_{i,\beta}$, $\nabla_{i,\gamma} = \sum_{\gamma=1}^n \frac{\partial z_j^i}{\partial z_i^1} \nabla_{i,\gamma}$ (see [21], p. 108).

We prove the case $p = 2$. The proofs of other cases are similar. We set

$$\xi_i = \sum_{k=1}^n \tilde{\xi}^k \left( \frac{\partial}{\partial z_i^k} \right) = \sum_{k=1}^n \left( \sum_{\beta=1}^n c_i^{-1} \nabla_{i,\gamma} \varphi_{i,\beta} \cdot \varphi_i^\beta \right) \left( \frac{\partial}{\partial z_i^k} \right)$$

on $U_i$.

Since $\{\varphi_{i,\alpha}\}$ (resp. $\{\varphi_i^{\kappa}\}$) is a $C^\infty$-section of $F \otimes TM \otimes TM^*$ (resp. $F \otimes TM \otimes TM$), we have

$$\varphi_{i,\alpha} = f_{i,\alpha} \sum_{a,\kappa=1}^n \frac{\partial z_i^a}{\partial z_i^\kappa} \varphi_{i,\kappa} \varphi_{i,\alpha}$$

on $U_i \cap U_j$.

Then, using (1.29), we have

$$\xi_i = \sum_{k,\kappa} \nabla_{i,\kappa} \varphi_{i,\kappa} \varphi_i^{\kappa} \left( \frac{\partial}{\partial z_i^\kappa} \right)$$

$$= c_i^{-1} |f_{i,\alpha}|^2 \sum_{k,\kappa} \left( \sum_{\beta=1}^n \frac{\partial z_i^\beta}{\partial z_i^k} \nabla_{i,\beta} \varphi_{i,\beta} \varphi_i^\beta \right) \left( \sum_{a,\kappa=1}^n \frac{\partial z_i^a}{\partial z_i^\kappa} \varphi_{i,\kappa} \varphi_{i,\alpha} \right)$$

$$\times \left( \sum_{a,\kappa=1}^n \frac{\partial z_i^a}{\partial z_i^\kappa} \varphi_i^{\kappa} \varphi_i^{\alpha} \right) \left( \sum_{r,\kappa=1}^n \frac{\partial z_i^r}{\partial z_i^{\kappa}} \varphi_i^{\kappa} \varphi_i^r \right)$$

$$= c_i^{-1} \sum_{k,\kappa} \left( \sum_{a,\kappa=1}^n \frac{\partial z_i^a}{\partial z_i^k} \varphi_{i,\kappa} \varphi_i^{\kappa} \right) \left( \sum_{a,\kappa=1}^n \frac{\partial z_i^a}{\partial z_i^\kappa} \varphi_i^{\kappa} \varphi_i^{\alpha} \right) \left( \sum_{r,\kappa=1}^n \frac{\partial z_i^r}{\partial z_i^{\kappa}} \varphi_i^{\kappa} \varphi_i^r \right)$$
Hence we have \(\xi_i = \xi_j\) on \(U_i \cap U_j\). Therefore \(\xi = \{\xi_i^\beta\}\) is a vector field on \(M\). We set

\[
\eta_i = \sum_{k=1}^n \eta_i^k \left( \frac{\partial}{\partial z_i^k} \right) = \sum_{k=1}^n \left( \sum_{r=1}^n c_i^{-1} \nabla_{i_r} \phi_i \theta_{i_r} \phi_i^{-1} \right) \left( \frac{\partial}{\partial z_i^k} \right) \text{ on } U_i.
\]

Then we have \(\eta_i = \eta_j\) on \(U_i \cap U_j\) and so \(\eta = \{\eta_i^\beta\}\) is a vector field on \(M\).

Q.E.D.

**Proposition 1.5.** We have

\[
(1.30) \quad \|\bar{\nabla} \phi\|^2 + \|\partial \phi\|^2
\]

\[
= \int_X \left( c_i \text{grad} \phi \right)^{-1} \sum_{\beta, \gamma} \sum_{\beta' \gamma'} \sum_{\alpha, \beta, \gamma'} \sum_{\alpha', \beta', \gamma'} \phi_{\alpha, \alpha'} \phi_{\beta, \beta'} \phi_{\gamma, \gamma'} dS
\]

\[
+ \|\bar{\nabla} \phi\|^2 + \int_X c_i^{-1} \sum_{\alpha, \beta, \gamma} \sum_{\alpha', \beta', \gamma'} \left( \theta_{\alpha, \alpha'} - R_{\alpha, \alpha'} \right) \phi_{\alpha, \alpha'} \phi_{\beta, \beta'} \phi_{\gamma, \gamma'} dV
\]

for \(\phi \in B^{p, p}(X, F)\) \((p \geq 1)\), where \(\theta_{\alpha, \alpha'} = \sum_{\beta, \gamma} g_{\alpha, \beta} \phi_{\alpha, \beta} \phi_{\gamma, \gamma'}\), \(R_{\alpha, \alpha'} = \sum_{\beta, \gamma} g_{\alpha, \beta} \phi_{\alpha, \beta} \phi_{\gamma, \gamma'}\), and

\[
\|\bar{\nabla} \phi\|^2 = \int_X c_i^{-1} \sum_{\alpha, \beta, \gamma} \sum_{\alpha', \beta', \gamma'} g_{\alpha, \alpha'} \phi_{\alpha, \beta} \phi_{\gamma, \gamma'} dV.
\]

**Proof.** For an element \(\phi\) of \(B^{p, p}(X, F)\), we set

\[
\bar{\xi} = \{\xi^\beta = \sum_{D_{p-1} = (d_{p-1}, d_{p-1})} c_i^{-1} \nabla_{i_r} \phi_i \theta_{i_r} \phi_i^{-1} D_{p-1}, \xi^0 = 0; \beta = 1, 2, \ldots, n\}
\]

\[
\bar{\eta} = \{\eta^\gamma = 0; \eta^\gamma = \sum_{D_{p-1} = (d_{p-1}, d_{p-1})} c_i^{-1} \nabla_{i_r} \phi_i \theta_{i_r} \phi_i^{-1} D_{p-1}; \gamma = 1, 2, \ldots, n\}.
\]

Then, by Lemma 1.4, \(\bar{\xi}\) and \(\bar{\eta}\) are vector fields on a neighborhood of \(X\). We calculate the divergences of \(\bar{\xi}\) and \(\bar{\eta}\).

\[
\sum_{\beta = 1}^n \nabla \phi^\beta = \sum_{\beta = 1}^n \nabla \phi_i \left( c_i^{-1} \sum_{D_{p-1} \in (d_{p-1}, d_{p-1})} \nabla_{i_r} \phi_i \theta_{i_r} \phi_i^{-1} D_{p-1} \right)
\]

\[
= c_i^{-1} \sum_{D_{p-1} \in (d_{p-1}, d_{p-1})} \left( \nabla \phi_i - \partial \phi \log c_i \right) \nabla_i \phi_i \phi_i^{-1} D_{p-1}
\]

\[
+ c_i^{-1} \sum_{D_{p-1} \in (d_{p-1}, d_{p-1})} \nabla_i \phi_i \phi_i^{-1} D_{p-1} \nabla \phi_i^{-1} D_{p-1}
\]
We calculate the commutator. Since \( \nabla_{\beta} = \nabla_{\beta} - \partial_{\beta} \log \ c_{t} \) for every \( D_{p} = (d_{1}, \ldots, d_{p}) \), we have

\[
[\nabla_{\beta}, \nabla_{\tau}] \varphi_{t, D_{p}} = [\nabla_{\beta}, \nabla_{\tau}] \varphi_{t, D_{p}} + \theta_{t, \beta \tau} \varphi_{t, D_{p}}.
\]

Using (1.26), we have

\[
\nabla_{\beta} \nabla_{\tau} \varphi_{t, D_{p}} = \partial_{\beta} \partial_{\tau} \varphi_{t, D_{p}} = \sum_{p=1}^{\infty} \sum_{r=1}^{n} \Gamma_{t, \beta \tau}^{r} \partial_{\beta} \varphi_{t, \tilde{a}_{1}, \ldots, \tilde{a}_{p}}
\]

and

\[
\nabla_{\beta} \nabla_{\tau} \varphi_{t, D_{p}} = \partial_{\beta} \partial_{\tau} \varphi_{t, D_{p}} = \sum_{p=1}^{\infty} \sum_{r=1}^{n} \Gamma_{t, \beta \tau}^{r} \varphi_{t, \tilde{a}_{1}, \ldots, \tilde{a}_{p}}
\]

Hence

\[
[\nabla_{\beta}, \nabla_{\tau}] \varphi_{t, D_{p}} = - \sum_{p=1}^{\infty} \sum_{r=1}^{n} \Gamma_{t, \beta \tau}^{r} \varphi_{t, \tilde{a}_{1}, \ldots, \tilde{a}_{p}} + \theta_{t, \beta \tau} \varphi_{t, D_{p}}.
\] (Use (1.3).)

Therefore we have

\[
[\nabla_{\beta}, \nabla_{\tau}] \varphi_{t, D_{p}} = - \sum_{p=1}^{\infty} \sum_{r=1}^{n} \Gamma_{t, \beta \tau}^{r} \varphi_{t, \tilde{a}_{1}, \ldots, \tilde{a}_{p}} + \theta_{t, \beta \tau} \varphi_{t, D_{p}}
\]

for every \( D_{p} = (d_{1}, \ldots, d_{p}) \).

So

\[
* * *) = c_{t}^{-1} \sum_{\alpha, \beta, \tau} g_{t}^{\alpha \beta} \nabla_{\beta} \nabla_{\tau} \varphi_{t, D_{p}} \phi_{t, D_{p}}^{\alpha \tau}(1.12).
\]

\[
= - c_{t}^{-1} \sum_{\alpha, \beta, \tau} g_{t}^{\alpha \beta} \left( \sum_{r=1}^{n-1} \sum_{r=1}^{n} \Gamma_{t, \beta \tau}^{r} \phi_{t, \tilde{a}_{1}, \ldots, \tilde{a}_{p}} \phi_{t, D_{p}}^{\alpha \tau} \right) + c_{t}^{-1} \sum_{\alpha, \beta, \tau} g_{t}^{\alpha \beta} \theta_{t, \beta \tau} \phi_{t, D_{p}} \phi_{t, D_{p}}^{\alpha \tau}
\]

\[
= - c_{t}^{-1} \sum_{\alpha, \beta, \tau} g_{t}^{\alpha \beta} \left( \sum_{r=1}^{n-1} \sum_{r=1}^{n} \Gamma_{t, \beta \tau}^{r} \phi_{t, \tilde{a}_{1}, \ldots, \tilde{a}_{p}} \phi_{t, D_{p}}^{\alpha \tau} \right) + c_{t}^{-1} \sum_{\alpha, \beta, \tau} g_{t}^{\alpha \beta} \theta_{t, \beta \tau} \phi_{t, D_{p}} \phi_{t, D_{p}}^{\alpha \tau}
\]
From the Kähler property of $ds^2$, we have

$$R_{i\bar{a} \bar{b} \bar{c}} = \sum_{\beta} g_{\beta \bar{\beta}} R_{i \beta \bar{a} \bar{b} \bar{c}} = R_{i \bar{a} \bar{b} \bar{c}}$$

(see 1.8).

Moreover we remark that $\Phi_{i, \bar{a} \bar{b} \bar{c}, \ldots, \bar{a} \bar{b} \bar{c}, \ldots, \bar{a} \bar{b} \bar{c}} = \Phi_{i, \bar{a} \bar{b} \bar{c}, \ldots, \bar{a} \bar{b} \bar{c}, \ldots, \bar{a} \bar{b} \bar{c}}$. Hence the second term is zero.

So we have

$$** = c_i^{-1} \sum_{i \in I} (\sum_{\beta} g_{\beta \bar{\beta}} R_{i \beta \bar{a} \bar{b} \bar{c}}) \Phi_{i, \bar{a} \bar{b} \bar{c}, \ldots, \bar{a} \bar{b} \bar{c}} \bar{\Phi}_{i, \bar{a} \bar{b} \bar{c}, \ldots, \bar{a} \bar{b} \bar{c}}$$

On the other hand,

$$\sum_{\alpha, \beta = 1}^{n} g_{\alpha \beta} R_{i \alpha \beta \bar{a} \bar{b} \bar{c}} = \sum_{\alpha, \beta = 1}^{n} g_{\alpha \beta} g_{i \alpha \beta} R_{i \alpha \beta \bar{a} \bar{b} \bar{c}}$$

(Use (1.5) and (1.8).)

$$= \sum_{\alpha, \beta = 1}^{n} g_{\alpha \beta} g_{i \alpha \beta} R_{i \alpha \beta \bar{a} \bar{b} \bar{c}}$$

(Use (1.6).)

$$= R_{i \bar{a} \bar{b} \bar{c}}$$

Hence we have

$$** = (p-1)! c_i^{-1} \sum_{\alpha, \beta = 1}^{n} \sum_{\alpha, \beta = 1}^{n} (\theta_{i, \bar{a} \bar{b} \bar{c}} - R_{i \bar{a} \bar{b} \bar{c}}) \Phi_{i, \bar{a} \bar{b} \bar{c}, \ldots, \bar{a} \bar{b} \bar{c}} \bar{\Phi}_{i, \bar{a} \bar{b} \bar{c}, \ldots, \bar{a} \bar{b} \bar{c}}$$

Since $\bar{\partial} \Phi_{i, \bar{a} \bar{b} \bar{c}, \ldots, \bar{a} \bar{b} \bar{c}} = \sum_{p=0}^{p} (-1)^{p} \nabla_{\bar{a} \bar{b} \bar{c}} \Phi_{i, \bar{a} \bar{b} \bar{c}, \ldots, \bar{a} \bar{b} \bar{c}}$

$$(\bar{\partial} \Phi)_{\bar{a} \bar{b} \bar{c}, \ldots, \bar{a} \bar{b} \bar{c}}^{p} = \sum_{p=0}^{p} g_{i \bar{a} \bar{b} \bar{c}, \ldots, \bar{a} \bar{b} \bar{c}} \Phi_{i, \bar{a} \bar{b} \bar{c}, \ldots, \bar{a} \bar{b} \bar{c}}$$

(Use (1.12).)
\[
\langle \bar{\partial} \varphi, \bar{\partial} \varphi \rangle = c_t^{-1} \sum_{p \neq \tau, d_p \neq (d_{t_1}, \ldots, d_{t_p})} (\bar{\partial} \varphi)_{t_i, d_{p+1}} (\bar{\partial} \varphi)_{t_i} \text{d}_p \\
= c_t^{-1} \sum_{\alpha=0}^{2} (-1)^\alpha (\bar{\partial} \varphi)_{t_i, d_{p+1}} (\bar{\partial} \varphi)_{t_i} \\
\times (\sum_{\tau \neq \alpha}^{2} (-1)^\tau g_t^{\bar{\partial} \varphi_{t_i, d_{p+1}}} \text{d}_{\tau}^{-1} \varphi_{t_i, d_{p+1}}^{d_{p+1}} a_{t_1}^{d_{p+1}} d_{p+1}) \\
= c_t^{-1} \sum_{\beta \in \mathcal{D}_p \neq (d_{t_1}, \ldots, d_{t_p})} (\bar{\partial} \varphi)_{t_i, d_{p+1}} (\bar{\partial} \varphi)_{t_i} \\
\times (\sum_{\alpha=\beta}^{2} (-1)^{\beta+\tau} g_t^{\bar{\partial} \varphi_{t_i, d_{p+1}}} \text{d}_{\beta}^{-1} \varphi_{t_i, d_{p+1}}^{d_{p+1}} a_{t_1}^{d_{p+1}} d_{p+1}) \\
\text{ (the terms with } \mu = \tau) \\
+ c_t^{-1} \sum_{\beta \neq \tau} (-1)^{\beta+\tau} g_t^{\bar{\partial} \varphi_{t_i, d_{p+1}}} \text{d}_{\beta}^{-1} \varphi_{t_i, d_{p+1}}^{d_{p+1}} a_{t_1}^{d_{p+1}} d_{p+1} \\
\text{ (the terms with } \mu \neq \tau). 
\]

Since \(\mu + \tau + (\mu + \tau - 1) \equiv 1 \mod 2\), we have

\[
\langle \bar{\partial} \varphi, \bar{\partial} \varphi \rangle = c_t^{-1} \sum_{\beta \in \mathcal{D}_p \neq (d_{t_1}, \ldots, d_{t_p})} g_t^{\bar{\partial} \varphi_{t_i, d_{p+1}}} \text{d}_{\beta}^{-1} \varphi_{t_i, d_{p+1}}^{d_{p+1}} a_{t_1}^{d_{p+1}} d_{p+1} \\
- c_t^{-1} \sum_{\alpha=\beta}^{2} \sum_{\beta \neq \tau} (-1)^{\beta+\tau} g_t^{\bar{\partial} \varphi_{t_i, d_{p+1}}} \text{d}_{\beta}^{-1} \varphi_{t_i, d_{p+1}}^{d_{p+1}} a_{t_1}^{d_{p+1}} d_{p+1}. 
\]

Add terms with \(\beta \in \mathcal{D}_p\) to the first sum on the right-hand side and add terms with \(\alpha = \beta\) to the second sum, then the difference remains unchanged and we have

\[
\langle \bar{\partial} \varphi, \bar{\partial} \varphi \rangle = c_t^{-1} \sum_{\beta \in \mathcal{D}_p \neq (d_{t_1}, \ldots, d_{t_p})} g_t^{\bar{\partial} \varphi_{t_i, d_{p+1}}} \text{d}_{\beta}^{-1} \varphi_{t_i, d_{p+1}}^{d_{p+1}} a_{t_1}^{d_{p+1}} d_{p+1} \\
- c_t^{-1} \sum_{\beta \neq \tau} \sum_{\beta \neq \tau} (-1)^{\beta+\tau} g_t^{\bar{\partial} \varphi_{t_i, d_{p+1}}} \text{d}_{\beta}^{-1} \varphi_{t_i, d_{p+1}}^{d_{p+1}} a_{t_1}^{d_{p+1}} d_{p+1}. 
\]

We set

\[
\langle \bar{\nabla} \varphi, \bar{\nabla} \varphi \rangle = c_t^{-1} \sum_{\beta \in \mathcal{D}_p \neq (d_{t_1}, \ldots, d_{t_p})} g_t^{\bar{\partial} \varphi_{t_i, d_{p+1}}} \text{d}_{\beta}^{-1} \varphi_{t_i, d_{p+1}}^{d_{p+1}} a_{t_1}^{d_{p+1}} d_{p+1}. 
\]

Then we have

\[
\langle \bar{\partial} \varphi, \bar{\partial} \varphi \rangle = \langle \bar{\nabla} \varphi, \bar{\nabla} \varphi \rangle - \sum_{\beta \neq \tau} \sum_{\beta \neq \tau} \nabla_{t_i, \beta} \varphi_{t_i, d_{p+1}} \text{d}_{\beta}^{-1} \varphi_{t_i, d_{p+1}}^{d_{p+1}} a_{t_1}^{d_{p+1}} d_{p+1}, 
\]

and so

\[
*) = (p - 1)! \{ \langle \bar{\nabla} \varphi, \bar{\nabla} \varphi \rangle - \langle \bar{\partial} \varphi, \bar{\partial} \varphi \rangle \}. 
\]
Therefore
\[ \sum_{\beta=1}^{n} \nabla_{\theta}^{\delta_{\beta}} = c_i^{-1} \sum \nabla_{v} \nabla_{\beta}^{(\gamma)} \psi_{i,\beta,\mathcal{D}_{1},\mathcal{D}_{2},\mathcal{K}_{1},\mathcal{P}_{i}} + \cdots + (p-1)! \{c_i^{-1} \sum_{\mathcal{D}_{1},\mathcal{D}_{2},\mathcal{K}_{1}} (\theta_{i,\mathcal{D}_{1},\mathcal{D}_{2},\mathcal{K}_{1}} \psi_{i,\mathcal{D}_{1},\mathcal{D}_{2},\mathcal{K}_{1},\mathcal{P}_{i},\mathcal{P}_{i}} + \cdots + \}}.\]

On the other hand
\[ \sum_{i=1}^{n} \nabla_{v} \eta_{\theta} = c_i^{-1} \sum (\nabla_{v} - \overline{\partial}_{v} \log c_i) \nabla_{\beta}^{(\gamma)} \psi_{i,\beta,\mathcal{D}_{1},\mathcal{D}_{2},\mathcal{K}_{1},\mathcal{P}_{i}} + \cdots + (p-1)! \{c_i^{-1} \sum_{\mathcal{D}_{1},\mathcal{D}_{2},\mathcal{K}_{1}} (\theta_{i,\mathcal{D}_{1},\mathcal{D}_{2},\mathcal{K}_{1}} \psi_{i,\mathcal{D}_{1},\mathcal{D}_{2},\mathcal{K}_{1},\mathcal{P}_{i},\mathcal{P}_{i}} + \cdots + \}.\]

Hence
\[ \sum_{i=1}^{n} \nabla_{v} \eta_{\theta} = c_i^{-1} \sum \nabla_{v} \nabla_{\beta}^{(\gamma)} \psi_{i,\beta,\mathcal{D}_{1},\mathcal{D}_{2},\mathcal{K}_{1},\mathcal{P}_{i}} + \cdots + (p-1)! \{c_i^{-1} \sum_{\mathcal{D}_{1},\mathcal{D}_{2},\mathcal{K}_{1}} (\theta_{i,\mathcal{D}_{1},\mathcal{D}_{2},\mathcal{K}_{1}} \psi_{i,\mathcal{D}_{1},\mathcal{D}_{2},\mathcal{K}_{1},\mathcal{P}_{i},\mathcal{P}_{i}} + \cdots + \}.\]

Therefore, we have
\[ \sum_{\beta=1}^{n} \nabla_{\theta}^{\delta_{\beta}} = \sum_{i=1}^{n} \nabla_{v} \eta_{\theta} = (p-1)! \{c_i^{-1} \sum_{\mathcal{D}_{1},\mathcal{D}_{2},\mathcal{K}_{1}} (\theta_{i,\mathcal{D}_{1},\mathcal{D}_{2},\mathcal{K}_{1}} \psi_{i,\mathcal{D}_{1},\mathcal{D}_{2},\mathcal{K}_{1},\mathcal{P}_{i},\mathcal{P}_{i}} + \cdots + \}.\]

We apply (1.16) of Lemma 1.1 to \( \sum_{\beta=1}^{n} \nabla_{\theta}^{\delta_{\beta}} = \sum_{i=1}^{n} \nabla_{v} \eta_{\theta} \). We have

\[ i) \int_{x} c_i^{-1} \sum_{\mathcal{D}_{1},\mathcal{D}_{2},\mathcal{K}_{1}} (\theta_{i,\mathcal{D}_{1},\mathcal{D}_{2},\mathcal{K}_{1}} \psi_{i,\mathcal{D}_{1},\mathcal{D}_{2},\mathcal{K}_{1},\mathcal{P}_{i},\mathcal{P}_{i}} + \cdots + \} dV \]

and
\[ \| \nabla \psi \|^2 = \int_{x} \langle \nabla \psi, \nabla \psi \rangle dV. \]
From $\varphi \in B^{0,p}(X, F)$, we have
\[
\text{ii) } \sum_{\beta=1}^{n} n_{\beta} \partial_{\beta} f = (c_{d}|\text{grad } \Theta|)^{-1} \sum_{\beta=1}^{n} \nabla_{\beta} \varphi_{\xi_{\beta} D_{\beta-1}} \overline{\varphi_{\xi_{\beta} D_{\beta-1}}} \partial_{\xi_{\beta} \Theta} \Theta
\]
\[= 0 \text{ on } \partial X \quad \text{(See (1.25).)}
\]
and
\[
\sum_{\beta=1}^{n} \xi_{\beta} \partial_{\xi_{\beta} f} = (c_{d}|\text{grad } \Theta|)^{-1} \sum_{\beta=1}^{n} \nabla_{\beta} \varphi_{\xi_{\beta} D_{\beta-1}} \overline{\varphi_{\xi_{\beta} D_{\beta-1}}} \partial_{\xi_{\beta} \Theta} \Theta \
\]

While, in a neighborhood of every point of $\partial X$, for any multi-index $D_{p-1} = (d_{1}, \ldots, d_{p-1})$, there exists a $C^{\omega}$-section $\psi_{D_{p}}$ of $T^{\omega} \overline{M}^{*}$ such that
\[
\sum_{\beta=1}^{n} \varphi_{\xi_{\beta} D_{\beta-1}} \partial_{\xi_{\beta} \Theta} \Theta = \Phi \psi_{D_{p-1}}.
\]

Since
\[
\sum_{\beta=1}^{n} \varphi_{\xi_{\beta} D_{\beta-1}} \partial_{\xi_{\beta} \Theta} \Theta = f_{i_{1}} \sum_{\beta=1}^{n} \varphi_{\xi_{\beta} D_{\beta-1}} \partial_{\xi_{\beta} \Theta} \Theta \text{ on } U_{i_{1}} \cap U_{j} \cap W, \text{ where } W \text{ is the neighborhood of } \partial X \text{ taken in (1.13)}, \varphi_{D_{p}} = \{ \varphi_{\xi_{\beta} D_{\beta-1}} = \sum_{\beta=1}^{n} \varphi_{\xi_{\beta} D_{\beta-1}} \partial_{\xi_{\beta} \Theta} \Theta \}
\]
is an element of $\Phi_{\omega}(W, F)$ for any multi-index $D_{p-1}$. Hence we can operate the covariant differentiation to $\varphi_{D_{p-1}}$. Then we have
\[
\sum_{\beta=1}^{n} \nabla_{\beta} \varphi_{\xi_{\beta} D_{\beta-1}} \partial_{\xi_{\beta} \Theta} \Theta + \sum_{\beta=1}^{n} \varphi_{\xi_{\beta} D_{\beta-1}} \partial_{\xi_{\beta} \Theta} \Phi
\]
\[= \psi_{D_{p}} \partial_{\xi_{\beta} \Theta} \Theta + \Phi \nabla_{\beta} \psi_{D_{p-1}} \text{ on } W.
\]

We multiply it by $\overline{\varphi_{\xi_{\beta} D_{\beta-1}}}$ and sum up with respect to the index $\tau$. Since $\varphi \in B^{0,p}(X, F)$, we obtain
\[
\sum_{\beta=1}^{n} \nabla_{\beta} \varphi_{\xi_{\beta} D_{\beta-1}} \partial_{\xi_{\beta} \Theta} \Theta \overline{\varphi_{\xi_{\beta} D_{\beta-1}}} + \sum_{\beta=1}^{n} \partial_{\xi_{\beta} \Theta} \Theta \partial_{\xi_{\beta} \Theta} \overline{\varphi_{\xi_{\beta} D_{\beta-1}}} = 0
\]
on $\partial X$. And so
\[
\text{iii) } \sum_{\beta=1}^{n} \xi_{\beta} \partial_{\xi_{\beta} \Theta} \Theta = - (c_{d}|\text{grad } \Theta|)^{-1} \sum_{\beta=1}^{n} \partial_{\xi_{\beta} \Theta} \Theta \varphi_{\xi_{\beta} D_{\beta-1}} \overline{\varphi_{\xi_{\beta} D_{\beta-1}}}
\]
on $\partial X$.

Finally, from i) ii), and iii), we obtain
\[
\|\partial \varphi\|^{2} + \|\partial \Phi\|^{2} = \int_{\partial X} (c_{d}|\text{grad } \Theta|)^{-1} \sum_{\beta=1}^{n} \partial_{\xi_{\beta} \Theta} \Theta \varphi_{\xi_{\beta} D_{\beta-1}} \overline{\varphi_{\xi_{\beta} D_{\beta-1}}} dS
\]
\[+ \|\nabla \varphi\|^{2} + \int_{\partial X} c_{d}^{-1} \sum_{\beta=1}^{n} \partial_{\xi_{\beta} \Theta} \Theta \varphi_{\xi_{\beta} D_{\beta-1}} \overline{\varphi_{\xi_{\beta} D_{\beta-1}}} dV
\]
for any \( \varphi \in B^{p,q}(\mathbb{X}, F) \).

Q.E.D.

Remark 1.3 Dr. A. Fujiki has pointed out that the equality of Proposition 1.5 had already been obtained by P. Griffiths in [14], p. 429, Theorem 7.2 and recently, the author noticed that A. Andreotti and E. Vesentini also had obtained some inequality for the elements of \( B^{p,q}(\mathbb{X}, F) \) in [6]. In their formulae, the base metric \( ds^2 \) need not to be a Kähler metric.

§1.2. Identity of Weak and Strong Extensions of \( \tilde{\theta} \)-Operator and Its Formal Adjoint \( \theta \)

Let \( X \) be a relatively compact domain with smooth boundary on a complex manifold \( M \) and let \( F \rightarrow M \) be a holomorphic line bundle on \( M \). We fix a hermitian metric \( d\sigma^2 \) on \( M \) and a hermitian metric \( \{a_t\} \) along the fibres of \( F \). We use notations \( ||, ||, \|, \|, \vartheta \) and \( \vartheta_F \) and so on as in Section 1.1 with respect to \( d\sigma^2, \{a_t\} \) and \( \Psi \in C^{0,0}(X) \).

Let \( L^{p,q}(X, F) \) \( \text{(resp. } L^{p,q}(X, F, \Psi) \text{)} \) be the completion of \( C_{p,q}^{\text{c}}(\mathbb{X}, F) \) with respect to the norm \( \|, \| \) \( \text{(resp. } \|, \|_{\Psi} \text{)} \). Then, since \( \|, \| \) and \( \|, \|_{\Psi} \) are equivalent on \( X \), \( L^{p,q}(X, F, \Psi) \) coincides as a topological vector space with \( L^{p,q}(X, F) \). In other words, \( L^{p,q}(X, F, \Psi) \) is the Hilbert space with \( L^{p,q}(X, F) \) as the underlying space and \((,)_F \) as the inner product. Thus \( L^{p,q}(X, F, \Psi) \) is understood as the pair \( \{L^{p,q}(X, F), (,)_F\} \).

Let \( \tilde{\varphi}: L^{p,q}(X, F, \Psi) \rightarrow L^{p,q+1}(X, F, \Psi) \) be the maximal closed extension of the original \( \varphi \) and let \( \vartheta_F: L^{p,q+1}(X, F, \Psi) \rightarrow L^{p,q}(X, F, \Psi) \) be the maximal closed extension of the original \( \vartheta_F \) i.e. when we represent the domain of \( \tilde{\varphi} \) \( \text{(resp. } \vartheta_F \text{)} \) by \( D^{p,q}_\Psi \) \( \text{(resp. } D^{p,q}_\Psi \text{)} \),

(1.31) i) \( \varphi \in D^{p,q}_\Psi \subset L^{p,q}(X, F, \Psi) \) if and only if there exists an element \( u \in L^{p,q+1}(X, F, \Psi) \) such that \( (\vartheta_F \psi, \varphi)_F = (\psi, u)_F \) for every \( \psi \in C_{p,q+1}^{\text{c}}(X, F) \),

ii) \( \varphi \in D^{p,q}_\Psi \subset L^{p,q}(X, F, \Psi) \) if and only if there exists an element \( v \in L^{p,q+1}(X, F, \Psi) \) such that \( (\tilde{\varphi} \vartheta, \varphi)_F = (\vartheta, v)_F \) for every \( \vartheta \in C_{p,q-1}^{\text{c}}(X, F) \).

Let \( L^{p,q}(M, F) \) \( \text{(resp. } L^{p,q}(M, F, \Psi) \text{)} \) be the completion of
with respect to the norm \( \| \cdot \| \) (resp. \( \| \cdot \|_r \)). Then differential operators \( \partial \) and \( \partial_r \) in \( L^r_{\infty}(M, F, \Psi) \) are defined similarly.

From (1.23), we have \( C^p_{\infty}(X, F) \subset D^p_{\infty} \cap D^p_{\infty} \). Hence \( \partial \) and \( \partial_r \) defined as above are, in the weak sense, closed densely defined operators. Next we consider the closure of the graph of \( \partial : C^p_{\infty}(X, F) \to C^{p+1}_{\infty}(X, F) \) in \( L^p_{\infty}(X, F, \Psi) \times L^{p+1}_{\infty}(X, F, \Psi) \). Then, from the general theory of linear operators in Hilbert spaces (cf. [46], p. 70, Theorem 4.15), there exists a unique linear operator \( T : L^p_{\infty}(X, F, \Psi) \to L^{p+1}_{\infty}(X, F, \Psi) \) such that i) the graph \( G(T) \) of \( T \) is equal to the closure of the graph of \( \partial|_{C^p_{\infty}(X, F)} \), ii) the domain \( D^p_{\infty} \) of \( T \) in \( L^p_{\infty}(X, F, \Psi) \) is the image of \( G(T) \) by the projection to the first factor.

**Remark 1.4.** Since \( L^p_{\infty}(X, F, \Psi) \) coincides with \( L^p_{\infty}(X, F) \) as a topological vector space) for any \( \Psi \in C^p_0(X) \), \( T \) is determined independently of \( \Psi \). \( T \) is called the closure of \( \partial|_{C^p_{\infty}(X, F)} \).

Since \( C^p_{\infty}(X, F) \) is contained in \( D_T \), \( T \) is a closed densely defined operator and from ii) it is clear that

\[(1.32) \varphi \in D^p_{\infty} \text{ if and only if there exists a sequence } \{\varphi_n\} \text{ of } C^p_{\infty}(X, F) \text{ such that } \|\varphi_n - \varphi\|_\Psi \text{ and } \|T\varphi_n - T\varphi\|^2_\Psi \text{ tend to zero as } n \to +\infty.\]

Since \( D^p_{\infty} \) is dense in \( L^p_{\infty}(X, F, \Psi) \), if for a given \( g \in L^{p+1}_{\infty}(X, F, \Psi) \), there exists an element \( g^* \in L^p_{\infty}(X, F, \Psi) \) such that \( (T\varphi, g) = (\varphi, g^*)_\Psi \) for any \( \varphi \in D^p_{\infty} \), \( g^* \) is uniquely determined by \( g \). Hence the adjoint operator \( T^*_g : L^{p+1}_{\infty}(X, F, \Psi) \to L^p_{\infty}(X, F, \Psi) \) is determined by \( T^*_g g = g^* \). Then from (1.32), we have

\[(1.33) g \text{ is contained in the domain } D^p_{\infty} \text{ of } T^*_g \text{ if and only if there exists a positive constant } C \text{ such that } |(\partial\varphi, g)_\Psi| \leq C \|\varphi\|_\Psi \text{ for any } \varphi \in C^p_{\infty}(X, F).\]

Since \( C^p_{\infty+1}(X, F) \) is contained in \( D^p_{\infty+1} \), \( T^*_g \) is a closed densely defined operator. Hence it holds that \( T^{**}_g = T \) (see [46], p. 90, Theorem 5.3) and moreover, from Lemma 1.3 and (1.32), we have
(1.34) \[ C^{p,q}(X,F) \cap D_{\tilde{\partial}}^{p,q} = B^{p,q}(X,F) \quad \text{and} \]
\[ T_{\tilde{\partial}} = \tilde{\partial} \quad \text{on} \quad B^{p,q}(X,F). \]

Similarly, we can consider the adjoint operator \( \tilde{\partial}^* \) of the maximal closed extension \( \tilde{\partial} \), then we have

(1.35) \( h \) is contained in the domain \( D_{\tilde{\partial}^*}^{p,q+1} \) of \( \tilde{\partial}^* \) if and only if there exists a positive constant \( C \) such that \( |(\tilde{\partial} \varphi, h)_{\Psi}| \leq C \| \varphi \|_{\Psi} \) for any \( \varphi \in D^{p,q}_{\tilde{\partial}}. \)

Hence \( \tilde{\partial}^* \) is a closed densely defined operator and so \( \tilde{\partial}^{p,q*} = \tilde{\partial}. \) \( T \) and \( \tilde{\partial}^* \) are called the minimal (or strong) extensions of the original \( \tilde{\partial} \) and its formal adjoint \( \tilde{\partial}. \) From (1.31), (1.32), (1.33) and (1.35), we have

(1.36) \( D^{p,q}_0 \subset D^{p,q}_{\tilde{\partial}} \subset D^{p,q}_{\tilde{\partial}^*} \subset D^{p,q}_{\tilde{\partial}} \) in \( L^{p,q}(X,F,\Psi) \) for any \( \Psi \in C^{0,q}(X). \)

**Remark 1.5.** If \( T \) is the closure of \( \tilde{\partial}|_{C^{0,q}(X,F)}, \) it holds that \( T^{p,q*} = \tilde{\partial} \) and \( \tilde{\partial}^{p,q*} \) is the closure of \( \tilde{\partial}|_{C^{0,q+1}(X,F)}. \)

The following Proposition is due to Hörmander [17] Propositions 1.2.3 and 1.2.4 which summarize results of [10], [16] and [30].

**Proposition 1.6.** i) If \( \varphi \in D^{p,q}_{\tilde{\partial}} \subset L^{p,q}(X,F,\Psi) \) and \( \text{supp} \varphi, \text{supp} \vartheta \subseteq X, \) then \( \varphi|_{\tilde{\partial}} \in D^{p,q}_{\tilde{\partial}} \subset L^{p,q}(X,F,\Psi) \) i.e. \( (\tilde{\partial} \varphi)|_{\tilde{\partial}} = \tilde{\partial}^{p,q*}(\varphi|_{\tilde{\partial}}) \) in \( L^{p,q-1}(X,F,\Psi). \)

ii) \( C^{p,q}(X,F) \) is dense in \( D^{p,q}_{\tilde{\partial}} \) with respect to the graph norm \( (\| \cdot \|_{\tilde{\partial}}, \| \cdot \|_{\tilde{\partial}})^{1/2}. \)

iii) \( B^{p,q}(X,F) \) is dense in \( D^{p,q}_{\tilde{\partial}} \) (resp. \( D^{p,q}_{\tilde{\partial}^*} \)) with respect to the graph norm \( (\| \cdot \|_{\tilde{\partial}} + \| \vartheta \|_{\tilde{\partial}})^{1/2} \) (resp. \( (\| \cdot \|_{\tilde{\partial}} + \| \vartheta \|_{\tilde{\partial}})^{1/2}. \)

As a consequence of Proposition 1.6, we have the following.

**Proposition 1.7.** i) \( D^{p,q}_{\tilde{\partial}} = D^{p,q}_{\tilde{\partial}} \) in \( L^{p,q}(X,F,\Psi) \) i.e. \( T = \tilde{\partial} \)
ii) \( D^{p,q}_{\delta^q \psi} = D^{p,q}_T \) in \( L^{p,q}(X, F, \psi) \) i.e. \( T^{p,q}_\psi = \delta^q \psi \\

iii) \( \delta^q \psi \) (\( = T^{p,q}_\psi \)) is the restriction of the maximal closed extension \( \partial_T \) to \( D^{p,q}_{\delta^q \psi} \).

iv) \( D^{p,q}_{\delta^q \psi} = D^{p,q}_{\delta^q \psi} \) in \( L^{p,q}(X, F) \) for any \( \psi \in C^{0,0}(X) \).

**Proof.** i), ii) and iii) follow from (1.32), (1.33), (1.35), (1.36) and Proposition 1.6 ii), iii) immediately. From (1.34), it holds that \( C^{p,q}(X, F) \cap D^{p,q}_{\delta^q \psi} = C^{p,q}(X, F) \cap D^{p,q}_{\delta^q \psi} \) for any \( \psi \in C^{0,0}(X) \). Hence, by Proposition 1.6 iii), we obtain iv). Q.E.D.

We denote the range and nullity of \( \partial \) in \( L^{p,q}(X, F, \psi) \) by \( R^{p,q}_\psi \) and \( N^{p,q}_\psi \) respectively. \( R^{p,q}_\psi \) and \( N^{p,q}_\psi \) are defined similarly. We set

\[
R^{p,q}_{\psi} = N^{p,q}_\psi \cap N^{p,q}_{\delta^q \psi} \quad \text{in} \quad L^{p,q}(X, F, \psi).
\]

§ 1.3. Basic Fact from Functional Analysis and Application

Let \((H_i, \|\cdot\|_i) \ (i = 1, 2, 3)\) be Hilbert spaces and let \( H_i \rightarrow H_2 \rightarrow H_3 \) be closed densely defined operators satisfying \( S \circ T = 0 \). We denote the adjoint operator of \( T \) (resp. \( S \)) by \( T^* \) (resp. \( S^* \)). Then, as mentioned in Section 1.2, \( T^* \) and \( S^* \) are also closed densely defined operators and satisfy \( T = T^{**} \) and \( S = S^{**} \). We use notations \( D_T, R_T \) and \( N_T \) etc. as defined in Section 1.2.

**Proposition 1.8** (cf. [17], Theorems 1.1.1 and 1.1.2 and [25], p. 210–p. 216). A necessary and sufficient condition for \( R_T \) and \( R_{S^*} \) both to be closed is that there exists a positive constant \( C \) such that

\[
\|f\|_i^2 \leq C (\|Sf\|_i^2 + \|T^*f\|_i^2) \quad \text{for any} \quad f \in D_S \cap D_T,
\]

and that \( f \) is orthogonal to the space \( N = N_S \cap N_T \). When this is the case, we have the strong orthogonal decomposition of \( H_2 \)

\[
H_2 \cong R_T \oplus N \oplus R_{S^*}.
\]

Furthermore, the operator \( L = TT^* + S^*S \) whose domain is \( D_L = \{f \in D_S \cap D_T : Sf \in D_S \cap D_T \} \), is self-adjoint and has a closed range in \( H_2 \). And (1.38) can be written more explicitly.
(1.39) \[ H_\mathcal{S} = R_{\mathcal{T}R} \oplus N \oplus R_{\mathcal{S}S} \] where \( R_{\mathcal{T}R} = R_{\mathcal{T}T} \) and \( R_{\mathcal{S}S} = R_{\mathcal{S}S} \).

**Remark 1.6.** From the definition of \( D_L \), the nullity of the operator \( L \) is equal to \( N = N_S \cap N_T \).

**Proposition 1.9** (cf. [17], Theorem 1.1.3). Assume that from every sequence \( f_k \in D_S \cap D_T^* \) with \( \| f_k \|_S \) bounded and \( \| T^* f_k \|_T^* \to 0 \) in \( H_\mathcal{T} \), \( \| S f_k \|_S^2 \to 0 \) in \( H_S \) as \( k \to +\infty \). one can select a strongly convergent subsequence. Then (1.37) holds and \( N \) is finite dimensional.

**Proposition 1.10** (cf. [17], Theorem 1.1.4). Let \( P \) be a closed linear subspace of \( H_\mathcal{T} \) containing \( R_T \). Assume that there exists a positive constant \( C \) such that

(1.40) \[ \| f \|_S^2 \leq C \{ \| S f \|_S^2 + \| T^* f \|_T^* \} \] for any \( f \in D_S \cap T_T^* \cap P \).

Then i) for every \( g \in P \) satisfying \( Sg = 0 \), there exists \( h \in D_T \subset H_\mathcal{T} \) such that \( Th = g \),

ii) \( R_T^* \) is closed in \( H_\mathcal{T} \) and for every \( g \in R_T^* \), there exists \( h \in D_T \subset H_\mathcal{T} \) such that \( T^* h = g \) and \( \| h \|_S^2 \leq C \| g \|_T^*^2 \).

Let \( M, X \) and \( F \) be as in Section 1.2.

**Proposition 1.11.**

I) If there exist in the degree \((p, q)\)

1) a hermitian metric \( ds^2 \) on \( M \) and a hermitian metric \( \{ c_i \} \) along the fibres of \( F \),

2) a positive constant \( C_1 \),

and

3) a proper compact subset \( K \) of \( X \) such that

(1.41) \[ \| \varphi \|_X^2 \leq C \{ \| \bar{\partial} \varphi \|_X^2 + \| \bar{\partial}^* \varphi \|_X^2 + \varphi ^2 \} \]

for any \( \varphi \in D^{p,q}_{\mathcal{S}} \cap D^{p,q}_{\mathcal{T}} \subset L^{p,q}(X,F) \), then

i) \( \dim_N N^{p,q} < +\infty \)

ii) \( L^{p,q}(X,F) \cong R^{p,q}_{\mathcal{S}S} \oplus N^{p,q} \oplus R^{p,q}_{\mathcal{T}T} \).
II) If there exist in the degree \((p, q)\)

1) a hermitian metric \(ds^2\) on \(M\) and a hermitian metric \(\{c_i\}\) along the fibres of \(F\).

2) a positive constant \(C_i\),

3) a proper compact subset \(K\) of \(X\), which does not contain any connected component of \(X\), such that

\[
\|\varphi\|_{L^2(X)}^2 \leq C_1 (\|\bar{\partial}\varphi\|^2 + \|\bar{\partial}^*\varphi\|^2)
\]

for any \(\varphi \in D^{p,q}_0 \cap D^{p,q}_2 \subset L^{p,q}(X, F)\), then there exists a positive constant \(C_2\) such that

\[
\|\varphi\|_{L^2(X)}^2 \leq C_2 (\|\bar{\partial}\varphi\|^2 + \|\bar{\partial}^*\varphi\|^2)
\]

for any \(\varphi \in D^{p,q}_0 \cap D^{p,q}_2 \subset L^{p,q}(X, F)\).

\textbf{Proof.} Take any sequence \(\{\varphi_m\}_{m \geq 1}\) such that \(\varphi_m \in D^{p,q}_0 \cap D^{p,q}_2\), \(\|\varphi_m\|^2 \leq 1\) and \(\|\bar{\partial}\varphi_m\|^2, \|\bar{\partial}^*\varphi_m\|^2 \to 0\) as \(m \to +\infty\). Then we assert that there exists a subsequence \(\{\varphi_{m_k}\}\) of \(\{\varphi_m\}\) which converges strongly on \(X\). From Proposition 1.6 iii), we may assume \(\varphi_m \in B^{p,q}(X, F)\). Let \(\chi\) be a \(C^\infty\)-function on \(M\) with compact support in \(X\) and \(\chi = 1\) on \(K\). Since \(\chi \varphi_m \in C^{p,q}(X, F)\), we have that

\[
(\bar{\partial} (\chi \varphi_m), \bar{\partial} (\chi \varphi_m)) + (\bar{\partial} (\chi \varphi_m), \bar{\partial} (\chi \varphi_m)) = (\chi \varphi_m, \chi \varphi_m)
\]

is bounded by the assumption. From coerciveness of elliptic differential operator \(\Box = -\bar{\partial}\partial + \partial\bar{\partial}\) on \(C^{p,q}_0(X, F)\) (see for example [9], (2.2.1) Theorem) and Rellich’s lemma (see for example [9], Appendix (A.1.6) Proposition and p.122, 2 Sobolev norms on manifolds), it follows that \(\{\varphi_m\}\) has a sequence \(\{\varphi_{m_k}\}\) which is strongly convergent on the compact subset \(K\) of \(X\). By (1.41), we conclude that \(\{\varphi_{m_k}\}\) converges strongly on \(X\). Therefore we can apply Propositions 1.8 and 1.9 for \(H_i = L^{p,q+i-2}(X, F)\) \((i=1, 2, 3)\), \(S = \bar{\partial}\) and \(T^* = \bar{\partial}^*\). Hence we obtain i) and ii) of I).

In the case of II), the same assertion of I) holds. In particular, from Propositions 1.8 and 1.9, we have that there exists a constant
\( C_\geq 0 \) such that
\[
\| \varphi \|^2 \leq C_\{ \| \bar{\partial} \varphi \|^2 + \| \partial^* \varphi \|^2 \}
\]
for any \( \varphi \in D^p_q \cap D^q_p \) with \( \varphi \perp N^{p,q} \), while, each element \( \varphi \) in \( N^{p,q} \) is a solution of the Laplace-Beltrami operator \( \Box = \bar{\partial} \partial + \partial^* \bar{\partial} \). Namely \( \varphi \) is a harmonic form with valued in \( F \). In general, when \( E \) is a hermitian vector bundle over a connected complex hermitian manifold \( M \), a harmonic form \( \varphi \in H^{p,q}(E) \) vanishes identically on \( M \) if it vanishes on a non-empty open subset of \( M \) (cf. [7], [38]). Now, from (1.42), \( \varphi \) vanishes identically on \( X \setminus K \). Since any connected component of \( X \) is not contained in \( K \), by the above unique continuation property, \( \varphi \) vanishes on each connected component and so \( \varphi \) vanishes identically on \( X \). Hence \( N^{p,q} \) is the null space. Combining this with (1.44), the proof is completed.

**Remark 1.7.** Since, from Proposition 1.7, \( L = \bar{\partial} \partial^* + \partial^* \bar{\partial} \) is the restriction of the Laplace-Beltrami operator \( \Box = \bar{\partial} \partial + \partial^* \bar{\partial} \) to the domain of \( L \), \( N^{p,q} \) may not coincide with the space of all harmonic forms in \( L^{p,q}(X, F) \).

**Chapter II. Cohomology Groups on Domains with Pseudoconvex Boundaries**

**§ 2.1. Definitions**

Let \( X \) be a relatively compact domain with boundary \( \partial X \) on a complex manifold \( M \) of complex dimension \( n \).

**Definition 2.1.** X is said to be a domain with pseudoconvex boundary \( \partial X \) if for any point \( p \) of \( \partial X \), there exist a neighborhood \( U \) of \( p \) and a real valued \( C^\infty \)-function \( \lambda \) on \( U \) such that i) \( U \cap X = \{ x \in U \mid \lambda(x) < 0 \} \), ii) the gradient of \( \lambda \) nowhere vanishes on \( \partial X \cap U \), iii) the complex Hessian of \( \lambda \) is positive semi-definite when restricted to the complex tangent space of \( \partial X \cap U \).
Remark 2.1. The relation between the above definition and other ones is the following.

The next three conditions are equivalent to one another:

I) $X$ is a domain with pseudoconvex boundary $\partial X$.

II) There exist a neighborhood $W$ of $\partial X$ and a real valued $C^\infty$-function $\Phi$ on $W$ such that i) $X \cap W = \{ x \in W | \Phi(x) < 0 \}$, ii) the gradient of $\Phi$ nowhere vanishes on $W$, iii) the complex Hessian of $\Phi$ is positive semi-definite when restricted to the complex tangent space of $\partial X$.

III) $X$ is a locally Stein domain with smooth boundary $\partial X$. Here we say $X$ is locally Stein if for any point $p$ of $\partial X$, there exists a neighborhood $U$ of $p$ such that $U \cap X$ is Stein.

Outline of the proof. We can show I) $\Rightarrow$ II) easily (see [15], the proof of p. 263, 4. Proposition). For II) $\Rightarrow$ III), since III) is a local property, we may assume that $W = U$ is an open ball centered at the origin of $\mathbb{C}^n$ and the boundary $B = \{ x \in U | \Phi(x) = 0 \}$ contains the origin and satisfies the property of II). Then the Euclidean metric function $d(x)$ from $x \in \{ \Phi < 0 \}$ to $B$ is $C^\infty$ near the boundary $B$ by the implicit function theorem and moreover, from [18], Theorems 2.6.7 and 2.6.12, $-\log d$ is $C^\infty$-plurisubharmonic on $V \cap \{ \Phi < 0 \}$, where $V = \{ |z|^2 < \varepsilon \}$ is an open ball contained in $U$. Then we may assume that $-\log d \geq 0$. Hence $1/(\varepsilon - |z|^2) - \log d$ is a $C^\infty$-strictly plurisubharmonic exhaustion function on $V \cap \{ \Phi < 0 \}$ and so $V \cap \{ \Phi < 0 \}$ is Stein. If $X$ is locally Stein, III) $\Rightarrow$ I) is due to E. E. Levi (see [15], p. 264, 1. Proposition).

Let $ds^2 = \sum_{a, \beta=1}^{n} g_{t,a\beta} dz^a_t d\overline{z}^\beta_t$ be a hermitian metric on $M$ and let $\sigma = \sum_{a, \beta=1}^{n} \theta_{t,a\beta} dz^a_t \wedge d\overline{z}^\beta_t$ be a $C^\infty$-(1, 1) form on $M$ whose matrix of coefficients $\theta_{t,a\beta}$ is a hermitian matrix. We set

$$
G_t = (g_{t,a\beta}) \quad \text{and} \quad \Theta_t = (\theta_{t,a\beta}).
$$

If $J_{ij}$ are the transition functions of the tangent bundle $TM$, then on $U_i \cap U_j$, $G_t = J_{ji} G_p J_{ij}$ and $\Theta_t = J_{ji} \Theta_p J_{ij}$ so that the coefficients of the characteristic polynomial $\det (G_t^{-1} \Theta_t - \lambda E)$ are $C^\infty$-functions on $M$. The eigenvalues of $G_t^{-1} \Theta_t$ at each point $x$ are real, let them be
(2.2) \( \varepsilon_i(x) \geq \varepsilon_j(x) \geq \cdots \geq \varepsilon_n(x) \),
so that each \( \varepsilon_n \) is a real valued continuous function on \( M \).

**Definition 2.2.** A holomorphic line bundle \( F \to M \) is said to be
\( q \)-positive (resp. \( q \)-semi-positive) with respect to a given hermitian metric
\( ds^2 \) on a subset \( Y \) of \( M \) if there exist a coordinate cover \( \{ U_i \}_{i \in I} \) such
that \( \pi^{-1}(U_i) \) are trivial and a hermitian metric \( \{ a_i \} \) along the fibres of
\( F \) such that for \( G_t = (g_t, a_t \Theta_t) \) and
\( \Theta_t = \left( \frac{\partial^2 \log a_i}{\partial z_i^* \partial \bar{z}_i^*} \right) \), \( \varepsilon_{n-q+1}(x) + n \inf (0, \varepsilon_n(x)) \)
is positive (resp. non-negative) on \( Y \).

Here \( \varepsilon_n(x) \) should be understood as in (2.2).

**Remark 2.2.** If \( F \) is \( q \)-positive (resp. \( q \)-semi-positive) with respect
to \( ds^2 \) on \( Y \), \( G_t^* \Theta_t \) has at least \( n - q + 1 \) positive (resp. non-negative)
eigenvalues on \( Y \). In particular, if \( F \) is 1-positive (resp. 1-semi-positive)
with respect to \( ds^2 \) on \( Y \), the hermitian matrix \( \Theta_t \) is positive-definite
(resp. positive semi-definite) on \( Y \). Since the inverse is true for any
hermitian metric \( ds^2 \), we say simply that \( F \) is positive (resp. semi-
positive) on \( Y \) instead that \( F \) is 1-positive (resp. 1-semi-positive).

\( \S \ 2.2. \) **Basic Estimates**

Let \( X \) be a (connected) relatively compact domain with smooth
boundary \( \partial X \) and let \( F \to M \) be a holomorphic line bundle on \( M \). Let
\( K_M \) be the canonical line bundle on \( M \). We set a Kähler metric \( ds^2 \) on
\( M \)
\[ (2.3) \quad ds^2 = \sum_{a, \bar{a}}^n g_{t, \bar{a}} dz_t^* d\bar{z}_t^* \]
and a hermitian metric of \( F \) and its curvature tensor
\[ (2.4) \quad \Theta_t = \partial_a \partial_{\bar{a}} \log a_t, \]

Then, from (1.21),
\[ (2.5) \quad \{ c_t = a_t \cdot g_t \} \quad \text{where} \quad g_t = \det (g_{t, \bar{a}}) \]
is a Hermitian metric of $F \otimes K_M$. With respect to $(2.3)$ and $(2.4)$, an inner product $(\cdot, \cdot)$ and a norm $\| \cdot \|$ are defined as in Section 1.1. Since the curvature tensor of $\{c_t\}$ is the sum of $\theta_t = (\theta_t, \theta^2)$ and the Ricci curvature with respect to $(2.3)$, we have from Proposition 1.5

\[(2.6) \quad \| \tilde{\partial} \varphi \|^2 + \| \tilde{\partial} \varphi \|^2 = \int_{\partial X} (c_t | \text{grad} \ \Phi |)^{-1} \sum_{\delta_1 < \cdot < \delta_p} \sum_{\delta_1 \cdot \cdot \cdot \cdot \cdot} \partial_\delta \partial_\gamma \Phi \cdot \delta_0 + \delta_1 \cdot \cdot \cdot \cdot \cdot$$dS$$

+ \| \tilde{\nabla} \varphi \|^2 + \int X \sum_{\delta_1 < \cdot < \delta_p} \sum_{\delta_1 \cdot \cdot \cdot \cdot \cdot} \partial_\delta \partial_\gamma \Phi \cdot \delta_0 + \delta_1 \cdot \cdot \cdot \cdot \cdot dV$$

for any $\varphi \in B^p(X, F \otimes K_M)$.

**Proposition 2.1.** Let $X$ be a (connected) relatively compact domain with the pseudoconvex boundary $\partial X$ on a complex manifold $M$ and let $F \pi M$ be a holomorphic line bundle on $M$.

I) If there exists a Kähler metric $d\sigma^2$ on a neighborhood $W$ of $\partial X$ and $F \pi M$ is $q$-positive with respect to $d\sigma^2$ on $W$, then there exist a proper compact subset $K_1$ of $X$ and a positive constant $C_1$ such that

\[(2.7) \quad \| \varphi \|^2_{X, K_1} \leq C_1 \{ \| \tilde{\partial} \varphi \|^2 + \| \tilde{\partial} \varphi \|^2 + \| \varphi \|_{K_1}^2 \}$$

for any $\varphi \in D_0^p \cap D_0^p \subset L^p(X, F \otimes K_M)$ and $p \geq q$.

II) If $M$ is provided with a Kähler metric $d\sigma^2$ and $F \pi M$ is $q$-semi-positive with respect to $d\sigma^2$ on a neighborhood $V$ of $\overline{X} = X \cup \partial X$ and $q$-positive with respect to $d\sigma^2$ on $V \setminus K$, where $K$ is a proper compact subset of $X$, then there exist a proper compact subset $K_2$ of $X$ with $K \subset \text{Int} \ K_2$ and a positive constant $C_2$ such that

\[(2.8) \quad \| \varphi \|^2_{X, K_2} \leq C_2 \{ \| \tilde{\partial} \varphi \|^2 + \| \tilde{\partial} \varphi \|^2 \}$$

for any $\varphi \in D_0^p \cap D_0^p \subset L^p(X, F \otimes K_M)$ and $p \geq q$.

**Proof.** We prove II) first. Let $\{a_t\}$ be the Hermitian metric of $F$ with respect to the covering $\{U_t\}_{t \in \mathcal{I}}$ of $M$ corresponding to the assumption. To obtain the required estimate, by Proposition 1.6 iii), it suffices to show that the estimate holds for any element of $B_0^p(\overline{X}, F \otimes K_M)$. From Remark 2.1 II), there exists a defining $C^\infty$-function $\Phi$ for $\partial X$
such that the complex Hessian of $\Phi$ is positive semi-definite when restricted to the complex tangent space of $\partial X$. Hence, by using notations (2.3), (2.4) and (2.5), it follows that the first term of the right-hand side of (2.6) is non-negative. Hence we have

\[
(2.9) \quad \int_X c_t^{-1} \sum_{p+1 \leq \alpha < \beta \leq p} \sum_{a=1}^n \Theta_{t,a} \frac{\partial^a \Phi}{\partial \bar{V}_{\beta}} \frac{\partial^a \Phi}{\partial V_{\alpha}} dV \\
\leq \|\bar{\phi}\|^2 + \|\bar{\phi}^*\|^2 \quad \text{for any } \phi \in B^{h,p}(X, F \otimes K_\beta).
\]

Let $\{\xi_{\alpha}\}_{1 \leq \alpha \leq n}$ be the eigenvalues of $G_{t}^{-1}\Theta_t$, where $G_t = (g_{t,a})$, and choose a system of local coordinates $(z_1^1, \ldots, z_n^p)$ around $x_0 \in X$ as follows:

\[
(2.10) \quad G_t(x_0) = (\delta_{\alpha \beta}) \quad \text{and} \quad \Theta_t(x_0) = (\xi_{\alpha}(x_0) \delta_{\alpha \beta}).
\]

By the assumption, for a suitable proper compact subset $K_2$ of $X$ satisfying $K \subset \text{Int } K_2$, there exists a positive constant $\varepsilon$, independent of the choice of $x_0 \in X \setminus K_2$, satisfying

\[
(2.11) \quad \xi_{n-q+1}(x_0) + \inf (0, \xi_{\alpha}(x_0)) \geq \varepsilon > 0.
\]

We apply (2.10) and (2.11) to (2.9), then at $x_0 \in X \setminus K_2$

\[
(2.12) \quad \sum_{d_1 < \cdots < d_p} \sum_{a=1}^n \Theta_{t,a} \frac{\partial^a \Phi}{\partial \bar{V}_{\beta}} \frac{\partial^a \Phi}{\partial V_{\alpha}} dV \\
\geq \sum_{d_1 < \cdots < d_p} \sum_{a=1}^n \xi_{\alpha}(x_0) |\Phi_{t,a} \bar{V}_{\beta}|^2 \\
+ \inf (0, \xi_{\alpha}(x_0)) \sum_{d_1 < \cdots < d_p} \sum_{a=n-q+2}^{n} |\Phi_{t,a} \bar{V}_{\beta}|^2.
\]

If $p \geq q$, then $p + n - q + 1 \geq n + 1$, thus any block $D_p$ of $p$-indices taken from $\{1, 2, \ldots, n\}$ must contain one of the indices $\{1, 2, \ldots, n - q + 1\}$ i.e., one of the indices corresponding to the positive eigenvalues $\xi_{t}(x_0)$, $\ldots$, $\xi_{n-q+1}(x_0)$. Then we have

\[
(2.13) \quad \sum_{d_1 < \cdots < d_p} \sum_{a=1}^{n-q+1} |\Phi_{t,a} \bar{V}_{\beta}|^2 \geq \sum_{d_1 < \cdots < d_p} |\Phi_{t,\beta} \bar{V}_{\alpha}|^2 \\
\quad \text{and} \\
\quad n \sum_{d_1 < \cdots < d_p} |\Phi_{t,\beta} \bar{V}_{\alpha}|^2 \geq \sum_{d_1 < \cdots < d_p} \sum_{a=n-q+2}^{n} |\Phi_{t,a} \bar{V}_{\beta}|^2.
\]
Hence, from (2.11), (2.12) and (2.13), we have

\[
(2.14) \quad \sum_{d_i < \ldots < d_{p-1}} \sum_{a_i = 1}^n \theta_{d_i, a} \partial \bar{\partial} (\varphi_i a \bar{\varphi}_{i}^{d_{p-1}}) \geq k \sum_{d_i < \ldots < d_p} |\varphi_i a|^2 \quad \text{at} \quad x_0 \in X \setminus K_2.
\]

Since \( \varepsilon_{n-q+1} + n \inf (0, \varepsilon_a) \geq 0 \) on \( K_2 \), from (2.9) and (2.14), if we take \( C_2 = 1/k \), we obtain

\[
\| \varphi \|_{X \setminus K} \leq C_2 \{ \| \bar{\partial} \varphi \|^2 + \| \partial \varphi \|^2 \}
\]

for any \( \varphi \in B^{p,q}(X, F \otimes K_M) \) with \( p \geq q \). This completes the proof of II).

Next we prove I). From the assumption and Remark 2.1 II), we may assume that there exists a neighborhood \( W \) of \( \partial X \) such that i) the gradient of the defining function \( \Phi \) nowhere vanishes on \( W \), ii) there exists a Kähler metric \( d\sigma^2 \) on \( W \), iii) there exist a coordinate cover \( \{ U_i \} \) of \( M \), for which \( \pi^{-1}(U_i) \) are trivial and \( U_i \subseteq W \) if \( U_i \cap \partial X \neq \emptyset \), and a hermitian metric \( \{ a_i \} \) of \( F \) satisfying the \( q \)-positivity with respect to \( d\sigma^2 \) on \( W \). Hence by the same reason as in the proof of II), we have only to show the required estimate for elements of \( B^{p,q}(X, F \otimes K_M) \). Take neighborhoods \( W_i \) of \( \partial X \) with \( \partial X \subseteq W_1 \subset W_2 \subset W_3 \subset W \). We may assume that \( M = W_1 \cup X \). Then we take a hermitian metric \( ds^2 \)

\[
(2.15) \quad ds^2 = \sum_{a,b = 1}^n g_{a,b} d\bar{z}_a d\bar{z}_b
\]
on \( M \) such that \( ds^2 = d\sigma^2 \) on \( W \).

We take a \( C^\infty \)-real valued function \( \chi \) on \( M \) such that i) \( \chi = 1 \) on \( (M \setminus \bar{X}) \cup W_5 \), ii) supp \( \chi \subseteq W_1 \). Then, for any \( \varphi \in B^{p,q}(X, F \otimes K_M) \), \( \chi \varphi \) is again contained in \( B^{p,q}(X, F \otimes K_M) \) and supp \( \chi \varphi \) is contained in \( W_1 \). We set

\[
K_1 = X \setminus (X \cap W_1).
\]

Since the metric \( ds^2 \) is Kähler on \( W_1 \), from (1.16)' of Remark 1.1, the equation (2.6) holds for elements \( \chi \varphi \), where \( \varphi \in B^{p,q}(X, F \otimes K_M) \). Hence by using notations in (2.4), (2.5) and (2.15), we obtain the following inequality in the same way as in the proof of II),
We estimate the both sides of this inequality. By the assumption, we apply (2.11), (2.12), (2.13) and (2.14) in the proof of 1) to $\chi \varphi$ and $K_i$. Then the left-hand side may be estimated in the following way.

If $p \geq q$, then

\begin{equation}
\|\varphi\|^2_{L^p_{\chi;K}} \leq \|\chi \varphi\|^2_{L^q_{\chi;K}},
\end{equation}

for some positive constant $\kappa$ as which is taken in (2.11).

As for the right-hand side,

\begin{align*}
\|\bar{\partial} (\chi \varphi)^* + |\bar{\partial}^* (\chi \varphi)|^2
&= \|\bar{\partial} \chi \wedge \varphi + \chi \bar{\partial} \varphi\|^2 + \|c_i \bar{\partial} \bar{\partial} \chi \wedge \varphi\|^2 \\
&= \|\bar{\partial} \chi \wedge \varphi + \chi \bar{\partial} \varphi\|^2 + \|\chi \bar{\partial}^* \varphi - \ast (\bar{\partial} \chi \wedge \varphi)\|^2 \\
&= \|\bar{\partial} \chi \wedge \varphi\|^2 + \|2 \Re (\bar{\partial} \chi \wedge \varphi, \chi \bar{\partial} \varphi) + \|\chi \bar{\partial} \varphi\|^2 \\
&+ \|\chi \bar{\partial}^* \varphi\|^2 + \|2 \Re (\chi \bar{\partial}^* \varphi, -\ast (\bar{\partial} \chi \wedge \varphi)) + \|\chi \bar{\partial} \varphi\|^2\|^2 \\
&\leq 2 \left\{ \|\bar{\partial} \chi \wedge \varphi\|^2 + \|\chi \bar{\partial} \varphi\|^2 + \|\bar{\partial} \chi \wedge \varphi\|^2 + \|\chi \bar{\partial}^* \varphi\|^2 \right\} \\
&\leq 2 \left\{ \|\bar{\partial} \varphi\|^2 + \|\bar{\partial}^* \varphi\|^2 + 2c_0 \left( \int_K |\text{grad} \chi|^2 \langle \varphi, \varphi \rangle dV \right) \right\}.
\end{align*}

Here we used [45], p.18, Lemma A and the fact that the star operator $\ast$ is isometric with respect to the pointwise inner product $\langle,\rangle$ and $c_0$ is a positive constant depending only the dimension of $M$ and $|\text{grad} \chi|^2$ is the length of $\text{grad} \chi$ with respect to $ds^2$.

Therefore, if

\begin{equation}
C_1 \geq \frac{4}{\kappa} \max \{1, c_0 \sup_{x \in M} |\text{grad} \chi|^2 (x)\},
\end{equation}

we have, from (2.16), (2.17) and (2.18),

\begin{equation*}
\|\varphi\|^2_{L^p_{\chi;K}} \leq C_1 \{ \|\bar{\partial} \varphi\|^2 + \|\bar{\partial}^* \varphi\|^2 + \|\varphi\|^2_{L^2_{\chi;K}} \}
\end{equation*}

for any $\varphi \in B^{0,p}(X, F \otimes K_M)$ with $p \geq q$. This completes the proof of 1).
Let $X$ be a relatively compact domain with smooth boundary $\partial X$ and let $\Phi$ be a defining function for $X$ on a neighborhood $V$ of $\partial X$. Then we may assume that the gradient of $\Phi$ nowhere vanishes on $V$. We set
\begin{equation}
X_t = \{x \in V | \Phi(x) < t\} \cup X, \quad \partial X_t = \{x \in V | \Phi(x) = t\}
\end{equation}
for any $t$ with $0 \leq t \leq \sup_{x \in V} \Phi(x)$, and in particular
\[X_0 = X, \quad \partial X = \partial X_0.\]

Let $F^\pi M$ be a holomorphic line bundle on $M$. Then, if we fix a hermitian metric on $M$ and a hermitian metric of $F$ for each $t \geq 0$, we can consider Hilbert spaces $L^{2,\pi}(X_t, F \otimes K_M)$, which are the completions of $C_{0,\pi}(X_t, F \otimes K_M)$ with respect to the norm $\| \cdot \|_t^2 = \int_{X_t} \langle \cdot, \cdot \rangle dV$, and operators $\partial_t$, $\bar{\partial}_t^\pi$ and $\partial_t$ in $L^{2,\pi}(X_t, F \otimes K_M)$.

**Proposition 2.2.** Let $\{X_t\}_{t \geq 0}$ and $F$ be as above. If there exist i) a positive constant $\delta$ such that for any $t$ ($0 \leq t \leq \delta$) $X_t$ is a domain with pseudoconvex boundary $\partial X_t$, and ii) a Kähler metric $d\sigma^t$ on a neighborhood $V$ of $\partial X$ such that $F^\pi M$ is $q$-positive with respect to $d\sigma^t$ on $V$, then there exist a positive constant $C_3$ independent of $t$ and a proper compact subset $K_3$ such that for each $t$ ($0 \leq t \leq \delta$)
\begin{equation}
\|\varphi\|_{t, t ; X_t, F}^2 \leq C_3 \{\|\partial_t \varphi\|_t^2 + \|\bar{\partial}_t^\pi \varphi\|_t^2 + \|\varphi\|_t^2\}
\end{equation}
for any $\varphi \in D_t^p \cap D_{t}^{\pi, p} \subset L^{p, q}(X_t, F \otimes K_M)$ with $p \geq q$.

The proof is similar to that of Proposition 2.1 I). In fact, $C_3$ in that proposition depends only on the length of the gradient of $C^\infty$ function on $M$ and the lower bound of $\epsilon_{n-\pi^t+1} + n \inf(0, \epsilon_n)$ with respect to $G_t^{-1} \Theta_t$ (see (2.11) and (2.18)).

§ 2.3. Weak Finiteness and Vanishing Theorems

**Theorem 2.3.** Let $X$ be a (connected) domain with pseudoconvex boundary $\partial X$ on a complex manifold $M$ and let $F^\pi M$ be a holomorphic line bundle on $M$.

I) If there exists a Kähler metric $d\sigma^t$ on a neighborhood of $\partial X$ and
$F \rightarrow M$ is $q$-positive with respect to $d\sigma$ on its neighborhood, then it holds that if $p \geq q$,

i) $L^{b,p}(X, F \otimes K_M) \cong R^{\alpha,b,p}_{\phi} \oplus N^{\alpha,b,p} \oplus R^{0,b,p}_{\phi}$,

ii) $\dim N^{\alpha,b,p} < +\infty$

and so

iii) the image of the restriction homomorphism

$$r: H^p(M, \mathcal{O}(F \otimes K_M)) \rightarrow H^p(X, \mathcal{O}(F \otimes K_M))$$

has finite dimension.

II) If $M$ is provided with a Kähler metric $d\sigma$ and $F \rightarrow M$ is $q$-semi-positive with respect to $d\sigma$ on a neighborhood $V$ of $\overline{X}$ and $q$-semi-positive with respect to $d\sigma$ on $V \setminus K$, where $K$ is a proper compact subset of $X$, then if $p > q$, for any $f \in L^{b,p}(X, F \otimes K_M)$ with $\bar{\partial} f = 0$, there exists $g \in L^{b,p-1}(X, F \otimes K_M)$ satisfying $f = \bar{\partial} g$ and so the natural homomorphism

$$r_1: H^p(M, \mathcal{O}(F \otimes K_M)) \rightarrow H^p(X, \mathcal{O}(F \otimes K_M))$$

$$r_2: H^p_\ast(X, \mathcal{O}(F \otimes K_M)) \rightarrow H^p(X, \mathcal{O}(F \otimes K_M))$$

are zero maps for $p \geq q$, where $H^p_\ast(X, \mathcal{O}(F \otimes K_M))$ denotes the $p$-th cohomology group with compact supports.

In particular, we obtain

**Corollary 2.4.** Let $X, M$ and $F$ be as above.

I) If $F \rightarrow M$ is positive on a neighborhood of $\partial X$, it holds that if $p \geq 1$,

i) $L^{b,p}(X, F \otimes K_M) \cong R^{\alpha,b,p}_{\phi} \oplus N^{\alpha,b,p} \oplus R^{0,b,p}_{\phi}$,

ii) $\dim N^{\alpha,b,p} < +\infty$

and so

iii) the image of the restriction homomorphism

$$r: H^p(M, \mathcal{O}(F \otimes K_M)) \rightarrow H^p(X, \mathcal{O}(F \otimes K_M))$$

has finite dimension.

II) If $M$ is provided with a Kähler metric $d\sigma$ and $F \rightarrow M$ is semi-positive on a neighborhood $V$ of $\overline{X}$ and positive on $V \setminus K$, where $K$
is a proper compact subset of $X$, then if $p \geq 1$, for any $f \in L^{\infty, p}(X, F \otimes K_M)$ with $\bar{\partial} f = 0$, there exists $g \in L^{\infty, p-1}(X, F \otimes K_M)$ satisfying $f = \bar{\partial} g$ and the natural homomorphisms

$$r_1: H^p(M, \mathcal{O}(F \otimes K_M)) \to H^p(X, \mathcal{O}(F \otimes K_M))$$

$$r_2: H^p(X, \mathcal{O}(F \otimes K_M)) \to H^p(X, \mathcal{O}(F \otimes K_M))$$

are zero maps for $p \geq 1$.

**Proof of Theorem 2.3.** I). i) and ii) of I) follow from Proposition 1.11 I) and Proposition 2.1 I). While, if we let $L^{\infty, p}_{\text{loc}}(Y, F \otimes K_M)$ denote the set of the locally square integrable $(0, p)$ forms on $Y$ with values in $F \otimes K_M$ for any open subset $Y$ of $M$, then there is a natural isomorphism

$$(2.21) \quad H^p(Y, \mathcal{O}(F \otimes K_M))$$

$$\cong \frac{\{f \in L^{\infty, p}_{\text{loc}}(Y, F \otimes K_M) : \bar{\partial} f = 0\}}{\{f \in L^{\infty, p}_{\text{loc}}(Y, F \otimes K_M) : f = \bar{\partial} g \text{ for some } g \in L^{\infty, p-1}_{\text{loc}}(Y, F \otimes K_M)\}}.$$  

Here we consider the operator $\bar{\partial}$ in the sense of distribution. Hence, there is the following factorization of the homomorphism $r$

$$H^p(M, \mathcal{O}(F \otimes K_M))$$

$$\downarrow$$

$$N^{\infty, p}_{\text{loc}} / R^{\infty, p}_{\text{loc}}$$

$$\circlearrowright$$

$$r$$

$$H^p(X, \mathcal{O}(F \otimes K_M))$$

From ii) of I) and $N^{\infty, p}_{\text{loc}} / R^{\infty, p}_{\text{loc}} \cong N^{\infty, p}$, Im $r$ is finite dimensional.

II). The former assertion follows from Propositions 1.10 i), 1.11 II) and 2.1 II). Combining this with (2.21), the latter one follows immediately.

Q.E.D.

**Remark 2.3.** Such finiteness theorems of weak type as Theorem 2.3, I) and Corollary 2.4, I) were treated in [9], [22], [23], [24] and [28] etc., from the viewpoint of boundary regularity of the $\bar{\partial}$-operator. The basic estimates used in these papers are more precise than ours in the following sense.

Let $X$ be a relatively compact domain with smooth boundary $\partial X$ on
a complex manifold $M$ and let the defining function $\Phi$ of $X$ be strongly $q$ pseudoconvex on a neighborhood $V$ of $\partial X$. For any holomorphic line bundle $F$ on $M$ and a fixed hermitian metric $\{a_i\}$ of $F$, we consider $\{a_i = a_i' \exp(\Phi)\}$ as a new hermitian metric of $F$, where $\Phi$ is a $C^\infty$-function on $M$ and coincides with the original $\Phi$ on a neighborhood of $\partial X$. Then, there exist a hermitian metric $ds^2$ on $M$ and a positive constant $C_*$ such that

i) for $\{a_i\}$, $F$ is $q$-positive with respect to $ds^2$ on a neighborhood of $\partial X$.

\[(*)\]

ii) if $s \geq 1$ and $t \geq q$,

$$a_i^{-1} \sum_{\alpha_i, \beta_i = 1}^n \partial_\alpha \partial_\beta \Phi \varphi_i \bar{\varphi}_i = \bar{A} \bar{B}$$

$$\geq C_* a_i^{-1} \sum_{\alpha_i, \beta_i = 1}^n \varphi_i \bar{\varphi}_i \bar{A} \bar{B}$$

on $\partial X$ for every $\varphi \in \mathcal{B}^{q,t}(X, F)$.

(See [5], §5.16, Lemmas 18, 19 and 20 and §5.17, Lemma 21.)

In general, the metric $ds^2$ is not Kähler near the boundary $\partial X$, nevertheless, we can prove that there exist a positive constant $C_{**}$ and a compact subset $K$ of $X$ such that, if $\varphi \in \mathcal{B}^{q,t}(X, F)$ and $s \geq 1$, $t \geq q$,

i) $\|\varphi\|_{X,K} \leq C_{**} \{\|\bar{\partial} \varphi\|^2 + \|\bar{\partial}^* \varphi\|^2 + \|\varphi\|^2\}$

\[(**)\]

ii) $\|\bar{\nabla} \varphi\|^2 + \|\varphi\|^2 + \int_{\partial X} \varphi \bar{\partial} \varphi dS \leq C_{**} \{\|\bar{\partial} \varphi\|^2 + \|\bar{\partial}^* \varphi\|^2 + \|\varphi\|^2\}$

where the integral $\|\bar{\nabla} \varphi\|^2$ is defined as same as in Proposition 1.5 with respect to the covariant differentiation $\bar{\nabla}$ of type $(0, 1)$ associated to the connection of $ds^2$.

The above inequalities are not new and, essentially, due to Hörmander [17], Theorem 3.2.5 and Proposition 3.4.4, Kohn and Rossi [29], 3.12 Proposition and 5.8 Theorem, and Andreotti and Vesentini [6]. In particular, the estimate $\text{(**) (ii)}$ is a crucial one. In fact, the estimate $\text{(**) (ii)}$ implies that the graph norm $Q(\varphi, \varphi) = (\bar{\partial} \varphi, \bar{\partial} \varphi) + (\bar{\partial}^* \varphi, \bar{\partial}^* \varphi) + (\varphi, \varphi)$ is completely continuous on the space $\mathcal{B}^{q,t}(X, F)$ (see [22], 6.2 Theorem and 6.16 Proposition). In this connection, the graph norm $Q(\ , \ )$ is always completely continuous on the space $C^{q,t}_c(X, F)$ by the
ellipticity of the operator $\Box = \overline{\partial} \partial + \partial \overline{\partial}$. By using this property of the graph norm $Q(\cdot, \cdot)$, Kohn and Nirenberg proved boundary regularity of the operator $\Box = \overline{\partial} \partial + \partial \overline{\partial}$, which is, in their sense, strongly elliptic operator of second order, i.e. in the situation as above, if $s \geq 1$ and $t \geq q$,

i) any element $\varphi$ in $N^{k,t} = N^k \cap N^t \subset L^{k,t}(X,F)$, which is the completion of $C^{k,t}(X,F)$ by the norm associated with the above $\{a_t\}$ and $ds^2$, can be taken as an element of $C^{k,t}(X,F)$

ii) any element $\varphi \in C^{k,t}(X,F)$ with $\partial \varphi = 0$ and $\varphi \perp N^{k,t}$, there exists an element $\psi \in C^{k-1,t}(X,F)$ with $\overline{\partial} \psi = \varphi$.

(For detailed descriptions, see [22] and [9] 3.1.11 Proposition and 3.1.15 Proposition.)

But in our situation, the quadratic form $\sum_{p,q=1}^{n} \sum_{\beta, \gamma=1}^{n} \partial_\beta \overline{\partial}_\gamma \varphi_{\alpha} \partial_\beta \overline{\partial}_\gamma \varphi_{\alpha} = \sum_{p,q=1}^{n} \partial_\beta \overline{\partial}_\gamma \varphi_{\alpha} \partial_\beta \overline{\partial}_\gamma \varphi_{\alpha}$ is at most non-negative on every point of $\partial X$ (compare with (*) (ii)). This obstruction cannot be covered by only the curvature condition of bundles and actually, if in Proposition 2.1 the boundary $\partial X = \{ x \in X | \Phi(x) = 0 \}$ is Levi-flat, local boundary regularity breaks down (see [27], §9, Propagation of singularities for $\partial$).

While, vanishing theorems of weak type as Theorem 2.3, II) and Corollary 2.4, II) were treated in [3], [4], [5] and [36] etc. In these works, the completeness with respect to Kahler metrics was an important ground for the proof. In the situation of Theorem 2.3, II), the actual situation is that a complete Kahler metric $d\sigma^a$ exists on $X$. $d\sigma^a$ is constructed as follows. Let $d(x)$ be the distance from $x \in X$ to $\partial X$ with respect to the given Kahler metric $ds^2$ on $M$ and $d$ is a $C^\infty$-function near $\partial X$. Then $\inf_{x \in X} \sum_{a,b=1}^{n} \partial_\beta ( - \log d ) \partial_\alpha \lambda_{\alpha \beta}$ is bounded from below with some constant, which may not be non-negative generally, near $\partial X$ uniformly (see [8], Principal lemma). We extend $d$ to a $C^\infty$-function $\tilde{d}$ on $X$ without changing the original near $\partial X$ in a suitable manner. If we take a positive constant $\kappa$ large enough, $d\sigma^a = \sum_{a,b=1}^{n} \partial_\beta ( - \log \tilde{d} ) \partial_\alpha \lambda_{\alpha \beta} + \kappa ds^2$ is, again, a Kahler metric on $X$ and since $- \log \tilde{d}$ is an exhaustion function on $X$ and the gradient of $\tilde{d}$ nowhere vanishes near $\partial X$, $d\sigma^a$ is a complete Kahler metric on $X$ (see [31], Proposition 1).
Therefore, if $\partial X$ is pseudoconvex and there exists a Kähler metric on a neighborhood of $\partial X$, then there exists a complete hermitian metric on $X$ which is Kähler near $\partial X$. But, in our situation, the existence of these complete metrics does not seem to be so useful. Because, the boundedness from below of the eigenvalues of the curvature of $F$ with respect to these complete metrics can not be easily verified. But pseudoconvexity and the existence of complete Kähler metrics have a deep relation. (See [36], [47], [49].)

In these points, our estimates in Section 2.2 by using the formula (1.30) seem to be most appropriate.

§ 2.4. A Stability Theorem for Spaces of Harmonic Forms

Through this section, we use notations in (2.19) and Proposition 2.2, and set ourselves in the following situation:

i) There exists a positive constant $\delta$ such that for any $t$ ($0 < t < \delta$) $X_t$ is a domain with pseudoconvex boundary $\partial X_t$.

ii) There exist a Kähler metric $d\sigma^z$ on a neighborhood $V$ of $\partial X$ and a holomorphic line bundle $F^\pi_M$ such that $F$ is $q$-positive with respect to $d\sigma^z$ on $V$.

Then, since we may assume that $\partial X_t \subseteq V$, from Proposition 2.2, weak finite theorem holds for each $t$ ($0 < t < \delta$), i.e., if $t$ ($0 < t < \delta$) and $\nu \geq q$, then

i) $L^h,p(X_t, F \otimes K_M) \cong R^h,p_{\delta t} \oplus N^h,p_{\delta t} \oplus R^h,p_{\delta t}$, where $N^h,p_{\delta t} = N^h,p_{\delta t} \cap N^h,p_{\delta t}$,

ii) $\dim_{C} N^h,p_{\delta t} < + \infty$.

In this way, we obtain a family $(N^h,p_{\delta t})_{0 < t < \delta}$ of spaces of harmonic forms parametrized by the defining function $\theta$. By composing the following natural holomorphisms,

restriction map $r^{i}_{t_1}: L^h,p(X_{t_1}, F \otimes K_M) \rightarrow L^h,p(X_{t_2}, F \otimes K_M)$

$$(0 \leq t_2 < t_1 \leq \delta)$$

orthogonal projection $H_{t}: L^h,p(X_t, F \otimes K_M) \rightarrow N^h,p_{t}$
we obtain a homomorphism

\[ \rho_{t_t}^*: N_{\delta+t_t}^{\sigma,\rho} \rightarrow N_{\delta}^{\sigma,\rho} \quad (0 \leq t \leq \delta), \]

We denote other two orthogonal projections by

\[ R_t: L^{\sigma,\rho}(X, F \otimes K_N) \rightarrow R_{\delta+t_t}^{\sigma,\rho}, \]
\[ R_t^*: L^{\sigma,\rho}(X, F \otimes K_N) \rightarrow R_{\delta+t_t}^{\sigma,\rho} \quad (p \geq q \text{ and } 0 \leq t \leq \delta). \]

The main result of this section is the following stability theorem.

**Theorem 2.5.** *In the above situation, there exists a positive constant \( \delta_* \) (\( \delta_* \leq \delta \)) such that for any \( t \) (\( 0 < t \leq \delta_* \)) and \( p \geq q \), the homomorphism \( \rho_t: N_{\delta}^{\sigma,\rho} \rightarrow N_{\delta+t_t}^{\sigma,\rho} \) is an isomorphism.*

**Remark 2.4.** If \( \Phi \) is plurisubharmonic, a stronger result hold (see Chapter III, Theorem 3.5). Our method, which will be used to prove Theorem 2.5, seems to be an interesting one in the theory of \( \bar{\partial} \)-operator with boundary condition.

The remainder of this section is devoted to the proof of Theorem 2.5. In the first place, we prove the following.

**Proposition 2.6.** *Under the circumstance mentioned at the beginning of this section, there exist positive constants \( \delta_\ast \) (\( \delta_* \leq \delta \)) and \( C_1 \) such that, for any \( t \) (\( 0 \leq t \leq \delta_\ast \)),

\[ \| \varphi \|^2 \leq C_1 \{ \| \bar{\partial} \varphi \|^2 + \| \bar{\partial}^* \varphi \|^2 \}, \]

if \( \varphi \in D_{\delta_\ast}^{\sigma,\rho} \cap D \subset L_{\delta_\ast}^{\sigma,\rho}(X, F \otimes K_N) \) and \( (r_\delta(\varphi), h)_0 = 0 \) for all \( h \in N_{\delta}^{\sigma,\rho} \subset L^{\sigma,\rho}(X, F \otimes K_N) \).

**Proof.** Assume that the assertion is false. For any \( C \), we can find arbitrarily small \( t \) for which \((2.20)\) does not hold. If necessary, take a subsequence and we may assume that there exist sequences \( \{t_n\}_{n \geq 1} \) and \( \{\varphi_n\}_{n \geq 1} \) such that

1) \( t_n > 0 \), \( t_n > t_{n+1} \) and \( t_n \rightarrow 0 \) as \( n \rightarrow +\infty \),
ii) \( \varphi_n \in D^p_{\delta n} \cap D^p_{\theta n} \subset L^p(X, F \otimes K_N) \)

(2.23) \[ \| \varphi_n \|_{t_n}^2 = 1 \text{ and } \| \tilde{\varphi}_n \|_{t_n}, \| \tilde{\varphi}_n^* \|_{t_n}^2 \rightarrow 0 \text{ as } n \rightarrow +\infty, \]

iii) \( (\varphi_n, h)_0 = 0 \) for all \( h \in N^0_{t} \subset L^0_p(X, F \otimes K_N) \).

From (2.23) ii), we may assume that \( \{\varphi_n\} \) is weakly convergent to some \( \varphi \) in \( L^0_p(X, F \otimes K_N) \). On the other hand, if \( \psi \in C^{p-1}_0(X, F \otimes K_N) \), we have

\[
| (\tilde{\varphi}, \varphi)_0 | = \lim_{n \rightarrow +\infty} | (\tilde{\varphi}_n, \varphi)_0 | = \lim_{n \rightarrow +\infty} | (\tilde{\varphi}_n, \varphi_n)_0 - (\tilde{\varphi}_n, \varphi_n)_{t_n} \| X, \chi_{i, n} \|

\leq \lim_{n \rightarrow +\infty} | (\tilde{\varphi}_n, \varphi_n)_0 | + \lim_{n \rightarrow +\infty} | (\tilde{\varphi}_n, \varphi_n)_{t_n} \| X, \chi_{i, n} \|

\leq \lim_{n \rightarrow +\infty} \| \tilde{\varphi}_n \|_{t_n} \| \varphi_n \|_{t_n} + \lim_{n \rightarrow +\infty} \| \tilde{\varphi}_n \|_{X, t_n} = 0
\]

and

\[
\lim_{n \rightarrow +\infty} \| \tilde{\varphi}_n \|_{t_n}^2 = \lim_{n \rightarrow +\infty} \| \tilde{\varphi}_n \|_{t_n}^2 = 0.
\]

Hence, by Proposition 1.6 ii), we have that \( \varphi \in D^0_p \subset L^0_p(X, F \otimes K_N) \) and \( \tilde{\varphi}^* \psi = 0 \) in \( L^0_p(X, F \otimes K_N) \) and \( \tilde{\varphi} = 0 \) in \( L^0_p(X, F \otimes K_N) \). So \( \varphi \) is contained in \( N^0_{t} \). But each \( \varphi_n \) is orthogonal to \( N^0_{t} \). Therefore we obtain \( \varphi = 0 \). While, by the same argument in the proof of Proposition 1.11 i), we may assume that \( \{\varphi_n\}_{n \geq 1} \) is strongly convergent to zero on \( K_t \). From (2.20) and (2.23) ii), we have

\[ 1 = \| \varphi_n \|_{t_n}^2 \leq C_5 \| \tilde{\varphi}_n \varphi_n \|_{t_n}^2 + \| \tilde{\varphi}_n \varphi_n \|_{t_n}^2 + (C_5 + 1) \| \varphi_n \|_{t_n}^2 \rightarrow 0 \]

as \( n \rightarrow +\infty \). This is a contradiction. Q.E.D.

Proof of Theorem 2.5. Step I. If \( p \geq q \) and \( 0 < t \leq \delta_0 \), \( \rho^*_t : N^0_{t} \rightarrow N^p_{t} \) is injective. If \( \varphi \in N^0_{t} \) and \( \rho^*_t (\varphi) = 0 \), then \( r^*_t (\varphi) \in R^p_{t} \). Hence \( r^*_t (\varphi) \) is orthogonal to \( N^0_{t} \). Therefore, from Proposition 2.6 (2.22), we have \[ \| \varphi \|_{t}^2 \leq C_4 \| \tilde{\varphi} \|_{t}^2 + \| \tilde{\varphi}^* \psi \|_{t}^2 = 0. \] Hence \( \varphi = 0 \).

Step II. There exists a positive constant \( \delta^* \) (\( \delta^*_* \leq \delta_0 \)) such that if \( p \geq q \) and \( 0 < t \leq \delta^*_t \), \( \rho^*: N^0_{t} \rightarrow N^p_{t} \) is surjective.

We remark that the following diagram is commutative: if \( t_1 > t_2 > 0 \),
To see this, it suffices to show that $H_0 \circ r_{t_1} = H_0 \circ r_{t_2} \circ H_1$, i.e. $\rho_{t_1} = \rho_{t_2} \circ H_{t_2}$. Take $\varphi \in N_{t_1}^0$, then $r_{t_2}^* (\varphi) = R_0 \circ r_{t_1}^* (\varphi) + H_0 \circ r_{t_2}^* (\varphi) = R_0 \circ r_{t_1}^* (\varphi) + H_0 (r_{t_2}^* \circ R_{t_1} (\varphi) + r_{t_2}^* \circ H_{t_2} (\varphi)) = R_0 \circ r_{t_2}^* (\varphi) + H_0 \circ r_{t_2}^* \circ H_{t_2} (\varphi)$. Hence $\rho_{t_1} = \rho_{t_2} \circ H_{t_2}$ on $N_{t_1}^0$. We come back to the beginning. Assume that the assertion is false. From the injectivity of $\rho_{t_2}$ and the above remark, it holds that

i) if $t < t'$, $\rho_{t_2} (N_{t_2}^0)$ is contained in $\rho_{t_1} (N_{t_1}^0)$ as a closed subspace,

ii) $\dim_c \rho_{t_2} (N_{t_2}^0) < \dim_c N_{t_2}^0$ for $0 < t \leq t_0$.

These imply that there exists a positive constant $\delta_t$ such that if $0 < t \leq \delta_t$, $\rho_{t_2} (N_{t_2}^0) = \rho_{t_1} (N_{t_1}^0)$ and $\dim_c \rho_{t_2} (N_{t_2}^0) < \dim_c N_{t_2}^0$. Hence there exists an element $u \neq 0$ of $N_{t_2}^0$ such that if $0 < t \leq \delta_t$, $(2.24) \quad (u, h)_0 = 0$ for all $h \in N_{t_2}^0$.

We shall show that $(u, g)_0 = 0$ for all $g \in N_{t_2}^0$ in this situation. This means $u = 0$ and we arrive at a contradiction. Take a sequence $\{t_n\}_{n \geq 1}$ such that $t_n \leq \delta_t$, $t_n > t_{n+1}$ and $t_n \to 0$ as $n \to +\infty$ and extend the definition of $u$ by setting $u = 0$ on $M \setminus X$. We denote it by $u'$. Then, from (2.24), we have for any element $\varphi$ of $N_{t_n}^0$

$$
(u', \varphi)_{t_n} = (u, r_{t_n}^* (\varphi))_0 = (u, R_0 \circ r_{t_1}^* (\varphi) + H_0 \circ r_{t_2}^* (\varphi))_0 \\
= (u, R_0 \circ r_{t_1}^* (\varphi))_0 + (u, H_0 \circ r_{t_2}^* (\varphi))_0 \\
= (u, \rho_{t_2}^* (\varphi))_0 = 0.
$$

Hence $u'$ is orthogonal to $N_{t_n}^0$ ($n \geq 1$) and so $u' \in R_{t_n}^{0,p+1}$ ($n \geq 1$). From Proposition 1.10 ii) and Proposition 2.6, there exists an element $v_n \in L_{t_n}^{0,p+1}$ ($X_{t_n}, F \otimes K_{t_n}$) such that $\tilde{\delta}_{t_n} v_n = u'$ and $\|v_n\|_{t_n} \leq C_{t_n}^{1/2} \|u\|_0$ for any $n \geq 1$. Hence we may assume that $\{v_n\}_{n \geq 1}$ is weakly convergent to some $v \in L_{t_2}^{0,p+1}$ ($X, F \otimes K_{t_2}$). If $\psi \in C_0^{0,p}(\overline{X}, F \otimes K_{t_2})$, we have

$$
|\tilde{\delta}_{t_n} v_n|_0 \leq \lim_{n \to +\infty} |\tilde{\delta}_{t_2} v_n|_0.
$$
\[= \lim_{n \to +\infty} |(\tilde{\partial}_t \psi_n, \varphi_n)_t - (\tilde{\partial}_t \psi_n, \psi_n)_{X_{t+1}}|\]
\[\leq \lim_{n \to +\infty} |(\psi_n, \tilde{\partial}_t^* \varphi_n)_t| + \lim_{n \to +\infty} |(\tilde{\partial}_t \psi_n, \varphi_n)_{X_{t+1}}|\]
\[\leq \|\varphi\|_0 \|u\|_0 + C_{t^p}(\lim_{n \to +\infty} \|\tilde{\partial}_t \psi_n\|_{X_{t+1}})
\[= \|\psi\|_0 \|u\|_0 .\]

Hence \(v \in D^{p+1}_p \subset L^{0,p+1}(X, F \otimes K_H)\). While, for any \(\varphi \in C^b_0(X_0, F \otimes K_N)\), we have
\[\langle \varphi, u \rangle_0 = \lim_{n \to +\infty} \langle \varphi, u \rangle_t = \lim_{n \to +\infty} \langle \varphi, \tilde{\partial}_t^* \psi_n \rangle_t
\[= \lim_{n \to +\infty} \langle \tilde{\partial}_t \varphi, \psi_n \rangle_t = (\tilde{\partial} \varphi, \varphi)_0 .\]

Hence \(\delta v = u\). Since \(v \in D^{p+1}_p\), we have \(\tilde{\partial}^* v = u\) (see Proposition 1.7 iii). Therefore, if \(g \in N^{p}_{\delta}\), we have \(\langle u, g \rangle_0 = \langle \tilde{\partial}^* v, g \rangle_0 = \langle v, \tilde{\partial} g \rangle_0 = 0\).

This completes the proof of Theorem 2.5.

**Remark 2.5.** The author does not know whether we can derive the finite dimensionality of \(H^p(X_t, \mathcal{O}(F \otimes K_H))\) for \(p \geq q\) and some \(\epsilon > 0\) in the situation of this section. To see this, at least, we need some approximation theorem of the Runge type (for example, see Proposition 3.3 and the proof of Theorem 3.6 in Chapter III). But, at present, the author can only prove the following approximation theorem of weak type.

**Under the situation of this section, if \(p \geq q - 1\), the closure of the space \(\bigcup_{0 \leq t \leq \delta} r_0(N^p_{\delta})\) in \(L^{0,p}(X, F \otimes K_H)\) coincides with the closed subspace \(N^p_{\delta}\) in \(L^{0,p}(X, F \otimes K_H)\), where \(\delta\) is the positive constant taken in Proposition 2.6, i.e., for any \(\varepsilon > 0\) and \(\varphi \in N^p_{\delta} \subset L^{0,p}(X, F \otimes K_H)\), there exist a positive constant \(\delta (\delta \leq \delta_0)\) depending on \(\varphi\) and \(\varepsilon\), and \(\tilde{\varphi} \in N^p_{\delta} \subset L^{0,p}(X_0, F \otimes K_H)\) satisfying \(\|\tilde{\varphi}\|_{X} - \|\varphi\| < \varepsilon\).

**Proof.** If there exists a non-zero element \(u \in N^p_{\delta} \subset L^{0,p}(X, F \otimes K_H)\) such that, for every \(t \ (0 < t \leq \delta_0)\),
\[\langle u, f \rangle_0 = 0 \quad \text{for any} \quad f \in N^p_{\delta} \subset L^{0,p}(X_t, F \otimes K_H),\]
then we shall show that
(u, g)_0 = 0 \quad \text{for any} \quad g \in N^p_\delta^p \subset L^p(X, F \otimes K_M).

This leads us to a contradiction. We set \( u' = u \) on \( X \) and \( 0 \) on \( M \setminus X \). Then \( u' \) is contained in \( L^{0,p}(X, F \otimes K_M) \) and orthogonal to \( N^p_\delta^p \) for every \( 0 < t \leq \delta_0 \). Hence \( u' \in R_{\delta_0}^{p, q} = R_{\delta_0}^{p, q} \) (in particular, when \( p = q - 1 \), this equality is secured by the closedness of the range \( R_{\delta_0}^{p, q} \)). Take a sequence \( \{t_n\} \) such that \( t_n \leq \delta_0 \) \( (n \geq 1) \), \( t_n > t_{n-1} \) and \( t_n \to 0 \) as \( n \to +\infty \). Then, by the same reason in Step II of the proof of Theorem 2.5, there exists a sequence \( \{v_n\}_{n \geq 1} \) such that \( v_n \in L_\delta^{p+1} (X, F \otimes K_M) \), \( \delta_n^\ast v_n = u' \) and \( \|v_n\|_{t_n} \leq C \|u\|_0 \) for every \( n \geq 1 \). Hence, by the similar manner in Step II of the proof of Theorem 2.5, there exists an element \( v \in L^{0,p+1}(X, F \otimes K_M) \) satisfying \( \delta^\ast v = u \) in \( L^{0,p}(X, F \otimes K_M) \). Therefore, if \( g \in N^p_\delta^p \subset L^p(X, F \otimes K_M) \), we have \( (u, g)_0 = 0 \). Q.E.D.

**Chapter III. Cohomology Groups on Weakly 1-Complete Manifolds**

§ 3.1. Definitions and the Basic Estimate

Let \( X \) be a connected complex manifold of dimension \( n \).

**Definition 3.1.** A complex manifold \( X \) is said to be weakly 1-complete if there exists a \( C^\infty \)-exhausting plurisubharmonic function \( \Phi \) on \( X \). \( \Phi \) is called an exhaustion function on \( X \).

**Remark 3.1.** If \( c \) is a non-critical value of \( \Phi \), plurisubharmonicity of \( \Phi \) implies that \( X_c = \{ x \in X \mid \Phi(x) < c \} \) is a domain with pseudoconvex boundary \( \partial X_c = \{ x \in X \mid \Phi(x) = c \} \).

In Sections 3.1 and 3.2, we set the following situation.

(3.1) i) \( X \) is a weakly 1-complete manifold with respect to \( \Phi \).

ii) There exists a Kähler metric \( d\sigma^\ast \) on \( X \setminus K_\ast \).

iii) \( F_\Phi \to X \) is a holomorphic line bundle on \( X \) which is \( q \)-positive with respect to \( d\sigma^\ast \) on \( X \setminus K_\ast \), where \( K_\ast \) is a proper compact subset.
of $X$.

We choose compact subsets $K''$, $K'$ and $K$ of $X$ such that $K_4 \subset \text{Int } K''$, $K'' \subset \text{Int } K'$ and $K' \subset \text{Int } K$. We set

\begin{equation}
\{a_i\}
\end{equation}

the hermitian metric of $F$ with respect to the coordinate cover $\{U_t\}_{t \in T}$ corresponding to the assumption. Let $K_X$ be the canonical line bundle of $X$. Then we define a hermitian metric $ds^2$ on $X$ and a hermitian metric of $F \otimes K_X$ as follows:

\begin{equation}
(3.3)
\begin{align*}
\text{i)} & \quad ds^2 = \sum_{a, \beta=1}^n g_{t, a\beta} dz^a_t d\bar{z}_t^\beta, \\
\text{ii)} & \quad ds^2 = d\sigma^2 \quad \text{on } X \backslash K', \\
\text{iii)} & \quad c_t = a_t g_t \quad \text{where } g_t = \det(g_{t, a\beta}).
\end{align*}
\end{equation}

We set

\begin{equation}
\begin{align*}
\text{i)} & \quad \theta_t = (\theta_t, a\beta) \quad \text{where } \theta_t, a\beta = \partial_a \partial_\beta \log a_t, \\
\text{ii)} & \quad G_t = (g_{t, a\beta}).
\end{align*}
\end{equation}

For any open subset $Y$ of $X$, let $L^{a,p}(Y, F \otimes K_X)$ be the completion of $C_c^{a,p}(Y, F \otimes K_X)$ by the norm $\| \cdot \|_Y$ defined by (3.3). We sometimes omit the symbol $Y$ if it is clear. Linear operators $\partial, \partial^*$ (resp. $\partial^*_p$) and $\vartheta$ (resp. $\vartheta_p$) are defined in the same way as in Chapter I, Section 1.2.

We fix non-critical values $c_0$ and $c_1$ of $\Phi$ such that $c_0 > \sup_{x \in K} \Phi(x)$. We set

\[ X_i = \{ x \in X | \Phi(x) < c_i \} \quad \text{and} \quad \partial X_i = \{ x \in X | \Phi(x) = c_i \} \quad \text{for } i = 0, 1. \]

We take a $C^\infty$-increasing convex function $\lambda$ on $(-\infty, \infty)$ such that

\begin{equation}
\lambda(t) \begin{cases} 
=0 & \text{if } t \leq c_0 \\
>0 & \text{if } t > c_0.
\end{cases}
\end{equation}

We define hermitian metrics of $F$ and $F \otimes K_X$ respectively as follows:

\begin{equation}
(3.5)
\begin{align*}
\text{i)} & \quad a_{m,t} = a_t \exp(m\lambda(\Phi)) \\
\text{ii)} & \quad c_{m,t} = a_{m,t} g_t
\end{align*}
\end{equation}

for every non-negative integer $m$. 

We set
\[ \theta_{m,t} = (\theta_{m,t,u}) \quad \text{where} \quad \theta_{m,t,u} = \bar{a}_m \log a_{m,t} \quad (m \geq 1). \]

With respect to (3.3) i) and (3.5), we set
\[ (\varphi, \psi)_m = \int_{X_1} \langle \varphi, \psi \rangle_m dV \]
\[ \| \varphi \|^2_m = (\varphi, \varphi)_m \quad \text{for any} \quad m \geq 0 \]
where \( \langle \varphi, \psi \rangle_m = (\varphi, \psi)_m e^{-m/(\varphi^2)}. \)

For any \( m \), the completion of \( C^0(X_1, F \otimes K_X) \) by the norm \( \| , \|_m \) coincides with \( L^0(X_1, F \otimes K_X) \). We denote the adjoint operator of the maximal closed extension \( \bar{\delta} \) in \( L^{\prime\prime}(X_1, F \otimes K_X) \) with respect to the inner product \( ( , )_m \) by \( \bar{\delta}^*_m \). Then from Proposition 1.7 iv), we have \( D^0_{\bar{\delta}^*_m} = D^0_{\bar{\delta}^*_m} \) in \( L^{\prime\prime}(X_1, F \otimes K_X) \) for any \( m \geq 1 \) and \( p \geq 1 \).

**Proposition 3.1.** There is a positive constant \( C_4 \) such that
\[ \| \varphi \|^2_{m,X_1} \leq C_4 \left\{ \| \bar{\delta} \varphi \|^2_m + \| \bar{\delta}^*_m \varphi \|^2_m + \| \varphi \|^2 \right\} \]
if \( m \geq 0, \varphi \in D^0_{\bar{\delta}^*_m} \cap D^0_{\bar{\delta}^*_m} \) and \( p \geq q \).

**Proof.** From (1.34) and Proposition 1.6 iii), it suffices to show that the estimate holds for any element of \( B^{0,p}(X_1, F \otimes K_X) \). Let \( \chi \) be a \( C^\infty \)-function on \( X \) such that \( \chi = 1 \) on \( X \setminus K \) and \( \chi \subseteq X \setminus K' \). From Remark 3.1, we obtain the following inequality in the same way as in the proof of Proposition 2.1 1).

\[ \int_{X \setminus K'} C_{m,t}^{-1} \sum_{d_1, \ldots, d_{p-1}} \sum_{a_1, \ldots, a_{p-1}} \theta_{m,t,a_1} \bar{\varphi}_{d_1, \ldots, d_{p-1}} (\chi \varphi)_{a_1\ldots a_{p-1}} dV \]
\[ \leq \| \bar{\delta} (\chi \varphi) \|^2_m + \| \bar{\delta}^*_m (\chi \varphi) \|^2_m \]
for \( \varphi \in B^{0,p}(X_1, F \otimes K_X) \).

In the first place, we estimate the left hand side. Let \( \{ \varepsilon_{m,a} \}_{1 \leq a \leq n} \) (resp. \( \{ \varepsilon_a \}_{1 \leq a \leq n} \)) be the eigenvalues of the matrix \( G_i^{-1} \Theta_{m,t} \) (resp. \( G_i^{-1} \Theta_i \)). Since \( G_i^{-1} \Theta_i \leq G_i^{-1} \Theta_{m,t} \) by the minimum-maximum principle for the eigenvalues, we have \( \varepsilon_{m,n-q+1} \geq \varepsilon_{n-q+1} \) and \( \varepsilon_{m,n} \geq \varepsilon_{n} \) on \( X \setminus K'' \). Hence at any point \( x_0 \in X \setminus K' \), we have the following similarly to the proof of Prop-
osition 2.1 II): If $p \geq q$, then

$$
\sum_{a_i=1}^n \sum_{\rho_{p_i}} \theta_{m_i, i} \overline{(x \varphi)} \cdot \overline{(x \varphi)} + \overline{(x \varphi)} \cdot \overline{(x \varphi)} \geq \left( \varepsilon_{m_i}^{n-q+1}(x) + n \inf (0, \varepsilon_{m_i}^{n}(x)) \right) \sum_{\rho_{p_i}} |(x \varphi) \cdot \overline{(x \varphi)} |^2
$$

$$
\geq \left( \varepsilon_{n-q+1}(x) + n \inf (0, \varepsilon_{n}^{n}(x)) \right) \sum_{\rho_{p_i}} |(x \varphi) \cdot \overline{(x \varphi)} |^2
$$

$$
\geq \kappa \sum_{\rho_{p_i}} |(x \varphi) \cdot \overline{(x \varphi)} |^2,
$$

where $\kappa = \inf_{x \in X \setminus \text{Int} K} (\varepsilon_{n-q+1}(x) + n \inf (0, \varepsilon_{n}^{n}(x))) > 0$.

Hence we have

(3.8) \[ \| \varphi \|_{m, \rho, \rho'} \geq \| \varphi \|_{m, \rho, \rho'} \]

where $C_1$ is a positive constant and independent of $m$ (see (2.18)).

Next, repeating the discussion of (2.17) in the proof of Proposition 2.1 I), we can estimate the right hand side of (3.7) in the following way:

(3.9) \[ \frac{1}{\kappa} \int_{X \setminus \text{Int} K} \left( |(x \varphi) \cdot \overline{(x \varphi)} |^2 \right) \]

From (3.7), (3.8) and (3.9), we obtain the required estimate (3.6) for elements of $B_{\mu}^p(X, F \otimes K)$. Q.E.D.

§ 3.2. Finiteness, Isomorphic and Representation Theorems

Let our situation be the same as in Section 3.1. From (3.1) and Remark 3.1, for any non-critical value $c$ of $\Phi$, $X_c = \{ x \in X | \Phi(x) < c \}$ and $F^\Phi X$ satisfy the conditions of Theorem 2.3 I) and so we have

(3.10) \[ L^0_p(X_c, F \otimes K) \cong R^0_p \oplus N^0_p \oplus R^0_p \]

and

\[ \dim C N^0_p < + \infty, \quad \text{for} \quad p \geq q. \]

Proposition 3.2. There exist a positive constant $C_2$ and an
integer $m_0$ such that for any $m \geq m_0$ and $p \geq q$

\begin{equation}
\|\varphi\|_m^2 \leq C_2 (\|\varphi\|_m^2 + \|\varphi_m\|_m^2)
\end{equation}

for $\varphi \in \mathcal{D}_0^{\beta} \cap \mathcal{D}_0^{\beta} \subset L^{0, p}(X, F \otimes K_x)$, and

$$(\varphi, h)_{x_0} = 0 \quad \text{for all} \quad h \in \mathcal{N}_{e_0} \subset L^{0, p}(X_0, F \otimes K_x).$$

**Proof.** Assume that the assertion were false. There would be a

sequence $\{\varphi_m\}$ such that

\begin{equation}
\begin{aligned}
i) & \quad \varphi_m \in \mathcal{D}_0^{\beta} \cap \mathcal{D}_0^{\beta} , \\
ii) & \quad \|\varphi_m\|_m^2 = 1 \quad \text{and} \quad \|\varphi_m\|_m^2, \|\varphi_m\|_m^2 \to 0 \quad \text{as} \quad m \to + \infty \quad \text{and} \\
iii) & \quad (\varphi_m, h)_{x_0} = 0 \quad \text{for all} \quad h \in \mathcal{N}_{e_0} \subset L^{0, p}(X_0, F \otimes K_x).
\end{aligned}
\end{equation}

Let $g_m = e^{-m \langle \theta \rangle \varphi_m}$, then we have

\begin{equation}
\begin{aligned}
i) & \quad \partial g_m = e^{-m \langle \theta \rangle \partial \varphi_m} , \\
ni) & \quad \|\partial g_m\| = \|\varphi_m\| = 1 .
\end{aligned}
\end{equation}

By (3.12), we have

\begin{equation}
\lim_{m \to + \infty} \|g_m\| = \|g_m\| = 1 .
\end{equation}

Therefore, choosing a subsequence if necessary, we may assume that

$\{g_m\}$ has a weak limit $g$ in $L^{0, p}(X, F \otimes K_x)$. From (3.12) ii) and

(3.13) ii), we have

\begin{equation}
\lim_{m \to + \infty} \|\partial g_m\| = \lim_{m \to + \infty} \|\varphi_m\| = 0 .
\end{equation}

On the other hand, for every $\varepsilon > 0$,

$$\int_{x \in X \mid (\theta (x) > e_0 + \varepsilon)} e^{-m \langle \theta \rangle} \langle g_m, g_m \rangle dV \leq 1 ,$$

and so we have

$$e^{-m \langle \theta >e_0 + \varepsilon} \int_{x \in X \mid (\theta (x) \geq e_0 + \varepsilon)} \langle g_m, g_m \rangle dV \leq 1 .$$

It follows that $\int_{x \in X \mid (\theta (x) \geq e_0 + \varepsilon)} \langle g_m, g_m \rangle dV$ tends to zero and hence $g_m \to 0$ almost everywhere in $\{x \in X \mid (\theta (x) \geq e_0 + \varepsilon)$. Hence $g = 0$ on $\{x \in X \mid (\theta (x) \geq e_0 + \varepsilon\}$ for every $\varepsilon > 0$. Combining this with (3.14), we have
By Proposition 1.6 i), we have
\[(3.16)\] \(\bar{\partial}^* (g|_{X_0}) = 0.\]

By (3.12) ii), we have
\[(3.17)\] \(\bar{\partial}^* (g|_{X_0}) = 0.\)

From (3.12) iii), (3.16) and (3.17), we have
\[(3.18)\] \(g = 0.\)

Since \(\bar{\partial}^* = \bar{\partial}^*_m\) on \(X_0\) for every \(m\), by the same argument as in the proof of Proposition 1.11 I), we may assume that \(\{g_m\}\) converges strongly on \(K\). Since \(g_m = \varphi_m\) on \(K\), from (3.6), (3.12) ii) and (3.18), we have
\[
1 = \|\varphi_m\|^2 \leq C_4 \{\|\bar{\partial}^* \varphi_m\|^2 + \|\bar{\partial}^*_m \varphi_m\|^2\} + (C_4 + 1) \|\varphi_m\|^2_\infty \rightarrow 0
\]
as \(m \rightarrow + \infty\). This is a contradiction. \(\Box\)

**Proposition 3.3.** If \(\psi \in L^{6,\rho}(X_0, F \otimes K_x)\) with \(\bar{\partial} \psi = 0\) and \(\rho \geq q - 1\), then for any \(\varepsilon > 0\), there exists an element \(\tilde{\psi}\) of \(L^{6,\rho}(X_0, F \otimes K_x)\) such that \(\|\tilde{\psi}|_{X_0} - \psi\|^2_{X_0} < \varepsilon\) and \(\bar{\partial} \tilde{\psi} = 0.\)

**Proof.** It suffices to show that if \(u \in L^{6,\rho}(X_0, F \otimes K_x)\) and
\[(3.19)\] \((f, u)_{X_0} = 0\)
for any \(f \in L^{6,\rho}(X_1, F \otimes K_x)\) with \(\bar{\partial} f = 0\), then we have
\[(3.20)\] \((g, u)_{X_0} = 0\)
for \(g \in L^{6,\rho}(X_0, F \otimes K_x)\) with \(\bar{\partial} g = 0.\)

We extend \(u\) by setting \(u = 0\) on \(X \setminus X_0\). We denote it by \(u'\). Then, from (3.19), \(u'\) is orthogonal to \(N^{6,\rho}_{\bar{\partial}} \subset L^{6,\rho}(X_1, F \otimes K_x)\). Hence we have \(u' \in R^{6,\rho}_{\bar{\partial}K_x}\) for any \(m \geq 0\). While Proposition 3.1 implies that \(R^{6,\rho}_{\bar{\partial}K_x}\) is closed for every \(\rho \geq q - 1\) and \(m \geq 0\). Hence from Proposition 1.10 ii) and Proposition 3.2, for any \(m \geq m_0\), there exists an element \(v_m \in D^{6,\rho+1}_{\bar{\partial}K_x}\) such that
\[(3.21)\] \(u' = \partial^*_m v_m\) and \(\|v_m\|^2_m \leq C_8 \|u\|^2_{X_0} .\)

We set
\[
\delta v_m = e^{-mZ(\theta)} v_m \quad \text{for} \quad m \geq m_0 .
\]
Then we have

\begin{equation}
(3.22) \quad \text{i) } \|w_m\| \leq C \|u\| \chi_x, \quad \text{ii) } \vartheta w_m = u'.
\end{equation}

Hence \{w_m\} has a subsequence which is weakly convergent to some \( w \) in \( L^{b,p+1}(X, F \otimes K_x) \). From (3.22), we have

\begin{equation}
(3.23) \quad \vartheta w = u' \quad \text{and} \quad \text{supp } w \subseteq \overline{X}_0.
\end{equation}

By Proposition 1.6 i), we have

\begin{equation}
(3.24) \quad \bar{\partial}^*(w|\chi_x) = u.
\end{equation}

Therefore, if \( g \in L^{b,p}(X_\circ, F \otimes K_x) \) with \( \bar{\partial} g = 0 \), then \( (u, g)_{\chi_x} = (\bar{\partial}^*(w|\chi_x), g)_{\chi_x} = (w|\chi_x, \bar{\partial} g)_{\chi_x} = 0 \). Q.E.D.

For any real number \( c \) with \( c > \sup_{x \in X} \Phi(x) \), we set

\[ H^{b,p}(X_\circ, F \otimes K_x) = N^b_p / R^b_p, \]

where \( X_\circ = \{ x \in X \mid \Phi(x) < c \} \) and \( L^{b,p}(X_\circ, F \otimes K_x) = R^b_p \oplus N^b_p \).

If \( c \) is a non-critical value of \( \Phi \), from (3.10), we have

\begin{equation}
(3.25) \quad H^{b,p}(X_\circ, F \otimes K_x) = N^b_p / R^b_p \simeq N^b_p
\end{equation}

and

\[ \dim H^{b,p}(X_\circ, F \otimes K_x) < +\infty \quad \text{for } p \geq q. \]

**Proposition 3.4.** The restriction homomorphism

\[ r: H^{b,p}(X, F \otimes K_x) \to H^{b,p}(X_\circ, F \otimes K_x) \]

is an isomorphism for \( p \geq q \).

**Proof.** Step I. The homomorphism \( r: H^{b,p}(X, F \otimes K_x) \to H^{b,p}(X_\circ, F \otimes K_x) \) is injective.

Take an element \( f \) in \( L^{b,p}(X, F \otimes K_x) \) such that \( \bar{\partial} f = 0 \) and \( f = \bar{\partial} g \) in \( L^{b,p}(X_\circ, F \otimes K_x) \) for some \( g \in L^{b,p-1}(X_\circ, F \otimes K_x) \). Then \( f \) satisfies the relation \( (f, h)_{\chi_x} = 0 \) for all \( h \in N^b_p \). Hence, from Proposition 1.10 i) and Proposition 3.2, there exists an element \( \bar{g} \) in \( L^{b,p-1}(X, F \otimes K_x) \) such that \( \bar{\partial} \bar{g} = f \) in \( L^{b,p}(X, F \otimes K_x) \).
Step II. The homomorphism \( r : H^0_p(X, F \otimes K_x) \to H^0_p(X_0, F \otimes K_x) \) is surjective.

From Proposition 3.3, \( \text{Im } r \) is dense in \( H^0_p(X_0, F \otimes K_x) \). Since \( r \) is injective and \( H^0_p(X_0, F \otimes K_x) \) is a finite dimensional vector space in view of (3.25), \( r \) is surjective. Q.E.D.

In particular, we obtain the following theorem from (3.25) and Proposition 3.4.

**Theorem 3.5.** (Compare with Theorem 2.5.) The homomorphism

\[
\rho^c_i : N^0_{c_i} \to N^0_{c_i}
\]

is an isomorphism \((\text{p} \geq q)\).

**Theorem 3.6** (Finiteness Theorem). The restriction homomorphism

\[
r : H^p(X, \mathcal{O}(F \otimes K_x)) \to H^0_p(X_0, F \otimes K_x)
\]

is an isomorphism and so

\[
\dim \mathcal{O} H^p(X, \mathcal{O}(F \otimes K_x)) < +\infty \ (\text{p} \geq q).
\]

**Proof.** By Sard’s theorem, we can choose a sequence \( \{c_i\}_{i \in \mathbb{Z}} \) of real numbers such that

\[
(3.26) \quad \text{i) } \ c_i > c_1, \\
\text{ii) } \ c_{i+1} > c_i \quad \text{and} \quad c_i \to +\infty \ \text{as} \quad \nu \to +\infty, \\
\text{iii) } \ \text{The boundary } \partial X_{c_i} \ \text{of} \ \{x \in X \mid \Phi(x) \leq c_i\} \ \text{is smooth for any} \ \nu \geq 2.
\]

We set

\[
X_{\nu} = \{x \in X \mid \Phi(x) < c_{\nu}\} \ \text{for} \ \nu \geq 2.
\]

For any pair \( (c_{\nu+1}, c_\nu) \ (\nu \geq 1) \), we can apply Proposition 3.2 and so Proposition 3.3 holds, i.e., if \( \nu \geq 0 \) and \( p \geq q - 1 \), the restriction homomorphism
(3.27) \( r_{\nu}^{r_{\nu}^{-1}} : \mathcal{N}_{\nu}^{p} \to \mathcal{N}_{\nu}^{p} \) has a dense image with respect to the norm \( \| \cdot \|_{\nu} \), where \( \mathcal{N}_{\nu}^{p} \subseteq L^{p}(\mathbb{R}^{n}, F \otimes K_{x}) \).

Moreover, from Proposition 3.4, we have the following: if \( \nu \geq 0 \) and \( p \geq q \), the restriction homomorphism

(3.28) \( r_{\nu}^{r_{\nu}^{-1}} : H^{b,p}(\mathbb{R}^{n+1}, F \otimes K_{x}) \to H^{b,p}(\mathbb{R}^{n}, F \otimes K_{x}) \)

is an isomorphism.

Let \( L^{p}_{\text{loc}}(X, F \otimes K_{x}) \) be the set of the locally square integrable \((0, p)\) forms on \( X \) with values in \( F \otimes K_{x} \). Then, there is a natural isomorphism such that for any \( p \geq 1 \) and open subset \( Y \) of \( X \),

(3.29) \( H^{p}(Y, \mathcal{O}(F \otimes K_{x})) \)

is an isomorphism such that for any \( \nu \geq 0 \) and \( f = \overline{\partial} g \) for some \( g \in L^{p}_{\text{loc}}(X, F \otimes K_{x}) \) such that \( dg = f \).

We set \( f_{\nu} = f \restr{X_{\nu}} \) for every \( \nu \geq 0 \).

Then from (3.28), there exists \( g_{\nu}' \in L^{p-1}(\mathbb{R}^{n}, F \otimes K_{x}) \) such that \( \overline{\partial} g_{\nu}' = f_{\nu} \) for every \( \nu \geq 1 \).

Let us show that we can choose, by induction, a sequence \( \{g_{\nu}\} \) so that

(3.30) i) \( g_{\nu} \in L^{p-1}(\mathbb{R}^{n}, F \otimes K_{x}) \),

ii) \( \overline{\partial} g_{\nu} = f_{\nu} \),

iii) \( \|g_{\nu} \restr{X_{\nu+1}} - g_{\nu} \|_{2}^{2} \leq \frac{1}{2^{\nu+1}} \).
We set
\[ g_0 = g'_0. \]

Suppose \( g_0, \ldots, g_{s-1} \) are chosen. Then \( g'_0|_{x_{s-1}} \in D_0^{p-1} \subset L^{p-1}_0(X_{s-1}, F \otimes K_X) \) and \( \bar{\partial}(g'_0|_{x_{s-1}} - g_{s-1}) = 0 \). Therefore, by (3.27) there exists \( g'' \in L^{p}(X_{s}, F \otimes K_X) \) such that \( \|g'_0|_{x_{s-1}} - g_{s-1} - g''|_{x_{s-1}}\|_{x_{s-1}}^2 < \frac{1}{2^{p+1}} \) and \( \bar{\partial}g'' = 0 \). We set \( g_s = g'_0 - g'' \). Then it is clear that \( g_s \) has the required properties (3.30). Hence we have obtained the sequence \( \{g_s\}_{s \geq 0} \). From (3.30), for any \( v \), \( \{g_s\}_{s \geq 0} \) converges with respect to the norm \( \| \cdot \|_{x_{s}} \) and clearly the limit is the same as the restriction of \( \lim_{s \to +\infty} g_s \) for any \( \nu \geq v + 1 \). Thus we can define an element \( g \) of \( L^{p-1}_{0}(X, F \otimes K_X) \) by \( g = \lim_{s \to +\infty} g_s \). For every \( \nu \geq 1 \),
\[ \lim_{s \to +\infty} g_s = g \quad \text{in} \quad L^{p-1}_{0}(X, F \otimes K_X) \]
\[ \lim_{s \to +\infty} \bar{\partial}g_s|_{x_{s}} = f_s \quad \text{in} \quad L^{p}(X_{s}, F \otimes K_X). \]

Since \( \bar{\partial} \) is a closed operator in \( L^{p-1}_{0}(X_{s}, F \otimes K_X) \) for every \( \nu \geq 1 \), we have for any \( \nu \geq 0 \)
\[ \bar{\partial}g = f_s \quad \text{in} \quad L^{p}_{0}(X_{s}, F \otimes K_X). \]

Hence we have \( f = \bar{\partial}g \) in \( L^{p}_{0}(X, F \otimes K_X) \).

**Step II.** The homomorphism \( r: H^{p}(X, \otimes (F \otimes K_X)) \to H^{p}_{\bar{\partial}}(X, F \otimes K_X) \) is surjective.

In the first place, we prove the following.

(3.31) The restriction map \( r: \{f \in L^{p}_{0}(X, F \otimes K_X)| \bar{\partial}f = 0 \} \to N^{p}_{\bar{\partial}} \) has a dense image \( (p \geq q - 1) \).

Take an element \( \phi \in N^{p}_{\bar{\partial}} \). From (3.27), we can choose a sequence \( \{\phi_s\}_{s \geq 0} \) such that for any \( \varepsilon > 0 \),

(3.32) i) \( \phi_0 = \phi \),
ii) \( \phi_s \in N^{p}_{\bar{\partial}} \),
iii) \( \|\phi_s|_{x_{s-1}} - \phi_{s-1}\|_{x_{s-1}}^2 < \frac{\varepsilon}{2} \quad (\nu \geq 1) \).
By (3.32) iii), for any \( v \), \( \{ \varphi_v \}_{v \in \mathbb{N}} \) converges with respect to the norm \( \| \cdot \|_{X_v} \). Hence, by the same argument as in Step I, we can define an element \( \varphi \) of \( L^2_{\epsilon}\text{loc}(X, F \otimes K_x) \) with \( \delta \varphi = 0 \). Then it is clear that \( \| \varphi|_{X_v} - \varphi \|_{X_v}^2 < \varepsilon \). This completes the proof of (3.31). Next, by (3.29) and (3.31), we see that the image of \( r \) is dense in \( H^{n,p}(X_0, F \otimes K_x) \). Since \( r \) is injective and \( H^{n,p}(X, F \otimes K_x) \) is finite dimensional vector space, \( r \) is surjective. Q.E.D.

**Theorem 3.7 (Isomorphic Theorem).** The restriction homomorphism
\[
r: H^{p}(X, \mathcal{O}(F \otimes K_x)) \to H^{p}(X, \mathcal{O}(F \otimes K_x))
\]
is an isomorphism for \( p \geq q \), where \( c > \sup_{x \in K_x} \Phi(x) \).

**Proof.** We may assume \( c > c_0 \). Then we have the following factorization:

\[
\begin{array}{c}
H^{p}(X, \mathcal{O}(F \otimes K_x)) \\
\downarrow r_1 \\
H^{n,p}(X_0, F \otimes X)
\end{array}
\begin{array}{c}
H^{n,p}(X_0, F \otimes X) \\
\downarrow r_3 \\
H^{p}(X_0, \mathcal{O}(F \otimes K_x))
\end{array}
\begin{array}{c}
\downarrow r_2 \\
H^{p}(X, \mathcal{O}(F \otimes K_x))
\end{array}
\]

Since, by Theorem 3.6, the homomorphism \( r_1 \) is an isomorphism, \( r_2 \) is injective. Similarly since the homomorphism \( r_3 \) is an isomorphism, \( r_3 \) is surjective. Q.E.D.

**Theorem 3.8 (Representation Theorem).** There is a natural isomorphism
\[
\rho_c: H^{p}(X_0, \mathcal{O}(F \otimes K_x)) \to N^{n,p}_c \text{ for } p \geq q
\]
where \( c \) is a non-critical value of \( \Phi \) with \( c > \sup_{x \in K_x} \Phi(x) \).

**Proof.** For any \( c' \) with \( c' > c \), from Theorems 3.6 and 3.7, we obtain isomorphisms \( r_1 \) and \( r_2 \)
\[
\begin{array}{c}
H^{p}(X, \mathcal{O}(F \otimes K_x)) \\
\downarrow r_1 \\
H^{n,p}(X_0, F \otimes X)
\end{array}
\begin{array}{c}
\downarrow r_2 \\
H^{p}(X, \mathcal{O}(F \otimes K_x))
\end{array}
\begin{array}{c}
\downarrow \rho_c \\
H^{n,p}(X_0, F \otimes K_x) \ldots \to H^{n,p}(X, F \otimes K_x).
\end{array}
\]
We define a morphism \( \rho'_c : \mathcal{H}^p(X, \mathcal{O}(F \otimes K_X)) \to \mathcal{H}^{a,p}(X, F \otimes K_X) \) by \( \rho'_c = r_c \circ r_c^{-1} \). It is clear that \( \rho'_c \) is well defined and does not depend on the choice of \( c' \). \( \rho'_c \) is an isomorphism from \( \mathcal{H}^p(X, \mathcal{O}(F \otimes K_X)) \) to \( \mathcal{H}^{a,p}(X, F \otimes K_X) \). The composition of \( \rho'_c \) and an isomorphism \( \mathcal{H}^{a,p}(X, F \otimes K_X) \cong N^{a,p}_c \) gives an isomorphism \( \rho_c : \mathcal{H}^p(X, \mathcal{O}(F \otimes K_X)) \to N^{a,p}_c \).

Q.E.D.

In particular for \( q = 1 \), we obtain the following.

**Theorem 3.9.** Let \( X \) be a weakly 1-complete manifold with respect to an exhaustion function \( \Phi \) and let \( F \rightarrow X \) be a holomorphic line bundle which is positive on \( X \setminus K \). Then with respect to the hermitian metric \( \{a_t\} \) of \( F \) corresponding to the assumption and the hermitian metric \( ds^2 \) on \( X \) induced by the curvature of \( \{a_t\} \), it holds that, for any two non-critical value \( c \) and \( c' \) with \( c' > c > \sup_{x \in K} \Phi(x) \) and \( \rho \geq 1 \),

1) \( L^{a,p}(X, F \otimes K_X) \cong R^{a,p}_c \oplus N^{a,p}_c \oplus R^{a,p}_c \) and \( \dim_c N^{a,p}_c < + \infty \),
2) the homomorphism \( \rho'_c : N^{a,p}_c \rightarrow N^{a,p}_c \) is isomorphic,
3) \( \dim_c H^p(X, \mathcal{O}(F \otimes K_X)) < + \infty \),
4) the restriction homomorphism \( r: H^p(X, \mathcal{O}(F \otimes K_X)) \rightarrow H^p(X, \mathcal{O}(F \otimes K_X)) \) is isomorphic,
5) there is an isomorphism \( \rho_c : \mathcal{H}^p(X, \mathcal{O}(F \otimes K_X)) \to N^{a,p}_c \).

**Remark 3.2.** In Theorem 3.9, 3) and 4) were proved by T. Ohsawa in [34]. He reduced these problems to the \( \bar{\partial} \)-operator theory without boundary conditions.

§ 3.3. Vanishing Theorems

**Theorem 3.10.** Let \( X \) be a connected weakly 1-complete Kähler manifold and let \( F \rightarrow X \) be a holomorphic line bundle on \( X \) which is \( q \)-semi-positive on \( X \) and \( q \)-positive on \( X \setminus K \) with respect to the given
Kähler metric, where $K$ is a proper compact subset. Then
\[ H^p(X, \mathcal{O}(F \otimes K_X)) = 0 \text{ for } p \geq q. \]

**Proof.** For a suitable non-critical value $c$ of the exhaustion function $\Phi$ on $X$, we obtain, from Theorems 3.7 and 3.8, the isomorphism $H^p(X, \mathcal{O}(F \otimes K_X)) \cong N^p_\Phi$ for $p \geq q$. But, from the assumption, $N^p_\Phi$ is the null space (see the proof of Proposition 1.11 II)). This completes the proof.

**Remark 3.3.** Any connected compact complex manifold is weakly 1-complete, any real constant function being an exhaustion function. Therefore we obtain the following.

Let $X$ be a connected compact complex Kähler manifold and let $F \rightarrow X$ be $q$-semi-positive on $X$ and $q$-positive at least one point of $X$ with respect to the given Kähler metric. Then
\[ H^p(X, \mathcal{O}(F \otimes K_X)) = 0 \text{ for } p \geq q. \]

**Remark 3.4.** With respect to the difference between weakly 1-completeness and pseudoconvexity, Diedrich and Fornaess showed that there are domains with pseudoconvex boundaries $\Omega$ on $\mathbb{C}^n$ such that $\overline{\Omega}$ does not have a Stein neighborhood basis (see [48]). In this situation, if the defining function $\Phi$ of $\Omega$ is plurisubharmonic on a neighborhood $W$ of $\partial \Omega$, domains \{\$\Omega_r = \Omega \cup \{x \in W \mid \Phi(x) < r\}\$ for $r > 0$\} consist of a Stein basis of $\overline{\Omega}$ since any domain with pseudoconvex boundary on $\mathbb{C}^n$ is Stein.

**Note added.** As mentioned in Remark 2.3, in the situation of Chapters II and III, the local boundary regularity does not always hold. But recently the author has shown that the global boundary regularity holds in the following sense if according to the degree of the required boundary regularity, we take the tensor product of the line bundle $F$, which is positive on a neighborhood of $\partial X$, sufficiently many times, we can solve the $L^1-\overline{\partial}$-Neumann problem satisfying the required boundary regularity. As a consequence of this regularization theorem, $H^p(X, \mathcal{O}(F \otimes K_X))$ can be represented by the space of harmonic forms being $C^k$-class ($0 \leq k < \infty$) up to $\partial X$, if $m$ is large enough. See a forthcoming paper “Global regularity and spectra of Laplace-Beltrami operators on pseudoconvex domains".
References


