Liouville Type Theorem for a System
\{P(D), B_j(D), j = 1, \ldots, p\}
of Differential Operators with Constant
Coefficients in a Half Space

By

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§ 1. Introduction

Concerning the behavior at infinity of solutions of a partial differential
equation \( Pu = f \), theorems of following three type are known. (1) The theorem of
Liouville type claims that if the function \( u(x) \) is a solution of \( Pu = 0 \) such that
\( u(x) = O(|x|^d) \) as \( |x| \to \infty \) (or \( \lim_{R \to \infty} \int_{R<|x|<2R} |u(x)|^2 \, dx = 0 \) for some real \( d \independent of u \), then \( u(x) \) must vanish identically (see, for example, Agmon [1],
Hörmander [7], Littman [9], Murata [10], Rellich [12] and so on). (2) The theorem of Rellich type claims that if \( Pu \) has compact support, then \( u \) has
compact support (see, for example, Rellich [12], Agmon [2], Littman [8, 9],
Murata [11], Hörmander [7], Trèves [16] and so on). (3) The theorem of Sommerfeld type gives conditions at infinity which derive the unique solution of \( Pu = f \)
(see, for example, Gruśin [4], Agmon and Hörmander [3], Vainberg [15] and so on). Recently the study on (1) and (2) has been completed by Hörmander [7]
in the constant coefficient and the whole space case and the study on (3) has
been remarkably promoted by Agmon and Hörmander [3].

The purpose of this paper is to study the problem of type (1) for a system
\{\( P(D), B_j(D), j = 1, \ldots, p \)\} of differential operators with constant coefficients in
a half space (boundary value problem) and to give almost corresponding results
to those obtained by Hörmander [7] in the whole space case. In order to state
results more precisely, let us first of all introduce certain notations. Let \( \mathbb{R}^{n+1} \)}
denote the \( n \)-dimensional Euclidean space, \( \mathbb{B}^{*+1} \) its dual space and write \((x, y)\) for the coordinate \((x_1, \ldots, x_n, y)\) in \( \mathbb{R}^{n+1} \) and \((\xi, \lambda)\) for the dual coordinate \((\xi_1, \ldots, \xi_n, \lambda)\). We denote by \( \mathbb{R}^{*+1} \) the half-space \( \{(x, y) \in \mathbb{R}^{n+1}; y > 0\} \). 

For differentiation we use the symbol \( D = i^{-1}(\partial/\partial x_1, \ldots, \partial/\partial x_n, \partial/\partial y) \), \( D_x = i^{-1}(\partial/\partial x_1, \ldots, \partial/\partial x_n) \), \( D_y = i^{-1}\partial/\partial y \). We denote by \( \mathcal{S}(\mathbb{R}^{*+1}) \) the space of restrictions to \( \mathbb{R}^{*+1} \) of all elements in \( \mathcal{S}(\mathbb{R}^{n+1}) \) and denote by \( \mathcal{S}'(\mathbb{R}^{*+1}) \) the space of all temperate distributions in \( \mathbb{R}^{*+1} \) (see Hörmander [6]). For a positive number \( \delta \) we put \( C^\infty([0, \delta); \mathcal{S}'(\mathbb{R}^n)) = \{u \in \mathcal{S}'(\mathbb{R}^{n+1}); \langle u(\cdot, y), \phi(\cdot) \rangle \text{ is a } C^\infty \text{ function of } y \text{ in } [0, \delta) \text{ for any } \phi(x) \in \mathcal{S}(\mathbb{R}^n)\} \). Let \( \sigma(y) \) be a \( C^\infty((-\delta, 0)) \) function with \( \sigma(y) = 1 \) for \( y \in [-\delta/2, \delta/2] \). Put

\[
\langle u, v \rangle = \int_0^\infty \langle \sigma(y)u(x, y), v(x, y) \rangle dy + \langle (1-\sigma(y))u(x, y), v(x, y) \rangle
\]

for \( u \in C^\infty([0, \delta); \mathcal{S}'(\mathbb{R}^n)) \) and \( v \in \mathcal{S}'(\mathbb{R}^{*+1}) \), where \( \langle \cdot, \cdot \rangle \) denotes the duality between \( \mathcal{S}'(\mathbb{R}^n) \) and \( \mathcal{S}(\mathbb{R}^n) \) and \( \langle \cdot, \cdot \rangle \) denotes the duality between \( \mathcal{S}'(\mathbb{R}^{*+1}) \) and \( \mathcal{S}(\mathbb{R}^{*+1}) \). Let

\[
P(D) = P(D_x, D_y) = \sum_{j=0}^{n} a_j(D_x)D_y^j
\]

be a differential operator with constant coefficients and \( B_j(D), j = 1, \ldots, p, \) be some other differential operators with constant coefficients of order \( r_j \) where \( a_j(D_x) \) is a differential operator in \( D_x \) with constant coefficients. We consider solutions \( u(x, y) \in C^\infty([0, \delta); \mathcal{S}'(\mathbb{R}^n)) \) of the following equations:

\[
P(D)u = 0 \quad \text{in } \mathbb{R}^{*+1},
\]

\[
B_j(D)u|_{y=0} = 0, \quad j = 1, \ldots, p, \quad \text{in } \mathbb{R}^n.
\]

We make two assumptions when \( m \geq 1 \). The following is an assumption about the number \( p \) of boundary conditions:

(A-1) The number of roots with positive imaginary part of the equation \( P(\xi, \lambda) = 0 \) in \( \lambda \) is less than or equal to \( p \) whenever \( \xi \in \mathbb{E}^n \).

Write \( P(\xi, \lambda) = \prod_{j=1}^{p} P_j(\xi, \lambda) \) and \( \bar{P}(\xi, \lambda) = \prod_{j=1}^{p} \bar{P}_j(\xi, \lambda) \) where all \( P_j(\xi, \lambda) \) are irreducible polynomials. Let us denote by \( Q(\xi) \) the resultant of \( \bar{P}(\xi, \lambda) \) and \( \bar{P}(\xi, \lambda)(\lambda, \xi) \). Put \( A_Q = \{\xi \in \mathbb{E}^n; Q(\xi) = 0\} \) and \( A_{a_m} = \{\xi \in \mathbb{E}^n; a_m(\xi) = 0\} \). Note that \( A_Q \) and \( A_{a_m} \) are empty or real analytic sets in this case. We decompose \( \mathbb{E}^n - (A_Q \cup A_{a_m}) \) into open connected components \( V_j \), that is,

\[
\mathbb{E}^n - (A_Q \cup A_{a_m}) = \bigcup_{\text{finite}} V_j
\]
where \( V_j \cap V_j' = \emptyset \) if \( j \neq j' \). Write \( V_j = V \) for the sake of simplicity. Let us denote by \( \lambda_j(\xi), j = 1, \ldots, m, \) the roots of the equation \( P(\xi, \lambda) = 0 \) in \( \lambda \) when \( \xi \in V \). We have that the imaginary part of \( \lambda_j(\xi) \) (denoting them by \( \text{Im} \lambda_j(\xi) \)) is a real analytic function of \( \xi \). Without loss of generality, we may assume that \( \text{Im} \lambda_j(\xi), j = 1, \ldots, \mu, (\mu \geq 0) \) do not vanish identically in \( V \) and \( \text{Im} \lambda_j(\xi), j = \mu + 1, \ldots, m, \) vanish identically in \( V \). Put
\[
A_{V, \text{Im}} = \{ \xi \in V; \text{Im} \lambda_t(\xi) = 0 \quad \text{for some} \ t \in \{1, \ldots, \mu\} \}.
\]
It is obvious that \( A_{V, \text{Im}} \) is either empty or a real analytic set. \( V - A_{V, \text{Im}} \) may be decomposed into open connected components \( W_{V,j} \), that is,
\[
V - A_{V, \text{Im}} = \bigcup W_{V,j}
\]
where \( W_{V,j} \cap W_{V,j'} = \emptyset \) if \( j \neq j' \). Write \( W_{V,j} = W \) for the sake of simplicity. Thus we have
\[
\Xi^n - (A Q \cup A_m \cup (\bigcup_j A_{V,j, \text{Im}})) = \bigcup W.
\]
Moreover, when \( \xi \in W \), the roots of the equation \( P(\xi, \lambda) = 0 \) in \( \lambda \) have constant multiplicity and split into three classes: real roots, those with positive imaginary part and those with negative imaginary part. We denote those by \( \lambda^0_j(\xi), j = 1, \ldots, a, \lambda^+_j(\xi), j = 1, \ldots, b, \) and \( \lambda^-_j(\xi), j = 1, \ldots, c, \) where \( \text{Im} \lambda^0_j(\xi) = 0, \text{Im} \lambda^+_j(\xi) > 0 \) and \( \text{Im} \lambda^-_j(\xi) < 0 \). Thus we have
\[
P(\xi, \lambda) = a_m(\xi) \prod_{j=1}^a (\lambda - \lambda^0_j(\xi))^{\alpha_j} \cdot \prod_{j=1}^b (\lambda - \lambda^+_j(\xi))^{\beta_j} \cdot \prod_{j=1}^c (\lambda - \lambda^-_j(\xi))^{\gamma_j}, \quad \xi \in W.
\]
Put
\[
P^0(\xi, \lambda) = \prod_{j=1}^a (\lambda - \lambda^0_j(\xi))^{\alpha_j} = \sum_{j=0}^{\alpha} a_j(\xi) \lambda^j,
\]
\[
P^+(\xi, \lambda) = \prod_{j=1}^b (\lambda - \lambda^+_j(\xi))^{\beta_j} = \sum_{j=0}^{\beta} b_j(\xi) \lambda^j,
\]
\[
P^-(\xi, \lambda) = a_m(\xi) \prod_{j=1}^c (\lambda - \lambda^-_j(\xi))^{\gamma_j} = \sum_{j=0}^{\gamma} c_j(\xi) \lambda^j, \quad \xi \in W,
\]
where \( a = \sum_{j=1}^a \alpha_j, b = \sum_{j=1}^b \beta_j, c = \sum_{j=1}^c \gamma_j \). Note that \( b \leq p \) under the assumption (A-1). Put
\[
L_{W, \sigma}(\xi) = \det((2\pi i)^{-1} \int_{\gamma(\xi)} B_{\sigma}(\xi, \lambda) \lambda^{k-1}(P^+(\xi, \lambda))^{-1} d\lambda)_{j,k=1,\ldots,\bar{b}}.
\]
where \( \gamma(\xi) \) is a simple closed curve in the complex upper half \( \lambda \)-plane which surrounds all \( \lambda^+_j(\xi), j = 1, \ldots, b \) when \( \xi \in W \) and we denote by \( \sigma = (\sigma_1, \ldots, \sigma_5) \) a set consisting of \( \bar{b} \) elements of \( \{1, \ldots, p\} \). When \( \bar{b} > 0 \) and \( m \geq 1 \) we make the following assumption on linear independence of boundary conditions:
(A-2) \( L^\sigma_{\mu}(\xi) \) does not vanish identically in \( W \) for some \( \sigma \subset \{1, \ldots, p\} \).

**Main Theorem.** Let \( u \) be a solution of the equations (1.1) and (1.2) which belongs to \( C^\infty([0, \delta); S'(\mathbb{R}^n) \cap L^2_{\text{loc}}(\mathbb{R}^{n+1}_+) \) for some positive number \( \delta \). We make the assumptions (A-1) and (A-2) when \( m \geq 1 \). Then there exist an open cone \( \Gamma \) in \( \mathbb{R}^{n+1} \) and a natural number \( N \) such that if \( u \) satisfies the condition:

\[
\lim_{R \to \infty} R^{-N} \int_{\Gamma_R} |u(x, y)|^2 \, dx \, dy = 0
\]

then \( u = 0 \). Here \( \Gamma \) and \( N \) are independent of \( u \), \( \Gamma_R = \{(x, y) \in \Gamma; y \geq 0, \ R < |(x, y)| < 2R\} \) and \( |(x, y)| = (\sum_{j=1}^n x_j^2 + y^2)^{1/2} \).

Moreover, in the case where \( m \geq 1 \), if at least one of (A-1) and (A-2) is not fulfilled, there exists a solution of the equations (1.1) and (1.2) which belongs to \( S(\mathbb{R}^{n+1}_+) \).

**Remark.** In the case where \( m = 0 \) and \( A_\infty = \{\xi \in \mathcal{B}^n; P(\xi, \lambda) = a_0(\xi) = 0\} \) is empty, if \( u \) satisfies equations (1.1) and (1.2) and belongs to \( C^\infty([0, \delta); S'(\mathbb{R}^n) \) then \( u = 0 \). In the case where \( m \geq 1 \) and the system \( \{P(\xi, \lambda), B_j(\xi, \lambda), \ j = 1, \ldots, p\} \) satisfies the following conditions:

(i) \( \{\xi \in \mathcal{B}^n; a_m(\xi) = a_{m-1}(\xi) = \cdots = a_0(\xi) = 0\} \) is empty,

(ii) for each \( \xi^0 \in \mathcal{B}^n \) all roots of the equation \( P(\xi^0, \lambda) = 0 \) have negative imaginary part or the degree \( d \) of \( P(\xi^0, \lambda) \) is equal to or less than \( p \) and

\[
\det((2\pi i)^{-1} \int_{\gamma(\xi^0)} B_{ij}(\xi^0, \lambda)\lambda^{a_{j-1}-(p(\xi^0, \lambda)P^+(\xi^0, \lambda))^{-1}d\lambda})_{j,k=1,\ldots,d} \neq 0
\]

for some \( \sigma = (a_1, \ldots, a_d) \subset \{1, \ldots, p\} \), if \( u \) satisfies the equations (1.1) and (1.2) and belongs to \( C^\infty([0, \delta); S'(\mathbb{R}^n) \cap L^2_{\text{loc}}(\mathbb{R}^{n+1}_+) \) then \( u = 0 \). Here denoting by \( \tau_j, j = 1, \ldots, \mu, \) and \( \tau_j, j = 1, \ldots, \nu \) the roots whose imaginary parts are zero and positive, respectively, of the equation \( P(\xi^0, \lambda) = 0 \) in \( \lambda \), we wrote \( P(\xi^0, \lambda) = \prod_{j=1}^\mu (\lambda - \tau_j^0) \) and \( P^+(\xi^0, \lambda) = \prod_{j=1}^\nu (\lambda - \tau_j^0) \), and \( \sigma = (a_1, \ldots, a_d) \) is a subset consisting of \( d \) elements of \( \{1, \ldots, p\} \). Thus in the statement of the Main Theorem we put \( \Gamma = \emptyset \) and interpret that the condition (1.3) is satisfied automatically under the situation which is stated above.

We will state more details on \( \Gamma \) and \( N \) in the proof of the Main Theorem.

On the other hand, we can show that the system \( \{P(\xi, \lambda), B_j(\xi, \lambda), \ j = 1, \ldots, p\} \) does not satisfy the condition stated in above Remark, there exists a solution \( u \in S'(\mathbb{R}^{n+1}_+) \cap C^\infty(\mathbb{R}^{n+1}_+) \) of the equations (1.1) and (1.2) with
In general case we have $N' \supseteq N$ and $N=N'$ for certain class of systems of differential operators with constant coefficients containing $\{P(D) = \mathcal{A} + k, B(D) = 1\}$ for which the result was given in Rellich [12] or Agmon [1] where $\mathcal{A}$ is the Laplacian operator and $k$ is a positive number. But we can not show that $N'=N$ in general case.

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§2. A Condition in Order That the Support of $\hat{u}(\xi, \eta)$ Is Contained in a Real Analytic Set of Higher Codimension

When $m=0$, put $A_0 = \mathcal{B}^n$. When $m \geq 1$, let us denote by $A_1$ an open set contained in $W$, by $A_{r+1} (1 \leq r \leq n-1)$ a real analytic manifold which is defined by $\xi' = \mu(\xi')$ where $\xi' \in \mathcal{Q} \subset \mathcal{B}^{n-r}$ and $\mu(\xi')$ is a real analytic function in $\mathcal{Q}$. $A_{n+1}$ denotes a set consisting of finite many points in $\mathcal{B}^n$. Further, when $1 \leq r \leq n+1$, we assume that $A_r$ is contained in $X_r$ where

\[
X_n = \{ \xi \in \mathcal{B}^n ; a_m(\xi) \neq 0 \}, \\
X_e = \{ \xi \in \mathcal{B}^n ; a_m(\xi) = \cdots = a_{e+1}(\xi) = 0, \ a_e(\xi) \neq 0 \ e = 1, \ldots, m-1 \}, \\
X_0 = \{ \xi \in \mathcal{B}^n ; a_m(\xi) = \cdots = a_1(\xi) = 0 \}.
\]

Let $u$ be a solution of the equations (1.1) and (1.2) which belongs to $C^\infty([0, \delta]; \mathcal{S}'(\mathcal{R}^n))$ such that the support of $\phi(\xi)\hat{u}(\xi, y) \overset{1)}{=} 0$ is contained in $A_r \times \mathcal{R}_1^1 (0 \leq r \leq n+1)$ for some $\phi \in C_c^\infty(\mathcal{B}^n)$. In this section we study what conditions imply that there is a real analytic set $B \subset A_r$ such that $\text{codim } B \geq \text{codim } A_r$ and $\text{supp } \phi(\xi)\hat{u}(\xi, y) \subset B \times \mathcal{R}^n_1$. We need the following lemma.

**Lemma 2.1.** Let $f(\xi, y)$ belong to $C^\infty([0, \delta]; \mathcal{S}(\mathcal{B}^n))$. If the support of $f$ is contained in the plane $\xi' = 0$, then $f$ has the form

\[
f(\xi, y) = \sum_{|\alpha| \leq \delta} f_\alpha(\xi', y) \otimes D_{\xi'}^\alpha, \delta
\]

1) We denote by $\hat{u}(\xi, y)$ the partial Fourier transform of $u(x, y)$ with respect to $x$. 

\[\int_{|x, y| < R} |u(x, y)|^2 dx dy = 0(R^N).\]
where \( f_\alpha \in C^\infty(0, \alpha); S'(E^{n-r}_\xi) \), \( \delta \) is the Dirac measure at the origin in \( E^{n-r}_\xi \), \( \alpha = (\alpha_{n-r+1}, \ldots, \alpha_0) \) and \( D_\xi \alpha = (D_{\xi_{n-r+1}}, \ldots, D_{\xi_n}) \). Here we wrote \( \xi' = (\xi_1, \ldots, \xi_{n-r}) \) and \( \xi'' = (\xi_{n-r+1}, \ldots, \xi_n) \).

Proof. From a theorem due to Seeley [13] it follows that \( f \) has the form

\[
f(\xi, y) = \sum_{|\alpha| \leq q} D_\alpha^\alpha g_\alpha(\xi, y) \quad \text{when } y \geq 0
\]

where the \( g_\alpha(\xi, y) \) are continuous functions of polynomial growth, \( F(\xi, y) = \sum_{|\alpha| \leq q} D_\alpha^\alpha g_\alpha(\xi, y) \in C^\infty((0, \alpha); S'(E^n)) \) and the support of \( F(\xi, y) \) is contained in the plane \( \xi'' = 0 \). Thus, by a slight modification of the proof of the fact that every distribution having the point \( x_0 \) as support has the form

\[
f = \sum_{|\alpha| \leq q} a_\alpha D_\alpha \delta (x - x_0)
\]

we have that for \( \phi(\xi', y) \in S(E^{n-r}_\xi \times \mathbb{R}^r) \) and \( \rho(\xi'') \in C_0^\infty(E^{n-r}_\xi) \)

\[
(F, \phi \otimes \rho) = \sum_{|\alpha| \leq q} (F, \phi \otimes (h(\xi')(\xi'')^\alpha/\alpha!)(D_\alpha \rho)(0))
\]

where \( h(\xi'') \in C_0^\infty(E^{n-r}_\xi) \) equals to 1 in a neighborhood of the origin in \( E^{n-r}_\xi \). Since

\[
(F, \phi \otimes h(\xi'')^\alpha/\alpha!)
\]

\[
= \sum_{|\beta| \leq q} \int (-D_\xi \gamma) \beta \phi(\xi', y) d\xi'' dy \int (D_\xi \gamma)^\beta [h(\xi'')(\xi'')^\alpha/\alpha!] g_\alpha(\xi, y) d\xi''
\]

\[
= \langle \sum_{|\beta| \leq q} D_\beta^\beta, \int (-D_\xi \gamma)^\beta [h(\xi'')(\xi'')^\alpha/\alpha!] g_\alpha(\xi, y) d\xi'' \rangle,
\]

if we put

\[
F_\alpha(\xi', y) = \sum_{|\beta| \leq q} D_\beta^\beta, \int (-D_\xi \gamma)^\beta [h(\xi'')(\xi'')^\alpha/\alpha!] g_\alpha(\xi, y) d\xi''
\]

we have

\[
F = \sum_{|\alpha| \leq q} F(\xi', y) \otimes D_\xi \delta,
\]

where \( D_{\xi'} = (D_{\xi_1}, \ldots, D_{\xi_{n-r}}, D_\gamma) \) and \( D_{\xi''} = (D_{\xi_{n-r+1}}, \ldots, D_{\xi_n}) \). Since

\[
\int (-D_\xi \gamma)^\beta [h(\xi'')(\xi''')^\alpha/\alpha!] g_\alpha(\xi, y) d\xi''
\]

is a continuous function in \((\xi', y)\) of polynomial growth, \( F_\alpha(\xi', y) \) belongs to \( S'(E^{n-r}_\xi \times \mathbb{R}^r) \). Further, choosing \( \rho \in C_0^\infty(E^{n-r}_\xi) \) with \( (D_\xi^\beta \rho)(0) = 1 \) and \( (D_\xi^\beta \rho)(0) = 0 \) for \( \beta \neq \alpha \) and \( |\beta| \leq q \), we have, when \( y < \alpha \),
\[
\langle F_a(\xi', y), \phi(\xi') \rangle_{y'} = \sum_{|a| \leq s} \langle F_a(\xi', y) \otimes D_{y'} \phi(\xi') \otimes \rho(\xi'') \rangle
\]
for any \( \phi \in \mathcal{S}(\mathbb{S}^{n-r}_{\xi'}) \), which shows that \( F_a(\xi', y) \in C^\infty((-\infty, 0); \mathcal{S}'(\mathbb{S}^{n-r}_{\xi'})) \).

Q.E.D.

First of all we consider the case where the support of \( \phi(\xi) \hat{\mu}(\xi, y) \) is contained in \( A_0 \) or \( A_r \) \((2 \leq r \leq n+1)\) which is contained in \( X_0 \). Let us denote by \( A \) such an \( A_r \). When the codimension of \( A \) is positive, that is, \( A_r = A \cap (2 \leq r \leq n+1) \) and \( A \subset X_0 \), we denote by \( v(\xi, y) \) the composition of \( \phi(\xi) \mu(\xi, y) \) and the map \( \xi \mapsto (\xi', \xi'' + \mu(\xi')) \) (defined arbitrarily for all \( \xi' \in \Omega \)). It is obvious that the support of \( v(\xi, y) \) is contained in the plane \( \xi'' = 0 \). Thus, by Lemma 2.1, we can write \( v \) as a finite sum:

\[
v(\xi, y) = \sum_{|a| \leq s} v_a(\xi', y) \otimes D^a_{\xi'} \delta,
\]
where \( \delta \) is the Dirac measure at 0 in \( \mathbb{S}^{n-r}_{\xi'} \), \( \alpha = (\alpha_{n-r+2}, \ldots, \alpha_n) \) and \( v_a(\xi', y) \in C^\infty([0, 3]; \mathcal{S}'(\mathbb{S}^{n-r+1}_{\xi'} \times \mathbb{R}^1_+)) \). Let us fix \( \alpha (|\alpha| = s) \) and let \( \psi \) be a \( C^\infty(\mathbb{S}^{n-r+1}_{\xi'}) \) function with \( (D^a \psi)(0) = 1 \) and \( (D^b \psi)(0) = 0 \) for \( \beta \neq \alpha \) and \( |\beta| \leq s \). Hence we have

\[
\langle P(\xi', \mu(\xi'), D_j) v_a(\xi', y), \chi(\xi') \rho(y) \rangle = \langle P(\xi, D_j) \phi(\xi) \hat{\mu}(\xi, y), \chi(\xi') \psi(\xi'' - \mu(\xi')) \rho(y) \rangle = 0
\]
for any \( \chi(\xi') \in C^\infty(\Omega) \) and \( \rho(y) \in \mathcal{S}(\mathbb{R}^1_+) \). This shows that

\[
(2.1) \quad \langle a_0(\xi', \mu(\xi')) v_a(\xi', y), w(\xi', y) \rangle = 0 \tag{2.1}
\]
for any \( w(\xi', y) \in \mathcal{S}_0(\Omega \times \mathbb{R}^1_+) \). Here we wrote for any open set \( \Omega \) in \( \mathbb{S}^k \mathcal{S}_0(\Omega \times \mathbb{R}^1_+) = \{ \phi \in C^\infty(\mathbb{S}^k \times \mathbb{R}^1_+); \text{there is a } \hat{\phi} \in \mathcal{S}(\mathbb{S}^k \times \mathbb{R}^1) \text{ with } \text{supp } \hat{\phi} \subset \Omega \times \mathbb{R}^1 \text{ such that } \phi = \hat{\phi} \mid_{\Omega} \} \). (2.1) shows the support of \( v_a(\xi', y) (|\alpha| = s) \) is contained in \( \{ \xi' \in \Omega; a_0(\xi', \mu(\xi')) = 0 \} \times \mathbb{R}^1_+ \). Since

\[
0 = \langle P(\xi, D_j) \phi(\xi) \hat{\mu}(\xi, y), \chi(\xi') \psi(\xi'' - \mu(\xi')) \rho(y) \rangle = \langle a_0(\xi', \mu(\xi')) v_a(\xi', y), \chi(\xi') \rho(y) \rangle
\]
for any \( \chi(\xi') \in C^\infty_0(\Omega; a_0(\xi', \mu(\xi')) 
eq 0) \) where \( \psi \in C^\infty_0(\mathbb{S}^{n-r+1}_{\xi'}) \) with \( (D^a \psi)(0) = 1 (|\alpha| = s - 1) \) and \( (D^b \psi)(0) = 0 \) for \( \beta \neq \alpha \) and \( |\beta| \leq s \), we have that the support of \( v_a(\xi', y) \) \((|\alpha| = s - 1)\) is contained in \( \{ \xi' \in \Omega; a_0(\xi', \mu(\xi')) = 0 \} \times \mathbb{R}^1_+ \). By repeating the argument we conclude that the support of \( v_a(\xi', y) \) is contained in \( \{ \xi' \in \Omega; a_0(\xi', \mu(\xi')) = 0 \} \times \mathbb{R}^1_+ \) for all \( \alpha (|\alpha| \leq s) \), which shows that
the support of $\phi(\xi)\hat{u}(\xi, y)$ is contained in \{$(\xi', \mu(\xi'))$; $\xi' \in B_{r-1}$ $\} \times R^n_+$ where $B_{r-1} = \{\xi' \in \Omega; a_0(\xi', \mu(\xi')) = 0\}$. Consider the case: $m=0$, that is, $A = A_0$. Since $\langle a_0(\xi)\phi(\xi)\hat{u}(\xi, y), v(\xi, y)\rangle = 0$ for any $v \in S'_0(\mathbb{R}^n \times R^n_+)$, we have that the support of $\phi(\xi)\hat{u}(\xi, y)$ is contained in $B_0 \times R^n_+$, where $B_0 = \{\xi \in \mathbb{R}^n; a_0(\xi) = 0\}$. When $B_r (0 \leq r \leq n)$ is empty, it is obvious that $\phi(\xi)\hat{u}(\xi, y) = 0$. On the other hand, when $B_r (0 \leq r \leq n)$ is not empty, we have, using a theorem due to Hörmander [see Theorem A p-3 in Appendix], the following

**Lemma 2.2.** Let $u \in C^\infty([0, \delta); S'((\mathbb{R}^n)') \cap L^2_{loc}(\mathbb{R}^{n+1}_+))$ be a solution of the equations (1.1) and (1.2). Assume that $\text{supp } \phi(\xi)\hat{u}(\xi, y) \subset A_r (r=0 \text{ or } 2 \leq r \leq n+1)$ for some $\phi(\xi) \in C^\infty(\mathbb{R}^n)$, $A_r$ is contained in $X_0$ when $2 \leq r \leq n+1$ and that $B_r (0 \leq r \leq n)$ is not empty. Put $N_r = \text{the codimension of } B_r \text{ in } \mathbb{R}^n$. Set

$$\Gamma_{r, R} = \{(x, y) \in \Gamma_r; y \geq 0, R < |(x, y)| < 2R\}$$

where $\Gamma_r$ is an open cone in $\mathbb{R}^{n+1}$ which for every analytic manifold $M_r$ and $\xi_0 \in M_r$ contains $(n(\xi_0), 0)$. Here when $r = 0$, $M_0$ is contained $B_0$ and when $1 \leq r \leq n$, $M_r$ is contained in \{$(\xi', \mu(\xi'))$; $\xi' \in B_r$ $\} \text{ and } n(\xi_0)$ denotes some normal of $M_r$ at $\xi_0$ in $\mathbb{R}^n_+$. If $u$ satisfies the condition:

$$\lim_{R \to \infty} R^{-N_r+r} \int_{\Gamma_{r, R}} |u(x, y)|^2 dxdy = 0,$$

then $\mathcal{F}[\phi u] = 0$. Here $\mathcal{F}$ denotes the inverse partial Fourier transform with respect to $\xi$.

**Remark.** (i) The codimension of \{$(\xi', \mu(\xi'))$; $\xi' \in B_r$ $\} \text{ is } N_r+r \text{ in } \mathbb{R}^n$. (ii) When $r = 0$, Lemma 2.2 shows the Main Theorem.

**Proposition 2.3.** Assume that the set \{$(\xi \in \mathbb{R}^n; a_m(\xi) = a_m-1(\xi) = \cdots = a_0(\xi) = 0)$ $\text{is not empty. Let } \Gamma \text{ be an open cone in } \mathbb{R}^{n+1} \text{ and } N' \text{ an integer such that every } w \in S'((\mathbb{R}^{n+1}_+)^n) \cap C^\infty(\mathbb{R}^{n+1}_+) \text{ which is a solution of the equations (1.1) and (1.2) with the condition:

(2.2) $$\lim_{R \to \infty} R^{-N'} \int_{\Gamma_R} |w(x, y)|^2 dxdy = 0$$

is equal to 0. If $M \subset \mathbb{B}^n$ is a $C^\infty$ manifold where $a_m(\xi) = \cdots = a_0(\xi) = 0$ and if $\xi_0 \in M$, then it follows that the closure of $\Gamma$ contains $(n(\xi_0), 0) \neq 0$ and that $N' \leq \text{codim } M$. Here $n(\xi_0)$ denotes some normal of $M$ at $\xi_0$ in $\mathbb{B}^n$.}
Proof. We may assume that $M$ is defined by $\xi'' = \phi(\xi')$ where $\xi' \in \omega \subset \mathcal{B}^{n-1}$, $\phi \in C^\infty(\omega)$ and $\xi_0 = (0, \phi(0))$. Here $l$ denotes the codimension of $M$. For $x \in C^0_0(\omega)$ we put

$$w(x, y) = \int \exp \{i(x' \cdot \xi' + \phi(\xi') \cdot x')\} x(\xi') d\xi' \cdot y^h \rho(y),$$

where $h = \max \{r_j + 1; 1 \leq j \leq p\}$ and $\rho \in C^\infty((-2, 2))$ with $\rho(y) = 1$ for $-y \in [1, 1]$. Since

$$D_k^j(y^h \rho(y)|_{y=0} = 0, 0 \leq k \leq \max \{r_j; 1 \leq j \leq p\},$$

$$a_j(\xi', \phi(\xi')) = 0, 0 \leq j \leq m, \xi' \in \text{supp } x,$$

we have

$$P(D)w(x, y) = \sum_{j=0}^m \int a_j(\xi', \phi(\xi')) \exp \{i(x' \cdot \xi' + \phi(\xi') \cdot x')\} x(\xi') d\xi' \cdot D_k^j(y^h \rho(y)) = 0$$

and

$$B_j(D)w(x, y)|_{y=0} = \sum_{k=0}^r \int b_j(\xi', \phi(\xi')) \exp \{i(x' \cdot \xi' + \phi(\xi') \cdot x')\}$$

$$\cdot x(\xi') d\xi' \cdot D_k^j(y^h \rho(y))|_{y=0} = 0 \quad j = 1, \ldots, p.$$}

Moreover, the condition (2.2) follows from $(A_p-1)$ for $w$ if $\text{codim } M < N'$, which gives a contradiction. If $\mathcal{F}$ contains no normal which has the form $(\mathcal{N}(\xi_0), 0)$ at $\xi_0$ and if $\text{supp } x$ is sufficiently close to $\xi_0$, the condition (2.2) follows from $(A_p-2)$ for any $N$, which completes the proof.

Remark. Proposition 2.3 shows that Lemma 2.2 is very precise.

Next we consider the case where the support of $\phi(\xi)\mathcal{W}(\xi, y)$ is contained in $A_{r+1}$ ($r = 0, 1, \ldots, n$) where $A_{r+1}$ is contained in $X_e$ for some $e \neq 0$ (i.e. $1 \leq e \leq m$) if $1 \leq r \leq n$. We need the following fact due to Wakabayashi [16, Lemma 2.10].

Lemma 2.4. Put

$$P(\xi, \lambda) = \lambda^l + a_0(\xi) \lambda^{l-1} + \cdots + a_l(\xi)$$

where $a_0(\xi)$ is real analytic in a connected open set $V (\subset \mathcal{B}^n)$. Then there exists a real analytic function $D(\xi)$ ($\equiv 0$) in $V$ such that the roots of $p(\xi, \lambda) = 0$ in $\lambda$ have constant multiplicities for $\xi$ and are real analytic functions of $\xi$ in each connected component of $\{\xi \in V; D(\xi) \neq 0\}$.

When $1 \leq r \leq n$, by assumption we have $A_{r+1} \subset X_e$ ($e \neq 0$) and $P(\xi', \mu(\xi')), \lambda$$= \sum_{l=0}^r (a_j(\xi', \mu(\xi')) \lambda^j, a_j(\xi', \mu(\xi'))) \neq 0$ for $\xi' \in \mathcal{O}$. So it follows from Lemma 2.4 that there exists a real analytic function $D(\xi')$ ($\equiv 0$) in $\mathcal{O}$ such that the roots of
$P(\xi',\mu(\xi'),\lambda)=0$ in $\lambda$ have constant multiplicities and are real analytic functions of $\xi'$ in each connected component of $\{\xi' \in \Omega; D(\xi') \neq 0\}$. Thus denoting by $V_0$ each connected component of $\{\xi' \in \Omega; D(\xi') \neq 0\}$, we have that the imaginary parts of the roots of the equation $P(\xi',\mu(\xi'),\lambda)=0$ in $\lambda$ are identically zero or real analytic functions of $\xi'$ in $V$. Denote the latter by $\text{Im} \lambda_j(\xi')$, $j=1,\ldots,k$, and put
\[
A_{V_0,\text{Im}} = \{\xi' \in V_0; \text{Im} \lambda_j(\xi') = 0 \text{ for some } j \in \{1,\ldots,k\}\}.
\]
It is obvious that $A_{V_0,\text{Im}}$ is empty or a real analytic set. $V_0-A_{V_0,\text{Im}}$ may be decomposed into connected components $\{W_{0,j}\}$. We write $W_{0,0}=W_0$ for the sake of simplicity. Thus when $\xi' \in W_0$, the roots of the equation $P(\xi',\mu(\xi'),\lambda)=0$ in $\lambda$ have constant multiplicity and split into three classes: real roots, those with positive imaginary part and those with negative imaginary part. We denote those by $\lambda_0^j(\xi'),j=1,\ldots,a$, $\lambda_j^+(\xi'),j=1,\ldots,b$ and $\lambda_j^-(\xi'),j=1,\ldots,c$ where $\text{Im} \lambda_0^j(\xi')=0$, $\text{Im} \lambda_j^+(\xi')>0$ and $\text{Im} \lambda_j^-(\xi')<0$, and then we have
\[
P(\xi',\mu(\xi'),\lambda) = \sum_{j=0}^a a_j(\xi',\mu(\xi'))\lambda^j,
\]
where $\sum_{j=0}^a a_j(\xi',\mu(\xi'))=0, \xi' \in W_0$.

Put
\[
P^0(\xi',\lambda) = \prod_{j=1}^a (\lambda-\lambda_0^j(\xi'))^{\alpha_j} = \sum_{j=0}^a a_j^0(\xi')\lambda^j,
\]
\[
P^+(\xi',\lambda) = \prod_{j=1}^b (\lambda-\lambda_j^+(\xi'))^{\beta_j} = \sum_{j=0}^b a_j^+(\xi')\lambda^j,
\]
\[
P^-(\xi',\lambda) = \prod_{j=1}^c (\lambda-\lambda_j^-(\xi'))^{\gamma_j} = \sum_{j=0}^c a_j^-(\xi')\lambda^j,
\]
where $\bar{a}=\sum_{j=1}^a \alpha_j$, $\bar{b}=\sum_{j=1}^b \beta_j$, $\bar{c}=\sum_{j=1}^c \gamma_j$.

Let $Z$ be any open set such that $Z \cap A_{r+1}$ is contained in $A_{W_0} = \{\xi',\mu(\xi'); \xi' \in W_0\}$ when $1 \leq r \leq n$. Let $\phi(\xi)$ be a $C_\sigma^0(Z)$ function and $v(\xi,y)$ be the composition of $\phi(\xi)\phi(\xi)\delta(\xi,y)$ and the map $\xi \mapsto (\xi',\xi''+\mu(\xi'))$ (defined arbitrarily for $\xi' \in W_0$). Since $\text{supp} \phi(\xi)\phi(\xi)\delta(\xi,y) \subset A_{W_0} \times \mathbb{R}^a$, the support of $v(\xi,y)$ is contained in the plane $\{\xi \in \mathbb{R}^a; \xi''=0\} \times \mathbb{R}^a$, and we can write $v$ as a finite sum:
\[
v(\xi,y) = \sum_{|\alpha|=s} v_\alpha(\xi',y) \otimes D^{\alpha} \delta,
\]
using Lemma 2.1. Let us fix $\alpha (|\alpha|=s)$. Let $\psi$ be a $C_\sigma^0(\mathbb{R}^a)$ function with $(D^\alpha \psi)(0)=1$ and $(D^\beta \psi)(0)=0$ for $\beta \neq \alpha$ and $|\beta| \leq s$. We have
\[
\begin{align*}
(2.3) & \quad \langle P(\xi', \mu(\xi'), D_\lambda) v_\omega(\xi', y), \chi(\xi') \rho(y) \rangle \\
& = \langle P(\xi, D_\lambda) \phi(\xi) \phi(\xi) \Delta(\xi, y), \chi(\xi'') \rho(\xi'' - \mu(\xi')) \rho(y) \rangle = 0 , \\
(2.4) & \quad \langle B_j(\xi', \mu(\xi'), D_\lambda) v_\omega(\xi', y) \mid_{y=0}, \chi(\xi') \rangle \\
& = \langle B_j(\xi, D_\lambda) \phi(\xi) \phi(\xi) \Delta(\xi, y) \mid_{y=0}, \chi(\xi'') \rho(\xi'' - \mu(\xi')) \rangle = 0 \\
& \quad j = 1, \ldots, p ,
\end{align*}
\]

for any \( \chi(\xi') \in C_c^0(W_0) \) and \( \rho(y) \in S(\mathbb{R}^4_+) \). In order to simplify the notation,

\[
\begin{align*}
\phi(\xi', y) &= \begin{cases} 
\phi(\xi) \Delta(\xi, y) & \text{when } r = 0 \\
v_\omega(\xi', y) & \text{when } 1 \leq r \leq n
\end{cases} \\
W &= \begin{cases} 
A_1 & \text{when } r = 0 \\
W_0 & \text{when } 1 \leq r \leq n
\end{cases} \\
P^0(\xi', D_\lambda) &= \begin{cases} 
P^0(\xi, D_\lambda) & \text{when } r = 0 \\
P^0(\xi', D_\lambda) & \text{when } 1 \leq r \leq n
\end{cases} \\
P^\pm(\xi', D_\lambda) &= \begin{cases} 
P^\pm(\xi, D_\lambda) & \text{when } r = 0 \\
P^\pm(\xi', D_\lambda) & \text{when } 1 \leq r \leq n
\end{cases} \\
B_j(\xi', D_\lambda) &= \begin{cases} 
B_j(\xi, D_\lambda) & \text{when } r = 0 \\
B_j(\xi', \mu(\xi'), D_\lambda) & \text{when } 1 \leq r \leq n
\end{cases} .
\end{align*}
\]

Lemma 2.5. Let \( \psi, W, P^0 \) and \( P^\pm \) be as in the above statement. Then

\[
(2.5) \quad \langle P^0(\xi', D_\lambda) P^+(\xi', D_\lambda) \psi(\xi', y), w(\xi', y) \rangle = 0
\]

for any \( w(\xi', y) \in S_0(W \times \mathbb{R}^4_+) \).

Proof. Repeating integration by parts, we have

\[
(2.6) \quad 0 = \langle P(\xi', \mu(\xi'), D_\lambda) \psi(\xi', y), \chi(\xi') \rho(y) \rangle \\
= i \sum_{k=0}^{\tilde{\gamma}} \left( \sum_{k=0}^{\tilde{\gamma}} a_j(\xi') D_j^{\tilde{\gamma}-k} P^0(\xi', D_\lambda) P^+(\xi', D_\lambda) \psi(\xi', y) \right) \mid_{y=0} \\
\chi(\xi') D_j^{\tilde{\gamma}} \rho(y) \mid_{y=0} + \langle P^0(\xi', D_\lambda) P^+(\xi', D_\lambda) \psi(\xi', y) \rangle .
\]

Here we wrote

\[
P^-(\xi', D_\lambda) = \sum_{j=0}^{\tilde{\gamma}} a_j(\xi') D_j .
\]

Since the roots of the equation \( P^-(\xi', \lambda) = 0 \) in \( \lambda \) have all positive imaginary part when \( \xi \in W \), we can define the Lopatinski determinant of the system \( \{ P^-(\xi' D_\lambda), D_j^{\tilde{\gamma}-1}, j = 1, \ldots, \tilde{\gamma} \} \) by


\[ L^- (\zeta') = \det ((2\pi i)^{-1} \int \lambda^{j+k-2j} (P^{-}(\zeta', \lambda))^{-1} d\lambda)_{j,k=1, \ldots, \varepsilon} = 1. \]

Since \( L^- (\zeta') = 1 \neq 0 \) in \( W \), for any \( w(\zeta', y) \in S_0(W \times \mathbb{R}_+^\varepsilon) \) where exists a \( S_0(W \times \mathbb{R}_+^\varepsilon) \) function \( z(\zeta', y) \) such that

\[ P^{-}(\zeta', D_j)z(\zeta', y) = w(\zeta', y), \quad D_j z(\zeta', y)|_{y=0} = 0, \quad j = 0, \ldots, \varepsilon - 1. \]

Q.E.D.

**Lemma 2.6.** Let \( \psi, W, P^0 \) and \( P^+ \) be as in Lemma 2.5. If \( \psi \) satisfies the formula:

\[ (2.7) \quad \left< P^0(\zeta', D_j)P^+(\zeta', D_j)\psi(\zeta', y), w(\zeta', y) \right> = \left< \psi(\zeta', y), P^0(\zeta', -D_j)P^+(\zeta' - D_j)w(\zeta', y) \right>, \]

for any \( w \in S_0(U \times \mathbb{R}_+^\varepsilon) \) then

\[ \left< \psi(\zeta', y), z(\zeta')\rho(y) \right> = 0 \]

for any \( z(\zeta') \in C_0^\infty(U) \) and \( \rho \in C_0^\infty(\mathbb{R}_+^\varepsilon) \). Here \( U \) is any open subset of \( W \).

**Proof.** Let \( \bar{U} \) be any open subset of \( U \) such that the closure of \( \bar{U} \) is contained in \( U \) and compact. It is sufficient to prove that

\[ (2.8) \quad \left< \psi(\zeta', y), z(\zeta')\rho(y) \right> = 0, \]

for any \( z(\zeta') \in C_0^\infty(\bar{U}) \) and \( \rho(y) \in C_0^\infty(\mathbb{R}_+^\varepsilon) \). From (2.7) and Lemma 2.5 we have

\[ (2.9) \quad \left< \psi(\zeta', y), P^0(\zeta', -D_j)P^+(\zeta', -D_j)w(\zeta', y) \right> = 0, \]

for any \( w(\zeta', y) \in S_0(U \times \mathbb{R}_+^\varepsilon) \). Since we can choose \( \tau \) so large that \( P^0(\zeta', \lambda - i\tau)P^+(\zeta', \lambda - i\tau) \neq 0 \) when \( \zeta' \in \bar{U} \) and \( \lambda \) is real, we can put

\[ w(\zeta', y) = \chi(\zeta') \int \exp \left\{-i(\lambda - i\tau) \cdot y\right\} \hat{\omega}[\rho_0(\lambda)] [P^0(\zeta', \lambda - i\tau)P^+(\zeta', \lambda - i\tau)]^{-1} d\lambda \]

\[ y \geq 0, \]

for any \( \chi(\zeta') \in C_0^\infty(\bar{U}) \) and \( \rho \in C_0^\infty(\mathbb{R}_+^\varepsilon) \) where \( \rho_0 \) is compactly supported \( C^\infty \) extension of \( \rho \) to \( y < 0 \) (see Seeley [13]), and then \( w(\zeta', y) \) belongs to \( S_0(\bar{U} \times \mathbb{R}_+^\varepsilon) \).

We obtain, using (2.9),

\[ \left< \psi(\zeta', y), e^{-\gamma y} \chi(\zeta')\rho(y) \right> = 0 \]

for any \( \chi(\zeta') \in C_0^\infty(\bar{U}) \) and \( \rho(y) \in C_0^\infty(\mathbb{R}_+^\varepsilon) \).

Q.E.D.

2) Here \( \hat{\omega} \) denotes the inverse Fourier transform with respect to \( y \).
Next lemma which is inspired by an idea due to Hörmander [7] plays an essential role in this section.

**Lemma 2.7.** Let \( \hat{\nu}, W, P^0 \) and \( P^+ \) be as in Lemma 2.5. Put \( M_j = \{ (\xi', \mu(\xi'), \lambda_j^0(\xi'); \xi' \in W) \}, j = 1, \ldots, a \). Let \( \delta_j, j = 1, \ldots, a, \) be non-negative integer such that \( \delta_j \leq \alpha_j \). Assume that \( \theta_j = (\theta^0_j, \ldots, \theta^{a+1}_j) \in \mathbb{R}^{a+1} \) is a normal of \( M_j \) at \( (\xi_0', \mu(\xi_0), \lambda_j^0(\xi_0)) \) \((\xi_0' \in W \text{ and } \delta_j = 1)\) and \( \epsilon > 0 \). If \( u \) belongs to \( L_{\text{loc}}^2(\mathbb{R}^{a+1}) \) and satisfies the condition:

\[
\lim_{|\xi,y| \to \infty} R^{-2(\delta_j - \delta_j + r + 1)} \int_{\xi,y \neq 0} |u(x, y)|^2dydx = 0 \quad \text{for } \delta_j \geq 1 \, ,
\]

then there exists a small neighborhood \( \omega \) of \( \xi_0 \) contained in \( W \) such that

\[
\sum_{i=0}^{d} \sum_{k=0}^{i-1} \frac{(i-1-k)!}{(i-1-k-\nu)!} b_i(\xi') \lambda_j^0(\xi')^{i-1-k-\nu} D_j^\nu Q(\xi', \nu(\xi', y), y) \big|_{y=0},
\]

for any \( \chi(\xi') \in \mathbb{C}_0^\infty(\omega) \) and \( 0 \leq \nu \leq \delta_j - 1 \) where we interpret that \( \lambda_j^0(\xi')^{i-1-k-\nu} \equiv 0 \) if \( i - k - 1 - \nu < 0 \). Here \( u \) is as in the first place of this section and

\[
d = \sum_{j=1}^{a} \delta_j, \quad \prod_{j=1}^{a} (\lambda - \lambda_j^0(\xi'))^{h_j} = \sum_{j=0}^{a} b_j(\xi') \lambda_j^0, \quad b_j(\xi') = 1, \quad Q(\xi', \lambda) = \prod_{j=1}^{a} (\lambda - \lambda_j^0(\xi'))^{a_j - \delta_j} \cdot P^+(\xi', \lambda), \quad \xi' \in W.
\]

**Proof.** Let us write \( \alpha_j = \alpha, \delta_j = \delta, \theta_j = \theta_k \) and \( \lambda_j^0(\xi') = \lambda(\xi') \). Let \( \omega_1 \) be an open neighborhood of \( \xi_0' \) with \( \omega_1 \subset \subset W \). Put

\[
m = \text{Max} \{ \text{sup} (|\partial x_k/\partial x_k(\xi')|), \text{sup} (|\partial x_k/\partial x_k(\xi')|) ; h = 1, \ldots, r, j = n-r+1, \ldots, n, i = 1, \ldots, a \}.
\]

Let \( \epsilon_1 \) be a positive number determined later on and \( \theta'' = (\theta'_{n+1}, \ldots, \theta'_r) \). If \( \theta_{n+1} = 0 \), we choose \( \rho \in C^\infty_0(\mathbb{R}^r) \) so that \( \text{supp} \rho \subset \{ y \in \mathbb{R}^r ; |y| < \epsilon_1 \} \) and \( (D_{\nu}^r \rho)(0) = 0 \) for \( \nu \geq 1 \). If \( \theta_{n+1} > 0 \), we choose \( \rho \in C^\infty_0(\mathbb{R}^r) \) so that \( \mathbb{R}^r \cap \text{supp} (D_{\nu}^r \rho) \subset \{ y \in \mathbb{R}^r ; |y - \theta_{n+1}| < \epsilon_1 \} \) for \( \nu \geq 1 \) and \( \rho(0) = 1 \). In the latter case, we choose \( \epsilon_1 \) so small that \( \theta_{n+1} - \epsilon_1 > 0 \) and then \( (D_{\nu}^r \rho)(0) = 0 \) for \( \nu \geq 1 \). Let \( \epsilon_2 \) be some positive number determined later on such that

\[
\epsilon_2^2 - 2((r - 1)m^2 + \sum_{j=n-r+1}^{a+1} |\theta_j|) \epsilon_2 = 2c > 0.
\]

Then using the inequality:

3) The notation: \( \omega_1 \subset \subset W \), means that the closure of \( \omega_1 \) is compact and contained in \( W \).
\[ |a + b|^2 > 2^{-1} |a|^2 - |b|^2 , \]

and the formula:

\[ \theta_k + \sum_{j=n-r+1}^{n} \partial_{\mu_j} / \partial \xi_k(\xi^j) \theta_j + \partial \lambda / \partial \xi_k(\xi^j) \theta_{n+1} = 0 , \quad h = 1, \ldots, r , \]

we have

\[ \sum_{k=1}^{n-r} (x_k + \sum_{j=n-r+1}^{n} \partial_{\mu_j} / \partial \xi_k(\xi^j)x_j + \partial \lambda / \partial \xi_k(\xi^j)y)^2 \geq 2^{-1}|(x, y)|^2 \{ \sum_{k=1}^{n-r} x_k/|(x, y) - \theta_k|^2 - \cdots |\} , \quad \varepsilon' \in \omega_1 . \]

Here

\[ |\cdots|^2 \leq 2 \sum_{k=1}^{n-r} \{ \sum_{j=n-r+1}^{n} |\partial_{\mu_j} / \partial \xi_k(\xi^j)|^2 |x_j/|(x, y)| - \theta_j|^2 \]
\[ + \sum_{j=n-r+1}^{n} |\partial \mu_j / \partial \xi_k(\xi^j) - \partial \mu_j / \partial \xi_k(\xi^j)|^2 |\theta_j|^2 \]
\[ + |\partial \lambda / \partial \xi_k(\xi^j)|^2 |y/|(x, y)| - \theta_{n+1}|^2 + |\partial \lambda / \partial \xi_k(\xi^j) - \partial \lambda / \partial \xi_k(\xi^j)|^2 |\theta_{n+1}|^2 \} . \]

If we choose \( \omega \subset \subset \omega_1 \) so that when \( \varepsilon' \in \omega \)

\[ |\partial \lambda / \partial \xi_k(\xi^j) - \partial \lambda / \partial \xi_k(\xi^j)| < \varepsilon_1 , \quad |\partial \mu_j / \partial \xi_k(\xi^j) - \partial \mu_j / \partial \xi_k(\xi^j)| < \varepsilon_1 , \]
\[ j = n-r+1 , \ldots, n , \]

and put

\[ V'' = \{(x'', y) \in \mathbb{R}^{r+1} ; |(x'', y)|/|(x, y)| - (\theta', \theta_{n+1})| < \varepsilon_1 \} , \]
\[ V = \{(x, y) \in \mathbb{R}^{r+1} ; |x'|/|(x, y)| - \theta'| < \varepsilon_2 , (x'', y) \in V'' \} , \]

we have

\[ |\cdots|^2 \leq 2(n-r)((r+1)m^2 + \sum_{j=n-r+1}^{n+1} |\theta_j|^2 \varepsilon_1^2 , \quad (x'', y) \in V'' . \]

Thus, we have, using (2.12),

\[ \sum_{k=1}^{n-r} (x_k + \sum_{j=n-r+1}^{n} (\partial \mu_j / \partial \xi_k(\xi^j)x_j + (\partial \lambda / \partial \xi_k(\xi^j)y)^2 \geq c |(x, y)|^2 , \]

when \((x'', y) \in V'', (x, y) \in V\) and \( \varepsilon' \in \omega \). It is obvious that \( V \) is a conic neighborhood of \( \theta \) in \( \mathbb{R}^{r+1} \). If we choose \( \varepsilon_1 \) and \( \varepsilon_2 \) so that \( \varepsilon_1 \) and \( \varepsilon_2 \) satisfy (2.12) and \( 0 < \varepsilon_1 , \varepsilon_2 < \varepsilon/2\sqrt{n+1} \), then

\[ \int \exp \{ -i(x' \cdot \varepsilon'' + x'' \cdot \xi(\varepsilon'') + y \cdot \lambda(\varepsilon')) \} x(\xi')d\xi' (D_{\theta'}^0 \sigma)(x''/\mathbb{R})(D_{\theta''}^0 \rho)(y/\mathbb{R}) \]

is rapidly decreasing for \( r \geq 1, x \in C_0^\infty(\omega) \) and \( \sigma \in C_0^\infty(\{x'' \in \mathbb{R}^r ; |x'' - \theta''| < \varepsilon_1 \}) \)

when \(|(x, y)|/R - \theta | > \varepsilon/2 \). When \( r \geq 1, \) let \( \sigma \) be a \( C_0^\infty(\mathbb{R}^r) \) function such that \( \text{supp} \sigma \subset \{x'' \in \mathbb{R}^r ; |x'' - \theta''| < \varepsilon_1 \} \) and

\[ (1/2\pi)^r \int \left( -x'' \right)^n \sigma(x')dx'' = 1 , \]

(2.13)
Repeating integration by parts, we have

(2.15) \[ 0 = \langle P^0(\xi', D_x)P^+(\xi', D_x)\psi(\xi', y), \chi(\xi')\rho(y/R)(-iy)^{\gamma}e^{-i\lambda(\xi'y)} \rangle \]

\[ = \sum_{j=1}^{d} \sum_{k=1}^{r} \left( \begin{array}{c} j-1-k \vspace{1mm} \\ j-1-k-\nu \end{array} \right) \lambda(\xi')^{j-1-k-\nu}D_{\xi}^{j}Q(\xi', D_x)\psi(\xi', y) \bigg|_{y=0} \chi(\xi') \]

\[ + \langle Q(\xi', D_x)\psi(\xi', y), \Pi \sum_{j=1}^{d} \left( -D_x - \lambda(\xi') \right)^{\nu} \chi(\xi') \rho(y/R)(-iy)^{\gamma}e^{-i\lambda(\xi'y)} \rangle. \]

Since

\[ \Pi \sum_{j=1}^{d} \left( -D_x - \lambda(\xi') \right)^{\nu} \chi(\xi') \rho(y/R)e^{-i\lambda(\xi'y)}(-iy)^{\gamma} = \chi(\xi') \Pi \sum_{j=1}^{d} \left( -D_x - \lambda(\xi') \right)^{\nu} \sum_{\delta=1}^{\nu} c_{\delta}(-iy/R)^{\gamma-\lambda}(D_{\xi}^{\delta-1}\rho) \]

\[ \cdot (y/R)(1/R)^{\gamma-\lambda}e^{-i\lambda(\xi'y)} \]

and

\[ (D_{\xi}^{\delta-1}\rho)(0) = 0, \quad \delta - I \geq 1, \]

where \( c_{\delta} \) is some constant, we have

(2.16) \[ \langle Q(\xi', D_x)\psi(\xi', y), \Pi \sum_{j=1}^{d} \left( -D_x - \lambda(\xi') \right)^{\nu} \chi(\xi') \rho(y/R)(-iy)^{\gamma}e^{-i\lambda(\xi'y)} \rangle \]

\[ = \langle \psi(\xi', y), P^0(\xi', -D_x)P^+(\xi', -D_x)[\chi(\xi')\rho(y/R)(-iy)^{\gamma}e^{-i\lambda(\xi'y)}] \rangle. \]

When \( r \geq 1 \), noting that \( \psi = \nu \) and writing

\[ \partial(\xi') = (1/2\pi)^{r} \int e^{-ix'\xi'} \sigma(x')dx', \]

\[ P^0(\xi', -D_x)P^+(\xi', -D_x) = T(\xi', -D_x), \]

\[ (\partial/\partial \lambda) T(\xi', \lambda) = T^{(\lambda)}(\xi', \lambda), \]

we have, using (2.13) and (2.14),

\[ \langle \psi(\xi', y), T(\xi', -D_x)\chi(\xi')e^{-i\lambda(\xi'y)}(-iy)^{\gamma}\rho(y/R) \rangle \cdot R^{[\sigma]} \]

\[ = \sum_{r=1}^{\nu} \left( r^{-1} \langle \nu_{\sigma}(\xi', y), T^{(\sigma)}(\xi', \lambda(\xi'))\chi(\xi')D_{\xi}(\xi', \lambda(\xi'))(\frac{-iy}{R})^{\gamma-1} \right) \]

\[ \cdot (y/R)(1/R)^{\gamma-\lambda}e^{-i\lambda(\xi'y)} \cdot R^{[\sigma]} \]

\[ = \sum_{\sigma} \sum_{r=1}^{\nu} \left( r^{-1} \cdot \sum_{\delta=1}^{\nu} \langle \nu_{\delta}(\xi', y) \otimes D_{\xi}^{\delta}, c_{r}T^{(\nu)}(\xi', \lambda(\xi'))\chi(\xi') \cdot (-iy/R)^{\gamma-1}(D_{\xi}^{\delta-1}\rho)(0) \right) \]

\[ \cdot (y/R)(1/R)^{\gamma-\lambda}e^{-i\lambda(\xi'y)} \cdot R^{[\sigma]} \]

\[ = \sum_{r=1}^{\nu} \sum_{\sigma} \left( r^{-1} \cdot \sum_{\delta=1}^{\nu} \langle \nu_{\delta}(\xi', y) \otimes D_{\xi}^{\delta}, c_{r}T^{(\nu)}(\xi', \lambda(\xi'))\chi(\xi') \cdot (-iy/R)^{\gamma-1}(D_{\xi}^{\delta-1}\rho)(0) \right) \]

\[ \cdot (y/R)(1/R)^{\gamma-\lambda}e^{-i\lambda(\xi'y)} \cdot R^{[\sigma]} \]

\[ = \sum_{r=1}^{\nu} \sum_{\sigma} \left( r^{-1} \cdot \sum_{\delta=1}^{\nu} \langle \nu_{\delta}(\xi', y) \otimes D_{\xi}^{\delta}, c_{r}T^{(\nu)}(\xi', \lambda(\xi'))\chi(\xi') \cdot (-iy/R)^{\gamma-1}(D_{\xi}^{\delta-1}\rho)(0) \right) \]

\[ \cdot (y/R)(1/R)^{\gamma-\lambda}e^{-i\lambda(\xi'y)} \cdot R^{[\sigma]} \]

\[ = \sum_{r=1}^{\nu} \sum_{\sigma} \left( r^{-1} \cdot \sum_{\delta=1}^{\nu} \langle \nu_{\delta}(\xi', y) \otimes D_{\xi}^{\delta}, c_{r}T^{(\nu)}(\xi', \lambda(\xi'))\chi(\xi') \cdot (-iy/R)^{\gamma-1}(D_{\xi}^{\delta-1}\rho)(0) \right) \]

\[ \cdot (y/R)(1/R)^{\gamma-\lambda}e^{-i\lambda(\xi'y)} \cdot R^{[\sigma]} \]
where \(c_i\) are some constants. Therefore, since \(|\alpha| \geq 0\), it is sufficient to show that

\[
\lim_{R \to \infty} \langle \phi(\xi) \phi_i(\xi) \hat{u}(\xi, y), T^{(\alpha)}(\xi', \lambda(\xi')) \chi(\xi') \sigma(-R(\xi'' - \mu(\xi'))e^{-i\lambda(\xi')})
\cdot (-iy/R)^{-\gamma-i}(D_{-\xi'}^{-1} \rho)(y/R)(1/R)^{-\gamma} \rangle = 0.
\]

When \(r = 0\), it is sufficient to show that

\[
\lim_{R \to \infty} \langle \phi(\xi) \hat{u}(\xi, y), T^{(\alpha)}(\xi, \lambda(\xi)) \chi(\xi) e^{-i\lambda(\xi)}(-iy/R)^{-\gamma-i}(D_{-\xi}^{-1} \rho)(y/R)(1/R)^{-\gamma} \rangle
\]

\[= 0.\]

Note that \(\tau - l \geq 1, \tau - l \geq \alpha_j - \nu \geq \alpha_j - (\beta_j - 1) \geq 1\). Let us show (2.17) and (2.17)'. We denote by \(H_0\) the left-hand side of (2.17) and (2.17)'. Put

\[
\begin{align*}
\mathcal{F}[\hat{u}](\xi, y) &= \frac{\mathcal{F}_{\xi}[\phi \hat{u}](\xi, y)}{\mathcal{F}_{\xi}[\phi \hat{u}](\xi, y)} \quad y \geq 0 \\
&= 0 \quad y < 0
\end{align*}
\]

where \(\phi = \phi(\xi) \phi_i(\xi)\) when \(1 \leq r \leq n\). Define \(v\) by \(\mathcal{F}[v](\xi, \lambda) = \phi(\xi) \psi(\xi, \lambda) \mathcal{F}[u_0](\xi, \lambda)\), where \(\psi \in C_0^\infty(\mathbb{S}^{n+1})\) is equal to 1 in a neighborhood of \((\xi_0, \mu(\xi_0), \lambda_0^2(\xi_0))\). Put \(I = I_1 + I_2\) where

\[
\begin{align*}
I_1 &= \langle \mathcal{F}[\hat{v}](\xi, \lambda) \mathcal{F}[u_0](\xi, \lambda), T^{(\alpha)}(\xi', \lambda(\xi')) \chi(\xi') \sigma(-R(\xi'' - \mu(\xi'))R(\lambda - \lambda(\xi'))) \rangle \\
I_2 &= \langle \mathcal{F}[\hat{v}](\xi, \lambda) (1 - \psi(\xi, \lambda)) \mathcal{F}[u_0](\xi, \lambda), T^{(\alpha)}(\xi', \lambda(\xi')) \chi(\xi') \sigma(-R(\xi'' - \mu(\xi'))R(\lambda - \lambda(\xi'))) \rangle.
\end{align*}
\]

Since we can choose \(\omega(\subset \subset \omega_1)\) and \(\psi\) so that there exists a positive constant \(c\) such that

\[|\lambda - \lambda(\xi')| \geq c\] when \((\xi, \lambda) \in \text{supp}[1 - \psi(\xi, \lambda)] \cap \omega \times \mathbb{S}^{n+1}_{\lambda^2}\),

we obtain, using the fact that \(\mathcal{F}[\hat{v}] \in S\),

4) \(\mathcal{F}_{\xi}[\hat{u}] = (1/2\pi)^n \int e^{i\xi \cdot \xi} \hat{u}(\xi, y) d\xi\).

5) \(\mathcal{F}[v](\xi, \lambda) = \int_{-\infty}^{+\infty} e^{-i\lambda(\xi')} e^{i\xi \cdot \xi} v(x, y) dx dy\).
for any $N$ and some positive constants $C_N$ and $C$ when $(\xi, \lambda) \in \text{supp}[1 - \psi(\xi, \lambda)] \cap \omega \times B_{\xi, \lambda}^{\pm 1}$. Therefore we have
\[
\lim_{R \to \infty} I_2 = 0 .
\]
On the other hand, since $\phi(\xi)\psi(\xi, \lambda)\mathcal{H}[\mathcal{H}(\xi, \lambda) \in \mathcal{E}'(\mathcal{L}_1^{k+1})$, we have
\[
(2.18) \quad |v(x, y)| \leq C(1 + |x| + |y|)^k,
\]
for some positive constants $C$ and $k$, and $v \in L^2_{\text{loc}}(\mathcal{L}_1^{k+1})$. Setting
\[
J(x, y) = \int \exp \left\{ -i(x' \cdot \xi' + x' \cdot \mu(\xi') + y \cdot \lambda(\xi')) \right\} T^{(\gamma)}(\xi', \mu(\xi')x(\xi')d\xi',
\]
we have
\[
\prod T^{(\gamma)}(\xi', \lambda(\xi'))x(\xi')d(-R(\xi'' - \mu(\xi'))\mathcal{H}[\mathcal{H}(\xi - \lambda(\xi'))]e^{-(x+\xi+y\lambda)}d\xi d\lambda
= R^{-(r+1)} J(x, y)(-iy/R)^{r-1}(D_{y}^{-1} \rho)(y/R)\sigma(x'/R).
\]
We obtain, using Cauchy-Schwarz's inequality,
\[
(2.19) \quad |I_1| < (1/R)^{r+1} \left( \prod \int_{|x, y| < R^\theta < \pi/2} |v(x, y)|^2 dxdy \right)^{1/2}
\]
\[
\cdot \left( \prod \int_{|x, y| < R^\theta < \pi/2} |J(x, y)|^2 |(y/R)^{r-1}(D_{y}^{-1} \rho)(y/R)\sigma(x'/R)|^2 dxdy \right)^{1/2}
\]
\[
+ (1/R)^{-\gamma} \prod \int_{|x, y| < R^\theta < \pi/2} |v(x, y)| \cdot |J(x, y)(y/R)^{r-1}(D_{y}^{-1} \rho)(y/R)\sigma(x'/R)| dxdy
\]
\[
= I_3(R) + I_4(R).
\]
It follows from Theorem $A_{\nu-1}$ and Lemma $A_{\nu-2}$ that
\[
\prod \int_{|x, y| < R^\theta < \pi/2} |J(x, y)|^2 |(y/R)^{r-1}(D_{y}^{-1} \rho)(y/R)\sigma(x'/R)|^2 dxdy \leq CR^{r+1},
\]
\[
\lim_{R \to \infty} R^{-k} \prod \int_{|x, y| < R^\theta < \pi/2} |v(x, y)|^2 dxdy \leq C \lim_{R \to \infty} R^{-k} \prod \int_{|x, y| < R^\theta < \pi/2} |u(x, y)|^2 dxdy
\]
for every $k \in \mathcal{R}$. Hence we have, using the fact: $\gamma - \nu \geq \alpha + \beta + 1$,
\[
I_3(R) \leq C \{ R^{-(2(\alpha + \beta) + r+1)} \prod \int_{|x, y| < R^\theta < \pi/2} |v(x, y)|^2 dxdy \}^{1/2},
\]
and then it follows from (2.10) that \( \lim_{R \to \infty} I_3(R) = 0 \). On the other hand, using (2.18), we have
\[
|v(x, y)J(x, y)(-iy/R)^{\gamma-1}(D_x^{\gamma-1}\rho)(y/R)\sigma(x''/R)| \leq CR^{-1}(1 + |x| + |y|)^{-(\alpha+2)}
\]
when \( |(x, y)/R - \theta| \geq \epsilon/2 \). Therefore we have \( \lim_{R \to \infty} I(R) = 0 \). Q.E.D.

For later reference, we now present some facts concerning elementary algebra.

Lemma 2.8. 1) Put \( f_j(\lambda) = a_m\lambda^j + a_{m-1}\lambda^{j-1} + \cdots + a_{m-j} \) and \( f^*_j(\lambda) = (d/d\lambda)^j f_j \). Then we have
\[
(2.20)
\]

\[
\begin{align*}
&f_0(\lambda), f_1(\lambda), \ldots, f_{q-1}(\lambda) \\
&f_0^{(\alpha)}(\lambda), f_1^{(\alpha)}(\lambda), \ldots, f_{q-1}^{(\alpha)}(\lambda) \\
&f_0(\lambda), f_1(\lambda), \ldots, f_{q-1}(\lambda) \\
&f_0^{(\alpha)}(\lambda), f_1^{(\alpha)}(\lambda), \ldots, f_{q-1}^{(\alpha)}(\lambda)
\end{align*}
\]

\[
= (-1)^{q+1} \sum_{i=1}^{q+1} \alpha_i \{ \Pi_{j=1}^q (j-1)!/\Pi_{i+1}^q (\alpha_i - j+1)! \}
\cdot a^*_m \prod_{i=1}^q \prod_{j=i+1}^q (\lambda_i - \lambda_j)^{\alpha_i}. \]

Here \( q = \sum_{i=1}^q \alpha_i \), \( 0 < q \leq m \) and \( a_m \neq 0 \).

2) (cf. Hörmander [5]) Let \( k(\lambda) = \lambda^\mu + a_{\mu-1}\lambda^{\mu-1} + \cdots + a_0 \) be a polynomial with constant coefficients of order \( \mu \). Let \( q_\nu(\lambda), \nu = 1, \ldots, \mu \), be some other polynomials with constant coefficients. Let \( \lambda_1, j = 1, \ldots, l \), be the roots of the equation \( k(\lambda) = 0 \) with multiplicity \( \alpha_j \). Then we have
\[
(2.21)
\]

\[
\det ((2\pi i)^{-1} \int q_\nu(\lambda)^{\gamma-1}(k(\lambda))^{-1}d\lambda)_{\nu, \sigma=1, \ldots, \mu}
\]

\[
= \frac{q_1(\lambda_1), q_1^{(\alpha_1-1)}(\lambda_1), \ldots, q_1^{(\alpha_1-1)}(\lambda_1)}{q_\mu(\lambda_1), q_\mu^{(\alpha_1-1)}(\lambda_1), \ldots, q_\mu^{(\alpha_1-1)}(\lambda_1)}
\]

\[
\prod_{j=1}^\mu \prod_{j<i} s_i \prod_{1 \leq k < j \leq l} (\lambda_j - \lambda_k)^{\alpha_k}.
\]

Lemma 2.9. Let \( k(\lambda), q_\nu(\lambda), \nu = 1, \ldots, \mu \) be as in Lemma 2.8. Assume that
\[
(2.22)
\]

\[
\det ((2\pi i)^{-1} \int q_\nu(\lambda)^{\gamma-1}(k(\lambda))^{-1}d\lambda)_{\nu, \sigma=1, \ldots, \mu} \neq 0.
\]

Put \( q_\nu(\lambda) = Q_\nu(\lambda)k(\lambda) + q'_\nu(\lambda), \nu = 1, \ldots, \mu \), where \( \deg q'_\nu(\lambda) \leq \mu - 1 \). Then the system
\[
\{ \lambda^{\gamma-1}k(\lambda), \nu = 1, \ldots, r \}, \{ q'_\nu(\lambda), \nu = 1, \ldots, \mu \}
\]
forms a base of polynomials of order less than \( r + \mu \) for any \( r > 0 \).

Proof. It is sufficient to show that \( \{ \lambda^{\nu-1} k(\lambda), \nu = 1, \ldots, r \}, \{ q_j(\lambda), \nu = 1, \ldots, \mu \} \) are linearly independent over the complex number field \( \mathbb{C} \). Assume that

\[
\sum_{j=1}^{r} \alpha_j \lambda^{j-1} k(\lambda) + \sum_{j=1}^{\mu} \beta_j q_j(\lambda) = 0
\]

for some \( \alpha_j, \beta_j \in \mathbb{C} \). Then we have

\[
\sum_{j=1}^{\mu} \beta_j (2\pi i)^{-1} \oint q_j(\lambda) \lambda^{j-1} (k(\lambda))^{-1} d\lambda = 0, \quad l = 1, \ldots, \mu.
\]

From (2.22) it follows that \( \beta_j = 0 \) for all \( j \). Thus we have

\[
(\sum_{j=1}^{r} \alpha_j \lambda^{j-1}) k(\lambda) \equiv 0.
\]

Since \( k(\lambda) \equiv 0 \), we have that \( \alpha_j = 0 \) for all \( j \). Q.E.D.

Lemma 2.10 (Green's formula). Let \( U \) be an open set in \( \mathbb{R}^k \) and \( U' \) be any open subset of \( U \) such that \( U' \subset U \). Let \( \lambda_1(\xi), j = 1, \ldots, a, \) be \( C^\infty \) functions in \( U \) such that \( \lambda_j(\xi) \neq \lambda_j(\xi) \) if \( j \neq j' \) when \( \xi \in U \) and let \( \alpha_j, j = 1, \ldots, a, \) be natural numbers. Set

\[
P(\xi, \lambda) = \prod_{j=1}^{a} (\lambda - \lambda_j(\xi))^{\alpha_j} = \sum_{j=0}^{\infty} p_j(\xi) \lambda^j, \quad \xi \in U.
\]

Let \( \beta_j, j = 1, \ldots, a, \) be non-negative integers such that \( \beta_j \leq \alpha_j, j = 1, \ldots, a. \) Put

\[
Q(\xi, \lambda) = \prod_{j=1}^{a} (\lambda - \lambda_j(\xi))^{\beta_j}, \quad m' = \sum_{j=1}^{a} \alpha_j - \beta_j,
\]

\[
\prod_{j=1}^{m'} (\lambda - \lambda_j(\xi))^{\beta_j} = \sum_{j=0}^{m'} q_j(\xi) \lambda^j.
\]

Let \( B_j(\xi, \lambda), j = 1, \ldots, m', \) be polynomials in \( \lambda \) of order \( r_j \) whose coefficients are \( C^\infty \) functions of \( \xi \) in \( U \). If

\[
\det ((2\pi i)^{-1} \oint B_j(\xi, \lambda) \lambda^{j-1} (Q(\xi, \lambda))^{-1} d\lambda)_{j=1, \ldots, m'} \neq 0 \text{ in } U,
\]

then there exist ordinary differential operators \( C_j(\xi, D_j), j = 0, \ldots, t-m, \)

\( E_j(\xi, D_j), j = 1, \ldots, m', F_{v,j}(\xi, D_j), \nu = 0, \ldots, \beta_j - 1, j = 1, \ldots, a, \) whose coefficients are \( C^\infty \) functions of \( \xi \) in \( U' \) such that

\[
(2.23) \quad \langle P(\xi, D_j)u(\xi, y), v(\xi, y) \rangle - \langle u(\xi, y), P(\xi, -D_j)v(\xi, y) \rangle
\]

\[
= \sum_{j=0}^{m'} \sum_{k=0}^{r_j} D_j^k P(\xi, D_j)u(\xi, y)|_{y=0} C_j(\xi, D_j)v(\xi, y)|_{y=0}
\]

\[
+ \sum_{j=1}^{m'} \sum_{k=0}^{r_j} B_j(\xi, D_j)u(\xi, y)|_{y=0} E_j(\xi, D_j)v(\xi, y)|_{y=0}
\]

\[
+ \sum_{j=1}^{m'} \sum_{k=0}^{r_j} \sum_{\nu=0}^{\beta_j-1} \sum_{l=0}^{l-1-k} q_{j\nu}(\xi) \lambda_j(\xi)^{l-1-k-\nu} D_j^\nu Q(\xi, D_j)u(\xi, y)|_{y=0} F_{v,j}(\xi, D_j)v(\xi, y)|_{y=0}
\]
for any \( u(\xi, y) \in C^\infty([0, \delta]; S'(\mathcal{E})) \) and \( v(\xi, y) \in S_0(U' \times \overline{R}_+^m) \). Here we interpret that \( \lambda_j(\xi)^{i-1-k-\nu} = 0 \) if \( i-1-k-\nu < 0 \), \( F_{j, \nu}(\xi, D_\nu) \equiv 0 \) if \( \beta_j = 0 \), \( C_j(\xi, D_j) \equiv 0 \) if \( t < m \), \( E_j(\xi, D_j) \equiv 0 \) if \( m' = 0 \), and \( t = \text{Max} \{ r_j; 1 \leq j \leq m' \} \).

**Proof.** Since \( \overline{U}' \) is compact, it follows from Lemmas 2.8 and 2.9 that there exist \( C^\infty \) functions in \( U' \): \( a_{j, \nu}^l(\xi), l = 1, \ldots, m - m', \nu = 0, \ldots, \beta_j - 1, j = 1, \ldots, a \), such that

\[
(2.24) \quad \lambda^{i-1} = \sum_{j=1}^{a} \sum_{\nu=0}^{\beta_j-1} a_{j, \nu}^l(\xi) \left( \sum_{l=0}^{m-m'} \sum_{k=0}^{i-1-k-\nu} \frac{(i-1-k)!}{(i-1-k-\nu)!} q_l(\xi) \lambda_j(\xi)^{i-1-k-\nu} k^l \right) \\
\xi \in U', \quad l = 1, \ldots, m - m'.
\]

(2.24) and repeated integrations by parts show that there exist ordinary differential operators \( G_{\lambda, j}(\xi, D_j) \), \( \nu = 0, \ldots, \beta_j - 1, j = 1, \ldots, a \), whose coefficients are \( C^\infty \) functions of \( \xi \) in \( U' \) such that

\[
(2.25) \quad \langle P(\xi, D_j)u(\xi, y), v(\xi, y) \rangle = \sum_{j=1}^{a} \sum_{\nu=0}^{\beta_j-1} \left( \sum_{l=0}^{m-m'} \sum_{k=0}^{i-1-k-\nu} \frac{(i-1-k)!}{(i-1-k-\nu)!} q_l(\xi) \lambda_j(\xi)^{i-1-k-\nu} \right) \cdot D^l_j Q(\xi, D_j)u(\xi, y) |_{y=0}, \quad G_{\lambda, j}(\xi, D_j)v(\xi, y) |_{y=0} \rangle \\
+ \langle Q(\xi, D_j)u(\xi, y), \sum_{j=1}^{a} (-D_j - \lambda_j(\xi))^{\nu} v(\xi, y) \rangle.
\]

We write, using Euclidean algorithm,

\[
(2.26) \quad B_j(\xi, \lambda) = R_j(\xi, \lambda)P(\xi, \lambda) + S_j(\xi, \lambda)Q(\xi, \lambda) + B_j'(\xi, \lambda), \\
\quad j = 1, \ldots, m', \quad \xi \in U',
\]

where \( \deg R_j(\xi, \lambda) \leq r - m, \deg S_j(\xi, \lambda) \leq m - m' - 1, \deg B_j'(\xi, \lambda) \leq m' - 1 \).

Since, by assumptions we have

\[
\text{det} \left( (2\pi i)^{-1} \oint B_j(\xi, \lambda) \lambda^{k-\nu}(Q(\xi, \lambda))^{-1} d\lambda \right)_{j, k = 1, \ldots, m'} = 0, \quad \xi \in \overline{U}',
\]

it follows from Lemma 2.9 that there exist functions \( b_{j, k}(\xi) \), \( j = 1, \ldots, m' \) which are infinitely differentiable in \( U' \) such that

\[
(2.27) \quad \lambda^{i-1} = \sum_{k=0}^{m'} b_{j, k}(\xi) B_j'(\xi, \lambda), \quad j = 1, \ldots, m', \quad \xi \in U'.
\]

Thus, repeating integrations by parts, we have from (2.27) that there exist ordinary differential operators \( E_j(\xi, D_j) \), \( j = 1, \ldots, m' \), such that
\[ \langle Q(\xi, D_2)u(\xi, y), \sum_{j=1}^{a} (-D_2 - \lambda_j(\xi))^{\nu}v(\xi, y) \rangle = \sum_{j=1}^{m} \langle B_j(\xi, -D_2)u(\xi, y) \mid y=0 \rangle \
abla \langle u(\xi, y), P(\xi, D_2)v(\xi, y) \rangle. \]

It follows from (2.26) that
\[ \sum_{j=1}^{m} \langle [B_j(\xi, D_2) - R_j(\xi, D_2)P(\xi, D_2) - S_j(\xi, D_2)Q(\xi, D_2)]u(\xi, y) \mid y=0 \rangle. \]

Since \( \deg R_j(\xi, \lambda) \leq t - m \), there exist ordinary differential operators \( C_j(\xi, D_2) \) \( j = 0, \ldots, t - m \) whose coefficients are \( C^\infty \) functions of \( \xi \) in \( U' \) such that
\[ \sum_{j=0}^{m} \langle C_j(\xi, D_2)u(\xi, y) \mid y=0 \rangle. \]

Since \( \deg S_j(\xi, \lambda) \leq m - m' - 1 \), it follows from (2.24) that there exist ordinary differential operator \( H_{\nu,j}(\xi, D_2) \), \( \nu = 0, \ldots, \beta_j - 1, j = 1, \ldots, a \), whose coefficients are \( C^\infty \) functions of \( \xi \) in \( U' \) such that
\[ \sum_{j=1}^{m'} \langle H_{\nu,j}(\xi, D_2)u(\xi, y) \mid y=0 \rangle. \]

Thus, we have
\[ \langle P(\xi, D_2)u(\xi, y), v(\xi, y) \rangle - \langle u(\xi, y), P(\xi, -D_2)v(\xi, y) \rangle \]
\[ = \sum_{j=0}^{m} \langle D_j^2P(\xi, D_2)u(\xi, y) \mid y=0 \rangle. \]

This shows the lemma.

Lemma 2.11. If the system \( \{P(\xi, \lambda); B_j(\xi, \lambda), j = 1, \ldots, p\} \) satisfies the conditions (i) and (ii) stated in Remark of Section 1 for each \( \xi \in W \), then \( \hat{\psi} = 0 \). Here \( \hat{\psi} \) is as in Lemma 2.5.

Proof. By assumption, we have \( \hat{\psi} = 0 \). Let \( \sigma = (\sigma_1, \ldots, \sigma_{\hat{\psi}}) \) be any subset consisting of \( \hat{\psi} + \hat{\xi} \) elements of \( \{1, \ldots, p\} \). Put
\[ L_\sigma(\xi') = \text{det} \left( (2\pi i)^{-1} \int_{\gamma(\xi')} B_\sigma(\xi', \lambda) \lambda^{r-1} (P^0(\xi', \lambda) P^+(\xi', \lambda))^{-1} d\lambda \right)_{j,k=1, \ldots, \tilde{a}+\tilde{b}}, \quad \xi' \in W, \]

where \( \gamma(\xi') \) is a simple closed curve in the complex \( \lambda \)-plane which surrounds all \( \lambda_{j}^{r}(\xi'), j=1, \ldots, a \) and all \( \lambda_{j}^{r}(\xi'), j=1, \ldots, b \). Since \( L_\sigma(\xi') \) vanishes identically in \( W \) or a real analytic function of \( \xi' \) in \( W \), putting \( A_{L_\sigma} = \{ \xi' \in W; L_\sigma(\xi') = 0 \} \), we have that \( A_{L_\sigma} = W \), \( A_{L_\sigma} \) is a real analytic set in \( W \) or \( A_{L_\sigma} \) is empty. By assumption we have that \( \bigcap_{\sigma} A_{L_\sigma} \) is empty where the intersection is taken over all \( \sigma \subset \{ 1, \ldots, \tilde{p} \} \). Let \( L_\sigma(\xi') \) not vanish identically in \( W \) and \( U \) be any open set whose closure is compact and contained in \( W - A_{L_\sigma} \). It is sufficient to show that

\[
\langle \psi(\xi', y), w(\xi', y) \rangle = 0
\]

for any \( w(\xi', y) \in S_0(U \times \mathbb{R}^+) \). It follows from Lemma 2.10 that there exist ordinary differential operators \( C_j(\xi', D_y), j=0, \ldots, t-(a+b), \quad E_j(\xi', D_y), j=1, \ldots, \tilde{a}+\tilde{b}, \) whose coefficients are \( C^\infty \) functions of \( \xi' \) in \( U \) such that

\[
\langle P^0(\xi', D_y) P^+(\xi', D_y) \psi(\xi', y), w(\xi', y) \rangle = \sum_{j=0}^{t-\tilde{a}-\tilde{b}} \psi(\xi', y) \bigg|_{y=0}, \quad C_j(\xi', D_y) w(\xi', y) \bigg|_{y=0} \]

for any \( w(\xi', y) \in S_0(U \times \mathbb{R}^+) \). Here \( t = \text{Max} \{ r_{\sigma_j}; 1 \leq j \leq \tilde{a}+\tilde{b} \} \). Since it follows from (1.2), (2.4) and Lemma 2.5 that

\[
\langle P^0(\xi', D_y) P^+(\xi', D_y) \psi(\xi', y), w(\xi', y) \rangle = \langle \psi(\xi', y), P^0(\xi', D_y) P^+(\xi', D_y) w(\xi', y) \rangle
\]

for any \( w(\xi', y) \in S_0(U \times \mathbb{R}^+) \), (2.28) follows from Lemma 2.6.

Now we shall prove the assertion: there exist an open cone \( \Gamma \), natural number \( N \) and real analytic set \( B \) contained in \( W \) such that if \( u \in L^2_{\text{loc}}(\mathbb{R}^{a+1}) \cap C^\infty([0, \tilde{b}]; S'(\mathbb{R}^a)) \) satisfies the condition:

\[
\lim_{R \to \infty} R^{-N} \int_{\Gamma \cap \mathbb{R}} |u(x, y)|^2 dx dy = 0,
\]

then the support of \( \psi(\xi', y) \) is contained in \( B \times \mathbb{R}^+ \), in the following cases.

**Case 1.** \( \tilde{a}=0 \) and \( \tilde{b}>0 \) in \( W \).

Since we have \( \tilde{b} \leq p \) by the assumption (A-1), we put
$L_q(\xi') = \det ((2\pi i)^{-1} \int B_{\sigma}(\xi', \lambda) \lambda^{k-1} (P^+(\xi', \lambda))^{-1} d\lambda)_{j=1,\ldots, \bar{k}}, \quad \xi' \in W$,

$A_\sigma = \{\xi' \in W; L_\sigma(\xi') = 0\}$

for $\sigma = (\sigma_1, \ldots, \sigma_{\bar{p}}) \subseteq \{1, \ldots, p\}$. Put $B = \bigcap A$ where the intersection is taken over all $\sigma \subseteq \{1, \ldots, p\}$. When $r=0$, we have that $B$ is empty or a real analytic set in $W$ by the assumption (A-2). But when $1 \leq r \leq n$, one of the following three cases may occur: (I) $B$ is empty, (II) $B$ is a real analytic set in $W$, (III) $B = W$. For the case (III) we have the following by Theorem A$p-3$.

**Lemma 2.12.** Let $u$ be as in the first place of this section and $\psi$ be as in Lemma 2.5. Assume that the hypotheses of Case 1 are fulfilled in $W$, $1 \leq r \leq n$ and $W = B$ (Case III). Set

$$
\Gamma_r^{(III)} = \{(x, y) \in \Gamma^{(III)}; y \geq 0, R < |(x, y)| < 2R\},
$$

where $\Gamma^{(III)}$ is an open cone in $\mathbb{R}^{n+1}$ which contains $(n(\xi'), 0)$ for every $\xi' \in W$. Here $n(\xi')$ denotes some normal of $\{(\xi', \mu(\xi')); \xi' \in W\}$ at $(\xi', \mu(\xi'))$. If $u$ belongs to $L^2_{\text{loc}}(\mathbb{R}^{n+1}) \cap C^\infty((0, \delta); S'({\mathbb{R}^n}))$ and satisfies the condition:

$$
\lim_{R \to \infty} R^{-r} \int_{\Gamma^{(III)}_R} |u(x, y)|^2 dx dy = 0
$$

then $\psi = 0$. Here $r$ is the codimension of $\{(\xi', \mu(\xi')); \xi' \in W\}$ in $\mathbb{B}^n$.

**Proof.** Put $u_0 = u$ when $y \geq 0$ and $u_0 = 0$ when $y < 0$. Since the support of $\phi(\xi)\phi_j(\xi)\chi_0(\xi, y)$ is contained in $\{(\xi', \mu(\xi')); \xi' \in W\} \times \mathbb{R}^1$, the support of $\phi(\xi)\phi_j(\xi)\mathcal{F}[u_0](\xi, \lambda)$ is contained in $\{(\xi', \mu(\xi')); \xi' \in W\} \times \mathbb{S}^1$. Let $\rho(\lambda)$ be any $C_0^\infty(\mathbb{S}^1)$ function. It follows from Lemma A$p-2$

$$
\lim_{R \to \infty} R^{-N} \int_{\Gamma_R'} |\mathcal{F}^{-1}[\phi(\xi)\phi_j(\xi)\chi_0](\xi, y)|^2 dx dy \lesssim C \lim_{R \to \infty} R^{-N} \int_{\Gamma_R''} |u(x, y)|^2 dx dy
$$

where $\Gamma_R' = \{(x, y) \in \Gamma'; R < |(x, y)| < 2R\}$ and $\Gamma''$ is another cone which satisfies the same condition as $\Gamma^{(III)}$. Then we have, using Theorem A$p-3$ that $\phi(\xi)\phi_j(\xi)\rho(\lambda)\mathcal{F}[u_0](\xi, \lambda) = 0$. This shows that $\phi(\xi)\phi_j(\xi)\chi_0(\xi, y) = 0$. Therefore, we have $\psi = 0$.

**Proposition 2.13.** Suppose that the hypotheses of Case 1 are fulfilled,
1 \leq r \leq n \text{ and } W = B. \text{ Let } \Gamma \text{ be an open cone in } \mathbb{R}^{n+1} \text{ and } N \text{ an integer such that if } w \in C^\infty(\mathbb{R}_+^{n+1}) \cap S'((\mathbb{R}_+^{n+1})) \text{ satisfies the equations (1.1) and (1.2) and the support of } \hat{w}(\xi, y) \text{ is contained in } \{(\xi', \mu(\xi')); \xi' \in W \} \times \mathbb{R}_+^{n+1}, \text{ with}

(2.30) \lim_{R \to \infty} R^{-N} \int_{\Gamma} |w(x, y)|^2dxdy = 0

then \(w = 0\). \text{ Then the closure } \Gamma \text{ contains } (n(\xi'), 0) \text{ for each } \xi' \in W \text{ and } N \leq r. \text{ Here } n(\xi') \text{ is as in Lemma 2.12.}

\textbf{Proof.} \text{ Since } W = B, L_x(\xi') \text{ vanishes identically in } W \text{ for all } \sigma \subseteq \{1, \ldots, p\}. \text{ So we have, using Lemma 2.8, that there exist } C^\infty \text{ functions } C_{j\nu}(\xi'), \nu = 0, \ldots, \beta_j - 1, j = 1, \ldots, b, \text{ of } \xi' \text{ in } W \text{ such that if we put}

\hat{w}(\xi', y) = \left(\sum_{j=1}^{b-1} \sum_{\nu=0}^{\beta_j - 1} C_{j\nu}(\xi')(iy) e^{i\lambda x} \psi(\xi') \right),

we have that \(\hat{w}(\xi', y)\) does not vanish identically in \(W \times \mathbb{R}_+^{n+1}\) and satisfies the equations:

\begin{align*}
P(\xi', \mu(\xi'), D_x)\hat{w}(\xi', y) &= 0, \quad y > 0, \\
B_j(\xi', \mu(\xi'), D_x)w(\xi', y)|_{y=0} &= 0, \quad j = 1, \ldots, p.
\end{align*}

Here \(\psi(\xi') \in C_0^\infty(W)\). \text{ Hence, putting}

\[w(x, y) = \int \exp \{i(x' \cdot \xi' + x'' \cdot \mu(\xi'))\} \hat{w}(\xi', y)d\xi',\]

we have that \(w(x, y)\) does not vanish identically in \(\mathbb{R}_+^{n+1}\), belongs to \(S'((\mathbb{R}_+^{n+1}) \cap C^\infty(\mathbb{R}_+^{n+1}) \text{ and satisfies the equations (1.1) and (1.2). (2.30) follows from Theorem A_p-1 if } N > r, \text{ which gives a contradiction. If } \Gamma \text{ contains no } (n(\xi'_0), 0) \text{ for some } \xi'_0 \text{ and if supp } \psi \text{ is sufficiently close to } \xi'_0 \text{ the condition (2.30) follows from Theorem A_p-1 for any } N, \text{ which gives a contradiction.}

\textbf{Lemma 2.14.} \text{ Let } u \text{ be as in the first place of this section and } \hat{v} \text{ be as in Lemma 2.5. Assume that the hypotheses of Case 1 are fulfilled in } W. \text{ (1) If } B \text{ is empty, then } \hat{v} = 0. \text{ (2) Suppose that } B \text{ is a real analytic set in } W. \text{ If } u \text{ belongs to } C^\infty([0, \delta]; S'(\mathbb{R}^n)) \cap L^2_{\text{loc}}(\mathbb{R}_+^{n+1}) \text{ and satisfies the condition;}

\lim_{R \to \infty} R^{-(N_r+r)} \int_{\Gamma^{(r)}} |u(x, y)|^2dxdy = 0,

then \(\hat{v} = 0\). \text{ Here } \Gamma^{(r)} \text{ is an open cone in } \mathbb{R}_+^{n+1} \text{ which contains } (n(\xi'), 0) \text{ for every analytic manifold } C \subseteq B \text{ and } \xi' \in C, \text{ when } n(\xi') \text{ denotes some normal of } \{(\xi', \mu(\xi')); \xi' \in C\} \text{ at } (\xi', \mu(\xi')), \Gamma^{(r)}_R = \{(x, y) \in \Gamma^{(r)}; y \geq 0, R < |(x, y)| < 2R\}.
and \( N_r \) is the codimension of \( B \) in \( \mathcal{E}_\xi^{g-r} \).

**Remark.** \( N_r + r \geq 1 \) for any \( r \in \{0, \cdots, n+1\} \).

**Proof.** In the same way as in Lemma 2.11, we have that the support of \( \nu(\xi', y) \) is contained in \( B \times R_1 \). If \( B = \emptyset \), then \( \hat{\nu} = 0 \). This shows (1). Since the codimension of the analytic set \( \{(\xi', \mu(\xi')); \xi' \in B\} \times R_1 \) is equal to \( N_r + r \), we have the assertion (2) in the same way as in Lemma 2.12.

**Proposition 2.15.** Suppose that \( B \) is a real analytic set. Let \( \Gamma \) be an open cone in \( \mathbb{R}^{n+1} \) and \( N \) an integer such that if \( w \in C^\infty(\mathbb{R}^{n+1}) \cap \mathcal{S}'(\mathbb{R}^{n+1}) \) satisfies the equations (1.1) and (1.2) and the support of \( \hat{\nu}(\xi, y) \) is contained in \( \{(\xi', \mu(\xi')); \xi' \in B\} \times R_1^{g-r} \) with

\[
\lim_{R \to \infty} R^{-N} \int_{R_0^2} |w(x, y)|^2 dxdy = 0
\]

then \( w = 0 \). If \( C \) is a \( C^\infty \) manifold contained in \( B \), then the closure of \( \Gamma \) contains \( \{(n(\xi'), 0); \xi' \in C \text{ and } N \leq \text{Codim } C + r\} \). Here \( n(\xi') \) is as in Lemma 2.14 and \( \text{Codim } C \) denotes the codimension \( C \) in \( \mathcal{E}_\xi^{g-r} \).

**Proof.** We may assume that \( C \) is defined by \( \eta'' = (\eta') \) where \( \eta' \in \omega \subset \mathcal{E}_\xi^{g-r-k} \) (\( k \) is codimension of \( C \) in \( \mathcal{E}_\xi^{g-r} \), \( \nu \) is a \( C^\infty \) function in \( \omega', \xi' = (\eta', \eta'') \) and \( (0, \nu(0)) = \xi_0' \). Put \( L = (f_{jk}(\eta'))_{j=1, \cdots, b, k=1, \cdots, t} \) where

\[
f_{jk}(\eta') = (2\pi i)^{-1} \oint_{B_2(\eta', \nu(\eta'), \lambda)} \lambda^{k-1}(P^+(\eta', \nu(\eta'), \lambda))^{-1} d\lambda.
\]

Since all minor of \( L \) of order \( b \) vanish identically in \( \omega \), we have that the rank \( l \) of \( L \) is less than \( b \) in \( \omega \). When \( l > 0 \), without loss of generality, we may assume that

\[
A(\eta') = \det(f_{jk}(\eta'))_{j,k=1, \cdots, t}
\]
does not vanish identically in \( \omega \). When \( l > 0 \), we put

\[
\hat{\omega}(\eta', y) = \left[ -\sum_{j,k=1}^t A_{jk}(\eta') f_{j+k}(\eta') (2\pi i)^{-1} \oint e^{i\lambda\lambda^{k-1}(P^+(\eta', \zeta(\eta'), \lambda))^{-1} d\lambda} \right. \\
\left. + A(\eta')(2\pi i)^{-1} \oint e^{i\lambda\lambda^i(P^+(\eta', \zeta(\eta'), \lambda))^{-1} d\lambda} \right] \varphi(\eta')
\]

where \( \varphi(\eta') \) is a \( C^\infty_0(\omega - \{\eta' \in \omega; A(\eta') = 0\}) \) function and \( A_{jk}(\eta') \) is the \((j, k)\) cofactor of \( (f_{jk}(\eta'))_{j,k=1, \cdots, t} \) and \( \zeta(\eta') = (\nu(\eta'), \mu(\eta', \nu(\eta'))) \). When \( l = 0 \), we put
\[ \hat{\psi}(\xi', y) = (2\pi i)^{-1} \int e^{i\lambda \cdot (P^+(\xi', \zeta(\eta'), \lambda))^{-1} \lambda \cdot \psi(\eta')} \]

where \( \psi(\eta') \) is a \( C_0^\infty(\omega) \) function. It is obvious that we have

\[ P(\eta', \zeta(\eta'), D_y)\hat{\psi}(\eta', y) = 0, \quad y > 0, \quad B_j(\eta', \zeta(\eta'), D_j)\hat{\psi}(\eta', y) |_{y=0} = 0, \quad j = 1, \ldots, p. \]

Thus, putting

\[ \psi(x, y) = \int e^{i(x', \eta' + \xi') \cdot (\psi(\eta'))} \hat{\psi}(\eta', y) d\eta', \quad x' = (x_{n-r-k+1}, \ldots, x_n), \]

we have that \( \psi(x, y) \) satisfies the equations (1.1) and (1.2). Since \( |\hat{\psi}(\eta', y)| \leq C_1 \cdot |\psi(\eta')| e^{-Cy^2} \) when \( y \geq 0 \) for some positive constants \( C_1 \) and \( C_2 \), we have

\[ \int_{|\xi'|, |\eta'| < R} \int_{R^2} |\psi(x, y)|^2 dx dy \leq CR^{r+k} \]

for some positive constant \( C \). So if \( N > \text{Codim } C + r \geq N_r + r \), then

\[ \lim_{R \to \infty} R^{-N} \int_{R^2} |\psi(x, y)|^2 dx dy = 0. \]

This gives a contradiction. Thus, we have \( N \leq \text{Codim } C + r \). If \( \Gamma \) does not contain any \( (n(\xi'), 0) \) and if \( \text{supp } \psi(\eta') \) is sufficiently close to 0

\[ \int_{|\xi'|, |\eta'| < R} \int_{R^2} |\psi(x, y)|^2 dx dy = 0, \]

for any \( N \). This gives a contradiction. Q.E.D.

Next we consider the case when \( a > 0 \). Put

\[ L^{(\delta', \delta, \sigma)} = \det ((2\pi i)^{-1} \oint B_{\sigma}(\xi', \lambda)\lambda^{b-1}(\prod_{j=1}^{p} (\lambda - \lambda_{\sigma}^{(j)}(\xi'')))^{a_j - \delta_j} P^+(\xi', \lambda))^{-1} d\lambda, \quad \xi' \in W, \]

if \( a > 0 \) where \( \delta = (\delta_1, \ldots, \delta_a) \) (0 \leq \delta_j \leq \alpha_j), \( b + 1 \leq p' \leq p \) and \( \sigma = (\sigma_1, \ldots, \sigma_{p'}) \) (\( \subset \{1, \ldots, p\} \)).

**Case 2.** \( a > 0 \) and there exist \( p', \delta, \sigma \), having the properties:

1. \( b + 1 \leq p' \leq p \),
2. \( \sum_{j=1}^{p'} (\alpha_j - \delta_j) + b = p' \),
3. if \( p' < p \), \( L^{(\delta', \sigma)}(\xi') \) vanishes identically in \( W \) for any \( \sigma''(p'' + 1 \leq p'' \leq p), \sigma'' = (\sigma'_1, \ldots, \sigma'_{p''})(\subset \{1, \ldots, p\}) \) and \( \delta'' = (\delta'_1, \ldots, \delta_{p''}) \) with \( 0 \leq \delta'_j \leq \alpha_j \).
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When \( r = 0 \), the hypotheses of Case 2 imply the assumption (A-2).

Put \( B = \{ (\xi', 0) \in W; L(\phi^{\xi', \delta}, \rho)(\xi') \neq 0 \} \). It is obvious that \( B \) is empty or a real analytic set in \( W \).

Lemma 2.16. Let \( \varphi \) be as in Lemma 2.5. If the hypotheses of Case 2 are fulfilled in \( W \) with \( \sum_{j=1}^{n} \alpha_j + b = p' \), then the support of \( \varphi \) is contained in \( B \times R_+ \).

Remark. If \( B = \emptyset \), then \( \varphi = 0 \).

Proof. Let \( U \) be any open set in \( W \) whose closure is compact and contained in \( W - B \). It is sufficient to show that

\[(2.31) \quad \langle \varphi(\xi', y), w(\xi', y) \rangle = 0 , \]

for any \( w(\xi', y) \in S_0(U \times R_+) \). We obtain, using Lemmas 2.5 and 2.10, (1.2) and (2.4), that

\[
\langle P^0(\xi', D_j)P^+(\xi', D_j)w(\xi', y), w(\xi', y) \rangle = \langle \delta(\xi', y), (P^0(\xi', D_j)P^+(\xi', - D_j)w(\xi', y) \rangle
\]

for any \( w(\xi', y) \in S_0(U \times R_+) \). Therefore, (2.31) follows from Lemma 2.6.

Lemma 2.17. Suppose that the hypotheses of case 2 are fulfilled with \( \delta_j > 0 \) for at least one \( j \), say, \( \delta_j > 0 \), \( j = 1, \ldots, k \). Set

\[ \Gamma_{j,R} = \{(x, y) \in \Gamma_j; y \geq 0, R < |(x, y)| < 2R\} , \]

where \( \Gamma_j \) is some open connected cone which for every \( \xi' \in W - B \) contains some normal of \( M_j = \{(\xi', \mu(\xi')), \lambda_0(\xi')\}; \xi' \in W - B \) at \( (\xi', \mu(\xi'), \lambda_0(\xi')) \). If \( u \) belongs to \( L^2_{loc}(R^{n+1}) \) with

\[(2.32) \quad \lim_{B \to \infty} R^{-n(x, y) + \delta_j + r + 1} \int_{\Gamma_{j,R}} |u(x, y)|^2 dx dy = 0 , \quad j = 1, \ldots, k , \]

then the support of \( \varphi(\xi', y) \) is contained in \( B \times R_+ \). Here \( u \) and \( \varphi \) are as in the first place of this section and Lemma 2.5, respectively.

Proof. Let \( U \) be an open set in \( W \) whose closure is compact and contained in \( W - B \). It is sufficient to show that
for any \(w(\xi', y) \in S_\delta(U \times \mathbb{R}^1_+)\). It follows from Lemma 2.10 that there exist ordinary differential operators \(C_j(\xi', D_y), j = 0, \ldots, t - (a + b), E_j(\xi', D_y), j = 1, \ldots, p, F_{\nu,j}(\xi', D_y), \nu = 0, \ldots, \delta_j - 1, j = 1, \ldots, k\), whose coefficients are \(C^\infty\) functions of \(\xi'\) in \(U\) such that

\[
\langle P^0(\xi', D_y)P^+(\xi', D_y)\psi(\xi', y), w(\xi', y) \rangle = 0,
\]

for any \(w(\xi', y) \in S_\delta(U \times \mathbb{R}^1_+)\). Here \(t = \text{Max} \{r_j; 1 \leq j \leq p\}\), \(d = \sum_{j=1}^{p} \delta_j\), \(\prod_{j=1}^{t}(\lambda - \lambda_0(\xi'))^{m_j} = \sum_{j=0}^{p} b_j(\xi')^{m_j}, Q(\xi') = \sum_{j=1}^{t} (\lambda - \lambda_0(\xi'))^{m_j - 1} \cdot P^+(\xi', \lambda)\). Since it follows from (2.32) and Lemma 2.7 that

\[
\sum_{j=0}^{p} \sum_{k=0}^{m_j - 1} \frac{(i - 1 - k)!}{(i - 1 - k - \nu)!} b_i(\xi')^{\lambda_0(\xi')}^{i - 1 - k - \nu} D_j Q(\xi', D_y)\psi(\xi', y) |_{y=0} = 0,
\]

for all \(\nu = 0, \ldots, \delta_j - 1, j = 1, \ldots, k\), and it follows from (1.2), (2.4) and Lemma 2.5 that

\[
\langle B_{\nu,j}(\xi', D_y)\psi(\xi', y) |_{y=0}, E_j(\xi', D_y)w(\xi', y) |_{y=0} \rangle = 0, \quad j = 1, \ldots, p,
\]

\[
\langle D_j^i P^0(\xi', D_y)P^+(\xi', D_y)\psi(\xi', y) |_{y=0}, C_j(\xi', D_y)w(\xi', y) |_{y=0} \rangle = 0,
\]

for all \(i = 0, \ldots, t - (a + b), j = 1, \ldots, p\), we have

\[
\langle P^0(\xi', D_y)P^+(\xi', D_y)\psi(\xi', y), w(\xi', y) \rangle = \psi(\xi', y), P^0(\xi', - D_y)P^+(\xi', - D_y)w(\xi', y) \rangle,
\]

for any \(w(\xi', y) \in S_\delta(U \times \mathbb{R}^1_+)\). Thus, (2.33) follows from Lemma 2.6.

Make the same assumption as in Lemma 2.17. Let \(n_j(\xi')\) denote some normal of \(M_j\) at \(\xi'\) and \(\theta \in \mathbb{R}^{t+1}_+\) be a normal of \(M_1\) at \(\xi_0 \in W - B\). Let \(D\) be a subset of \(\{2, \ldots, a\}\) and let \(\phi\) be a \(C^\infty_\delta\) \(\{(x, y) \in \mathbb{R}^{t+1}_+; \ |(x, y) - \theta | < 2\varepsilon\}\) with \(\phi = 1\) on \(|(x, y) - \theta | \leq \varepsilon\) for sufficiently small \(\varepsilon\). Assume that \(\theta = n_j(\xi_0) = n_j(\xi_0)\) for some \(n_j(\xi_0)\) when \(j \in D\) and \(\theta \neq n_j(\xi_0)\) for all \(n_j(\xi_0)\) when \(j \notin D\). Choose a
small neighborhood \( \omega \) of \( \xi_0 \) so that \( \{ n_j(\xi'), \xi' \in \omega \} \subset \{(tx, ty); t \neq 0, |(x, y)-\theta| < \varepsilon \} \) when \( j \in D \) and \( \{ n_j(\xi'); \xi' \in \omega \} \cap \{(tx, ty); t \neq 0, |(x, y)-\theta| < \varepsilon \} \) is empty when \( j \notin D \). Put

\[
Q(\xi', \lambda) = (\lambda - \lambda_j^0(\xi'))^{(a_j - \delta_j)^{-1}+1} \prod_{j=1}^d (\lambda - \lambda_j^0(\xi'))^{a_j - \delta_j} + P^+(\xi', \lambda)
\]

where \( a_j = \alpha_j - \delta_j \), \( \alpha_j = \alpha - \delta_j \), \( j = 2, \cdots, a \), \( \mu_j + a = \beta_j, j = 1, \cdots, b \), \( \tau_j(\xi') = \lambda_j^0(\xi'), j = 1, \cdots, a \), and \( \tau_j(\xi') = \lambda_j^0(\xi'), j = 1, \cdots, b \). Put

\[
L = \begin{bmatrix}
B_1(\xi, \tau_1(\xi')), \cdots, B_1(\mu_1 - 1)(\xi', \tau_1(\xi')), B_2(\xi', \tau_2(\xi')), \cdots, B_1(\mu_1 - 1)(\xi', \tau_2(\xi')) \\
\vdots \hspace{1cm} \vdots \hspace{1cm} \vdots \\
B_p(\xi', \tau_1(\xi')), \cdots, B_p(\mu_1 - 1)(\xi', \tau_1(\xi')), B_p(\xi', \tau_2(\xi')), \cdots, B_p(\mu_1 - 1)(\xi', \tau_2(\xi'))
\end{bmatrix}
\]

It follows from the hypotheses of Case 2 and Lemma 2.8 that the rank of \( L \) is equal to \( p' \) in \( W - B \) and that

\[
A(\xi') = \det \begin{bmatrix}
B_{s_1}(\xi', \tau_1(\xi')), \cdots, B_{s_1}(\mu_2 - 2)(\xi', \tau_1(\xi')), B_{s_1}(\xi', \tau_2(\xi')) \\
\vdots \hspace{1cm} \vdots \hspace{1cm} \vdots \\
B_{s_p}(\xi', \tau_1(\xi')), \cdots, B_{s_p}(\mu_2 - 2)(\xi', \tau_1(\xi')), B_{s_p}(\xi', \tau_2(\xi'))
\end{bmatrix}
\]

does not vanish in \( W - B \). So there exist real analytic functions \( C_{js}(\xi') \) \( s = 1, \cdots, \mu_j - 1, j = 1, \cdots, a + b \) in \( W - B \) such that \( C_{js}(\xi') \neq 0 \) and

\[
\psi(x', y) = \sum_{j=1}^t \sum_{k=0}^{p-1} C_{js}(\xi')(iy)^k e^{\xi'y}, \psi(\xi')
\]

satisfies the equations:

\[
P(\xi', D_j)w(x', y) = 0, y > 0 \quad \text{and} \quad B_j(\xi', D_j)w(x', y)|_{y=0} = 0, \quad j = 1, \cdots, p,
\]

where \( \psi(\xi') \in C_0^0(\omega) \) and \( P(\xi', D_j) = P^0(\xi', D_j)P^+(\xi', D_j)P^-(\xi', D_j) \). So setting

\[
w(x, y) = \int \psi(x', y) \exp \{ i(x' \cdot \xi' + x' \cdot \mu(\xi')) \} d\xi',
\]

we have that \( w(x, y) \) satisfies the equations (1.1) and (1.2) and that \( w(x, y) \) belongs to \( \mathcal{S}'(\mathbb{R}^{n+1}) \cap C^\infty(\mathbb{R}^{n+1}) \). Put

\[
N' = \max \{ s; C_{js}(\xi') \text{ does not vanish identically in } \omega \text{ when } j \in D \}.
\]

Note that if \( \{ 2, \cdots, k \} \cap D \) is not empty we may assume that \( N' \geq \alpha_i - \delta_i \geq \alpha_j - \delta_j \) for \( j \in \{ 2, \cdots, k \} \cap D \). Put \( N' = \mu_k - \nu_{j, k} \), \( k = 1, \cdots, t \) and \( C_{j, k} = \{(x, y) \in C; y \geq 0, R < |(x, y)| < 2R\} \) where \( C_1 \) is a small conic neighborhood of \( n_j(\xi_0) \).

Since \( \text{supp } \phi \subset \mathbb{R}^{n+1}_+ \), it follows from Theorem Ap-1 and Theorem Ap-4 due to Agmon-Hörmander [3 Theorem 3.1] that
\[
\lim_{R \to \infty} \int_{C_{1,R}} |w(x, y)|^2 dx dy / R^{r+1+2N'} \\
\geq \lim_{R \to \infty} \int |w(x, y)|^2 \phi((x, y)/R) dx dy / R^{r+1+2N'} \\
\geq \lim_{R \to \infty} \phi((x, y)/R) \cdot |\sum_{k=1}^N F_k(x, y)|^2 dx dy / R^{r+1},
\]

where

\[
F_k(x, y) = \int \exp \{i(x' \cdot \xi' + x'' \cdot \mu(\xi') + y \cdot \tau_j(\xi'))\} C_{i_k, j_k} \psi(\xi') d\xi'.
\]

Since \(\tau_j(\xi') \neq \tau_{j_k}(\xi')\) if \(j \neq j_k\), we have, using Corollary Ap-5,

\[
\lim_{R \to \infty} \int F_k(x, y)F_j(x, y)\phi((x, y)/R) dx dy / R^{r+1} = 0, \quad k \neq k'.
\]

Thus, we have, using Theorem Ap-4,

\[
\lim_{R \to \infty} \int_{C_{1,R}} |w(x, y)|^2 dx dy / R^{r+1+2N'} \\
\geq C \sum_{k=1}^N \int |C_{i_k, j_k} \psi(\xi')|^2 d\xi' > 0.
\]

This shows that

\[
\int_{C_{1,R}} |w(x, y)|^2 dx dy \geq CR^{2N'+r+1}.
\]

In the same way, we have, using Theorems Ap-1 and Ap-4,

\[
\int_{C_{1,R}} |w(x,y)|^2 dx dy \leq CR^{2N'+r+1}.
\]

Summing up, we have proved.

**Proposition 2.18.** Make the same assumption as in Lemma 2.17. Let \(\theta_j \in R^{\alpha+1}_{\pm} (j = 1, \cdots, k)\) be normal of \(M_j\) at \(\xi_0 \in W - B\) and \(C_j\) a small conic neighborhood of \(\theta_j\). Then there exists a solution \(w(x, y) \in C^\infty(R^{\alpha+1}_+ \cap S'(R^{\alpha+1}_+))\) of the equations (1.1) and (1.2) and a natural number \(N_j\) such that \(N_j \geq \alpha_j - \delta_j\) and \(w(x, y)\) satisfies for some positive constants \(d_1\) and \(d_2\)

\[
d_1 R^{2N_j+r+1} \leq \int_{C_{1,R}} |w(x, y)|^2 dx dy \leq d_2 R^{2N_j+r+1}
\]

where \(C_{j,R} = \{(x, y) \in C_j; y \geq 0, R < |(x, y)| < 2R\}\).

**Remark.** For example, if \(D\) is empty, then \(N_1 = \alpha_1 - \delta_1\).
Case 3. \( a > 0 \) and \( b = 0 \) and \( L(\psi^{a,b}(\xi')) \) vanishes identically for all \( p' \) \((1 \leq p' \leq p)\), \( \delta = (\delta_1, \ldots, \delta_p) \) \((0 \leq \delta_j \leq \alpha_j)\) and \( \sigma = (\alpha_1, \ldots, \alpha_p) \) \((\subset \{1, \ldots, p\}\)) in \( W \).

**Lemma 2.19.** Let \( u \) and \( \tilde{u} \) be as in the first place of this section and Lemma 2.5, respectively. Suppose that the hypotheses of Case 3 are fulfilled. If \( u \) belongs to \( L^2_{\text{loc}}(\mathbb{R}^{n+1}_+) \) and satisfies the condition:

\[
\lim_{R \to \infty} R^{-(r+1)} \int_{\Gamma_j} |u(x, y)|^2 dx dy = 0, \quad j = 1, \ldots, a,
\]

then \( \tilde{u} = 0 \). Here \( \Gamma_j \) and \( \Gamma_j \) are as in Lemma 2.17.

**Proof.** In the same way as in Lemma 2.17, the assertion follows from Lemmas 2.5, 2.6, 2.7 and 2.10 and formulas (1.2) and (2.4).

**Proposition 2.20.** Suppose that the hypotheses of Case 3 are fulfilled. Let \( \Gamma \) be an open cone in \( \mathbb{R}^{n+1} \) and \( N \) is an integer such that if \( \omega \in S'((\mathbb{R}^{n+1}_+) \cap C^\infty(\mathbb{R}^{n+1}_+) \) satisfies the equations (1.1) and (1.2) with

\[
|u(x, y)|^2 dx dy = 0
\]

and the support of \( \hat{u}(\xi, y) \) is contained in \( \{(\xi', \mu(\xi')) \in \mathbb{R}^1_+ \times \mathbb{R}^{n}_+ \text{ then } u = 0 \text{.} \) If \( (\xi_0', \mu(\xi_0'), \lambda_0(\xi_0')) \in M_j \) then it follows that the closure of \( \Gamma \) contains some normal \( \neq 0 \) of \( M_j \) at \( (\xi_0', \mu(\xi_0'), \lambda_0(\xi_0')) \) and that \( N \leq r + 1 \).

**Proof.** Let \( \psi(\xi') \) be a \( C^\infty(\mathbb{R}^n) \) function. It follows from the hypotheses of Case 3 that

\[
B_j(\xi', \lambda_j(\xi')) = \int B_j(\xi', \lambda)(\lambda - \lambda_j(\xi'))^{-1} d\lambda = L^{0,1}(\xi') = 0, \quad j = 1, \ldots, p,
\]

where \( \delta = (\alpha_1, \ldots, \alpha_{j-1}, \alpha_j - 1, \alpha_{j+1}, \ldots, \alpha_p) \). So, putting

\[
\psi(x, y) = \int \exp \{i(x' \cdot \xi' + x'' \cdot \mu(\xi') + y' \lambda_j(\xi'))\} \psi(\xi') d\xi',
\]

we have the assertion in the same way as in Proposition 2.13.

**Case 4.** \( a > 0 \) and \( b > 0 \) and \( L(\psi^{a,b}(\xi')) \) vanishes identically for all \( p' \) \((1 \leq p' \leq p)\), \( \delta = (\delta_1, \ldots, \delta_p) \) \((0 \leq \delta_j \leq \alpha_j)\) and \( \sigma = (\alpha_1, \ldots, \alpha_p) \) \((\subset \{1, \ldots, p\}\)).

Let \( B \) be as in Case 1. We have the followings in the same way.

**Lemma 2.21.** Let \( u \) and \( \tilde{u} \) be as in the first place of this section and Lemma
2.5, respectively. Suppose that the hypotheses of Case 4 are fulfilled, $1 \leq r \leq n$ and $B = W$. If $u$ belongs to $L^2_{x_1 \theta}(R^{n+1}_+)$ with
\[
\lim_{R \to \infty} R^{-r} \int_{\Gamma^{(I)\theta}} |u(x, y)|^2 \, dx \, dy = 0,
\]
then $v = 0$. Here $\Gamma^{(I)\theta}$ is as in Lemma 2.12.

**Proposition 2.22.** Suppose that $1 \leq r \leq n$, the hypotheses of case 4 are fulfilled and $B = W$. Let $\Gamma$ be an open cone in $R^{n+1}$ and $N$ an integer such that if $w \in C^\infty(R^{n+1}_+ \cap S'(R^{n+1}_+))$ satisfies the equations (1.1) and (1.2) and support of $\hat{w}(\xi, y)$ is contained in $\{(\xi', \mu(\xi'); \xi' \in W) \times R_+\}$ with
\[
\lim_{R \to \infty} R^{-N} \int_{\Gamma^{\theta}} |w(x, y)|^2 \, dx \, dy = 0,
\]
then $w = 0$. Then the closure $\Gamma$ contains $(n(\xi'), 0)$ for each $\xi' \in W$ and $N \leq r$. Here $n(\xi')$ is as in Lemma 2.12.

When $r = 0$, the assumption (A-2) implies that $B$ is empty or a real analytic set. Noting this, we have the followings in the same way.

**Lemma 2.23.** Let $u$ and $v$ be as in the first place of this section and Lemma 2.5, respectively. (1) Suppose that the hypotheses of Case 4 are fulfilled and $B$ is empty. If $u$ belongs to $L^2_{x_1 \theta}(R^{n+1}_+)$ with
\[
\lim_{R \to \infty} R^{-(r+1)} \int_{\Gamma^{(I)\theta}} |u(x, y)|^2 \, dx \, dy = 0, \quad j = 1, \ldots, a,
\]
then $v = 0$. (2) Suppose that the hypotheses of Case 4 are fulfilled and $B$ is a real analytic set in $W$. If $u$ belongs to $L^2_{x_1 \theta}(R^{n+1}_+)$ with

\[
\lim_{R \to \infty} R^{-(r+1)} \int_{\Gamma^{(I)\theta}} |u(x, y)|^2 \, dx \, dy = 0, \quad j = 1, \ldots, a,
\]
\[
\lim_{R \to \infty} R^{-(N+r+r)} \int_{\Gamma^{(I)\theta}} |u(x, y)|^2 \, dx \, dy = 0,
\]
then $v = 0$. Here $\Gamma^{(I)\theta}$ is as in Lemma 2.17 and $\Gamma^{(I)\theta}_N$ and $N$, are as in Lemma 2.14.

**Proposition 2.24.** Suppose that the hypotheses of Case 4 are fulfilled and $B$ is a real analytic set. Let $\Gamma$ be an open cone in $R^{n+1}$ and $N$ an integer such that if $w \in S'(R^{n+1}_+ \cap C^\infty(R^{n+1}_+))$ is a solution of the equations (1.1) and (1.2) with
\[
\lim_{R \to \infty} R^{-N} \int_{\mathbb{R}^2} |w(x, y)|^2 \, dx \, dy = 0
\]
and the support of \( w(x, y) \) is contained in \( \{(\xi', \mu(\xi')); \xi' \in W\} \times \mathbb{R}_+^2 \), then \( w = 0 \).

If \( C \subset B \) is a \( C^m \) manifold, then the closure of \( \Gamma \) contains \( (n(\xi'), 0) \) for each \( \xi' \in C \) and some normal \( \neq 0 \) of \( M_j \) at \( (\xi', \mu(\xi'), \lambda_j^2(\xi')) \) for each \( \xi' \in W - B \). Moreover, we have \( N \leq r + 1 \). Here \( n(\xi') \) is as in Lemma 2.14.

§3. Proof of the Main Theorem

Now we prove the Main Theorem. When \( m = 0 \), it follows immediately from Lemma 2.2 \( (r = 0) \). We may assume that \( m \geq 1 \). Let \( u \) be a solution of the equations (1.1) and (1.2) which belongs to \( C^m([0, \delta]; \mathcal{S}'(\mathbb{R}^n) \cap L_{\text{loc}}^1(\mathbb{R}^{n+1})) \) and satisfies the condition (1.3). For a while let \( W \) be as in Section 1. When the hypotheses of Case 1 are fulfilled in \( W \), we write \( W = X_{0,w} \) and choose \( N \) and \( \Gamma \) so that \( N \leq N_1 + 1 \) and \( \Gamma \supset \Gamma(II) \). Here if \( B = \emptyset \), we interpret that \( N_1 = \infty \) and \( \Gamma(II) = \emptyset \). When the hypotheses of Case 2 are fulfilled in \( W \), we denote by \( X_{0,w} \) each connected component of \( \{ \xi \in W, L_{\text{loc}}^1(\mathbb{R}^{n+1}) \} \) and choose \( N \) and \( \Gamma \) so that \( (\alpha_j - \delta_j) + 1 \geq N, j = 1, \ldots, k \) and \( \Gamma \supset \bigcup \{ \Gamma_j, 1 \leq j \leq k \} \). Here if the hypotheses of Lemma 2.16 are fulfilled, we interpret that \( \bigcup_{j=1}^k \Gamma_j = \emptyset \) and \( (\alpha_j - \delta_j) + 1 = \infty \). When the hypotheses of case 3 are fulfilled in \( W \), we write \( W = X_{0,w} \) and choose \( N \) and \( \Gamma \) so that \( N = 1 \) and \( \Gamma \supset \bigcup_{j=1}^k \Gamma_j \). When the hypotheses of Case 4 are fulfilled in \( W \), we write \( W = X_{0,w} \) and choose \( N \) and \( \Gamma \) so that \( N = 1 \) and \( \Gamma \supset \bigcup_{j=1}^k \Gamma_j \cup \Gamma(II) \). Here if \( B = \emptyset \), we interpret that \( \Gamma(II) = \emptyset \). Put \( Y_0 = \bigcup_{w} X_{0,w} \). Thus, using the notation in Section 1, we have

\[
E = Y_0 \cup A_0 \cup A_{s,m} \cup \bigcup \{ A_{\nu, \mu,m} \} \subseteq Y_0 \cup Y_1,
\]
where \( E = \bigcup \{ \xi \in W; L_{\text{loc}}^1(\mathbb{R}^{n+1}) (\xi) = 0 \} \) and it is obvious that \( Y_0 \) is open. Let \( U \) be any open set such that \( U \) is contained in \( W \cap Y_0 \) for some \( W \). Let \( \phi \) be any \( C^m_0(U) \) function. It follows from Section 2 that \( \phi(\xi) \partial(\xi, y) = 0 \). Thus we have that the support of \( \partial(\xi, y) \) is contained in \( Y_1 \times \mathbb{R}_+^2 \). Suppose that \( \{ \xi \in W; L_{\text{loc}}^1(\mathbb{R}^{n+1}) (\xi) = 0 \} \) is not empty for some \( W \). Let \( U \) be any small open set contained in \( W \) and let \( \phi \) be any \( C^m_0(\{ \xi \in U; (\partial/\partial \xi) L_{\text{loc}}^1(\mathbb{R}^{n+1}) (\xi) \neq 0 \}) \). \( A_2 = \{ \xi \in U; L_{\text{loc}}^1(\mathbb{R}^{n+1}) (\xi) = 0, (\partial/\partial \xi) L_{\text{loc}}^1(\mathbb{R}^{n+1}) (\xi) \neq 0 \} \) is an analytic manifold of codimension 1 or empty and the support of \( \phi(\xi) \partial(\xi, y) \) is contained in \( A_2 \times \mathbb{R}_+^2 \). If \( A_2 \) is not empty, \( A_2 \) may be defined by \( \xi_n = \mu(\xi') \) where \( \xi' \in \omega \subset E^{n-1} \), \( \mu \) is a real analytic function in \( \omega \) and \( \xi' = (\xi_1, \ldots, \xi_{n-1}) \). We can denote by \( f_j^2(\xi', \mu(\xi')) - f_j^2(\xi'), j = 1, \ldots, a, f_j^2(\xi', \mu(\xi')) = f_j^2(\xi'), j = 1, \ldots, b, \) and \( f_j^2(\xi', \mu(\xi')) = f_j(\xi') \),
When the hypotheses of Case 2 are fulfilled in $\omega$, we choose $N$ and $\Gamma$ so that $N \subseteq (\alpha_j - \delta_j)^2 + 2$, $j = 1, \ldots, k$, and $\Gamma \supset \{ \Gamma_j; 1 \leq j \leq k \}$. Then we have that the support of $\phi(\xi)\alpha(\xi, y)$ is contained in $\{ \xi \in \omega; L(\ell', \ell') \alpha(\xi') = 0 \}$ by Lemma 2.17. When the hypotheses of Case 3 are fulfilled in $\omega$, we choose $N$ and $\Gamma$ so that $N \subseteq 2$ and $\Gamma \supset \{ \Gamma_j; 1 \leq j \leq a \}$. Then we have $\phi(\xi)\alpha(\xi, y) = 0$ by Lemma 2.19. If the hypotheses of Case 4 are fulfilled in $\omega$, we choose $N$ and $\Gamma$ so that $N \subseteq 2$ and $\Gamma \supset \{ \Gamma_j; 1 \leq j \leq a \}$ and $\Gamma^{(\text{II})}$.

Here if $B$ stated in Lemma 2.23 is empty we interpret that $\Gamma^{(\text{II})} = \emptyset$. Then we have that $\phi(\xi)\alpha(\xi, y) = 0$ by Lemma 2.23. In the same way, we have the support of $\alpha(\xi, y)$ is contained in $A_0 \cup A_{a_m} \cup (\bigcup A_{\nu_j, 1_m})$. Let $U$ be any small open set in $V$ stated in Section 1. Let $\phi$ be any $C_0^\infty(\{ \xi \in U; (\partial/\partial \xi_n)\Omega(\xi) \neq 0, a_m(\xi) \neq 0 \}$ function. The support of $\phi(\xi)\alpha(\xi, y)$ is contained in $\{ \xi \in U; (\partial/\partial \xi_n)\Omega(\xi) \neq 0, a_m(\xi) \neq 0 \}$ which is a real analytic manifold of codimension 1 or empty. In the same way, we choose $N$ and $\Gamma$ so that $\phi(\xi)\alpha(\xi, y) = 0$. By repeating the argument, we have that the support of $\alpha(\xi, y)$ is contained in $A_0 \cup A_{a_m}$. Let $U$ be any small open set contained in $\{ \xi \in \mathbb{B}_n; (\partial/\partial \xi_n)\Omega(\xi) \neq 0, a_m(\xi) \neq 0 \}$. Let $\phi$ be any $C_0^\infty(U)$ function. The support of $\phi(\xi)\alpha(\xi, y)$ is contained in $\{ \xi \in U; (\partial/\partial \xi_n)\Omega(\xi) \neq 0, a_m(\xi) \neq 0 \}$ which is empty or a real analytic manifold of codimension 1. In the same way, we have that $\phi(\xi)\alpha(\xi, y) = 0$. We repeat above reasoning on $\{ \xi \in \mathbb{B}_n; a_m(\xi) \neq 0 \}$, and then we have that the support of $\alpha(\xi, y)$ is contained in $A_{a_m}$. Let $U$ be any small open set contained in $\{ \xi \in \mathbb{B}_n; (\partial/\partial \xi_n)\alpha_m(\xi) \neq 0, a_m(\xi) \neq 0 \}$. Let $\phi$ be any $C_0^\infty(U)$ function. The support of $\phi(\xi)\alpha(\xi, y)$ is contained in $\{ \xi \in \mathbb{B}_n; (\partial/\partial \xi_n)\alpha_m(\xi) \neq 0, a_m(\xi) \neq 0 \}$ which is a real analytic manifold of codim 1 or empty and contained in $X_{m-1}$. In the same way, we have, using the results of section 2, that $\phi(\xi)\alpha(\xi, y) = 0$. By repeating the argument we conclude that the support of $\alpha(\xi, y)$ is contained in $X_{m-2}$. Repeated arguments imply that the support of $\alpha(\xi, y)$ is contained in $X_0$. Therefore, we choose $N$ and $\Gamma$ so that $\alpha(\xi, y) = 0$ by Lemma 2.2.

Finally we show the last statement of the Main Theorem. First of all, we suppose that the assumption (A-1) is not fulfilled, that is, there exists a $\xi^0 \in \mathbb{B}_n$ such that the number of roots with positive imaginary parts of the equation $P(\xi^0, \lambda) = 0$ in $\lambda$ is greater than $p$. Since the union of all $W$ stated in Section 1 is dense in $\mathbb{B}_n$, any open neighborhood $U$ of $\xi^0$ intersects some $W$. Thus, $b > p$ for some $W$. Put
\[ A(\xi) = (f_{j,k}(\xi))_{j=1,\ldots,p}^{i=1,\ldots,q}, \quad \xi \in W, \]

where

\[ f_{j,k}(\xi) = (2\pi i)^{-1} \oint B_{j}(\xi, \lambda) \lambda^{k-1}(P^{*}(\xi, \lambda))^{-1} d\lambda. \]

Since \( b > p \), \( A(\xi) \) has rank \( r \) \((0 \leq r \leq p)\) in \( W \), that is, some minor \( A(\xi) \) of order \( r \) does not vanish identically in \( W \) when \( 1 \leq r \leq p \) and every \((r + 1), \ldots, p\)-rowed minor of \( A(\xi) \) vanish identically in \( W \) when \( 1 \leq r \leq p - 1 \). When \( r \neq 0 \), we may assume without loss of generality that

\[ A(\xi) = \det (f_{j,k}(\xi))_{j=1,\ldots,r}, \]

When \( 1 \leq r \leq p \), we put

\[ G(\xi, y) = \left\{ \begin{array}{ll}
-\sum_{j,k=1}^{r} A_{j,k}(\xi) f_{j,k}(\xi) \{ (2\pi i)^{-1} \oint e^{-y\lambda} \lambda^{k-1}(P^{*}(\xi, \lambda))^{-1} d\lambda \} \\
+ A(\xi)(2\pi i)^{-1} \oint e^{iy\lambda} (P^{*}(\xi, \lambda))^{-1} d\lambda \phi(\xi),
\end{array} \right. \]

where \( A_{j,k}(\xi) \) is the \((j, k)\) cofactor of \((f_{j,k}(\xi))_{j=1,\ldots,r}\) and \( \phi(\xi) \) is a \( C_{\infty}^{0}(\{\xi \in W; A(\xi) \neq 0\}) \) function. It is obvious that we have

\begin{align*}
(5.1) & \quad P(\xi, D_{j}) G(\xi, y) = 0, \quad y > 0, \\
(5.2) & \quad B_{j}(\xi, D_{j}) G(\xi, y) |_{y=0} = 0, \quad j = 1, \ldots, p.
\end{align*}

When \( r = 0 \), we put

\[ G(\xi, y) = (2\pi i)^{-1} \oint e^{iy\lambda} (P^{*}(\xi, \lambda))^{-1} d\lambda \phi(\xi) \]

where \( \phi(\xi) \) is a \( C_{\infty}^{0}(W) \) function. It is obvious that \( G(\xi, y) \) satisfies the equations \((5.1)\) and \((5.2)\). Put

\[ u(x, y) = (2\pi)^{-n} \int G(\xi, y) \exp (ix\xi) d\xi. \]

It is obvious that \( u(x, y) \) belongs to \( S(\mathbb{R}_{+}^{d+1}) \) and satisfies the equations \((1.1)\) and \((1.2)\), since \( G(\xi, y) \) satisfies the equations \((5.1)\) and \((5.2)\). Next, we suppose that the assumption \((A-1)\) is fulfilled but the assumption \((A-2)\) is not fulfilled for some \( W \). Then \( p \geq b \) and \( L^{+}_{\omega}(\xi) \) stated in Section 1 vanish identically for all \( \sigma = \{\sigma_{1}, \ldots, \sigma_{b}\} \subset \{1, \ldots, p\} \) in \( W \). Thus it follows from the same reason as the first case that there exists a solution \( u(x, y) \) of the equations \((1.1)\) and \((1.2)\) which belongs to \( S(\mathbb{R}_{+}^{d+1}) \). This completes the proof.
Remark. When we put $N=1$ and $\Gamma=\mathbb{R}^{n+1}$, for any system $\{P(D), B_j(D), j=1, \cdots, p\}$ which satisfies the assumptions (A-1) and (A-2) we have that if $u \in L^2_{\text{loc}}(\mathbb{R}^{n+1}) \cap C^\infty(\mathbb{R}^n)$ is a solution of the equations (1.1) and (1.2) with
\[
\lim_{R \to \infty} R^{-1} \int_{R < |(x, y)| < 2R} |u(x, y)|^2 \, dx \, dy = 0 ,
\]
then $u=0$.

Example 1 (Rellich [12] or Agmon [13]). We consider a solution $u \in C^\infty([0, \delta)S'(\mathbb{R}^n)) \cap L^2_{\text{loc}}(\mathbb{R}^{n+1})$ of the equations:
\begin{align*}
(5.3) \quad & Au + ku = 0 \quad \text{in} \quad \mathbb{R}^{n+1}, \\
(5.4) \quad & u|_{\partial \Omega} = 0 \quad \text{in} \quad \mathbb{R}^n ,
\end{align*}
where $A = -\sum_{j=1}^n D_j^2 - D_0^2$ and $k > 0$. Put $A^+ = \{\xi \in \mathbb{S}^n; |\xi|^2 > k\}$, $A_0^+ = \{\xi \in \mathbb{S}^n; |\xi|^2 < k, \xi, > 0\}$, $A_0^- = \{\xi \in \mathbb{S}^n; |\xi|^2 < k, \xi, < 0\}$. When $\xi \in A^+$, we can denote by $\lambda^\pm(\xi) = \pm i\sqrt{|\xi|^2 - k}$ the roots of the equation $\lambda^2 + |\xi|^2 - k = 0$. It follows from Lemma 2.5 that
\[
\langle (D_\gamma - \lambda^+(\xi))\hat{u}(\xi, y), v(\xi, y) \rangle = 0 ,
\]
for any $v(\xi, y) \in \mathcal{S}(A^+ \times R_\Omega)$. Since $u|_{\partial \Omega} = 0$, we have
\[
\langle (D_\gamma - \lambda^+(\xi))\hat{u}(\xi, y), v(\xi, y) \rangle = \langle i\langle \hat{u}(\xi, 0), v(\xi, 0) \rangle + \langle \hat{u}(\xi, y), (-D_\gamma - \lambda^+(\xi))v(\xi, y) \rangle, \langle (-D_\gamma - \lambda^+(\xi))v(\xi, y) \rangle \rangle .
\]
Then it follows from Lemma 2.6 that the support of $\hat{u}(\xi, y)$ is contained in $\{\xi \in \mathbb{S}^n; |\xi|^2 \leq k\}$. Put $\lambda_0^\pm(\xi) = \pm \sqrt{k - |\xi|^2}, \xi \in \mathbb{S}^n; |\xi|^2 < k\}$. Let $u$ satisfy the condition:
\[
\lim_{R \to \infty} R^{-1} \int_{\Gamma_R} |u(x, y)|^2 \, dx \, dy = 0
\]
where $\Gamma$ is an open cone which contains a normal of $M = \{(x, y) \in \mathbb{R}^{n+1}; |x|^2 + y^2 = k, |x|^2 \leq k$ and $x_1 \geq 0\}$ for every $(x, y) \in M$ and $\Gamma_R = \{(x, y)|\Gamma; y \geq 0, R < |(x, y)| < 2R\}$. When $\xi \in A_0^+$, $\Gamma$ contains an outer normal of $\{(\xi, \lambda_0^+(\xi)); \xi \in A_0^+\}$ at $\xi \in A_0^+$. It follows from Lemma 2.7 that
\[
\langle (D_\gamma - \lambda_0^+(\xi))\hat{u}(\xi, 0), \chi(\xi) \rangle = 0 ,
\]
for any \( x(\xi) \in C_0^\infty(A^0_+) \). Since \( u | y = 0 \), we have that
\[
\langle D_j \hat{u}(\xi), x(\xi) \rangle = 0, \quad j = 0, 1,
\]
for any \( x(\xi) \in C_0^\infty(A^0_+) \). Thus we have that the support of \( \hat{u}(\xi, y) \) is contained in \([\{\xi \in B^n; |\xi|^2 < k, \xi_1 = 0\} \cup \{\xi \in B^n; |\xi|^2 = k\}] \times R^1_+\). When \( \xi \in A_0^0 \), \( \Gamma \) contains an inner normal of \( \{(\xi, \lambda_0^k(\xi)); \xi \in A^0_+\} \). It follows from Lemma 2.7 that
\[
\langle (D_j - \lambda_0^k(\xi))u(\xi, 0), x(\xi) \rangle = 0,
\]
for any \( x(\xi) \in C_0^\infty(A^0_+) \). So we have that the support of \( \hat{u}(\xi, y) \) is contained in \([\{\xi \in B^n; |\xi|^2 < k, \xi_1 = 0\} \cup \{\xi \in B^n; |\xi|^2 = k\}] \times R^1_+\). Let \( \phi \) be any \( C_0^\infty(\{\xi \in B^n; |\xi|^2 < k\}) \) function. The support of \( \phi(\xi)\hat{u}(\xi, y) \) is contained in \( \{\xi \in B^n; |\xi|^2 < k, \xi_1 = 0\} \times R^1_+\) and \( \{\xi \in B^n; |\xi|^2 < k, \xi_1 = 0\} \) is a real analytic set of codimension 1. \( \Gamma \) contains some normal of \( \{(\xi \cdot \lambda_0^k(\xi)); \xi_1 = 0, |\xi|^2 < k\} \). \( \phi(\xi)\hat{u}(\xi, y) \) has the form:
\[
\phi(\xi)\hat{u}(\xi, y) = \sum_{|\alpha| \leq s} v_\alpha(\xi', y) \otimes D_{\xi_1} \delta(\xi_1),
\]
where \( \xi' = (\xi_2, \ldots, \xi_n) \). It follows from Lemma 2.7 that
\[
\langle (D_j^2 + |\xi'|^2 - k)v_\alpha(\xi', y), w(\xi', y) \rangle = 0,
\]
\[
\langle v_\alpha(\xi', y), x(\xi') \rangle = 0,
\]
\[
\langle (D_j - \lambda_0^k)(0, \xi')v_\alpha(\xi', y), x(\xi') \rangle = 0, \quad |\alpha| = s
\]
for any \( w(\xi', y) \in S_0(\{\xi \in B^{n-1}_n; |\xi'|^2 < k\} \times R^1_+)\) and any \( x(\xi') \in C_0^\infty(\{\xi \in B^{n-1}_n; |\xi'|^2 < k\}) \). Then we have that \( \phi(\xi)\hat{u}(\xi, y) \) has the form:
\[
\phi(\xi)\hat{u}(\xi, y) = \sum_{|\alpha| \leq s-1} v_\alpha(\xi', y) \otimes D_{\xi_1} \delta(\xi_1).
\]
Repeating this reasoning on \( v_\alpha (|\alpha| \leq s - 1) \), we obtain that \( \phi(\xi)\hat{u}(\xi, y) = 0 \). So we have that the support of \( \hat{u}(\xi, y) \) is contained in \( \{\xi \in B^n; |\xi|^2 = k\} \). Let \( U \) be any open set contained in \( \{\xi \in B^n; \xi_1 > 0, \xi_2 + \cdots + \xi_n < k\} \) and let \( \phi \) be any \( C_0^\infty(U) \) function. The support of \( \phi(\xi)\hat{u}(\xi, y) \) is contained in \( A_1 \times R^1_+ \) where \( A_1 = \{(\sqrt{k} - |\xi'|^2, \xi') \in B^n; \xi' = (\xi_2, \ldots, \xi_n), |\xi'|^2 < k\} \) which is real analytic manifold of codimension 1 in \( B^n \). When \( \xi \in A_1 \), the roots of the equation \( P(\xi, \lambda) = \lambda^2 + |\xi|^2 - k = 0 \) in \( \lambda \) is zero with multiplicity 2, that is, \( P(\xi, \lambda) = \lambda^2, \xi \in A_1 \). \( 0, \ldots, 0, 1 \) is normal of real analytic manifold \( \{(\xi, 0); \xi \in A_1\} \) where \( P(\xi, \lambda) = 0 \) and \( \Gamma \) contains \( 0, \ldots, 0, 1 \). Let \( v \) be the composition of \( \phi(\xi)\hat{u}(\xi, y) \) and the map \( \xi \mapsto (\xi_1 + \sqrt{k - |\xi'|^2, \xi'}) \). \( v(\xi, y) \) has the form:
\[
v(\xi, y) = \sum_{|\alpha| \leq s} v_\alpha(\xi', y) \otimes D_{\xi_1} \delta(\xi_1),
\]
for the support of $v(\xi, y)$ is contained in the plane $\xi_1 = 0$. It follows from Lemma 2.7 that
\[
\langle D_y^2 v_a(\xi', y), w(\xi', y) \rangle = 0 ,
\]
\[
\langle v_a(\xi', y)|_{y=0}, x(\xi') \rangle = 0 ,
\]
\[
\langle D_y v_a(\xi', y)|_{y=0}, x(\xi') \rangle = 0 ,
\]
for any $w(\xi', y) \in S_0(\{\xi' \in B^{a-1}; |\xi'|^2 < k\} \times \mathbb{R}^1)$ and $x(\xi') \in C_0(\{\xi' \in B^{a-1}; |\xi'|^2 < k\})$. In the same way, we have that the support of $u(\xi, y)$ is contained in $\{\xi \in \mathbb{B}; \xi_1 = 0, \xi_2^2 + \cdots + \xi_k^2 = k\}$. Repeating the argument we conclude that $u = 0$. Summing up, we have proved.

**Theorem** (Rellich [12] or Agmon [1]). Let $\Gamma$ be an open cone in $\mathbb{R}^{n+1}$ which contains a normal of $M = \{(x, y) \in \mathbb{R}^{n+1}; x^2 + y^2 = k, |x|^2 \leq k\}$ at every $(x, y) \in M$. Set
\[
\Gamma_R = \{(x, y) \in \Gamma; y \geq 0, R < |(x, y)| < 2R\}.
\]
If $u \in C^\infty((0, \delta); S'(\mathbb{R}^n)) \cap L^2_{\text{loc}}(\mathbb{R}^{n+1})$ is a solution of the equations (5.3) and (5.4) with
\[
\lim_{R \to \infty} R^{-1} \int_{\Gamma_R} |u(x, y)|^2 dxdy = 0 ,
\]
then $u = 0$.

**Remark.** Let $\phi(\xi)$ be a $C^\infty(\{\xi \in \mathbb{B}; |\xi|^2 < k, \xi_1 > 0\})$. Put
\[
v(x, y) = \int e^{i\alpha \xi} \{e^{i\alpha \xi - y_1 \xi} - e^{i\alpha \xi + y_1 \xi}\} \phi(\xi) d\xi .
\]
Then $v(x, y)$ is a solution of the equations (5.3) and (5.4) with
\[
C_1 R \leq \int_{\Gamma_R} |v(x, y)|^2 dxdy \leq C_2 R
\]
for some positive constants $C_1$ and $C_2$.

**Example 2.** We consider a solution $u \in C^\infty((0, \delta); S'(\mathbb{R}^n)) \cap L^2_{\text{loc}}(\mathbb{R}^{n+1})$ of the equations:
\[
P(D)u = (D^2_x + D^2_y - k)u = 0 , \quad \text{in} \quad \mathbb{R}^{n+1} ,
\]
\[
B_1(D)u|_{y=0} = (D_x - i(k-1/2))u|_{y=0} = 0 , \quad \text{in} \quad \mathbb{R}^n ,
\]
\[
B_2(D)u|_{y=0} = (D_x - iD_1 + i)u|_{y=0} = 0 , \quad \text{in} \quad \mathbb{R}^n ,
\]
where \( k \) satisfies \( ((k + 1)/2)^2 > k > 1 \). When \( \xi^2 > k \), we can denote by \( \lambda^\pm(\xi) = \pm i \sqrt{\xi^2 - k} \) the roots of the equations \( P(\xi, \lambda) = 0 \) in \( \lambda \). When \( \xi^2 > k \) and \( \xi_1 = (k + 1)/2 \), we have

\[
(2\pi i)^{-1} \oint B_j(\xi, \lambda)(\lambda - \lambda^+(\xi))^{-1} d\lambda = 0, \quad j = 1, 2.
\]

When \( \xi^2 > k \) and \( \xi_1 = (k + 1)/2 \), we have

\[
(2\pi i)^{-1} \oint B_j(\xi, \lambda)(\lambda - \lambda^+(\xi))^{-1} d\lambda = 0, \quad j = 1, 2.
\]

Where \( \xi^2 < k \), the roots of the equation \( P(\xi, \lambda) = 0 \) are all real and we have

\[
det ((2\pi i)^{-1} \oint B_j(\xi, \lambda)\lambda^{k-1}(P(\xi, \lambda))^{-1} d\lambda)_{j,k=1,2}
= det \begin{bmatrix} 1, & -i(k-1)/2 \\ 1, & -i\xi_1 + i \end{bmatrix} = -i(\xi_1 - (k + 1)/2) \neq 0.
\]

In the same way, when \( \xi^2 = k \), we have

\[
det ((2\pi i)^{-1} \oint B_j(\xi, \lambda)\lambda^{k-1}(P(\xi, \lambda))^{-1} d\lambda)_{j,k=1,2} = 0.
\]

Then the support of \( \hat{u}(\xi, y) \) is contained in the plane \( \{\xi \in \mathbb{S}^n; \xi_1 = (k + 1)/2\} \times \mathbb{R}_1^+ \). This is real analytic manifold of codimension 1 in \( \mathbb{S}^n \) and \( (1, 0, \ldots, 0) \) is its normal. Let \( \Gamma \) be an open conic neighborhood of \( (1, 0, \ldots, 0) \) in \( \mathbb{R}^{n+1} \) and set

\[
\Gamma_R = \{(x, y) \in \Gamma; \ y \geq 0, \ R < |(x, y)| < 2R\}.
\]

If \( u \) satisfies the condition:

\[
\lim_{R \to \infty} R^{-1} \int_{\Gamma_R} |u(x, y)|^2 dx dy = 0,
\]

then \( u = 0 \).

Remark. Let \( \psi \) be a \( C^\infty_0(\mathbb{S}^{n-1}) \) function. Put

\[
w(x, y) = \int \exp \{i((k + 1)/2 \cdot x + x'' \cdot \xi''') - (k - 1)/2 \cdot y\} \cdot \psi(\xi'') d\xi''
\]

where \( \xi'' = (\xi_2, \ldots, \xi_n) \). \( w(x, y) \) satisfies the equations (5.5)–(5.7) and there exist positive constants \( C_1 \) and \( C_2 \) such that

\[
C_1 R \leq \int_{\{(x, y) \in \Gamma; \ y \geq 0\}} |w(x, y)|^2 dx dy \leq C_2 R,
\]

for \( R > 0 \).
for small positive $\varepsilon$.

**Example 3.** Let us consider a solution $u \in C^\infty([0, \delta); \mathcal{S}'(\mathbb{R}^n)) \cap L^1_{\text{loc}}(\mathbb{R}^n_+) \cap L^1_{\text{loc}}(\mathbb{R}^n_+)$ of the equations (1.1) and (1.2) for a system:

$$
P(D) = (D_x - D_y)(D_x - D_y - D_1 - D_3)^2, \quad B_1(D) = 1, \quad B_2(D) = D_y,$$

$$
B_3(D) = D_y^2, \quad B_4(D) = D_y^2 + \sum_{j=1}^n D_jD_y, \quad B_6(D) = D_y^4 - (4D_1 + 2)D_y^2.
$$

When $\xi_1 = 1/6$, we have

$$
\det \left((2\pi i)^{-1} \int B(\xi, \lambda)\lambda^{k-1}(\lambda - \xi_1)^{3}(\lambda - \xi_1 - 1)^{-2}d\lambda\right)_{j,k=1,\ldots,5} = 0,
$$

$$
\det \left((2\pi i)^{-1} \int B(\xi, \lambda)\lambda^{k-1}(\lambda - \xi_1)^{4}(\lambda - \xi_1 - 1)^{-1}d\lambda\right)_{j,k=1,\ldots,5} = 0,
$$

$$
\det \left((2\pi i)^{-1} \int B_{\sigma}(\xi, \lambda)\lambda^{k-1}(\lambda - \xi_1)^{-2}(\lambda - \xi_1 - 1)^{-2}d\lambda\right)_{j,k=1,\ldots,4} = 0
$$

for all $\sigma = (\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ with $\sigma_5 = 5$,

$$
\det \left((2\pi i)^{-1} \int B(\xi, \lambda)\lambda^{k-1}(\lambda - \xi_1)^{3}(\lambda - \xi_1 - 1)^{-1}d\lambda\right)_{j,k=1,\ldots,4} = 0,
$$

for all $\sigma = (\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ with $\sigma_5 = 5$,

$$
\det \left((2\pi i)^{-1} \int B_{\sigma}(\xi, \lambda)\lambda^{k-1}(\lambda - \xi_1)^{-2}(\lambda - \xi_1 - 1)^{-2}d\lambda\right)_{j,k=1,\ldots,4} = 0
$$

for all $\sigma$.

Since the normal of $\{((\xi, \lambda); \lambda = \xi_1, \xi_1 = 1/6\}$ and $\{((\xi, \lambda); \lambda = \xi_1 + 1, \xi_1 = 1/6\}$ is $(-1, 0, \ldots, 0, 1)$, it follows from Lemma 2.17 that if $u$ satisfies the condition:

$$
\lim_{R \to \infty} R^{-3} \int_{|x|,|y| < R^{-\frac{1}{2}} \cdot 1, \ldots, 0, 0, 1} |u(x, y)|^2 dx dy = 0,
$$

then the support of $u(\xi, y)$ is contained in $\{\xi \in \mathbb{R}^n; \xi_1 = 1/6\} \times \mathbb{R}^5$. On the other hand, there exist $C^\infty(\{\xi \in \mathbb{R}^n; \xi_1 = 1/6\})$ functions $C_{\sigma}(\xi)$ such that

$$
v(x, y) = \sum_{\sigma=0}^{1} C_{\sigma}(\xi)(iy)^{\sigma}(e^{i(x+y)\xi}) + \sum_{\sigma_0=0}^{1} C_{\sigma_0}(\xi)(iy)^{\sigma_0}(e^{i(x+y)\xi})\phi(\xi) d\xi
$$

is non-trivial solution of the equations (1.1) and (1.2) where $C_{1,2}(\xi) = 1$ and $
\phi(\xi) \in C^\infty(\{\xi \in \mathbb{R}^n; \xi_1 = 1/6\})$. Further, we have

$$
C_{1,2} R^5 \leq \int_{|x|,|y| < R^{-\frac{1}{2}} \cdot 1, \ldots, 0, 0, 1} |v(x, y)|^2 dx dy \leq C_2 R^5
$$
for some positive constants $C_1$ and $C_2$. This is an example for what we could
not take $N = \alpha_j - \delta_j$ in Proposition 2.18.

§4. Appendix

Here for the convenience of readers we state some results due to Hörmander [6] and Agmon-Hörmander [3] which is used in our paper without proof.

**Theorem A_p-1 ([6]).** If $u$ is a smooth density with compact support on a $C^\infty$ submanifold $M$ of $\mathbb{R}^n$ of codimension $k$, then

$$\int_{|\xi|<R} |\hat{u}(\xi)|^2d\xi \leq CR^k, \quad R>0.$$  \hspace{1cm} (A_p-1)

If $\Gamma$ is a closed cone in $\mathbb{R}^n$ which contains no element $\neq 0$ which is normal to $M$ at a point in $\text{supp } u$,

$$|\hat{u}(\xi)| < C_N(1 + |\xi|)^{-N}, \quad \xi \in \Gamma,$$  \hspace{1cm} (A_p-2)

for every integer $N$. Here $\hat{u}(\xi)$ denotes the Fourier transform of $u$.

**Theorem A_p-2 ([6]).** Let $u \in S'(\mathbb{R}^n)$, $\hat{u} \in L^2_{\text{loc}}, \quad \theta \in \mathbb{R}^n$ and $\epsilon > 0$. If $\chi \in C_0^\infty(\mathbb{R}^n)$ and $v = \chi u$, it follows that for every $k \in \mathbb{R}$

$$\lim_{R \to \infty} \int_{|\xi|<R} |\hat{v}(\xi)|^2d\xi \leq C \lim_{R \to \infty} \int_{|\xi|<2R} |\hat{u}(\xi)|^2d\xi,$$  \hspace{1cm} (A_p-3)

where $C = (2\pi)^{-n} \int |\chi|d\xi$.

**Theorem A_p-3 ([6]).** Let $u \in S'(\mathbb{R}^n)$ be supported by a real analytic set $A$ of codimension $k>0$, and assume that $\hat{u} \in L^2_{\text{loc}}$. Set

$$\Gamma_R = \{\xi \in \Gamma; \quad R < |\xi| < 2R\},$$

where $\Gamma$ is an open cone in $\mathbb{R}^n$ which for every analytic manifold $M \subset A$ and $x_0 \in M$ contains some normal of $M$ at $x_0$. If

$$\lim_{R \to \infty} \int_{\Gamma_R} |\hat{u}(\xi)|^2d\xi = 0,$$

it follows that $u = 0$.

**Theorem A_p-4 ([3]).** Let $\phi$ be a continuous function with compact support in $\mathbb{R}^n$. If $u \in S'$ and $\hat{u} = \hat{u}_d dS$ is an $L^2$ density with compact support on a $C^1$
manifold $M \subset \mathbb{R}^n$ of codimension $k$, then
\[
\lim_{R \to \infty} \int |u(x)|^2 \phi(x/R) dx/R^k = (2\pi)^{-k-n} \int_M |u_0(\xi)|^2 \left( \int_{N_\xi} \phi(x) d\sigma(x) \right) dS(\xi),
\]
where $dS$ is the Euclidean surface element on $M$ and $d\sigma$ is the Euclidean integration element in the normal plane $N_\xi$ of $M$ at $\xi$, passing through 0.

Modifying the proof of Theorem A$p$-4 slightly, we have the following.

**Corollary A$p$-5.** Let $U$ be an open set in $\mathbb{R}^{n-k}$, let $\mu_j(\xi'), j = 1, 2$, be $C^\infty(U)$ functions such that $\mu_1(\xi') = \mu_2(\xi')$ when $\xi' \in U$ and let $\chi$ be a $C^\infty(U)$ function. Put
\[
F_j(x) = \int \exp \{i(x' \cdot \xi' + x'' \cdot \mu_j(\xi'))\} \chi(\xi') d\xi', \quad j = 1, 2.
\]
Let $\phi$ be a $C^\infty(\mathbb{R}^n)$ function. Then it follows that
\[
\lim_{R \to \infty} \int F_1(x)F_2(x)\phi(x/R) dx/R^k = 0.
\]
Here $\mu_1(\xi') = \mu_2(\xi')$ means that $|\mu_1(\xi') - \mu_2(\xi')| > 0$ when $\xi' \in U$, $x' = (x_1, \ldots, x_{n-k})$, $x'' = (x_{n-k+1}, \ldots, x_n)$ and $\mu_j(\xi') = (\mu_{j-2k+1}(\xi'), \ldots, \mu_{j-k}(\xi'))$ ($j = 1, 2$).

**Proof.** Let $\phi(x) = \Phi(-x)$, thus $\Phi \in \mathcal{S}$. The Fourier transform of $R^{-k}\Phi(R\xi)$ and $F_j(x)$ are $R^{-k}\Phi(-x/R)$ and $\chi(\xi') \otimes \delta(\xi'' - \mu_j(\xi'))$, respectively, so the Fourier transform of $F_1(x)\phi(x/R)R^{-k}$ is
\[
(2\pi)^{-n} \int \Phi(R(\xi' - \eta')), R(\xi'' - \mu_1(\eta')) \chi(\eta') d\eta'.
\]
Hence it follows from Parseval's equality that
\[
\int F_1(x)F_2(x)\phi(x/R) dx/R^k = (2\pi)^{-n} \int \Phi(R(\xi' - \eta')), R(\mu_2(\xi') - \mu_1(\eta')) \chi(\xi') \chi(\eta') d\xi' d\eta'
\]
\[
= (2\pi)^{-n} \int \Phi(\eta', R(\mu_2(\xi') - \mu_1(\xi' - \eta'/R))) \chi(\xi' - \eta'/R) \chi(\xi') d\eta' d\xi' .
\]
Since
\[
|\chi(\xi' - \eta'/R)| < C,
\]
\[
|\Phi(\eta', R(\mu_2(\xi') - \mu_1(\xi' - \eta'/R)))| < C_N(1 + |\eta'|)^{-N} \quad \text{for any } N,
\]
for any $\varepsilon > 0$ there exists a large number $K$ such that
\[ \left| \iint_{|\xi'| \geq K} x(\xi') x(\xi' - \eta'/R) \Phi(\eta', R(\mu_2(\xi') - \mu_1(\xi' - \eta'/R))) d\xi' d\eta' \right| \leq C' \int |x(\xi')| d\xi' \cdot K^{-1} < \varepsilon , \]

where \( C' \) is independent of \( K \). On the other hand, we have

\[ R |\mu_2(\xi') - \mu_1(\xi' - \eta'/R)| > R |\mu_2(\xi') - \mu_1(\xi')| - |\mu_1(\xi')| |\eta'| - CR^{-1} , \]

when \( |\eta'| \leq K \) and \( \xi' \in \text{supp} \ x \), where \( C \) is some constant independent of \( R \). Since \( |\mu_2(\xi') - \mu_1(\xi')| \geq C' \) when \( \xi' \in \text{supp} \ x \), we have that

\[ |\Phi(\eta', R(\mu_2(\xi') - \mu_1(\xi' - \eta'/R)))| \leq C/(1 + C'R) , \]

\( \xi' \in \text{supp} \ x \) and \( |\eta'| \leq K \).

Therefore we have

\[ \lim_{R \to \infty} \iint x(\xi') x(\xi' - \eta'/R) \Phi(\eta', R(\mu_2(\xi') - \mu_1(\xi' - \eta'/R))) d\xi' d\eta' < \varepsilon , \]

for any \( \varepsilon > 0 \), that is,

\[ \lim_{R \to \infty} \iint x(\xi') x(\xi' - \eta'/R) \Phi(\eta', R(\mu_2(\xi') - \mu_1(\xi' - \eta'/R))) d\xi' d\eta' = 0 . \]

Q.E.D.

**References**


[16] Wakabayashi, S., Eigenfunction expansion for symmetric systems of first order in the half-space \( \mathbb{R}^n_* \), *Publ. RIMS, Kyoto Univ.*, 11 (1975), 67–147.