Homotopy Classification of Connected Sums of Sphere Bundles over Spheres, II

By

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Introduction

In the classification problems of manifolds, the connected sums of sphere bundles over spheres appear frequently. For example, we can see those in [6], [7], and [13]. Motivated by those, in the preceding paper [8], we classified the connected sums consisting of sphere bundles over spheres which admit cross-sections up to homotopy equivalence.

In this paper, as promised previously, we investigate the general case. And, under some assumptions on dimensions, i.e., in metastable range, we obtain a necessary and sufficient condition for two connected sums of sphere bundles over spheres to be homotopy equivalent, by extending the results of James-Whitehead [10] and using the handlebody theory of Wall [14] and Ishimoto [5]. Applications of the main theorem to special cases will appear in the subsequent paper.

Let $B_i$, $i=1, 2, \cdots, r$, be $p$-sphere bundles over $q$-spheres ($p, q > 1$), and let $\overline{B}_i$, $i=1, 2, \cdots, r$, be the associated $(p+1)$-disk bundles. It is understood that each $B_i$, or $\overline{B}_i$, also denotes the total space of the bundle and has the oriented differentiate structure induced from those of the fibre and the base space. If $p \geq q$, each $B_i$ admits a cross-section, and the homotopy classification of the connected sums of such bundles has been completed in [8]. So, we assume that $p < q - 1$. The torsion case that $p = q - 1$ is excluded from this paper and the problem is still open. We denote the characteristic element of $B_i$ by $\alpha(B_i)$ or simply by $\alpha_i$ and we put $e_i = \pi_a(x_i)$, where $\pi_a: \pi_{q-1}(SO_{p+1}) \to \pi_{q-1}(S^p)$ is the homomorphism induced from the projection $\pi: SO_{p+1} \to SO_{p+1}/SO_p = S^p$.

The boundary connected sum $\overline{\cup}_{i=1}^{r} \overline{B}_i$ can be considered as a handlebody of

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$(m+1, r, q), m = p + q,$ and the connected sum $\#_{i=1} B_i$ is its boundary. In general, $\#_{i=1} B_i$ may have various representations into the connected sums of $p$-sphere bundles over $q$-spheres up to diffeomorphism. In fact, we can observe it using the handlebody theory as follows.

Let $W$ be a handlebody of $(m+1, r, q), m = p + q,$ and assume that $2p > q > 1$. Let $\phi: H \times H \to \pi_q(S^{p+1}), H = H_q(W),$ be the pairing defined by Wall [14], and let $\alpha: H \to \pi_{q+1}(SO_{p+1})$ be the map assigning to each $x \in H \cong \pi_q(W)$ the characteristic element of the normal bundle of the imbedded $q$-sphere which represents $x$. $\alpha$ is a quadratic form with the associated homomorphism $\partial \alpha$, where $\partial: \pi_q(S^{p+1}) \to \pi_{q+1}(SO_{p+1})$ belongs to the homotopy exact sequence of the fibering $SO_{p+1} \to SO_{p+2} \to S^{p+1}$. ([14], p. 257). A base $\{w_1, w_2, \cdots, w_r\}$ of the free abelian group $H$ is called admissible if $\alpha(w_i, w_j) = 0$ for all $i, j (i \neq j)$. If $W$ has an admissible base $\{w_1, w_2, \cdots, w_r\}$, then $W$ can be represented as a boundary connected sum of $(p+1)$-disk bundles over $q$-spheres with the characteristic elements $\alpha(w_i), i = 1, 2, \cdots, r$. For, we can take the imbedded $q$-spheres which represent $w_i, i = 1, 2, \cdots, r$, to be disjoint (cf. Ishimoto [5]). Hence, by tying the tubular neighbourhoods of such imbedded $q$-spheres with thin bands in $W$, and by the $h$-cobordism theorem, we know that $W$ is diffeomorphic ($m > 4$) to such a boundary connected sum of disk bundles over spheres.

Thus, the representations of $W=\#_{i=1} B_i$ into the boundary connected sums of $(p+1)$-disk bundles over $q$-spheres correspond with the admissible bases of $H = H_q(W)$. Since $H_q(\partial W) \cong H_q(W)$ if $p \neq q - 1, q,$ we obtain various representations of $\partial W=\#_{i=1} B_i$ into the connected sums of $p$-sphere bundles over $q$-spheres associated with the admissible bases of $H \cong H_q(\partial W)$.

In Section 2, it is shown that Wall's pairing is a homotopy invariant of the boundary of the handlebody if $p \neq q - 1$. That is,

**Proposition 1.** Let $W, W'$ be handlebodies of $(p+q+1, r, q)$ and assume that $2p > q > 1$ and $p \neq q - 1$. If there exists a homotopy equivalence $f: \partial W \to \partial W'$ which preserves orientation, then for the isomorphism $h = i'_* \circ f_* \circ i_*^{-1}: H_q(W) \to H_q(W')$, we have $\phi = \phi' \circ (h \times h)$, where $\phi, \phi'$ are Wall's pairings of $W, W'$ and $i, i'$ are inclusion maps of $\partial W, \partial W'$ into $W, W'$, respectively.

If $p \geq q$, the proposition is trivial since $\phi = \phi' = 0$. Hence, it makes sense for $p \leq q - 1$. Note that $\alpha(x, x) = (E \circ \pi_a)(\alpha(x))$ by [14], where $E$ is the suspension homomorphism. Immediately we have the following.

**Corollary 2.** Under the above assumptions on $p, q$, if $\partial W$ has the homotopy
type of $\bigoplus_{i=1}^r B_i$, a connected sum consisting of $p$-sphere bundles over $q$-spheres, then $W$ is represented into a boundary connected sum of $(p+1)$-disk bundles over $q$-spheres, and hence $\partial W$ into a connected sum of $p$-sphere bundles over $q$-spheres. Furthermore, if $s$ bundles in $B_i$, $i=1, 2, \ldots, r$, admit cross-sections, $\partial W$ is represented into a connected sum of $p$-sphere bundles over $q$-spheres in which $s$ bundles admit cross-sections.

**Corollary 3.** Under the above assumptions on $p$, $q$, if $H=H_q(W)$ has no admissible bases, then $\partial W$ never has the homotopy type of a connected sum of $p$-sphere bundles over $q$-spheres.

Let $\omega$ be an element of $\pi_{q-1}(Sp)$. We have the following homomorphisms

$$
\pi_{p+q-1}(Sp_{p+1}) \longrightarrow \pi_{p+q-1}(Sp) \longrightarrow \pi_{q-1}(SO_p) \longrightarrow \pi_{q-1}(SO_{p+1}),
$$

where $\omega_\ast$ is defined by the composition with $\omega$, $J$ is the $J$-homomorphism, and $i_\ast$ is induced from the inclusion. Let $G(\omega)=i_\ast(J^{-1}(Im \omega_\ast))$ (James-Whitehead [9]).

Let $H$, $\phi$, $\alpha$ and $\epsilon=\pi_\ast \circ \alpha$ be the invariants of $W=\bigoplus_{i=1}^r B_i$, where $\epsilon$ is a quadratic form with the associated homomorphism $\pi_\ast \circ \partial \circ \phi$. We note that if $p \neq q-1$, $q$, then $(H; \phi, \alpha)$ is determined from $\partial W=\bigoplus_{i=1}^r B_i$. In fact, $H=H_q(W) \cong H_q(\partial W), \alpha(w_i)=\alpha(B_i)$, $\alpha(w_i)=\alpha(B_i)$, $i=1, 2, \ldots, r$, where $\{w_1, \ldots, w_r\}$ is the canonical basis of $H$ represented by zero cross-sections of $B_i$, $i=1, 2, \ldots, r$, and $\phi(w_i, w_j)=0$ if $i \neq j$, $\phi(w_i, w_i)=E\pi_\ast \alpha(B_i)$ for each $i, j$. Let $B_i', i=1, 2, \ldots, r'$, be another set of $p$-sphere bundles over $q$-spheres ($p \neq q-1$). If $\bigoplus_{i=1}^r B_i$ has the homotopy type of $\bigoplus_{i=1}^{r'} B_i'$, then $r=r'$ by those homological aspect. Therefore, we assume that $r=r'$ henceforth. Similarly define $H'$, $\phi'$, $\alpha'$, and $\epsilon'$ for $W'=\bigoplus_{i=1}^{r'} B_i'$. Let $\alpha_i, \alpha_i'$ be the characteristic elements of $B_i, B_i'$ respectively and put $\epsilon_i=\pi_\ast(\alpha_i), \epsilon_i'=\pi_\ast(\alpha_i'), i=1, 2, \ldots, r$. We obtain the following.

**Theorem 4.** Let $q/2 < p < q-1$. Then, the connected sums $\bigoplus_{i=1}^r B_i, \bigoplus_{i=1}^{r'} B_i'$ are of the same oriented homotopy type if and only if $\epsilon_i=\epsilon_i'$ and $\{\alpha_i\} = \{\alpha_i'\}$ in $\pi_{q-1}(SO_{p+1})/G(\epsilon_i)=\pi_{q-1}(SO_{p+1})/G(\epsilon_i')$ for $i=1, 2, \ldots, r$ “modulo representations”. More precisely, they are of the same oriented homotopy type if and only if there exist the admissible bases $\{w_1, \ldots, w_r\}, \{w_1', \ldots, w_{r'}\}$ of $H, H'$ respectively such that

1. $\epsilon(w_i)=\epsilon'(w_i)$, $i=1, 2, \ldots, r$, (i.e. $\epsilon \cong \epsilon'$) and
2. $\{\alpha(w_i)\} = \{\alpha'(w_i)\}$ in $\pi_{q-1}(SO_{p+1})/G(\epsilon(w_i))=\pi_{q-1}(SO_{p+1})/G(\epsilon'(w_i))$, $i=1, 2, \ldots, r$. 


If all $B_i, B'_i, i=1, 2, \cdots, r$, admit cross-sections, then $\phi=\phi'=0$. So, any bases of $H, H'$ are admissible, $\alpha, \alpha'$ are the homomorphisms, and $\epsilon=\epsilon'=0$. Furthermore, $G(0)=i_\ast J^{-1}(0)$ induces $i_\ast \pi_{q-1}(SO_p)/G(0)\cong J\pi_{q-1}(SO_p)/P\pi_q(S^p)$, where $P=[\ , \epsilon_p]$ and $\epsilon_p$ is the orientation generator of $\pi_p(S^p)$ (cf. [10], p. 152). Hence, we have Theorem 1 of [8] for $p<q-1$.

Proposition 1 is proved in Section 2, and using it Theorem 4 is proved in Section 4 and Section 5.

§ 1. Cell Structure and Linking Elements

Let $W=D^{m+1} \cup \{ \cup_{i=1}^r D_i \times D_i^{p+1} \}$ be a handlebody of $\mathcal{H}(m+1, r, q)$, $m=p+q$, $p, q>1$, where $\phi_i$: $D_i^p \times D_i^{p+1} \to D^{m+1}, i=1, 2, \cdots, r$, are the disjoint imbeddings. Let $Y=S^m \cup \{ \cup_{i=1}^r D_i \times S_i^p \}$, where $\phi_i=\phi_i|D_i^{p-1} \times S_i^p, i=1, 2, \cdots, r$. Let $\bar{S}_i^p \subset \text{Int} \ Y$ be the imbedded $p$-sphere slightly moved from $x_i \times S_i^p$, where $i=1, 2, \cdots, r$.

We join $\bar{S}_i^p, i=1, 2, \cdots, r$, by $r$ arcs in $\text{Int} \ Y$ from a fixed point and take a thin closed neighbourhood $N$. $N$ has the homotopy type of $\bigvee_{i=1}^r S_i^p$.

By the Alexander duality theorem, we have

$$H_i(N) \cong H_q(Y) \quad \text{if} \quad i<m-1,$$

and, since $N, Y$ are simply connected,

$$\pi_i(N) \cong \pi_q(Y) \quad \text{if} \quad i<m-2,$$

where the isomorphisms are induced from the inclusion map. So that, $H_i(Y, N)$ $\cong 0$ for $i<m-1$, and therefore by the homology exact sequence of $(\partial W, Y, N)$, we have

$$H_i(\partial W, N) \cong H_i(\partial W, Y) \quad \text{if} \quad i<m-1.$$

Here, by the excision theorem,

$$H_i(\partial W, Y) \cong \begin{cases} \mathbb{Z} + \cdots + \mathbb{Z} & \text{if} \quad i=q, m \\ 0 & \text{otherwise,} \end{cases}$$

and $[D_i^p \times y_i], y_i \in S_i^p, i=1, 2, \cdots, r$, form a basis of $H_q(\partial W, Y)$. Hence, noting that $N, Y$ and $\partial W$ are simply connected and $H_i(\partial W, N) \cong H_i(\partial W, Y) \cong 0$ for $i<q$, we know

$$\pi_q(\partial W, N) \cong H_q(\partial W, N),$$

$$\pi_q(\partial W, Y) \cong H_q(\partial W, Y),$$
by the Hurewicz isomorphism theorem.

Let $V = \partial W - \text{Int } D^m$ and we may assume that $N \subset \text{Int } V$. Then, by the homology exact sequence of $(\partial W, V, N)$,

$$H_i(V, N) \cong H_i(\partial W, N) \quad \text{if } i < m,$$

and similarly as above,

$$\pi_q(V, N) \cong H_q(V, N).$$

Thus, we have the following commutative diagram

$$\begin{array}{c}
H_q(V, N) \xrightarrow{\cong} H_q(\partial W, V, N) \xrightarrow{\cong} H_q(\partial W, Y) \\
\pi_q(V, N) \xrightarrow{\cong} \pi_q(\partial W, V, N) \xrightarrow{\cong} \pi_q(\partial W, Y) \\
\pi_{q-1}(N) \xrightarrow{\cong} \pi_{q-1}(Y) \\
\pi_{q-1}(\bigvee_{i=1}^r S_f^p),
\end{array}$$

where the horizontal isomorphisms are all induced from the inclusion maps.

We note that $H_{m-1}(\partial W, N) \cong 0$ by the homology exact sequence of $(\partial W, N)$ and $H_m(V, N) \cong 0$. So that, $H_q(V, N) \cong 0$ if $i \neq q$. Let $m > 5$. Then, by [12] or applying Theorem 7.6 and 7.8 of [11] to the triad $(V', \partial N, S^{m-1})$, where $V' = V - \text{Int } N$, we obtain the $q$-handles $T_i$ in $V'$, $i = 1, 2, \cdots, r$, such that the homology classes $[T_i]$, $i = 1, 2, \cdots, r$, form the basis of $H_q(V', \partial N) \cong H_q(V, N)$ which corresponds to the basis $\{[D_f \times y_i]; i = 1, 2, \cdots, r\}$ of $H_q(\partial W, Y)$. We may identify $V$ with $N \cup T_1 \cup T_2 \cup \cdots \cup T_r$. Henceforth, we assume that $2p > q > 1$ and $m > 5$. Let $\lambda_j = \sum_{i=1}^r \lambda_{ij} \in \pi_{q-1}(\bigvee_{i=1}^r S_f^p) = \sum_{i=1}^r \pi_{q-1}(S_f^p)$ be the linking element of the link $\{\cup_{i=1}^r \phi_i(S_f^p)\} \cup \phi_j(S_f^p) \subset S^m$ defined by $\phi_j(S_f^p) \subset S^m - \cup_{i=1}^r \phi_i(S_f^p) \times o \cong Y$. $\lambda_{ij} \in \pi_{q-1}(S_f^p)$ coincides with the linking element of the link $\phi_i(S_f^p) \times o \cup \phi_j(S_f^p) \times o \subset S^m$ defined by $\phi_j(S_f^p) \times o \subset S^m - \phi_i(S_f^p) \times o \cong S^m$ and is called the self-linking element of $\phi_j(S_f^p) \times o$. Note that $\lambda_{jj} = \pi_*(w_j)$, where $w_j$ is the basis element of $H_q(W) \cong H_q(W, D^{m+1})$ determined by $[D_f^j \times o]$. Let $v_j \in \pi_{q-1}(Y)$ be the homotopy class of $\phi_j|S_f^p \times y_j, j = 1, 2, \cdots, r$. Then, in the above diagram, $v_j$ corresponds to $\lambda_j$ for $j = 1, 2, \cdots, r$. Hence, by com-
mutativity of the diagram, the attaching map of the $q$-axis of $T_j$ is given by $\lambda_j, j=1, 2, \ldots, r$. Thus, we have

**Lemma 1.1.** Let $W=D^{m+1} \cup \bigcup_{i=1}^{r} \{ \cup_{i=1}^{r} D_i^q \times D_i^{p+1} \}$ be a handlebody of $\mathcal{M}(m+1, r, q)$, where $m=p+q$ and $\varphi_i: \partial D_i^q \times D_i^{p+1} \to \partial D^{m+1}$, $i=1, 2, \ldots, r$, are disjoint imbeddings. We assume that $2p>q>1$ and $(p, q) \neq (2, 3)$. Then, $\partial W$ has the homotopy type of

\[
(\bigcup_{i=1}^{r} S_i^p) \cup (\bigcup_{j=1}^{r} D_j^q) \cup D^m
\]

and the attaching map of each $D_j^q$ is given by $\lambda_j = \sum_{i=1}^{r} \lambda_{ij} \in \pi_q-1(\bigcup_{i=1}^{r} S_i^p) = \sum_{i=1}^{r} \pi_{q-1}(S_i^p)$, where each $\lambda_{ij}$ is the linking element of the link $\varphi_i(S_i^{p-1} \times o)$ $\cup \varphi_j(S_j^{p-1} \times o) \subset S^m$ ($i \neq j$) and $\lambda_{jj}$ is the self-linking element of $\varphi_j(S_j^{p-1} \times o) \subset S^m$, $i, j = 1, 2, \ldots, r$.

**Remark.** In each additional case for $m=4, 5$, the lemma holds trivially since $\partial W$ is represented as a connected sum of $p$-sphere bundles over $q$-spheres which admit cross-sections.

§ 2. **Proof of Proposition 1**

Let $W, W'$ be the handlebodies of $\mathcal{M}(m+1, r, q)$, $m=p+q$, and assume that $1<p<q-1$ or $p>q>1$. Let $W'=D^{m+1} \cup \bigcup_{i=1}^{r} \{ \cup_{i=1}^{r} D_i^q \times D_i^{p+1} \}$ be a representation, where $\varphi_i: \partial D_i^q \times D_i^{p+1} \to \partial D^{m+1}$, $i=1, 2, \ldots, r$, are disjoint imbeddings. By the assumption on $p, q$, we know that $H_q(\partial W') \cong 0$ if $k \neq 0$, $p, q, m, H_q(\partial W)$ has the basis $u_1= \big[ x_1^i \times S_i^p \big]$, $i=1, 2, \ldots, r$, and $H_q(\partial W')$ has the basis $v_j, j=1, 2, \ldots, r$, which corresponds to $[D_i^q \times o] \in H_q(W', D_i^{m+1})$, $j=1, 2, \ldots, r$, under the isomorphisms induced from the inclusion maps $H_q(\partial W') \cong H_q(W) \cong H_q(W', D^{m+1})$. We call $\{u_1, \ldots, u_r\}, \{v_1, \ldots, v_r\}$ to be the bases associated with the handles of $W'$.

**Lemma 2.1.** For any homotopy equivalence $f: \partial W \to \partial W'$ which preserves orientation, there exists a representation $W=D^{m+1} \cup \bigcup_{i=1}^{r} \{ \cup_{i=1}^{r} D_i^q \times D_i^{p+1} \}$, where $\varphi_i: \partial D_i^q \times D_i^{p+1} \to \partial D^{m+1}$, $i=1, 2, \ldots, r$, are disjoint imbeddings, such that $f_*(u_i)=u_i$, $f_*(v_j)=v_j$ for $i, j=1, 2, \ldots, r$. Here, $\{u_1, \ldots, u_r\}, \{v_1, \ldots, v_r\}$ are the bases of $H_q(\partial W), H_q(\partial W')$ respectively associated with the handles of $W$.

**Proof.** Let $\bar{u}_i=f_{\bar{u}}^{-1}(u_i), \bar{v}_j=f_{\bar{v}}^{-1}(v_j), i, j=1, 2, \ldots, r$, and let $\bar{w}_j=i_{\bar{w}}(\bar{v}_j), j=1, 2, \ldots, r$, where $i_{\bar{w}}: H_q(\partial W) \cong H_q(W)$ is induced from the inclusion map.
We represent $W$ by the basis $\{\tilde{w}_1, \ldots, \tilde{w}_r\}$ (cf. Milnor [11], Theorem 7.6). So, we have a representation $W = D^{m+1} \cup \{\cup_{i=1}^r D^q_i \times D^{p+1}_i\}$. Then, clearly $i_* (v_j) = \tilde{w}_j = i_* (\tilde{v}_j)$ and therefore $\tilde{v}_j = v_j$, $j = 1, 2, \ldots, r$. Furthermore, $\tilde{u}_i \cdot \tilde{v}_j = \delta_{ij}$, $i, j = 1, 2, \ldots, r$, and $u_i \cdot v_j = \delta[x_i \times D^{p+1}_i] \cdot \tilde{v}_j = [x_i \times D^{p+1}_i] \cdot (i_* (\tilde{v}_j)) = [x_i \times D^{p+1}_i] \cdot \tilde{w}_j = \delta_{ij}$, $i, j = 1, 2, \ldots, r$. Hence, $\tilde{u}_i = u_i$, $i = 1, 2, \ldots, r$.

Now, we prove Proposition 1. If $p \geq q$, the assertion holds trivially. So, we assume that $2p > q > 1$ and $p < q - 1$. Let $f: \partial W \to \partial W'$ be a homotopy equivalence which preserves orientation. Let $W = D^{m+1} \cup \{\cup_{i=1}^r D^q_i \times D^{p+1}_i\}$ be the representation given by Lemma 2.1. Then, by Lemma 1.1, we have the following diagram commutative up to homotopy.

It may be assumed that $g((\bigvee_{i=1}^r S^q_i) \cup (\bigcup_{j=1}^r D^p_j) \cup D^m) = (\bigvee_{i=1}^r S^q_i) \cup (\bigcup_{j=1}^r D^p_j)$ and each $g | S^q_i$ is the identity $(S^q_i, S^q_i)$ is copies of $S^p$ since $f_q (u_i) = u_i$, $i = 1, 2, \ldots, r$. Hence, we have the following commutative diagram, where we put $X = (\bigvee_{i=1}^r S^q_i) \cup (\bigcup_{j=1}^r D^p_j) \cup D^m$ and $X' = (\bigvee_{i=1}^r S^q_i) \cup (\bigcup_{j=1}^r D^p_j) \cup D^m$.

Note that each $v_j$, $v'_j$ correspond to $[D^p_j] \in H_q (X, \bigvee_{i=1}^r S^q_i)$, $[D^p_j] \in H_q (X', \bigvee_{i=1}^r S^q_i)$ respectively. $\{D^p_j\} \in \pi_q (X, \bigvee_{i=1}^r S^q_i)$, $\{D^p_j\} \in \pi_q (X', \bigvee_{i=1}^r S^q_i)$ correspond to $[D^p_j]$, $[D^p_j]$ under the Hurewicz isomorphisms. Then, since $f_* (v_i)$
Let $B_i$, $B'_i$, $i = 1, 2, \ldots, r$, be $p$-sphere bundles over $q$-spheres $(p, q > 1)$ with the characteristic elements $\alpha_i, \alpha'_i, i = 1, 2, \ldots, r$, respectively. Then, $\#_{i=1}^r B_i$ has the homotopy type of $X = (\bigvee_{i=1}^r S_f^p) \cup (\bigvee_{i=1}^r D_f^q) \cup D^{p+q}$, and $\#_{i=1}^r B'_i$ the homotopy type of $X' = (\bigvee_{i=1}^r S'_f^p) \cup (\bigvee_{i=1}^r D'_f^q) \cup D^{p+q}$, where each $D_f^q$, $D'_f^q$ are attached to $S_f^p$, $S'_f^p$ by $e = \pi_0 \alpha$, $e' = \pi_0 \alpha'$ respectively (cf. [8] §1). Let $p < q - 1$. Let $\{u_i; i = 1, 2, \ldots, r\}$ be the basis of $H_p(\#_{i=1}^r B_i)$ represented by the fibres of $B_i$, $i = 1, 2, \ldots, r$. Since $H_q(\#_{i=1}^r B_i) \cong H_q(\#_{i=1}^r B'_i)$, the zero cross-sections of $B_i$, $i = 1, 2, \ldots, r$, determine the basis $\{v_i; i = 1, 2, \ldots, r\}$ of $H_q(\#_{i=1}^r B_i)$. $u_i$ corresponds to $[S_f^p] \in H_p(X)$ and $v_i$ to $[D_f^q] \in H_q(X), i = 1, 2, \ldots, r$. Those are the bases associated with handles if we consider $\#_{i=1}^r B_i$ to be a handlebody. Similarly define $\{u'_i; i = 1, 2, \ldots, r\}$, $\{v'_i; i = 1, 2, \ldots, r\}$ for $\#_{i=1}^r B'_i$. Then, the above diagram and a similar argument will show the following, where $\pi_{q-1}(S_f^p)$, $\pi_{q-1}(S'_f^p)$ are direct summands of $\pi_{q-1}(\bigvee_{i=1}^r S_f^p)$, $\pi_{q-1}(\bigvee_{i=1}^r S'_f^p)$ respectively, $i = 1, 2, \ldots, r$.

**Lemma 2.2.** Let $1 < p < q - 1$. If there exists a map $f: \#_{i=1}^r B_i \to \#_{i=1}^r B'_i$ such that $f_*(u_i) = u'_i, f_*(v_i) = v'_i$, $i = 1, 2, \ldots, r$, then $e_i = e'_i$ for $i = 1, 2, \ldots, r$. Here, if $p \geq q$ the assertion is trivial.

## § 3. Difference of Bundles

Let $B_i, B'_i$ be $p$-sphere bundles over $q$-spheres with the characteristic elements $\alpha_i, \alpha'_i$ respectively, $i = 1, 2, \ldots, r$, and assume that $e_i = e'_i$, where $e_i = \pi_0 \alpha$, $e'_i = \pi_0 \alpha'$. Let $S_i^p$, $p_i$, and $S'_i$ be respectively the fixed fibre, the projection, and the base space of $B_i$. Define $S_i'^p$, $p'_i$, and $S'_i^q$ similarly for $B'_i$. In the disjoint union of $B_i$ and $B'_i$, identify $S_i^p$ with $S'_i^p$. Then, we have a $p$-sphere bundle over $S_i^p \vee S_i'^q$, where $S_i^p, S_i'^q$ are identified at $s_i = p(S_i^p) = p'(S_i^p)$. Since $B_i, B'_i$ are included in this bundle as subspaces, we may denote it by $B_i \cup B'_i$ (cf. [10], p. 156).
Let \( g_i: S^q \to S^q_i \vee S^q_i \) be a map representing \( \zeta^q_i - \zeta^q_i \in \pi_q(S^q_i \vee S^q_i) \), where \( \zeta^q_i, \zeta^q_i \) are the orientation generators of \( \pi_q(S^q_i) \), \( \pi_q(S^q_i) \) respectively. The induced bundle \( A_i = g_i^*(B_i \cup B_i') \) has the characteristic element \( \alpha_i - \alpha'_i \) and admits a cross-section since \( \alpha_i - \alpha'_i = 0 \) by the above assumption. Let \( h_i: A_i \to B_i \cup B_i' \) be the bundle map which covers \( g_i \). A fixed fibre \( S^q_A \) of \( A_i \) is oriented so that \( h_i|^{|S^q_A} : S^q_A \to S^q_i \) is of degree 1, and \( A_i \) is oriented by the orientations of \( S^q_i \) and \( S^q \).

Let \( S^q_{A_i} \) be the cross-section of \( A_i \) associated with \( \xi_i \in \pi_q-1(SO_p) \) satisfying \( i_\#(\xi_i) = \alpha_i - \alpha'_i \). Then, \( A_i = (S^q_A \vee S^q_A) \cup e_{p+q}^i \) and the attaching map is given by \( \partial \tau_i = \zeta^q_i \circ \eta_i + [\zeta^q_i, \zeta^q_i] \), where \( \eta_i = J \xi_i \) and \( \tau_i \) is the orientation generator of \( \pi_{p+q}(A_i, S^q_A \vee S^q_A) \) (cf. [9]). Hence, by Lemma 1.1 of [8],

\[
\frac{r}{i=1} A_i \simeq A = \left\{ \bigvee_{i=1}^r (S^q_A \vee S^q_A) \right\} \cup e_{p+q}^i,
\]

and the attaching map of the \((p+q)\)-cell is given by

\[
\partial \tau = \sum_{i=1}^r (\zeta^q_i \circ \eta_i + [\zeta^q_i, \zeta^q_i]),
\]

where \( \tau \) is the orientation generator of \( \pi_{p+q}(A, \bigvee_{i=1}^r (S^q_A \vee S^q_A)) \) and \( \pi_{p+q-1}(S^q_A \vee S^q_A), i = 1, 2, \ldots, r, \) are considered as direct summands of \( \pi_{p+q-1}(\bigvee_{i=1}^r (S^q_A \vee S^q_A)) \).

In \( A_1 \sharp A_2 \sharp \cdots \sharp A_r \), join every \((p+q-1)\)-sphere where connected sum is performed to the base points of the bundles neighboring at the \((p+q-1)\)-sphere by suitably chosen arcs. If we crush the \((p+q-1)\)-spheres and the arcs to a point, the yielding space can be considered as \( \bigvee_{i=1}^r A_i \). Let \( v: \bigvee_{i=1}^r A_i \to \bigvee_{i=1}^r A_i \) be the collapsing map. Then, we have a map

\[
h = \left( \bigvee_{i=1}^r h_i \right) \circ v: \frac{r}{i=1} A_i \longrightarrow \frac{r}{i=1} (B_i \cup B_i').
\]

\( \bigvee_{i=1}^r A_i \) can be replaced by the complex \( A \) of (3.1) and \( h \) may be assumed to preserve the base point. We denote the map by the same symbol \( h \).

\( B_i \) has the cell structure \( B_i = S^q_i \cup e^q_i \cup e_{p+q}^i \), where \( e^q_i \) is attached to \( S^q_i \) by \( \zeta^q_i \circ \sigma_i \). Here, \( \zeta^q_i \) is the orientation generator of \( \pi_p(S^q_i) \). Let \( \sigma_i \) be the orientation generator of \( \pi_{p+q}(B_i, S^q_i \cup e^q_i) \). Then, \( \partial \sigma_i \in \pi_{p+q-1}(S^q_i \cup e^q_i) \) is represented by the attaching map of \( e_{p+q}^i \). Similarly to Lemma 1.1 of [8], it is seen that

\[
\frac{r}{i=1} B_i \simeq B = \left\{ \bigvee_{i=1}^r (S^q_i \cup e^q_i) \right\} \cup e_{p+q}^i,
\]

where each \( e^q_i \) is attached by \( \zeta^q_i \circ \sigma_i \), and
\( \partial \sigma = \partial \sigma_1 + \partial \sigma_2 + \cdots + \partial \sigma_r, \)

where \( \sigma \) is the orientation generator of \( \pi_{p+q}(B, \cup_i S_i) \) and each \( \pi_{p+q-1}(S_i) \) is considered as a direct summand of \( \pi_{p+q-1}(S_i) \).

Let \( B_i = S_i \cup e_i \cup e_i'^q \) be the cell structure of \( B_i \). \( e_i'^q \) is attached to \( S_i \) by \( \epsilon_i'^q \), where \( \epsilon_i'^q \) is the orientation generator of \( \pi_p(S_i) \). Let \( \sigma_i \) be the orientation generator of \( \pi_{p+q}(B_i, S_i \cup e_i'^q) \).

Let \( K = \cup_i (S_i \cup e_i \cup e_i'^q) \) be the subcomplex of \( \cup_i (B_i \cup B_i') \), where \( e_i, e_i'^q \) are attached to \( S_i \) by \( \epsilon_i e_i \) and \( \epsilon_i'^q e_i' \) respectively. Then, it may be assumed that \( h(\cup_i (S_i \cup e_i \cup e_i'^q)) \subseteq K \) for the map \( h: A \to \cup_i (B_i \cup B_i') \). Let \( \overline{h}: (A, \cup_i (S_i \cup e_i \cup e_i'^q)) \to (\cup_i (B_i \cup B_i'), K) \). From the construction of \( h \), we know

\[
\overline{h}(\tau) = (\overline{\sigma}_1 - \overline{\sigma}_1') + (\overline{\sigma}_2 - \overline{\sigma}_2') + \cdots + (\overline{\sigma}_r - \overline{\sigma}_r'),
\]

where \( \overline{\sigma}_i \) is the image of \( \sigma_i \) by the homomorphism induced from the inclusion \( (B_i, S_i \cup e_i') = (B_i, S_i' \cup e_i'^q) \), \( i = 1, 2, \cdots, r \). \( \overline{\sigma}_1' \) is similar, and \( \pi_{p+q}(B_i, S_i \cup e_i'^q), S_i \cup e_i \cup e_i'^q), i = 1, 2, \cdots, r \), are considered as the direct summands of \( \pi_{p+q}(\cup_i (B_i \cup B_i'), K) \). Let \( \delta_i = \delta \sigma_i \), and let \( \delta_i \) be the image of \( \delta \sigma_i \) by the homomorphism induced from the inclusion \( S_i \cup e_i'^q \subseteq S_i \cup e_i \cup e_i'^q \), \( i = 1, 2, \cdots, r \). Define \( \delta_i, \delta_i' \) similarly, \( i = 1, 2, \cdots, r \). Here, \( \pi_{p+q-1}(S_i \cup e_i \cup e_i'^q), i = 1, 2, \cdots, r \), are understood as direct summands of \( \pi_{p+q-1}(\cup_i (S_i \cup e_i \cup e_i'^q)) \). Then,

\[
\partial \overline{h}_* \tau = \partial \sum_{i=1}^r (\overline{\sigma}_i - \overline{\sigma}_i') = \sum_{i=1}^r (\partial \overline{\sigma}_i - \partial \overline{\sigma}_i') = \sum_{i=1}^r (\delta_i - \delta_i') = \sum_{i=1}^r \delta_i - \sum_{i=1}^r \delta_i',
\]

and by (3.2),

\[
\partial \overline{h}_* \tau = h_\ast \partial \tau = \sum_{i=1}^r h_\ast (\epsilon_i \ast \eta_i + [\epsilon_i, \epsilon_i]).
\]

Hence, we have

\[
\sum_{i=1}^r \delta_i - \sum_{i=1}^r \delta_i' = \sum_{i=1}^r h_\ast (\epsilon_i \ast \eta_i + [\epsilon_i, \epsilon_i]).
\]

§ 4. Proof of the Necessity for Theorem 4

Let \( B_i, B_i', i = 1, 2, \cdots, r \), be \( p \)-sphere bundles over \( q \)-spheres with the characteristic elements \( \alpha_i, \alpha_i' \) respectively and assume that \( q/2 < p < q - 1 \). Let \( f: \#_{i=1}^r B_i \to \#_{i=1}^r B_i' \) be a homotopy equivalence which preserves orientation.

Assertion 1. There exists another expression of \( \#_{i=1}^r B_i \) into a connected
sum of p-sphere bundles over q-spheres \( \#_{i=1} B_i \) such that in the cell decompositions \( \#_{i=1} B_i \simeq B = \{ \vee_{i=1} (S_i^p \cup e_i^q) \} \cup \tilde{e}^p+q \) and \( \#_{i=1} B_i \simeq B = \{ \vee_{i=1} (S_i^p \cup e_i^q) \} \cup e^p+q, \) \( f_* : H_*(B) \rightarrow H_*(B') \) satisfies \( f_*([S_i^p]) = [S_i^p], \) \( i = 1, 2, \ldots, r \) and \( f_*([\tilde{e}^p]) = [e_i^q], \) \( j = 1, 2, \ldots, r, \) where \( f \) may be assumed to satisfy \( f(\vee_{i=1} S_i^p) \subset \vee_{i=1} S_i^p \) and \( f : (\tilde{B}, \vee_{i=1} S_i^p) \rightarrow (B', \vee_{i=1} S_i^p) \) is the relativization of \( f. \)

**Proof.** Let \( W = \vee_{i=1} B_i, W' = \vee_{i=1} B_i, \) and let \( \{u_1, \ldots, u_r, \}, \{v_1, \ldots, v_r, \} \) be the bases of \( H_q(\partial W), H_q(\partial W') \) respectively associated with the handles of \( W. \) Then, by Lemma 2.1, there exists a representation of \( W \) into such a handlebody that \( f_*(u_i) = u'_i, f_*(v_i) = v'_i, i = 1, 2, \ldots, r, \) where \( \{u_1, \ldots, u_r, \}, \{v_1, \ldots, v_r, \} \) are bases of \( H_\rho(\partial W), H_\rho(\partial W) \) respectively associated with the new handles of \( W. \) Of course, \( \{w'_j = i'_*v'_j; j = 1, 2, \ldots, r, \} \), the basis of \( H_q(W) \) is admissible since \( w'_j, j = 1, 2, \ldots, r, \) are represented by zero cross-sections of \( \tilde{B}_j, j = 1, 2, \ldots, r. \) Hence, by Proposition 1, the basis of \( H_q(W), \{w_j = i_\#v_j; j = 1, 2, \ldots, r, \} \) is admissible. Therefore, again \( W \) can be represented into a boundary connected sum of \( (p + 1) \)-disk bundles over q-spheres \( \#_{i=1} \tilde{B}_i \) and \( \partial W \) into a connected sum of p-sphere bundles over q-spheres \( \#_{i=1} B_i. \) We note that in the above cell-decompositions, \( u_1, v_j, u'_1, \) and \( v'_j \) correspond to \( [S_i^p], [\tilde{e}^p], [S'_i^p], \) and \( [e_i^q] \) respectively for each \( i, j. \) This completes the proof.

**Assertion 2.** Under the cell decompositions \( \#_{i=1} B_i \simeq B = \{ \vee_{i=1} (S_i^p \cup e_i^q) \} \cup e^p+q \) and \( \#_{i=1} B'_i \simeq B' = \{ \vee_{i=1} (S_i^p \cup e_i^q) \} \cup e^p+q, \) if \( f_* : H_*(B) \rightarrow H_*(B') \) satisfies \( f_*([S_i^p]) = [S_i^p], f_*([e_i^q]) = [e_i^q], \) for \( i = 1, 2, \ldots, r, \) then \( e_i = e_i', \) \( i = 1, 2, \ldots, r, \) where \( e_i = \pi_\#a_i, e_i' = \pi_\#a_i', \) and \( \{a_i\} = \{a_i'\}, i = 1, 2, \ldots, r. \) in \( \pi_{q-1}(S_i^p) = \pi_{q-1}(S_i^p'). \)

**Proof.** The former half of the assertion is known immediately from Lemma 2.2. To prove the latter half, we apply Section 3. By the assumption, we may assume that \( f \) maps each \( S_i^p \) identically onto \( S_i^p \) and \( S_i^p \) can be identified by means of \( \left( e_i^q \cup \rho(\tilde{e}_i^p) \right)^{-1} \) and \( f(\vee_{i=1} (S_i^p \cup e_i^q)) \subset \vee_{i=1} (S_i^p \cup e_i^q). \) Let \( f^0 = f|_{\vee_{i=1} (S_i^p \cup e_i^q)} \) and let \( \rho : K \rightarrow \vee_{i=1} (S_i^p \cup e_i^q) \) be the retraction defined by \( \rho|_{\vee_{i=1} (S_i^p \cup e_i^q)} = f_0 \) and \( \rho|_{\vee_{i=1} (S_i^p \cup e_i^q)} = \text{identity}. \) Then, we have the following commutative diagram.

\[
\begin{array}{ccccccccc}
\pi_q \left( \bigvee_{i=1} S_i^p \right) & \xrightarrow{k_i^p} & \pi_q(K) & \xrightarrow{\rho_0^q} & \pi_q(K, \bigvee_{i=1} S_i^p) \\
\downarrow \rho_{i=1} & & \downarrow \rho_+ & & \downarrow \rho_+ \\
\pi_q \left( \bigvee_{i=1} S_i^p \right) & \xrightarrow{i_i^q} & \pi_q \left( \bigvee_{i=1} (S_i^p \cup e_i^q) \right) & \xrightarrow{i_i^q} & \pi_q \left( \bigvee_{i=1} (S_i^p \cup e_i^q), \bigvee_{i=1} S_i^p \right)
\end{array}
\]
where \( k^0, l^0, k', \) and \( l' \) are inclusion maps and \( \tilde{\rho} \) is the relativization of \( \rho \).

Let \( j: (S^p_i \cup e^q_i, S^p_i) \to (\vee_{i=1} S^p_i \cup e^q_i), \vee_{i=1} S^p_i \to (K, \vee_{i=1} S^p_i) \) be inclusion maps and let \( j', \tilde{m} \) be similar for \((S^p_i \cup e^q_i, S^p_i)\). Let \( \kappa_q^i \in \pi_q(S^p_i \cup e^q_i, S^p_i) \cong H_q(S^p_i \cup e^q_i, S^p_i) \) be the generator corresponding to \([e^q_i]\) and define \( \kappa_q^i \in \pi_q(S^p_i \cup e^q_i, S^p_i) \) similarly. Then, we have

\[
(4.2) \quad \tilde{\rho} \circ \tilde{m} \cdot (\kappa_q^i) = j_q^i \kappa_q^i, \quad \tilde{\rho} \circ \tilde{m} \cdot (\kappa_q^i) = j_q^i \kappa_q^i.
\]

The former is clear since \( \tilde{\rho} \circ \tilde{m} = j_q' \). The latter is known from the following commutative diagram including the factorization of \( \tilde{m} \).

\[
\begin{array}{ccc}
\pi_q(S^p_i \cup e^q_i, S^p_i) & \xrightarrow{j_q^i} & \pi_q(K, (\vee_{i=1} S^p_i)) \\
\tilde{\rho} \circ \tilde{m} \cdot (\kappa_q^i) & \mapsto & \tilde{\rho} \circ \tilde{m} \cdot (\kappa_q^i)
\end{array}
\]

where \( j_q(\kappa_q^i) = j_q^i \kappa_q^i \) since \( j_q^i[e^q_i] = [e^q_i] \) by the assumption.

We apply the homomorphism \( h^*: \pi_q(\vee_{i=1} S^p_i, S^p_i) \to \pi_q(K) \). \( l^q_{0} h^* \zeta_q^i = \tilde{m} \kappa_q^i - \tilde{m} \kappa_q^i \) is clear from the definition of \( h \). Hence, by (4.1) and (4.2),

\[
l^q_{0} h^* \zeta_q^i = \tilde{\rho} \circ \tilde{m} \cdot (\kappa_q^i) - \tilde{\rho} \circ \tilde{m} \cdot (\kappa_q^i)
\]

So that,

\[
(4.3) \quad \rho^* h^* \zeta_q^i = k_q^i \theta_q^i \quad \text{for some} \quad \theta_q^i \in \pi_q(\vee_{j=1} S^f_j).
\]

Then, applying \( \rho^* \) to (3.6) and by (4.3),

\[
\begin{align*}
\sum_{i=1}^r \rho^* \delta_i - \sum_{i=1}^r \rho^* \delta_i &= \sum_{i=1}^r \rho^* h^* (\zeta_p^i \circ \eta_i + [\zeta_p^i, \zeta_p^i]) \\
&= \sum_{i=1}^r (\rho^* h^* \zeta_p^i \circ \eta_i + [\rho^* h^* \zeta_p^i, \rho^* h^* \zeta_p^i]) \\
&= \sum_{i=1}^r (k_q^i \zeta_p^i \circ \eta_i + [k_q^i \theta_q^i, k_q^i \zeta_p^i]) = \sum_{i=1}^r k_q^i (\zeta_p^i \circ \eta_i + [\theta_q^i, \zeta_p^i]).
\end{align*}
\]

On the other hand, let \( \sigma \in \pi_{p+q}(B, \vee_{i=1} (S^p_i \cup e^q_i)), \sigma' \in \pi_{p+q}(B', \vee_{i=1} (S^{p'}_i \cup e^{q'}_i)) \) be the orientation generators and let \( \delta = \partial \sigma, \delta' = \partial \sigma' \). Since \( f \) is of degree 1

\[
f^q_{0} \delta = f^q_{0} \partial \sigma = \partial f_{0} \sigma = \partial \sigma' = \delta',
\]

and therefore,

\[
\sum_{i=1}^r \rho^* \delta_i - \sum_{i=1}^r \rho^* \delta_i = \sum_{i=1}^r f^q_{0} \delta_i - \sum_{i=1}^r \delta_i = f^q_{0} (\sum_{i=1}^r \delta_i) - \sum_{i=1}^r \delta_i
\]

\[
=f^q_{0} \delta - \delta' = 0.
\]
Thus, we have

\begin{align*}
\sum_{i=1}^{r} k_a^* (\epsilon_p^i \ast \eta_i + [\theta_i^i, \epsilon_p^i]) = 0.
\end{align*}

(i) Now, we assume that \(2p > q + 1\). Then, \(\pi_q(\vee_{i=1}^r S_i^j) \cong \pi_q(S_i^p) \oplus \cdots \oplus \pi_q(S_i^p)\) and we have the unique summation \(\theta_i^j = \sum_{j=1}^r \theta_i^j, \theta_i^j \in \pi_q(S_i^j), j = 1, 2, \ldots, r\). Let \(\theta_i = \epsilon_p^j \ast \theta_i^j, \theta_i \in \pi_q(S_i^p)\), for \(j = 1, 2, \ldots, r\). Then,

\begin{align*}
\epsilon_p^i \ast \eta_i + [\theta_i^i, \epsilon_p^i] = & \epsilon_p^i \ast \eta_i + \sum_{j=1}^r [\theta_i^j, \epsilon_p^i] \\
= & \epsilon_p^i \ast \eta_i + \sum_{j=1}^r [\epsilon_p^j \ast \theta_i^j, \epsilon_p^i] \\
= & \epsilon_p^i \ast (\eta_i + [\theta_i^j, \epsilon_p^i]) + \sum_{j \neq i} [\epsilon_p^j \ast \theta_i^j, \epsilon_p^i],
\end{align*}

and by Barcus-Barratt [1] or G. W. Whitehead [15],

\[ [\epsilon_p^j \ast \theta_i^j, \epsilon_p^i] = [\epsilon_p^j, \epsilon_p^i] \circ (-1)^{p+q} E^{p-1} \theta_i^j, \]

where \(\theta_i^j, j = 1, 2, \ldots, r\), are the suspension elements. Hence, we have

\begin{align*}
(4.5) \quad \epsilon_p^i \ast \eta_i + [\theta_i^j, \epsilon_p^i] = & \epsilon_p^i \ast (\eta_i + [\theta_i^j, \epsilon_p^i]) + \sum_{j \neq i} [\epsilon_p^j, \epsilon_p^i] \circ (-1)^{p+q} E^{p-1} \theta_i^j.
\end{align*}

Let \(a_i = \epsilon_p^i \ast (\eta_i + [\theta_i^j, \epsilon_p^j]), \beta_{ji} = (-1)^{p+q} E^{p-1} \theta_{ij} + (-1)^{q} E^{p-1} \theta_{ji}\), and \(b_{ji} = [\epsilon_p^j, \epsilon_p^i] \circ \beta_{ji}\), where \(a_i \in \pi_{p+q-1}(S_i^j) \subset \pi_{p+q-1}(\vee_{i=1}^r S_i^j)\) and \(b_{ji} \in \pi_{p+q-1}(S_i^j \cup S_i^j) \subset \pi_{p+q-1}(\vee_{i=1}^r S_i^j), i, j = 1, 2, \ldots, r\). Then, by (4.4) and (4.5), we know

\begin{align*}
(4.6) \quad \sum_{i=1}^{r} k_a^* a_i + \sum_{i<j} k_a^* b_{ij} = 0, \quad \text{for} \quad k_a^*: \pi_n(\vee_{i=1}^r S_i^j) \longrightarrow \pi_n(\vee_{i=1}^r (S_i^j \cup e_i^q)), \quad n = p + q - 1.
\end{align*}

Here, each \(k_a^* a_i\) belongs to the direct summand \(\pi_{p+q-1}(S_i^j \cup e_i^q)\). Now, assume temporarily that every \(k_a^* b_{ij}\) belongs to another direct summand independent of \(\pi_{p+q-1}(S_i^j \cup e_i^q), i, j = 1, 2, \ldots, r\). This is the fact which will be shown in Assertion 3. Then, (4.6) yields \(k_a^* a_i = 0\) for \(i = 1, 2, \ldots, r\), and by the commutative diagram

\[
\begin{array}{ccc}
\pi_{p+q-1}(S^p) & \xrightarrow{k_a} & \pi_{p+q-1}(S^p \cup e_i^q) \\
\cong & \epsilon_p^i & \cong \mu_e \\
\pi_{p+q-1}(S_i^p) & \xrightarrow{k_a^*} & \pi_{p+q-1}(S_i^p \cup e_i^q),
\end{array}
\]

where \(k_a^*\) is induced from the inclusion map and \(\mu_e\) is the canonical isomorphism, we have
(4.7) \[ k_\alpha(\eta_i + [\theta_{ii}, \xi_p]) = 0, \quad i = 1, 2, \ldots, r, \]

where \( \eta_i = J\xi_{ii}, \xi_i \in \pi_{q-1}(SO_p) \), and \( i_\alpha\xi_i = \alpha_i - \alpha_{i'}, \quad i = 1, 2, \ldots, r \). Here, \( [\theta_{ii}, \xi_p] = -J_\theta\theta_{ii}, \partial : \pi_{q-1}(SO_p) \to \pi_{q-1}(SO_p) \). Let \( \xi_i = \xi_i - J\partial\theta_{ii} \in \pi_{q-1}(SO_p) \). Then, \( k_\alpha J\xi_i = k_\alpha(J\xi_i - J\partial\theta_{ii}) = k_\alpha(\eta_i + [\theta_{ii}, \xi_p]) = 0, \quad i = 1, 2, \ldots, r \). Since \( \text{Ker} k_\alpha = \text{Im} (\xi_i)_\alpha \), \( (\xi_i)_\alpha = \xi_i : \pi_{p+q-1}(S^{q-1}) \to \pi_{p+q-1}(SO_p) \) by (3.2) of James-Whitehead [10], \( J\xi_i \) belongs to \( \text{Im} (\xi_i)_\alpha \), and \( i_\alpha\xi_i = i_\alpha(\xi_i - J\partial\theta_{ii}) = i_\alpha\xi_i = \alpha_i - \alpha_{i'} \). Hence, we know that \( \alpha_i - \alpha_{i'} \in i_\alpha(J^{-1}(\text{Im} (\xi_i)_\alpha)) = G(\xi_i) \). That is, \( \{\alpha_i\} = \{\alpha_i\} \) in \( \pi_{q-1}(SO_{p+1})/G(\xi_i) = \pi_{q-1}(SO_{p+1})/G(\xi_i), \quad i = 1, 2, \ldots, r. \)

(ii) Let \( 2p = q + 1 \) (\( p, q > 1 \)). Then, \( \pi_q(\cup f_{i=1}^p S_i^{q-1}) = \bigoplus_{j<k} [\xi^j_p, \alpha^k_p] \pi_q(S^{2p-1}) \) by Hilton [4]. So that, we have the unique sum \( \theta_i = \bigoplus_{j=1}^p \xi^j_p \circ \theta_{ij} + \sum_{j<k} [\xi^j_p, \alpha^k_p] \circ \theta_{ijk} \), where \( \theta_{ij} \in \pi_q(S^{2p}), \quad \theta_{ijk} \in \pi_q(S^{2p-1}) \approx \mathbb{Z} \) for any \( i, j, k \). Therefore,

\[ [\theta_{i}, \xi_{p}] = \sum_{j=1}^p [\xi^j_p \circ \theta_{ij}, \alpha^k_p] + \sum_{j<k} [[\xi^j_p, \alpha^k_p] \circ \theta_{ijk}, \xi_p]. \]

By (7.4) of Barcus-Barratt [1],

\[ [\xi^j_p \circ \theta_{ij}, \alpha^k_p] = [\xi^j_p, \alpha^k_p] \circ (-1)^{p-1} E^{p-1} \theta_{ij} + [\xi^j_p, [\xi^j_p, \alpha^k_p]] \circ (-1)^{p-1} E^{p-1} H_0(\theta_{ij}), \]

where \( H_0 \) is the Hopf-Hilton homomorphism and the second term vanishes if \( p \) is odd since \( \theta_{ij} \) becomes a suspension element. And,

\[ [[\xi^j_p, \alpha^k_p] \circ \theta_{ijk}, \alpha^l_p] = [[\xi^j_p, \alpha^k_p], \alpha^l_p] \circ E^{p-1} \theta_{ijk} \]

\[ = [\xi^j_p, [\xi^j_p, \alpha^k_p]] \circ (-1)^{p-1} E^{p-1} \theta_{ijk}. \]

So that, we have

(4.8) \[ [\theta_{i}, \xi_{p}] = \xi^i_p \circ \theta_{ii}, \alpha^l_p] + \sum_{j \neq i} [\xi^j_p, \alpha^l_p] \circ (-1)^{p-1} E^{p-1} \theta_{ij} \]

\[ + \sum_{j<k} [\xi^j_p, [\xi^j_p, \alpha^k_p]] \circ (-1)^{p-1} E^{p-1} H_0(\theta_{ij}) \]

\[ + \sum_{j<k} [\xi^j_p, [\xi^j_p, \alpha^k_p]] \circ (-1)^{p-1} E^{p-1} \theta_{ijk}. \]

Every Whitehead product of weight 3 is a linear combination of the Whitehead products \( [\xi^i_p, [\xi^j_p, \alpha^k_p]] \) such that \( i \geq j < k \) by using the Jacobi identity (Hilton [4]). Hence,

(4.9) \[ \sum_{i=1}^r (\xi^i_p \circ \eta_i + [\theta_{ii}, \xi_p]) = \sum_{i=1}^r \xi^i_p \circ (\eta_i + [\theta_{ii}, \xi_p]) + \sum_{j<k} [\xi^j_p, [\xi^j_p, \alpha^k_p]] \circ \beta_{ij} \]

\[ + \sum_{i \geq j<k} [\xi^j_p, [\xi^j_p, \alpha^k_p]] \circ \gamma_{ijk}, \]

where \( \beta_{ji} \in \pi_{3p-2}(S^{2p-1}) (j < i) \) is defined as in (i) and \( \gamma_{ijk} \) (\( i \geq j < k \)) is a certain element of \( \pi_{3p-2}(S^{2p-2}) \approx \mathbb{Z} \).
Let \( a_i = \varepsilon_p^i \eta_i + [\theta_{ii}, \varepsilon_p^i] \), \( b_{ji} = [\varepsilon_p^i, \varepsilon_p^j] \beta_j \) as in (i), and let \( c_{ijk} = [\varepsilon_p^i, [\varepsilon_p^j, \varepsilon_p^k]] \). Then, by (4.4) and (4.9), we know

\[
\sum_{i=1}^r k'_* a_i + \sum_{i<j} k'_* b_{ij} + \sum_{i \leq j < k} k'_* c_{ijk} = 0,
\]

where \( k'_*: \pi_{p+q-1}(\vee_{t=1}^r S^p_t) \to \pi_{p+q-1}(\vee_{t=1}^r (S^p_t \cup e_t^q)) \), \( q = 2p-1 \). Therefore, if we show that every \( k'_* b_{ij} \) and every \( k'_* c_{ijk} \) belong to the direct summands independent of \( \pi_{p+q-1}(S^p_t \cup e_t^q) \), \( i = 1, 2, \ldots, r \), then \( k'_* a_i = 0 \) for \( i = 1, 2, \ldots, r \) by (4.10), and we can complete the proof similarly as in (i).

Thus, the following will conclude the proof of Assertion 2.

**Assertion 3.** Every \( k'_* b_{ij} \) \( (i < j) \) and every \( k'_* c_{ijk} \) \( (i \geq j < k) \) are included in a direct summand of \( \pi_{p+q-1}(\vee_{t'=1}^r (S^p_t \cup e_t^q)) \) which is independent of \( \pi_{p+q-1}(S^p_t \cup e_t^q) \), \( i = 1, 2, \ldots, r \).

**Proof.** Let \( X_t = S^p_t \cup e_t^q \), \( t = 1, 2, \ldots, r \). Then, \( \pi_n(\vee_{t'=1}^r X_{t'}) = \sum_{t=1}^r \pi_n(X_t) \otimes \partial \pi_n+1(\prod_{t=1}^r X_t, \vee_{t'=1}^r X_{t'}) \), \( n = p+q-1 \). We have the following commutative diagram.

\[
\begin{array}{cccc}
S^{p+q-1} \xrightarrow{\beta_{ij}} S^{2p-1} & \xrightarrow{[\varepsilon_p^i, \varepsilon_p^j]} & S^p_t \vee S^p_j & k' \xrightarrow{\pi_n} \vee_{t=1}^r X_t \\
\downarrow & \downarrow & \downarrow & \downarrow \\
D^{p+q} \xrightarrow{c(\beta_{ij})} D^{2p} & \xrightarrow{[\varepsilon_p^i, \varepsilon_p^j]} & S^p_t \times S^p_j & k' \xrightarrow{\pi_n} \vee_{t=1}^r X_t,
\end{array}
\]

where vertical maps and \( k' \) are inclusions. Hence, \( k'_* b_{ij} \) belongs to \( \partial \pi_n+1(\prod_{t=1}^r X_t, \vee_{t'=1}^r X_{t'}) \) which is independent of \( \pi_n(X_t) \), \( t = 1, 2, \ldots, r \).

Generally, every basic product of weight \( \geq 2 \) belongs to \( \partial \pi_n+1(\prod_{t=1}^r S^p_t, \vee_{t'=1}^r S^p_{t'}) \). In fact, in the splitting exact sequence

\[
0 \xrightarrow{i} \pi_n+1(\prod_{t=1}^r S^p_t) \xrightarrow{\varepsilon_P} \pi_n(\vee_{t=1}^r S^p_t) \xrightarrow{\pi_n(\vee_{t=1}^r S^p_t)} \pi_n(\prod_{t=1}^r S^p_t) \xrightarrow{\sum_{t=1}^r \pi_n(S^p_t)} 0,
\]

such Whitehead products are mapped to zero. So, for any basic product \( [\varepsilon_p^i, [\varepsilon_p^j, \varepsilon_p^k]] \) \( (i \geq j < k) \), there exists an element \( \chi \in \pi_n+1(\prod_{t=1}^r S^p_t, \vee_{t'=1}^r S^p_{t'}) \) such that \( [\varepsilon_p^i, [\varepsilon_p^j, \varepsilon_p^k]] = \partial \chi \). Therefore, we have the following commutative diagram.
Hence, $k'_*c_{ijk}$ belongs to $\partial \pi_{n+1}(\bigcap_{i=1}^n X_i, \bigvee_{i=1}^n X_i)$. This completes the proof.

**Assertion 4.** For "any" admissible basis $\{w'_1, \cdots, w'_r\}$ of $H'=Hq(\#_{i=1}^r B'_i)$, there exists an admissible basis $\{w_1, \cdots, w_r\}$ of $H=Hq(\#_{i=1}^r B_i)$ such that

(i) $e(w_i) = e'(w'_i)$, \hspace{1cm} i = 1, 2, \cdots, r.

(ii) $\{\alpha(w_i)\} = \{\alpha'(w'_i)\}$ in $\pi_{q-1}(SO_{p+1})/G(e(w_i)) = \pi_{q-1}(SO_{p+1})/G(e'(w'_i))$,
\hspace{1cm} i = 1, 2, \cdots, r.

**Proof.** There exists another expression of $\#_{i=1}^r B'_i$ into a connected sum of $p$-sphere bundles over $q$-spheres $\#_{i=1}^r B_i'$ such that in the cell decomposition $\#_{i=1}^r B'_i = \{\bigvee_{i=1}^r (\tilde{S}_i^p \cup \tilde{e}^g_i)\} \cup \tilde{e}^{p+q}$, each homology class $[\tilde{e}^g_i]$ corresponds to $w'_i$, $i = 1, 2, \cdots, r$. Hence, Assertion 1 and Assertion 2 conclude the proof.

This completes the proof of the necessity for Theorem 4.

§ 5. Proof of the Sufficiency for Theorem 4

Let $B_i$, $B'_i$ be $p$-sphere bundles over $q$-spheres $(2p>q>1)$ with the characteristic elements $\alpha_i$, $\alpha'_i$ respectively and let $e_i = \pi_{q}(\alpha_i)$, $e'_i = \pi_{q}(\alpha'_i)$, where $i = 1, 2, \cdots, r$. Let $\{w_1, \cdots, w_r\}$, $\{w'_1, \cdots, w'_r\}$ be the admissible bases of $H, H'$ respectively, satisfying

(i) $e(w_i) = e'(w'_i)$, \hspace{1cm} i = 1, 2, \cdots, r, and

(ii) $\{\alpha(w_i)\} = \{\alpha'(w'_i)\}$ in $\pi_{q-1}(SO_{p+1})/G(e(w_i)) = \pi_{q-1}(SO_{p+1})/G(e'(w'_i))$,
\hspace{1cm} i = 1, 2, \cdots, r.

By adopting the representations of the connected sums of given bundles using the admissible bases $\{w_1, \cdots, w_r\}$, $\{w'_1, \cdots, w'_r\}$, we may assume that $w_i$, $w'_i$ are represented by zero cross-sections of $B_i$, $B'_i$ respectively, $i = 1, 2, \cdots, r$. Then, $\alpha(w_i) = \alpha_i$, $\alpha'(w'_i) = \alpha'_i$, $e(w_i) = e_i$, and $e'(w'_i) = e'_i$, $i = 1, 2, \cdots, r$. Hence, the proof is accomplished by directly extending that of James-Whitehead [10] ((1.5), p. 163).

Since $\alpha_i - \alpha'_i \in G(e_i)$, there exists an element $\xi_i \in \pi_{q-1}(SO_p)$ such that $i_*\xi_i = \alpha_i - \alpha'_i$ and $J\xi_i \in \text{Im } (e_i)_{\#}, i = 1, 2, \cdots, r$. By (3.2) of [10], the sequence
\[ \pi_{p+q-1}(S^{q-1}) \xrightarrow{(e_i)_*} \pi_{p+q-1}(S^p) \xrightarrow{(k_i)_*} \pi_{p+q-1}(S^p \cup e^g) \]
is exact, where \( (k_i)_{\ast} \) is induced from the inclusion. Hence, \( J_{\xi} \in \text{Im} (e_{\ast}) \) 

\( = \text{Ker} (k_{\ast}) \), \( i = 1, 2, \cdots, r \). Let \( B_i = \mathbb{S}_p \cup e_i^q \cup e_i^{p+q} \), \( B'_i = \mathbb{S}_p \cup e_i^q \cup e_i^{p+q} \) be the cell-decompositions given by (3.3) of [9], where \( e_i^q \), \( e_i^{p+q} \) are attached by \( \zeta_p^{(q)}e_i \), \( \zeta_p^{(p+q)}e_i \) respectively. We identify \( \mathbb{S}_p \) canonically with \( \mathbb{S}_1 \), \( \mathbb{S}_1^q \) so that \( \zeta_p^{(q)} = \zeta_p^{(1)} \).

Since \( e_i = e_i', \) there exists a homotopy equivalence \( g_i : \mathbb{S}_p \cup e_i^q \to \mathbb{S}_p \cup e_i^{p+q} \) such that \( g_i \big| \mathbb{S}_p = \text{id} \). Let \( \sigma_i \in \pi_{p+q}(B_i, \mathbb{S}_p \cup e_i^q), \sigma_i' \in \pi_{p+q}(B'_i, \mathbb{S}_p \cup e_i^{p+q}) \) be the orientation generators and let \( \delta_i = \partial \sigma_i, \delta_i' = \partial \sigma_i' \). Then, by (3.3) and Lemma (3.8) of [10],

(i) \( (g_i)_{\ast} \delta_i - \delta_i' = \pi_q \xi_i' \) for some \( \xi_i' \in \pi_{q-1}(SO_p) \) such that \( i \ast \xi_i' = \alpha_i - \alpha_i' \), and

(ii) \( g_i \) can be chosen so that \( \beta_i \) is a given element in \( \pi_{q-1}(\alpha_i - \alpha_i') \).

Hence, by taking \( \xi_i \) as \( \xi_i' \), there exists a homotopy equivalence \( g_i : \mathbb{S}_p \cup e_i^q \to \mathbb{S}_p \cup e_i^{p+q} \) such that \( g_i \big| \mathbb{S}_p = \text{id} \) and \( (g_i)_{\ast} \delta_i = \delta_i' \), where \( i = 1, 2, \cdots, r \).

In the cell-decompositions \( \#_i \bigcirc B_i = \{ \bigvee_i (\mathbb{S}_1^q \cup e_i^q) \} \cup e_i^{p+q}, \#_i \bigcirc B'_i = \{ \bigvee_i (\mathbb{S}_1^p \cup e_i^{p+q}) \} \cup e_i^{p+q}, \) \( \sigma \in \pi_{p+q}(B, \bigvee_i (\mathbb{S}_1^q \cup e_i^q)), \sigma' \in \pi_{p+q}(B', \bigvee_i (\mathbb{S}_1^p \cup e_i^{p+q})) \) be the orientation generators, and let \( \delta = \partial \sigma, \delta' = \partial \sigma' \). Then,

\[
\delta = \delta_1 + \delta_2 + \cdots + \delta_r, \quad \delta' = \delta_1' + \delta_2' + \cdots + \delta_r',
\]

where it is understood that \( \sum_i \pi_{p+q-1}(\mathbb{S}_1^q \cup e_i^q) \subset \pi_{p+q-1}(\bigvee_i (\mathbb{S}_1^q \cup e_i^q)) \) and \( \sum_i \pi_{p+q-1}(\mathbb{S}_1^p \cup e_i^{p+q}) \subset \pi_{p+q-1}(\bigvee_i (\mathbb{S}_1^p \cup e_i^{p+q})) \). Now, let

\[
g = \bigvee_i (g_i)_{\ast} : \bigvee_i (\mathbb{S}_1^q \cup e_i^q) \longrightarrow \bigvee_i (\mathbb{S}_1^p \cup e_i^{p+q}).
\]

Then, \( g_{\ast} \delta = \sum_i (g_i)_{\ast} \delta_i = \sum_i \delta_i = \delta' \), that is, \( g_{\ast} \delta = \delta' \). Hence, \( g \) has an extension \( f : B \to B' \) of degree 1. \( f_{\ast} : H_\ast(B) \to H_\ast(B') \) is isomorphic for \( n = 0, p, p+q \), and for \( n = q \) as is shown by the following diagram

\[
\begin{align*}
H_\ast(B) & \xrightarrow{\sim} H_\ast(B') \\
\cong \downarrow D & \cong \downarrow D \\
H_\ast(B) & \xrightarrow{f_{\ast}} H_\ast(B'),
\end{align*}
\]

where \( f_{\ast} = (D \circ f_{\ast} \circ D^{-1}) = \text{id} \) and \( D \) is the Poincaré duality isomorphism. Since \( B, B' \) are simply connected, \( f \) is a homotopy equivalence. This completes the proof.

References


