On The Equivariant Isotopy Classes of Some Equivariant Imbeddings of Spheres

By

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§ 0. Introduction

The purpose of this paper is to study the G-isotopy classes of G-imbeddings of spheres into spheres, where the spheres are equipped with semi-free linear G-actions for a finite group G.

Let $V$ be an $m$-dimensional real $G$-module. Throughout this paper we shall assume that $V$ is a product module $V = R \oplus V_1$ of a trivial real $G$-module $R$ of positive dimension $n$ and an $(m-n)$-dimensional real $G$-module $V_1$ on the $G$-invariant unit sphere $S(V_1)$ of which $G$ acts freely. Let $W$ be a real $G$-module which contains $V$ as a direct summand. Let $S_v$ and $S_w$ denote the one-point compactifications of $V$ and $W$ respectively. Then $S_v$ and $S_w$ are spheres on which $G$ acts linearly. The direct sum of $d$ copies of $V$ will be denoted by $dV$.

**Theorem A.** Let $G$ be a cyclic group $Z_q$ and let $W = dV \oplus R^t$ for $k > m+1$. If $d \geq \max \{(n+3)/2, (m+2)/(m-n)\}$, then any $G$-imbedding of $S_v$ into $S_w$ is $G$-isotopic to the standard imbedding.

**Theorem B.** Let $G$ be a cyclic group $Z_q$ for $q \geq 2$ and let $W = dV \oplus R^t$ for $k > m+1$. Suppose that $d \geq (m+1)/(m-n)$ and $V_1$ is a direct sum of $(m-n)/2$ copies of an irreducible 2-dimensional real $G$-module.

(1) If $d = (m+1)/(m-n)$, then there are infinitely many $G$-
imbeddings of \( S^r \) into \( S^w \) which are not G-isotopic to each other, and

(2) if \( d > (m+1)/(m-n) \), then any G-imbedding of \( S^r \) into \( S^w \) is G-isotopic to the standard imbedding.

The paper is organized as follows. For any G-imbedding \( f: S^r \to S^w \), we shall show that, by G-isotopies, \( f|S^r \) can be deformed to be standard in § 1, \( f \) can be deformed to be linear on a neighborhood of \( S^* \) in § 2 and \( f \) can be deformed to be orthogonal on a neighborhood of \( S^* \) in § 3. Moreover we shall prove that, if two G-imbeddings of \( S^r \) into \( S^w \) are G-isotopic and are orthogonal on a neighborhood of \( S^* \), then there exists a G-isotopy between them which is orthogonal on a neighborhood of \( S^* \) in § 3. Then we see that the G-isotopy class of \( f \) is determined by the homotopy class of the orbit map of \( f|S^r - U \) relative to the boundary, where \( U \) is a neighborhood of \( S^* \). In § 4, using the obstruction theory, we shall prove Theorem A and Theorem B.

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§1. Imbeddings Can Be Deformed to Be Standard on the Fixed Point Set

In this paper we shall assume that all manifolds and all actions are differentiable of class \( C^\infty \). Until Section 3 the results are valid in the case of \( G \) a compact Lie group.

In this section we shall prove that any G-imbedding of \( S^r \) into \( S^w \) is G-isotopic to a G-imbedding which is standard on \( S^* \) (see Proposition 1.3), and if two G-imbeddings of \( S^r \) into \( S^w \), which are standard on \( S^* \), are G-isotopic, then there exists a G-isotopy between them which is standard on \( S^* \) (see Proposition 1.4).

Definition 1.1. Let \( M \) be a G-submanifold of a G-manifold \( N \). Let \( I \) denote the unit interval \([0,1]\) with trivial G-action. A smooth map (resp. smooth G-map) \( f: M \times I \to N \) is said to be an isotopy (resp. G-isotopy) if each \( f_t: M \to N \) is an imbedding (resp. G-imbed-
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ding), where \( f_i(x) = f(x, t) \), and \( f_i \) is independent of \( t \) in some neighborhood of 0 and in some neighborhood of 1 (see G. Bredon [1, Chapter VI, §3]). Two imbeddings (resp. G-imbeddings) \( f_i : M \to N \) \((i = 0, 1)\) are said to be isotopic (resp. G-isotopic) if there exists an isotopy (resp. G-isotopy) \( F : M \times I \to N \) with \( F_0 = f_0 \) and \( F_1 = f_1 \). If \( \partial M \) is not empty, we shall consider \( M \times I \) as a smooth manifold with corners.

Let \( I(S^r, S^w) \) denote the set of all G-isotopy classes of G-imbeddings \( f : S^r \to S^w \). Our purpose is to determine the set of \( I(S^r, S^w) \), provided that \( W = dV \oplus \mathbb{R}^q \) for \( k > m + 1 \).

Remarks. 1. It is easy to see that any G-map \( f : S^r \to S^w \) is G-homotopic to the standard imbedding.

2. Using the method of A. Wasserman [7, § 1], we can see that any G-imbedding \( f : S^r \to S^w \) is G-isotopic to the standard imbedding if \( d > 2m + 2 \).

The following lemma will be useful.

Lemma 1.2. Let \( N \) be a \( q \)-dimensional manifold on which \( G \) acts semi-freely and let \( M \) be a \( p \)-dimensional G-submanifold of \( N \). Let \( K \) denote \( I \) or \( I \times I \) and let \( L \) be a closed subset of \( K \) which contains \( \partial K \). Let \( f : M \times K \to N \) be a continuous G-map such that each \( f_k : M \to N \) is a G-imbedding, where \( f_k \) is defined by \( f_k(x) = f(x, k) \). If \( f \) is a smooth G-map on \( M \times L' \), where \( L' \) is a neighborhood of \( L \) in \( K \), then there exists a smooth G-map \( H : M \times K \to N \) such that each \( H_k \) is a G-imbedding and \( H = f \) on \( M \times L \), where \( H_k : M \to N \) is defined by \( H_k(x) = f(x, k) \).

Proof. We shall prove Lemma 1.2 by an equivariant version of J. Munkres' argument [4, Chapter I, §4]. Let \( \{ U_i \} \) (resp. \( \{ V_j \} \)) be a family of locally finite countable invariant open sets of \( M \) such that \( \bigcup_i U_i \subset M^o \) (resp. \( \bigcup_j V_j \supset N^o \)) and \( U_i \) (resp. \( V_j \)) is equivariant diffeomorphic to a \( p \)-dimensional disc or half disc (resp. \( q \)-dimensional
euclidean space or euclidean half space) with linear $G$-action, where $M^o$ and $N^o$ denote the fixed point set of $M$ and $N$ respectively. We can choose the family $\{U_i\}$ such that, for any $k \in K$ and for any $i$, $f_k(U_i)$ is contained in $V_j$ for some $j$ depending on $k$ and $i$. There exists a positive continuous function $\delta_i$ on $M$ as follows. For any continuous map $g : M \times K \rightarrow N$, such that $g_\epsilon$ is $\delta_i$-approximation to $f_k$ for each $k$, has these properties. Let $\{W_i\}$ be a family of invariant open sets of $M$ with $\bigcup_i W_i \supseteq M^o$.

Let $C_r(y)$ denote a closed $r$-neighborhood of $y$ in $K$ for a positive number $r$ and $y \in K$. There exists a sufficiently small positive number $r$ such that, for any $y \in C_r(y)$ and for any $y \in K$, $f_k(U_k)$ is contained in $V_j$ for some $j$ depending on $y$. Then we can find a finite number of $C_r(y)$, say $C_r = C_r(y)$ $(n=1, 2, \ldots, l)$, such that $\bigcup C_r \subseteq K - L'$. We can assume that $C_r \cap L = \emptyset$ for any $n$. Let $\phi_i : M \rightarrow I$ be an invariant smooth function on $M$ which equals 1 on $W_i$ and 0 outside of $U_i$. Let $A_n$ and $B_n$, $n=1, 2, \ldots, l$, be open sets of $K$ such that $\bar{A}_n \subseteq B_n \subseteq \bar{B}_n \subseteq \text{int} C_r$ and $\bigcup A_n$ contains $K - L'$. Let $\xi_n : K \rightarrow I$, $n=1, 2, \ldots, l$, be smooth functions on $K$ which equals 1 on $\bar{A}_n$ and 0 outside of $B_n$.

We shall identify $U_i$ and $V_j$ as euclidean spaces or euclidean half spaces with linear $G$-actions. For any $n$, we can find $j(n)$ such that $f_k(U_k) \subseteq V_j$ for any $z \in C_r$. Let $f_i : M \times C_r \rightarrow V_j$, be a $G$-map defined by $f_i(x) = \phi_i(x) \cdot f_k(x)$ for $x \in U_i$ and $z \in C_r$, and $f_i = 0$ outside of $U_i \times C_r$. Let $g_i : M \times K \rightarrow V_j$, be a $G$-map defined by $g_i(x) = \xi_n(x) \cdot f_i(x)$ for $x \in M$ and $z \in K$. Since $g_i = 0$ for $z \notin C_r$, we can extend $g_i$ trivially on $M \times R$ (resp. $M \times R^2$) if $K = I$ (resp. $I \times I$). Define a smooth $G$-map $h_i : M \times K \rightarrow V_j$ by

$$h_i(x) = \int_{C_r} \varphi_\ast(y) \cdot g_i(x) dy \text{ for } x \in M \text{ and } z \in K,$$

where $C(\epsilon_\ast)$ is a closed $\epsilon_\ast$-disc in $K$ and $\varphi_\ast$ is a smooth function on $R$ or $\mathbb{R}^2$ which is positive on int $C(\epsilon_\ast)$ and 0 outside of $C(\epsilon_\ast)$ and $\int_{C(\epsilon_\ast)} \varphi_\ast(y) dy = 1$. Choose the positive number $\epsilon_\ast$ less than the distance from $B_n$ to the complement of $C_r$. Then $h_i = 0$ outside of $U_i \times C_r$.

Let $F^{1,0} = f$. Assume that $F^{1,0} : M \times K \rightarrow N$ is defined such that
F^{i,-1}_n is smooth on $\bar{W}_1 \times (\bar{A}_1 \cup \ldots \cup \bar{A}_{i-1})$ and $F^{i,-1}_n = F^{i,-2}_n$ outside of $U_1 \times C_{i-1}$. Moreover we assume that $F^{i,-1}_n$ is a $\delta_i / 2^{i-1} + 1$-approximation to $F^{i,-2}_n$ for each $k \leq K$. Then $F^{i,-1}_n$ is a $\delta_i$-approximation to $f_i$.

Let $F^i : M \times K \rightarrow N$ be a $G$-map defined by

$$F^i_n(x) = F^{i,-1}_n(x) (1 - \psi_i(x) \xi_i(x)) + h_i(x),$$

for $x \in M$ and $z \in K$.

Since $F^i_n = h^i_n$ on $\bar{W}_i \times \bar{A}_n$, $F^i_n$ is smooth on $\bar{W}_i \times (\bar{A}_1 \cup \ldots \cup \bar{A}_n)$ (note that, if $F^i_n$ is smooth on a subset of $M \times K$, $F^i_n$ is smooth on the subset of $M \times K$). Since $h^i_n = 0$ outside of $U_i \times C_n$, $F^i_n = F^{i,-1}_n$ outside of $U_i \times C_n$. By the argument of J. Munkres [4, Chapter I, §4], we can choose the positive numbers $\varepsilon_i (i = 1, 2, \ldots, n)$ so small that $F^i_n$ is a $\delta_i / 2^{i-1} + 1$-approximation to $F^{i,-1}_n$ for each $k$. Then we can see that $F^i_n$ is a $\delta$-approximation to $f_i$ and $F^{i,-1}_n$ is defined. By the induction, we have a $G$-map $F^{i,-1} : M \times K \rightarrow N$ such that $F^{i,-1}$ is smooth on $\bar{W}_i \times (\cup \bar{A}_n)$ and $F^{i,-1} = F^{i,-1}_{n+1}$ outside of $U_i \times C_n$. Set $F^i = F^{i,-1}$. Since $\cup \bar{A}_n$ contains $K - L'$, $F^i$ is smooth on $\bar{W}_i \times K$. And since $C_n \cap L' = \emptyset$, $F^i : M \times L$.

There exists a positive continuous function $\delta \leq \delta_i$ on $M$ such that, for each $k \leq K$, any $C^1$-map from $M$ to $N$, which is a $\delta$-approximation in $C^1$-topology to $f_i$, is an imbedding (see J. Munkres [4, Chapter I, Theorem 3.10]). We can choose the positive numbers $\varepsilon_n, n = 1, 2, \ldots, l$, so small that $F^i_n$ is a $\delta / 2^{i-1} + 1$-approximation to $f_i$ in $C^1$-topology for each $k \leq K$.

By the induction we have $G$-maps $F^i : M \times K \rightarrow N \ (i = 2, 3, \ldots)$, which is smooth on $(\cup \bar{W}_i) \times K$, such that $F^i = f$ on $M \times L$ and $F^i = F^i_{i-1}$ outside of $U_i \times K$. Moreover we can choose $F^i_n$ is a $\delta / 2^i$-approximation to $F^i_{i-1}$ in $C^1$-topology for each $k \leq K$. Define a $G$-map $F : M \times K \rightarrow N$ by $F_i(x) = \lim_{i \rightarrow \infty} F^i_i(x)$; $F_i$ is well defined because $F^i_i = F^i_{i+1} = \ldots$ on some neighborhood of $x$, for sufficiently large $i$. $F : M \times K \rightarrow N$ is smooth on $(\cup \bar{W}_i) \times K$ and $F = f$ on $M \times L$. Moreover $F_i$ is a $\delta$-approximation to $f_i$ in $C^1$-topology, for each $k \leq K$.

Let $T$ be a closed invariant neighborhood of $M^o$ in $M$ such that $T$ is contained in $\cup \bar{W}_i$. $F_i(M - M^o)$ is contained in $N - N^o$, for
each $k$, since $F_k$ is a $G$-imbedding. Let $\tilde{F} : (M-M^o) / G \times K \to (N-N^o) / G$ be the orbit map of $F$. Then $\tilde{F}$ is a smooth map on a neighborhood of $(T-M^o) / G \times K$ and $\tilde{F} = \tilde{f}$ on $(M-M^o) / G \times L$, and $\tilde{F}$ is a $\delta$-approximation to $\tilde{f}_i$ for each $k \in K$, where $\tilde{f}_i$ is the orbit map of $f_i$. By the relative version of the argument of J. Munkres [4, Chapter I, §4], we have a smooth map $\tilde{H} : (M-M^o) / G \times K \to (N-N^o) / G$ such that $\tilde{H} = \tilde{F}$ on $(T-M^o) / G \times K$ and $\tilde{H}$ is homotopic to $\tilde{F}$ relative to $(T-M^o) / G \times L$. Moreover $\tilde{F}_i$ is a $\delta$-approximation to $\tilde{f}_i$ in $C^1$-topology, for each $k \in K$. By the covering homotopy property, we have a smooth $G$-map $H : (M-M^o) \times K \to N-N^o$ whose orbit map is $\tilde{H}$. Define $H = \tilde{F}$ on $T \times K$. Then $H : M \times K \to N$ is a smooth $G$-map such that $H_i$ is a $\delta$-approximation to $f_i$ in $C^1$-topology, for each $k \in K$, and $H = \tilde{f}$ on $M \times L$. This completes the proof of Lemma 1.2.

Let $f : S^v \to S^w$ be a $G$-imbedding. The fixed point set of $S^v$ and $S^w$ are $S^v$ and $S^{s+1}$ respectively. Let $f^o : S^v \to S^{s+1} \subset S^w$ denote an imbedding which is a restriction of $f$ to $S^v$. Let $j : S^v \to S^w$ be the standard imbedding.

**Proposition 1.3.** Let $f_0 : S^v \to S^w$ be a $G$-imbedding. Then there exists a $G$-isotopy $f : S^v \times I \to S^w$ between $f_0$ and $f_1$ with $f^v_0 = j$ on $S^v$.

**Proof.** Since $dn + k \geq 2n$, we have an isotopy $h : S^{s+1} \times I \to S^{s+1}$ such that $h_0 = 1$ and $h_1 \cdot f^o_0 = j$. By the isotopy extension theorem, there exists an isotopy $H : S^w \times I \to S^w$ such that $H_0 = 1$ and $H = h$ on $S^{s+1} \times I$. Using a result of G. Bredon [1, Chapter VI, Theorem 3.1], we have a $G$-isotopy $K : S^w \times I \to S^w$ such that $K_0 = 1$ and $K = H$ on $H^o \times I$, where $H^o = \{ x \in S^w \ ; \ H_t(g \cdot x) = g \cdot H_t(x) \}$ for any $t \in I$ and $g \in G$. Note that $S^{s+1} \subset H^o$. Let $f : S^v \times I \to S^w$ be a $G$-isotopy between $f_0$ and $f_1$ defined by $f_i = K \cdot f^o_i$. Then $f^v_0 = j$ and this completes the proof of Proposition 1.3.

**Proposition 1.4.** Let $f : S^v \times I \to S^w$ be a $G$-isotopy with $f^o_i = j$ for $i = 0, 1$. Then there exists a $G$-isotopy $h : S^v \times I \to S^w$ such that
$h_i = f_i$ for $i = 0, 1$ and $h^*_i = j$ for $0 \leq t \leq 1$.

**Proof.** Let $f : S^* \times I \to S^w \times I$ be a $G$-imbedding defined by $f(x, t) = (f_i(x), t)$. Let $f^* : S^* \times I \to S^{d+n+k} \times I$ be an imbedding which is a restriction of $f$ to $S^* \times I$. Let $E(S^*, S^{d+n+k})$ denote the set of all imbeddings of $S^*$ into $S^{d+n+k}$ with $C^\infty$-topology. By a result of J. Dax [2, Chapter VI, §3], $\pi_1(E(S^*, S^{d+n+k})) = 0$ since $dn + k > 2n + 2$. Then we have a continuous map $a : I \times I \to E(S^*, S^{d+n+k})$ such that, for a sufficiently small $\varepsilon > 0$,

$$a(t, s) = \begin{cases} f_i^* & \text{for } (t, s) \in I \times [0, \varepsilon] \\ j & \text{for } (t, s) \in [0, \varepsilon] \times I \cup I \times [1 - \varepsilon, 1] \cup [1 - \varepsilon, 1] \times I. \end{cases}$$

Using Lemma 1. 2, we may assume that $a : S^* \times I \times I \to S^{d+n+k} \times I$ is an isotopy, where $a(x, t, s) = (a(t, s)(x), t)$. Then we have an imbedding $\tilde{a} : S^* \times I \times R \to S^{d+n+k} \times I \times R$ defined by

$$\tilde{a}(x, t, s) = \begin{cases} \tilde{a}(x, t, s), s) & \text{for } 0 \leq s \leq 1 \\ \tilde{a}(x, t, 0), s) & \text{for } s < 0 \\ \tilde{a}(x, t, 1), s) & \text{for } s > 1. \end{cases}$$

$\tilde{a}(S^* \times I \times R)$ is a closed $G$-submanifold of $S^w \times I \times R$, and $\tilde{a}(S^* \times I \times R)$ intersects normally on $\partial(S^* \times I \times R)$ with respect to a product $G$-invariant Riemannian metric on $S^w \times I \times R$. By using the proof of G. Bredon [1, Chapter IV, Theorem 2.2] with respect to the Riemannian metric, we have an invariant open $\delta$-tubular neighborhood $N$ of $\tilde{a}(S^* \times I \times R)$, where $\delta$ is a $G$-invariant positive real valued function on $\tilde{a}(S^* \times I \times R)$.

The tangent vectors to the curves $\tilde{a}(x \times t \times R)$ define an invariant vector field $\tilde{X}$ on $\tilde{a}(S^* \times I \times R)$ of the form $\tilde{X}(\tilde{a}(x, t, s)) = (X(x, s, t), 0, 1) \in T_{\tilde{a}(x, t, s)}(S^w \times I \times R)$, where $T(S^w \times I \times R)$ is the tangent bundle of $S^w \times I \times R$. Identifying $N$ with a $G$-invariant normal bundle to $\tilde{a}(S^* \times I \times R)$ in $S^w \times I \times R$, we denote $p : N \to \tilde{a}(S^* \times I \times R)$ the bundle projection. Let $r : I \to R$ be a $C^\infty$-function such that $r(t) = 1$ for $0 \leq t \leq 1/3$, $0 < r(t) < 1$ for $1/3 < t < 2/3$ and $r(t) = 0$ for $2/3 \leq t \leq 1$. Let $Y$ be a $G$-invariant vector field on $S^w \times I \times R$ defined by $Y(v) = r(||v||/\delta(p(v))) \cdot X(p(v))$ for $v \in N$ and $Y = 0$ on the outside of $N$, where $|| \cdot ||$ denote the $G$-invariant metric of $S^w \times I \times R$. 
Since \( a(x, t, s) = (j(x), t, s) \) for \( 0 \leq t \leq \varepsilon \) and \( 1 - \varepsilon \leq t \leq 1 \), and since \( a(x, t, s) = (a(x, t, 0), s) \) for \( s \leq 0 \) and \( a(x, t, s) = (a(x, t, 1), s) \) for \( s \geq 1 \), \( \text{Supp}(Y) \) is contained in \( S^w \times [\varepsilon, 1 - \varepsilon] \times I \) which is compact. We can regard \( Y \) as a time-dependent \( G \)-invariant vector field on \( S^w \times I \), and \( Y \) generates a \( G \)-isotopy \( F : S^w \times I \times I \rightarrow S^w \times I \) (see M. Hirsch [3, Chapter 8, Theorem 1.1]). Since \( I \) component of \( Y \) is 0, each \( F_t : S^w \times I \rightarrow S^w \times I \) is level preserving. Let \( h : S^v \times I \rightarrow S^w \) be a \( G \)-isotopy defined by \( h = p_1 \cdot F_t \cdot f \), where \( p_1 : S^w \times I \rightarrow S^w \) is the projection on the first factor. Then \( h_t \vDash f_t \) for \( t = 0, 1 \) and \( h_t \vDash j \) on \( S^* \) for each \( t \). This completes the proof of Proposition 1.4.

§ 2. Linearity on a Neighborhood of the Fixed Point Set

In this section we shall prove that any \( G \)-imbedding of \( S^v \) into \( S^w \) is \( G \)-isotopic to a \( G \)-imbedding which is linear on a neighborhood of \( S^* \) (see Proposition 2.1), and if two \( G \)-imbeddings of \( S^v \) into \( S^w \), which are linear on a neighborhood of \( S^* \), are \( G \)-isotopic, then there exists a \( G \)-isotopy between them which is linear on a neighborhood of \( S^* \) (see Proposition 2.3).

Since the fixed point set of \( S^v \) is \( S^* \) and since \( S^v \) is a \( G \)-submanifold of \( S^w \), we can regard \( S^* \) as a \( G \)-submanifold of \( S^w \). Let \( U \) and \( N \) denote invariant open tubular neighborhoods of \( S^* \) in \( S^v \) and \( S^* \) in \( S^w \) respectively. We shall identify \( U \) and \( N \) with invariant normal bundles to \( S^* \) in \( S^v \) and to \( S^* \) in \( S^w \) respectively. Let \( f : S^v \rightarrow S^w \) be a \( G \)-imbedding with \( f^o \vDash j \). We shall assume that \( f(U) \) is contained in \( N \). Let \( f^o : U \rightarrow N \) be a \( G \)-bundle monomorphism defined by the differential of \( f \).

**Proposition 2.1.** Let \( f : S^v \rightarrow S^w \) be a \( G \)-imbedding with \( f^o \vDash j \). Then there exists a \( G \)-isotopy \( h : S^v \times I \rightarrow S^w \) such that \( h_0 = f \) and \( h_i = f^o \) on some invariant neighborhood of \( S^* \) in \( S^v \).

In order to prove Proposition 2.1, we start with the following lemma.
Lemma 2.2. Let $M$ be a $G$-submanifold of a $G$-manifold $N$. Let $f : M \times I \to N$ be a $G$-isotopy such that $f_t(\partial M) \subseteq \partial N$ and $f_t(M)$ intersects transversally on $\partial N$ for each $t$. Let $A$ be an invariant subspace of $M$ such that $\bar{A}$ is compact. Then there exists a $G$-isotopy $F : N \times I \to N$ such that $F_0 = 1$ and $F_t \cdot f_0 = f_t$ on $A$ for $0 \leq t \leq 1$.

Proof. Let $\tilde{f} : M \times R \to N \times R$ be a $G$-imbedding defined by

$$\tilde{f}(x, t) = \begin{cases} (f_t(x), t) & \text{for } 0 \leq t \leq 1 \\ (f_0(x), 0) & \text{for } t < 0 \\ (f_t(x), t) & \text{for } t > 1. \end{cases}$$

We can assume that $G$ acts by isometries in some product metric on $N \times R$. Let $\nu$ be an invariant normal bundle of $\tilde{f}(M \times R)$ in $N \times R$ and let $p : \nu \to \tilde{f}(M \times R)$ be the projection. Then the exponential map is defined on some neighborhood of $\tilde{f}(M \times R)$ in $\nu$ and is an equivariant immersion on a smaller invariant open neighborhood of $\tilde{f}(M \times R)$ (see the proof of G. Bredon [1, Chapter VI, Theorem 2.2]). Let $B$ be an invariant open neighborhood of $\bar{A}$ such that $\bar{B}$ is compact. Since $\bar{B}$ is compact, the exponential map is a $G$-imbedding on an invariant neighborhood of $\tilde{f}(B \times I)$ in $\nu \mid \tilde{f}(B \times I)$. By a method of the proof of G. Bredon [1, Chapter VI, Theorem 2.2], we have a $G$-imbedding $\varphi : \nu \mid \tilde{f}(B \times I) \to N \times R$. We shall identify $\nu \mid \tilde{f}(B \times I)$ as the image of $\varphi$.

The tangent vectors to the curves $\tilde{f}(x \times R)$ $(x \in M)$ define an invariant vector field $\tilde{X}$ on $\tilde{f}(M \times R)$ of the form $\tilde{X}(\tilde{f}(x, t)) = (X(x, t), 1) \in T_{f_t(x, t)}(N \times R)$. Note that $\text{Supp}(X)$ is contained in $\tilde{f}(M \times I)$. Take an invariant $C^\infty$-partition of unity subordinate to the covering $\{B, M - \bar{A}\}$ of $M$, and let $u$ be the invariant function correspondence to $B$. Let $X'$ be an invariant vector fields on $\tilde{f}(M \times R)$ defined by $X'(\tilde{f}(x, t)) = u(x) \cdot X(x, t)$ and $X' = 0$ outside of $\tilde{f}(B \times R)$. Then $\text{Supp}(X')$ is contained in $\tilde{f}(B \times I)$ and $X' = X$ on $\tilde{f}(\bar{B} \times R)$.

Let $r : R \to [0, 1]$ be a $C^\infty$-function such that $r(t) = 1$ for $t \leq 1$, $0 < r(t) < 1$ for $1 < t < 2$ and $r(t) = 0$ for $t \geq 2$. Let $Y$ be an invariant vector field on $N \times I$ defined by $Y(\nu) = r(||\nu||) \cdot X'(p(\nu))$ on $\nu \mid \tilde{f}(B \times I)$ and $Y = 0$ outside of $\nu \mid \tilde{f}(B \times I)$, where $|| \cdot ||$ is an invariant Rie-
mannian metric on \( \nu \). Then we can regard \( Y \) as a time-dependent invariant vector field on \( N \). Note that \( \text{Supp}(Y) \) is contained in \( \nu(2) \) \( |B \times I| \) which is compact, where \( \nu(2) = \{ \nu \in \nu; ||\nu|| \leq 2 \} \). Therefore \( Y \) generates a \( G \)-isotopy \( F : N \times I \rightarrow N \) such that \( F_{0} = 1 \) and \( F_{t} \cdot f_{0} = f_{t} \) on \( A \) for \( 0 \leq t \leq 1 \). This completes the proof of Lemma 2.2.

**Proof of Proposition 2.1.** Let \( g : U \times I \rightarrow N^{\nu} \rightarrow S^{w} \) be a homotopy of \( G \)-imbeddings defined by \( g_{t}(v) = 1/(1-t) \cdot f((1-t) v) \) for \( 0 \leq t < 1 \) and \( v \in U \), and \( g_{t} = f_{t} \). Note that \( g_{0} = f_{0}|U \), \( \lim_{t \rightarrow 1} g_{t} = f_{t} \) and \( g_{t} \) is a \( G \)-imbedding for each \( t \). By Lemma 1.2 we can assume that \( g \) is a \( G \)-isotopy between \( f|U \) and \( f \). By Lemma 2.2 there exists a \( G \)-isotopy \( G : S^{w} \times I \rightarrow S^{w} \) such that \( F_{0} = 1 \) and \( F_{s} \cdot g_{s} = g_{s} \) on some neighborhood of \( S^{\nu} \). Let \( h : S^{\nu} \times I \rightarrow S^{w} \) be a \( G \)-isotopy defined by \( h_{s} = F_{s} \cdot f_{s} \). Then \( h_{0} = f \) and \( h_{1} = f_{1} \) on some neighborhood of \( S^{\nu} \). This completes the proof of Proposition 2.1.

By Proposition 1.3 and Proposition 2.1, any element of \( I(S^{\nu}, S^{w}) \) is represented by a \( G \)-imbedding \( f : S^{\nu} \rightarrow S^{w} \) such that \( f_{0} = j \) and \( f = f_{t} \) on an invariant tubular neighborhood of \( S^{\nu} \).

**Proposition 2.3.** Let \( f : S^{\nu} \times I \rightarrow S^{w} \) be a \( G \)-isotopy such that \( f_{0} = j \) \( (0 \leq t \leq 1) \) and \( f_{i} = f_{i}^{'} \) \( (i = 0, 1) \) on an invariant tubular neighborhood \( U \) of \( S^{\nu} \). Then there exists a \( G \)-isotopy \( h : S^{\nu} \times I \rightarrow S^{w} \) such that \( h_{i} = f_{i} \) \( (i = 0, 1) \) and \( h_{i} = h_{i}^{'} \) on an invariant neighborhood of \( S^{\nu} \) for \( 0 \leq t \leq 1 \).

**Proof.** Let \( \bar{f} : S^{\nu} \times I \rightarrow S^{w} \times I \) be a \( G \)-imbedding defined by \( \bar{f}(x, t) = (f_{i}(x), t) \). We can assume that \( f_{i}(U) \) is contained in \( N \) for each \( t \). Let \( \bar{f} : U \times I \rightarrow N \times I \) be a \( G \)-imbedding defined by \( \bar{f}(v, t) = (f^{'}(v), t) \). Let \( F : U \times I \rightarrow N \times I \) be a \( G \)-map defined by \( F_{s}(v, t) = (1/(1-s) \cdot f_{i}((1-s)v), t) \) for \( 0 \leq s \leq 1 \) and \( F_{0} = f_{s} \). Then \( F_{0} = \bar{f} \) and \( \lim_{s \rightarrow 1} F_{s} = \bar{f}^{'} \) and \( F_{s} \) is a \( G \)-imbedding for each \( s \). Note that, by the definition of \( G \)-isotopy, there exists a positive number \( \varepsilon \) such that \( f_{i} = f_{i} \) for \( 0 \leq t \leq \varepsilon \) and \( f_{i} = f_{i}^{'} \) for \( 1 - \varepsilon \leq t \leq 1 \). Thus \( F_{s} = f_{s} \times 1 \) for \( 0 \leq t \leq \varepsilon \) and \( F_{s} = f_{i}^{'} \times 1 \) for \( 1 - \varepsilon \leq t \leq 1 \). By Lemma 1.2 we can assume that \( F \)
is a $G$-isotopy between $\tilde{f}|U \times I$ and $\tilde{f}$. Let $\tilde{F} : U \times I \times R \rightarrow N \times I \times R \rightarrow S^w \times I \times R$ be a $G$-imbedding defined by

$$\tilde{F}(x, t, s) = \begin{cases} (F(x, t, s), s) & \text{for } 0 \leq s \leq 1 \\ (F(x, t, 0), s) & \text{for } s < 0 \\ (F(x, t, 1), s) & \text{for } s > 1. \end{cases}$$

Let $U$ and $U_2$ be invariant open tubular neighborhoods of $S^w$ such that $U_1 \subseteq U_2 \subseteq U$. Let $\nu$ be an invariant normal bundle of $\tilde{F}(U \times I \times R)$ in $S^w \times I \times R$ and let $\rho : \nu \rightarrow \tilde{F}(U \times I \times R)$ be the projection. Similarly as the proof of Lemma 2.2, we have a $G$-imbedding $\varphi : \nu|\tilde{F}(U_1 \times I \times I) \rightarrow S^w \times I \times R$. We shall identify $\nu|\tilde{F}(U_2 \times I \times I)$ as the image of $\varphi$.

The tangent vectors to the curves $\tilde{F}(x \times t \times R)$ ($x \times t \in U \times I$) define an invariant vector field $\tilde{X}$ on $\tilde{F}(U \times I \times R)$ of the form $\tilde{X}(\tilde{F}(x, t, s)) = (X(x, t, s), 0, 1) \in T_{F(x, t, s)} (N \times I \times R)$. Note that $\text{Supp}(X)$ is contained in $\tilde{F}(U \times [\varepsilon, 1 - \varepsilon] \times I)$. Take an invariant partition of unity subordinate to the covering $\{U_2, U - U_1\}$ of $U$, and $u$ be the invariant $C^\infty$-function corresponding to $U_2$. Let $X' = \text{X}'(\tilde{F}(x, t, s)) = u(x) \cdot X(x, t, s)$ for $(x, t, s) \in \tilde{U}_2 \times I \times R$ and $X' = 0$ outside of $\tilde{F}(\tilde{U}_1 \times I \times R)$. Then $\text{Supp}(X')$ is contained in $\tilde{F}(\tilde{U}_2 \times [\varepsilon, 1 - \varepsilon] \times I)$ and $X' = X$ on $\tilde{F}(\tilde{U}_1 \times I \times R)$.

Let $r : R \rightarrow [0, 1]$ be a $C^\infty$-function such that $r(t) = 1$ for $t \leq 1$, $0 < r(t) < 1$ for $1 < t < 2$ and $r(t) = 0$ for $t \geq 2$. Let $Y$ be an invariant vector field on $S^w \times I \times R$ defined by $Y(v) = r(||v||) \cdot X'(\rho(v))$ for $v \in \nu|\tilde{F}(\tilde{U}_1 \times I \times I)$ and $Y = 0$ outside of $\nu|F(\tilde{U}_1 \times I \times I)$, where $|| ||$ is an invariant Riemannian metric on $\nu$. Then we can regard $Y$ as a time-dependent invariant vector field on $S^w \times I$. Note that $\text{Supp}(Y)$ is contained in $\nu(2) \mid \tilde{F}(\tilde{U}_1 \times [\varepsilon, 1 - \varepsilon] \times I)$, where $\nu(2) = \{v \in \nu : ||v|| \leq 2\}$. Then $Y$ generates a $G$-isotopy $H : S^w \times I \times I \rightarrow S^w \times I$ such that $H_0 = 1$ and $H_s F = F$, on $\tilde{U}_1 \times I$ for $0 \leq s \leq 1$. Since $I$ component of $Y$ is 0, each $H_s : S^w \times I \rightarrow S^w \times I$ is level preserving equivariant diffeomorphism. Let $h : S^w \times I \rightarrow S^w$ be a $G$-isotopy defined by $h = p \cdot H_t \cdot \tilde{f}$, where $p_t : S^w \times I \rightarrow S^w$ is the projection on the first factor. Then $h_0 = F$, for $i = 0, 1$ and $h_i = h_i$ on $U$. This completes the proof of
Proposition 2.3.

**Definition 2.4.** Let \( f_i : S^\nu \to S^\omega \) \((i=0, 1)\) be \( G \)-imbeddings such that \( f_i = f'_i \) on \( U \). \( f_0 \) and \( f_i \) are said to be equivalent if there exists a \( G \)-isotopy \( f : S^\nu \times I \to S^\omega \) between \( f_0 \) and \( f_i \) such that \( f_t = f'_i \) on some neighborhood of \( S^\nu \). Let \( I_i(S^\nu, S^\omega) \) denote the set of all equivalence classes of these \( G \)-imbeddings.

**Corollary 2.5.** The natural map \( i_i : I_i(S^\nu, S^\omega) \to (S^\nu, S^\omega) \) is bijective.

**Proof.** By Proposition 1.3 and Proposition 2.2, \( i_i \) is surjective. By Proposition 1.4 and Proposition 2.3, \( i_i \) is injective, and Corollary 2.5 follows.

§ 3. Orthogonality on a Neighborhood of the Fixed Point Set

In this section we shall prove that any \( G \)-imbedding from \( S^\nu \) into \( S^\omega \) is \( G \)-isotopic to a \( G \)-imbedding which is orthogonal on a neighborhood of \( S^\nu \). Moreover we shall prove that, if two \( G \)-imbeddings \( f_0 \) and \( f_i \), which are orthogonal on \( U \), coincide on \( U \), then there exists a \( G \)-isotopy \( f \) between \( f_0 \) and \( f_i \) such that \( f_t = f'_i \) \((0 \leq t \leq 1)\) on \( U_i \), where \( U \) and \( U_i \) are invariant neighborhood of \( S^\nu \).

As in § 2, let \( U \) and \( N \) be invariant normal bundles of \( S^\nu \) in \( S^\nu \) and to \( S^\omega \) in \( S^\omega \) respectively. Note that \( U \) and \( N \) are isomorphic to product bundles \( S^\nu \times V_1 \) and \( S^\nu \times (dV_1 \oplus R^{(d-1)*+1}) \) as a \( G \)-vector bundles over \( S^\nu \) respectively. Let \( f : S^\nu \to S^\omega \) be a \( G \)-imbedding with \( f_0 = j \). Then \( f : U \to N \) induces a continuous map

\[
f : S^\nu \to Mon^G(V_1, dV_1 \oplus R^{(d-1)*+1}),
\]

where \( Mon^G(V_1, dV_1 \oplus R^{(d-1)*+1}) \) is the set of all \( G \)-module monomorphisms from \( V_1 \) to \( dV_1 \oplus R^{(d-1)*+1} \) with usual topology. By Schur’s lemma, \( Mon^G(V_1, dV_1 \oplus R^{(d-1)*+1}) \) is isomorphic to \( Mon^G(V_1, dV_1) \).
Proposition 3.1. Let $f : S^v \to S^w$ be a $G$-imbedding with $f^0 = j$. Let $h : S^s \times I \to \text{Mon}^G(V_1, dV_1)$ be a homotopy with $h_0 = f$. Then there exists a $G$-isotopy $F : S^v \times I \to S^w$ such that $F_0 = f$ and $F_1 = h_1$.

Proof. Let $p : U \to S^s$ be the bundle projection. Let $F' : U \times I \to N$ be a homotopy of $G$-imbeddings defined by $F'_t(u) = h_t(p(u)) (u)$ for $u \in U$. Then, by Lemma 1.2, we can assume that $F'$ is a $G$-isotopy. By Lemma 2.2, we have a $G$-isotopy $H : S^v \times I \to S^w$ such that $H_0 = 1$ and $H_t \cdot f' = F'_t$ on some invariant neighborhood of $S^s$ for each $t$. Let $F : S^v \times I \to S^w$ be a $G$-isotopy defined by $F_t = H_t \cdot f$. Then $F_0 = f$ and $F_1 = h_1$, and this completes the proof of Proposition 3.1.

Let $O^G(V_1, dV_1)$ denote the set of all $G$-module orthogonal monomorphisms from $V_1$ to $dV_1$. Let $F$ denote the field of real numbers $R$, complex numbers $C$ or quaternionic numbers $H$. Let $U(q, F)$ denote the orthogonal group $O(n)$, the unitary group $U(n)$ or the symplectic group $Sp(n)$ in the case of $F = R$, $C$ or $H$ respectively. Let $\text{Hom}^G(V_1, V_1)$ denote the group of $G$-module endmorphisms of $V_1$. Let $V_{s,t}(F)$ denote the Stiefel manifold (over $F$) of $s$-frames in $F$.

Lemma 3.2. Suppose that $V_1$ is isomorphic to $\bigoplus_i k_i W_i$, where $W_i$ runs over the inequivalent irreducible real $G$-modules. Then

$$\text{Mon}^G(V_1, dV_1) = \prod_i V_{d^k_i, 1_i}(F_i)$$

and

$$O^G(V_1, dV_1) = \prod_i U(dk_i, F_i) / U((d-1)k_i, F_i),$$

where $F_i = R$, $C$ and $H$ when dim $\text{Hom}^G(W_i, W_i) = 1, 2$ and $4$ respectively.

Proof. If $W_i$ is a real restriction of an irreducible complex (resp. quaternionic) $G$-module $W_i$, then $\text{Hom}^G(W_i, W_i)$ is isomorphic to $C$ (resp. $H$) given by the scalar multiplication of $W_i$. Otherwise $\text{Hom}^G(W_i, W_i)$ is isomorphic to $R$ given by the scalar multiplication of $W_i$ (see J. -P. Serre [6, 13.2]). Therefore $\text{Mon}^G(k_i W_i, dk_i W_i)$.
and \( O^g(k_i W_i, dk_i W_i) \) are identified with \( V*_{a_i, t}(F_i) \) and \( U(dk_i, F_i)/U((d-1)k_i, F_i) \) respectively. By Schur's lemma \( \text{Hom}^g(V_i, dV_i) \) is isomorphic to \( \bigoplus_i \text{Hom}^g(k_i W_i, dk_i W_i) \). Then \( \text{Mon}^g(V_i, dV_i) \) and \( O^g(V_i, dV_i) \) are identified with \( \Pi \text{Mon}^g(k_i W_i, dk_i W_i) \) and \( \Pi O^g(k_i W_i, dk_i W_i) \) respectively. This completes the proof of Lemma 3.2.

**Proposition 3.3.** Let \( f : S^v \times I \to S^w \) be a \( G \)-isotopy such that \( f^0_t = j, f_t = f_1 \) on \( U \) for each \( t \) and \( f^0_1 = f_t \). If \( \pi_{n+1}(\text{Mon}^g(V_i, dV_i)) = 0 \), then there exists a \( G \)-isotopy \( h : S^v \times I \to S^w \) such that \( h_t = f_t \) for \( i = 0, 1 \) and \( h_1 = f_0 \) for \( 0 \leq t \leq 1 \).

**Proof.** Let \( a_f : S^v \times \partial(I \times I) \to \text{Mon}^g(V_i, V_i) \) be a continuous map defined by

\[
a_f(x, t, s) = \begin{cases} 
  f^0_t(x) & \text{for } s = 0 \text{ and } 0 \leq t \leq 1 \\
  f^1_t(x) & \text{for } s = 1 \text{ and } 0 \leq t \leq 1 \\
  f_t(x) & \text{for } t = 0, 1 \text{ and } 0 \leq s \leq 1.
\end{cases}
\]

Since \( \pi_{n+1}(\text{Mon}^g(V_i, dV_i)) = 0 \), the only obstruction to extend \( a_f \) to \( S^v \times I \times I \) is a well defined cohomology class \( o(a_f) \in H^n(S^v \times I \times I, S^v \times \partial(I \times I)) \); \( \pi_n(\text{Mon}^g(V_i, dV_i)) = \pi_n(\text{Mon}^g(V_i, dV_i)) \). If \( d \geq 3 \), \( \text{Mon}^g(V_i, dV_i) \) is 2-connected by Lemma 3.2, and \( o(a_f) = 0 \).

Now we will consider the case of \( d = 1 \). In this case \( \text{Mon}^g(V_i, dV_i) \) is a group \( A^g(V_i) \), where \( A^g(V_i) \) is the group of all \( G \)-module automorphisms of \( V_i \). Let \( b_f : \partial(I \times I) \to \text{Mon}^g(V_i, dV_i) = A^g(V_i) \) be a continuous map defined by \( b_f(x) = a_f(\ast, x) \) for \( x \in \partial(I \times I) \), where \( \ast \) is a point of \( S^v \). Then the above obstruction class \( o(a_f) \) is represented by \( b_f \). Note that an element of \( A^g(V_i) \) can be regarded as an equivariant linear diffeomorphism of \( S^w \) in the natural way. Let \( g : S^v \times I \to S^w \) be a \( G \)-isotopy between \( f_0 \) and \( f_1 \) defined by \( g_t = f^0_t(\ast) \cdot f_t(\ast)^{-1} \cdot f_t \). Then \( b_f(x) = f^0_t(\ast) \) for any \( x \in \partial(I \times I) \), and \( o(a_f) = 0 \). Replacing the \( G \)-isotopy \( f \) between \( f_0 \) and \( f_1 \) by \( g \), we can assume \( o(a_f) = 0 \).

We now turn to the case \( d = 2 \). If \( V_i \) is isomorphic to \( \bigoplus_i k_i W_i \), then \( \text{Mon}^g(V_i, 2V_i) = \Pi V*_{a_i, t}(F_i) \) by Lemma 3.2. Note that \( \pi_1(V*_{a_i, t}(F_i)) \) is 0 beside the case \( F_i \equiv R \) and \( k_i = 1 \). Let \( J \) be the set of index \( i \)
such that \( F_i = R \) and \( k_i = 1 \). Let \( p : \prod_{i \in J} V_{2i, i}^1(F_i) \to \prod_{i \in J} V_{2i, i}^1(R) \) be the natural projection. Then \( p_* : \pi_1(\prod_{i \in J} V_{2i, i}^1(F_i)) \to \pi_1(\prod_{i \in J} V_{2i, i}^1(R)) \) is isomorphic. Let \( r : I \to \prod_{i \in J} V_{2i, i}^1(R) \) be a continuous map defined by \( r(t) = p \cdot f_t(\ast) \). Since \( \pi : \prod_{i \in J} GL(2, R) \to \prod_{i \in J} F_{2i, i}^1 \) is a product bundle, there exists a continuous map \( \hat{r} : I \to \prod_{i \in J} GL(2, R) \) such that \( \pi \circ \hat{r} = r \) and \( \hat{r}(0) = \hat{r}(1) \). Note that, for each \( i \in J \), \( GL(2, R) \) can be regarded as the automorphism group \( A(2W_i) \) of \( G \)-module \( 2W_i \) whose element defines an equivariant linear diffeomorphism of \( S^n \).

Let \( q : U \times I \to S^n \) be a \( G \)-isotopy between \( f_0 \) and \( f_1 \) defined by \( q_t = f_t(\ast) \). Since \( \pi \) is identified with the natural map \( \pi_n^+ : \prod_{i \in J} A^0(2W_i) \to \prod_{i \in J} \text{Mon}^\sigma(W_i, 2W_i) \), \( p \cdot f_t(\ast) = f_t(\ast) \) and \( o(q_t) = 0 \). Replacing the \( G \)-isotopy \( f \) between \( f_0 \) and \( f_1 \) by \( g \), we can assume that \( o(a_j) = 0 \).

Therefore we can assume that \( a_j \) can be extended to \( S^n \times I \times I \). Let \( F : U \times I \times I \to N \times I \) be an equivariant map defined by \( F(v, t, s) = (a_j(q(v), t, s)(v), t) \), where \( q : U \to S^n \) is the bundle projection. Then each \( F(\ast, t, s) \) is a \( G \)-imbedding, and \( F_t(u, t) = (f_t(u), t) = (f_t(u), t) \) and \( F_t(u, t) = (f_t(u), t) = (f_t(u), t) \) for \( (u, t) \in U \times I \). By Lemma 1.2 we can assume that \( F \) is a \( G \)-isotopy. In the same way as the proof of Proposition 2.3, we have a \( G \)-isotopy \( h : S^n \times I \to S^n \) such that \( h_i = f_i \) \((i = 0, 1) \) and \( h_t = f_t(0 \leq t \leq 1) \) on some invariant neighborhood of \( S^n \). Therefore \( h_t = f_t(0 \leq t \leq 1) \) on some invariant neighborhood of \( S^n \). Therefore \( h_t = f_t \) for each \( t \), and this completes the proof of Proposition 3.3.

**Remark.** I don't know whether Proposition 3.3 is valid without the assumption \( \pi_{n+1}(\text{Mon}^\sigma(V_1, dV_1)) = 0 \).

Now we shall assume \( \pi_{n+1}(\text{Mon}^\sigma(V_1, dV_1)) = 0 \). Choose a continuous map \( a_j : S^s \to O^\sigma(V_1, dV_1) \), which represents an element \( \lambda \) for each element \( \lambda \) of \( \pi_s(O^\sigma(V_1, dV_1)) \). Let \( A = \{a_j : \lambda \in \pi_s(O^\sigma(V_1, dV_1))\} \)

**Definition 3.4.** Let \( f_i : S^n \to S^n \), \( i = 0, 1 \), be \( G \)-imbeddings, which represent elements of \( I_i(S^n, S^n) \), such that \( f_i \) \((i = 0, 1) \) are elements of \( A \). \( f_0 \) and \( f_1 \) are said to be equivalent if there exists a \( G \)-isotopy \( f : S^n \times I \to S^n \) between \( f_0 \) and \( f_1 \) such that \( f_t = f_0 \) for \( 0 \leq t \leq 1 \). Let
$I_2(S^r, S^w)$ denote the set of equivalence classes of these $G$-imbeddings.

**Corollary 3.5.** If $\pi_{n+1}(\text{Mon}^G(V_1, dV_1)) = 0$, the natural map $i_2 : I_2(S^r, S^w) \rightarrow I_1(S^r, S^w)$ is bijective.

**Proof.** Let $f : S^r \rightarrow S^w$ be a $G$-imbedding which represents an element of $I_1(S^r, S^w)$. By Lemma 3.2 $O^G(V_1, dV_1)$ is a deformation retract of $\text{Mon}^G(V_1, dV_1)$. Therefore, by Proposition 3.1, we can assume that $f$ is an element of $A$, and $i_2$ is surjective. By Proposition 3.3, $i_2$ is injective, and this completes the proof of Corollary 3.5.

§4. Proof of Theorem A and Theorem B

In this section we shall prove that, if $G$ is a finite group and $\pi_{n+1}(\text{Mon}^G(V_1, dV_1)) = 0$, then the $G$-isotopy class of a $G$-imbedding $f : S^r \rightarrow S^w$ is determined by the homotopy class of the orbit map of $f|_U(S^r - U)$ relative to the boundary, where $U$ is an invariant open neighborhood of $S^r$. And, using the obstruction theory, we shall prove Theorem A and Theorem B.

In this section we shall assume that $G$ is a finite group and $\pi_{n+1}(\text{Mon}^G(V_1, dV_1)) = 0$. Let $f_i : S^r \rightarrow S^w$, $i = 0, 1$, be $G$-imbeddings which represent elements of $I_1(S^r, S^w)$. Let $U$ be an invariant open $\varepsilon$-tubular neighborhood of $S^r$ in $S^r$. We can choose a sufficiently small positive number $\varepsilon$ such that $f_i = f_i^+U$ on $U$ and $f_i(S^r - U) \subset S^w - T$ for $i = 0, 1$. By Corollary 3.5, we have the following:

**Lemma 4.1.** With the above notations, $f_0$ and $f_1$ are $G$-isotopic if and only if there exists a $G$-isotopy $f : S^r \times I \rightarrow S^w$ such that $f_t(S^r - U)$ is contained in $S^w - T$ and $f_t = f_0^+U$, $0 \leq t \leq 1$, on $U$.

It is clear that free $G$-manifolds $S^r - U$ and $S^w - T$ are equivariant diffeomorphic to $S(V_1) \times D^{r+1}$ and $S(dV_1) \times D^{d+1}$ respectively. Let $L$ and $L'$ denote the orbit spaces $S(V_1)/G$ and $S(dV_1)/G$ respectively. Then the orbit spaces $(S^r - U)/G$ and $(S^w - T)G$ are diffeomorphic to $L \times D^{r+1}$ and $L' \times D^{d+1}$ respectively. Let $f_i : L \times D^{r+1} \rightarrow L' \times D^{d+1}$,
Proposition 4.2. With the above notations, $f_0$ and $f_i$ are $G$-isotopic if and only if $f_0$ and $f_i$ are homotopic relative to $L \times S^s$.

Proof. By Lemma 4.1, if $f_0$ and $f_i$ are $G$-isotopic, then $f_0$ and $f_i$ are homotopic relative to $L \times S^s$. Conversely if $f_0$ and $f_i$ are isotopic relative to $L \times S^s$, then $f_0$ and $f_i$ are isotopic relative to $L \times S^s$ because $\dim (L' \times D^{n+l}) > \dim (L \times D^{n+1}) + 1$. Since $G$ is a finite group, $S^s - U \to (S^s - U)/G$ and $S^w - T \to (S^w - T)/G$ are covering spaces. By the covering homotopy property, there exists a $G$-isotopy $h_t : S^s - U \to S^w - T$ $(0 \leq t \leq 1)$ such that $h_0 = f_0$ and $h_1 = f_i$ for $0 \leq t \leq 1$. Since $h_t \mid \partial (S^s - U) = f_t \mid \partial (S^s - U)$ and $h_t = f_i$, by the property of the covering space, we have $h_t = f_i$ on $S^s - U$. Therefore $f_0$ and $f_i$ are $G$-isotopic and this completes the proof of Proposition 4.2.

Proof of Theorem A. Suppose that $V_i = \bigoplus k_i W_i$, where $W_i$ runs over the inequivalent irreducible real $G$-modules. If $q > 2$, $\text{Mon}^g (V_i, dV_i) = \Pi_{k_i} V_{d_i, k_i} (C)$ by Lemma 3.2. Since $V_{d_i, k_i} (C)$ is $2(d-1)k_i$-connected, $\pi_{n+1} (\text{Mon}^g (V_i, dV_i)) = 0$ if $d \geq (n+3)/2$. If $q = 2$, $V_i = (m-n) W_i$ and $\text{Mon}^g (V_i, dV_i) = V_{d(m-n), n-1} (R)$, where $W_i$ is the non-trivial 1-dimensional real representation of $Z_2$. Since $V_{d(m-n), n-1} (R)$ is $(d-1)(m-n) - 1$-connected, $\pi_{n+1} (\text{Mon}^g (V_i, dV_i)) = 0$ if $d \geq (m+2)/(m-n)$. Therefore, combining Corollary 2.5 and Corollary 3.5, the set $I_2 (S^s, S^w)$ can be identified with the set $I_2 (S^s, S^w)$. Let $f_s : S^s \to S^w$ be the standard imbedding. Let $f : S^s \to S^w$ be a $G$-imbedding which represents an element of $I_2 (S^s, S^w)$. Since $\pi_s (\text{Mon}^g (V_s, dV_s)) = 0$, we can assume $f = f_s$. With the notation of Proposition 4.2, $f$ and $f_s$ are $G$-isotopic if and only if $f$ and $f_s$ are homotopic relative to $S^s \times L$.

Note that $L$ (resp. $L'$) is an $m-n-1$ (resp. $d(m-n)-1$)-dimensional lens space or real projective space. Since $d \geq (m+2)/(m-n)$, $\pi_i (L' \times D^{n+l}) = 0$ for $2 \leq i \leq m$. By the obstruction theory of P. Olum [5, Theorem 9.10 and Theorem 16.5], $f$ and $f_s$ are homotopic relative to $L \times S^s$. This completes the proof of Theorem A.
Proof of Theorem B. By Lemma 3.2, $\text{Mon}^G(V_1, dV_1) = V_d^{(m-n)/2, (m-n)/2}$ (C). Thus $\text{Mon}^G(V_1, dV_1)$ is $(d-1)(m-n)$-connected, and if $d \geq (m+1)/(m-n)$, $\pi_{*_t} (\text{Mon}^G(V_1, dV_1)) = 0$. Combining Corollary 2.5 and Corollary 3.5, the set $I(S^r, S^w)$ can be identified with $I_1(S^r, S^w)$.

Let $f : S^r \rightarrow S^w$ be a $G$-imbedding which represents an element of $I_1(S^r, S^w)$. Similarly as the proof of Theorem A, we can assume $f|L \times S^t = f_s|L \times S^t$, and in the case of $d > (m+1)/(m-n)$, $f$ is $G$-isotopic to the standard imbedding $f_s$.

Now consider the case of $d = (m+1)/(m-n)$. Since $H^i(L \times D^{d+1}, L \times S^t ; \pi_i(L' \times D^{d+1})) = H^{i-s-1}(L ; \pi_i(L')) = 0$ for $i < m$ and $H^{i-1}(L \times D^{d+1}, L \times S^t ; \pi_i(L' \times D^{d+1})) = H^{i-s-2}(L ; \pi_i(L')) = 0$ for $i < m$, by the obstruction theory, the homotopy classes of maps $f : L \times D^{d+1} \rightarrow L' \times D^{d+1}$ relative to $L \times S^t$ are in one to one correspondence with the elements of $H^m(L \times D^{d+1}, L \times S^t ; \pi_m(L' \times D^{d+1})) = \pi_m(L')$.

Since $\dim L' = d(m-n) - 1 = m$ in the case of $d = (m+1)/(m-n)$, by Proposition 4.2, we have $I(S^r, S^w) = \mathbb{Z}$. This completes the proof of Theorem B.

Remark. Suppose that $V_1 = \bigoplus k_i W_i$ and $k_i \geq 3$ for each $i$ if $\dim \text{Hom}^G(W_i, W_i) = 1$, where $W_i$ runs over the inequivalent irreducible real $G$-modules. Then Theorem A is valid when $G$ is a finite group.

References