Extensions of the Inner Automorphism Group of a Factor

By

Marie CHODA*

1. Introduction

Let $M$ be the crossed product $R(G, A, \alpha)$ of a von Neumann algebra $A$ by a locally compact group $G$ under a continuous action $\alpha$. By $\text{Aut}(M, A)$ we shall denote the group of all automorphisms of $M$, each of which is an extension of an automorphism of $A$. A systematic attempt to study $\text{Aut}(M, A)$ for a finite factor $M$ by the group measure space construction has been made in [11]. For the crossed product $M$ of a von Neumann algebra $A$ by a discrete countable group $G$ of freely acting automorphisms of $A$, some results concerning the structure of an element of $\text{Aut}(M, A)$, which is inner on $M$, have been obtained in [2], [3], [8] and [9], and generalized in [1]. Some relations between elements in $\text{Aut}(M, A)$ and two-cocycles on $G$ have been studied for a general crossed product of a von Neumann algebra $A$ by a discrete countable group, or a locally compact group $G$ under an action ([6], [10], [12], [14]).

In this paper, we consider this generalized crossed product in the form $M=R(G, A, \alpha, v)$ of a factor $A$ by a locally compact group $G$ under an action $\alpha$ with a factor set $\{v(g, h) ; g, h \in G\}$ (cf. Definition in below). In §2, we shall study the structure of the normal subgroup $K$ of $\text{Aut}(M, A)$, each element of which acts on $A$ as an inner automorphism. Under a certain condition, the group $K$ is isomorphic to the direct product of $\text{Int}(A)$ and $\chi(G)$, where $\text{Int}(A)$ is the group of natural extensions $\text{Ad}_u(u \in A)$ and $\chi(G)$ is the character

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Department of Mathematics, Osaka Kyoiku University, Osaka 543, Japan.
group of $G$ (Theorem 1 and Corollary 4). In § 3, we shall restrict our interest to a discrete countable group $G$ and study the structure of the normal subgroup $\text{Int}(M, A)$ of $\text{Aut}(M, A)$, each element of which is an inner automorphism of $M$. If the action under $\alpha$ of all elements in $G$ except the identity is outer on $A$, then $\text{Int}(M, A)$ is isomorphic to an extension group of $\text{Int}(A)$ by $G$ (Theorem 7).

2. Extensions of Inner Automorphisms

Let $A$ be a von Neumann algebra acting on a separable Hilbert space $H$. By $\text{Aut}(A)$ we shall denote the group of all automorphisms (*-preserving) of $A$ and by $\text{Int}(A)$ the group of all inner automorphisms of $A$. For a locally compact group $G$, we denote by $K(H; G)$ the vector space of all continuous $H$-valued functions on $G$ with compact support. Considering the inner product in $K(H; G)$ defined by

$$
(\xi, \eta) = \int_G (\xi(g), \eta(g)) \, dg, \quad \xi, \eta \in K(H; G),
$$

$K(H; G)$ is a pre Hilbert space, where $dg$ is a fixed left Haar measure of $G$. The completion of $K(H; G)$ with respect to this inner product is denoted by $L^2(H; G)$. A map $\alpha$ of $G$ into $\text{Aut}(A)$ is called an action of $G$ on $A$, if for each fixed $a$ in $A$, the map $g \in G \mapsto \alpha_g(a) \in A$ is $\sigma$-strongly *continuous and $\alpha$ satisfies the following condition (1);

$$
(i, g, h) = \alpha_g^* \alpha_h \in \text{Int}(A), \quad g, h \in G.
$$

For such a map $\alpha$, a family $\{v(g, h); g, h \in G\}$ of unitaries in $A$ is called a factor set associated with the action $\alpha$, if the map $(g, h) \in G \times G \mapsto v(g, h) \in A$ is $\sigma$-strongly *continuous and the following conditions (2) and (3) are satisfied;

$$
(i) \quad v(g, h) = Adv(g, h), \quad g, h \in G,
$$

$$
(ii) \quad v(g, hk) = v(gh, k) \alpha_{g^{-1}}(v(g, h)), \quad g, h, k \in G,
$$

where $Ad_u$ is an automorphism of $A$ such that $Ad_u(a) = uau^*$ for $a$ in $A$. In the sequel, we assume that $\alpha_1 = \iota$, where $1$ is the identity.
of $G$ and $\iota$ is the identity automorphism of $A$. On the Hilbert space $L^2(H; G)$, we shall denote by $\pi_\alpha$ the representation of $A$ such that

$$ (\pi_\alpha(a) \xi)(h) = a^{-1}\alpha(a) \xi(h), \quad h \in G, \xi \in L^2(H; G). $$

By $\rho$, we shall denote a map of $G$ into the unitary group on $L^2(H; G)$ such that

$$ (\rho(g) \xi)(h) = v(g, g^{-1}h) \xi(g^{-1}h), \quad h \in G, \xi \in L^2(H; G). $$

By the direct computation, we have that

$$ \rho(g) \rho(h) = \rho(gh) \pi_\alpha(v(g, h)), \quad g, h \in G $$

and $\pi_\alpha$ and $\rho$ satisfy the covariance relation;

$$ \rho(g) \pi_\alpha(a) \rho(g)^* = \pi_\alpha(\alpha(a)), \quad g \in G, a \in A. $$

The von Neumann algebra on $L^2(H; G)$ generated by $\pi_\alpha(A)$ and $\rho(G)$ is called the crossed product of $A$ by $G$ with the factor set $\{v(g, h) ; g, h \in G\}$ respect to $\alpha$ and denoted by $R(G, A, \alpha, v)$. If the action $\alpha$ is a representation of $G$ into $\text{Aut}(A)$ and the factor set $\{v(g, h) ; g, h \in G\}$ associated with the action $\alpha$ is the trivial set, that is, $v(g, h)$ is the identity for every $g, h$ in $G$, then $R(G, A, \alpha, v)$ is the usual crossed product ([16]), which we shall denote by $R(G, A, \alpha)$.

At first, we shall be concerned with the group of all extensions to $R(G, A, \alpha, v)$ of the inner automorphism group of a factor. Fix a von Neumann algebra $A$ equipped with an action $\alpha$ of a locally compact group $G$ and a factor set $\{v(g, h) ; g, h \in G\}$ associated with the action $\alpha$. Throughout this paper, we shall denote by $M$ the crossed product $R(G, A, \alpha, v)$. By $\text{Aut}(M, A)$, we shall denote the group of automorphisms of $M$ sending $\pi_\alpha(A)$ onto itself:

$$ \text{Aut}(M, A) = \{\beta \in \text{Aut}(M) ; \beta(\pi_\alpha(A)) = \pi_\alpha(A)\}. $$

It is clear that all inner automorphisms of $A$ admit natural extensions $\text{Ad} u (u \in u(A))$ to $M$ and the automorphisms $\alpha_a$ admit natural liftings $\text{Ad} \rho(g) (g \in G)$, where $u(A)$ is the group of unitaries in $A$. By the same notation $\text{Int}(A)$ and $\alpha(G)$ we shall denote the set of such automorphisms of $M$: 
Let $K$ be the group of all extensions to $M$ of the inner automorphism group of $A$:

$$K = \{ \beta \in \text{Aut}(M, A) : \beta \text{ is inner on } \pi_* (A) \}.$$

**Theorem 1.** Let $A$ be a factor equipped with an action $\alpha$ of a locally compact group $G$ and a factor set $\{v(g, h) ; g, h \in G\}$ associated with the action $\alpha$. If $\alpha$ is such that $\pi_* (A)' \cap M$ is the scalar multiples of the identity, then $K$ is isomorphic to the direct product of $\text{Int}(A)$ and $\chi(G)$, where $\chi(G)$ is the group of all continuous characters of $G$.

**Proof.** Take a $\beta$ in $K$. Let $u$ be a unitary in $\pi_* (A)$ such that $\beta(a) = uau^*$ for all $a$ in $\pi_* (A)$. Then, for each $a$ in $\pi_* (A)$ and $g$ in $G$, we have that

$$u\rho(g)a\rho(g)^*u^* = \beta(\rho(g)a\rho(g)^*) = \beta(\rho(g))\beta(a)\beta(\rho(g))^* = \beta(\rho(g))uau^*\beta(\rho(g))^*,$$

so that $\rho(g)^*u^*\beta(\rho(g))u$ is contained in $\pi_* (A)' \cap M$. Since $\pi_* (A)' \cap M$ is the scalar multiples of the identity $I$, we have a $\chi$ in $\chi(G)$ such that

$$\beta(\rho(g)) = \chi(g)u\rho(g)u^*.$$

In fact, put $\chi(g)I = \rho(g)^*u^*\beta(\rho(g))u$, then we have that

$$\chi(gh)I = \rho(gh)^*u^*\beta(\rho(gh))u = \rho(gh)^*u^*\beta(\rho(g)\rho(h)\pi_*(v(g, h)))u = \rho(gh)^*u^*\beta(\rho(g))\beta(\rho(h))u\pi_*(v(g, h)) = \chi(g)\rho(gh)^*u^*\beta(\rho(h))u\pi_*(v(g, h)) = \chi(g)\chi(h)I.$$

For each character $\chi$ of $G$, put

$$(u(\chi)\xi)(g) = \overline{\chi(g)}\xi(g), \quad g \in G, \xi \in L^1 (H; G),$$

where $\overline{\chi(g)}$ is the complex conjugate of $\chi(g)$. Then $u(\chi)$ is a unitary satisfying

$$u(\chi)a = au(\chi), \quad a \in \pi_* (A),$$
and

(10) \[ u(\chi)(\rho(g))u(\chi)^* = \overline{\chi(g)}\rho(g), \quad g \in G. \]

Let \( \delta(\chi) \) be an automorphism of \( M \) induced by \( u(\chi) \), then \( \delta(\chi) \)
belongs to the group \( K \).

For a \( \beta \) in \( K \), let \( u \) be a unitary in \( \pi_*(A) \) such that \( \beta(a) = uau^* \)
for all \( a \) in \( \pi_*(A) \). Take a \( \chi \) in \( \chi(G) \) satisfying the property (8),
then we have that

(11) \[ (\beta\delta(\chi))(a) = \beta(a) = \text{Ad}_u(a), \quad a \in \pi_*(A) \]
and

(12) \[ (\beta\delta(\chi))(\rho(g)) = \overline{\chi(g)}\beta(\rho(g)) = \text{Ad}_u(\rho(g)), \quad g \in G, \]
so that \( \beta \cdot \delta(\chi) \) belongs to \( \text{Int}(A) \). Thus every \( \beta \) in \( K \) has a form
\( \beta = \text{Ad}_u \cdot \delta(\chi) \) for some \( u \) in \( \pi_*(A) \) and \( \chi \) in \( \chi(G) \). Such a decom-
position is unique. In fact, if

\[ \text{Ad}_u \cdot \delta(\chi) = \text{Ad}_w \cdot \delta(\chi'), \quad u, w \in \pi_*(A), \chi, \chi' \in \chi(G), \]
then we have that on \( \pi_*(A) \), \( \text{Ad}_w u \) is the identity automorphism. Since \( A \) is a factor, it follows that \( w \) is a scalar multiple of \( u \), which implies that \( \text{Ad}_u = \text{Ad}_w \) on \( M \) and that \( \delta(\chi) = \delta(\chi') \).

By the property (10), we have that, for \( \chi \) and \( \chi' \) in \( \chi(G) \),
\( \delta(\chi) = \delta(\chi') \) if and only if \( \chi = \chi' \).

Therefore, defining a map \( \sigma \) of the direct product of \( \text{Int}(A) \) and
\( \chi(G) \) onto \( K \) by \( \sigma(\gamma, \chi) = \gamma \cdot \delta(\chi) \), \( (\gamma \in \text{Int}(A), \chi \in \chi(G)) \), we have an
isomorphism of \( K \) onto the direct product of \( \text{Int}(A) \) and \( \chi(G) \).

Let \( K_\alpha \) be the group of all extensions to \( M \) of the identity automorphism of \( A \):

\[ K_\alpha = \{ \beta \in \text{Aut}(M, A) ; \beta \text{ is the identity on } \pi_*(A) \}. \]

**Corollary 2.** Let \( A, \alpha, G \) and \( \{v(g, h) ; g, h \in G\} \) be as in
Theorem 1. The group \( K_\alpha \) is isomorphic to \( \chi(G) \).

Denote by \([G, G]\) the commutator group of \( G \), that is, \([G, G]\)
is the closed group generated by \([ghg^{-1}h^{-1}; g, h \in G]\). A group \(G\) is called perfect if \([G, G]\) coincides with \(G\).

**Corollary 3.** Let \(A, G, \alpha\) and \([v(g, h); g, h \in G]\) be as in Theorem 1. The following three statements are equivalent:

(a) \(K\) coincides with \(\text{Int}(A)\);
(b) \(K_\alpha\) is the trivial group \([e]\);
(c) \(G\) is perfect.

**Proof.** By Theorem 1 and Corollary 2, it is clear that the statements (a) and (b) are equivalent and that they are equivalent to the condition that \(\chi(G) = \{1\}\). On the other hand, \(\chi(G)\) is the group \(\text{Hom}(G, T)\) of all continuous homomorphism of \(G\) into \(T\), where \(T\) is the unit circle of the complex plane. Since \(T\) is an abelian group, it follows that for each \(\chi\) in \(\chi(G)\), \([G, G]\) is contained in the kernel of \(\chi\). Hence \(\chi(G)\) is isomorphic to \(\text{Hom}(G/[G, G], T)\). Thus the condition that \(\chi(G) = \{1\}\) is equivalent to \(G = [G, G]\), which is statement (c).

Especially, assume that \(G\) is a discrete countable group. If \(\alpha_g\) is an outer automorphism of \(A\) for all \(g\) in \(G\) except the unit, then by [5, Corollary 3], we have that \(\pi_* (A)' \cap \mathcal{M}\) is the scalar multiples of the identity. Therefore, we have the following corollary:

**Corollary 4.** Let \(A\) be a factor equipped with an action \(\alpha\) of a discrete countable group \(G\) and a factor set \([v(g, h); g, h \in G]\) associated with the action \(\alpha\). Assume that \(\alpha_g\) is an outer automorphism of \(A\) for all \(g\) in \(G\) except the unit. Then \(K\) is isomorphic to the direct product of \(\text{Int}(A)\) and \(\chi(G)\), so that \(K_\alpha\) is isomorphic to \(\chi(G)\). The three statements in Corollary 3 are equivalent.

3. Extensions as Inner Automorphisms.

In this section, we shall be concerned with extensions of automorphisms of \(A\) to \(M\) which are inner on \(M\).
Throughout this section, we shall treat a factor equipped with an action of a discrete countable group $G$ and a factor set $\{v(g, h) ; g, h \in G\}$ associated with the action $\alpha$. For $M = R(G, A, \alpha, v)$, we shall denote by $\text{Int}(M, A)$ the group of inner automorphisms of $M$ sending $\pi_s(A)$ into itself and by $u(M, A)$ the group of unitaries in $M$ normalizing $\pi_s(A)$:

$$\text{Int}(M, A) = \{\beta \in \text{Int}(M) ; \beta(\pi_s(A)) = \pi_s(A)\},$$

and

$$u(M, A) = \{u \in u(M) ; u\pi_s(A)u^* = \pi_s(A)\}.$$ 

We shall determine a relation among $\text{Int}(M, A)$, $\text{Int}(A)$ and $G$.

**Theorem 5.** Let $A$ be a factor equipped with an action $\alpha$ of a discrete countable group $G$ and a factor set $\{v(g, h) ; g, h \in G\}$ associated with the action $\alpha$. Then each $u$ in $u(M, A)$ has a form;

$$u = w\rho(g), \quad w \in u(\pi_s(A)), \quad w' \in u(\pi_s(A)' \cap M), \quad g \in G. \tag{13}$$

By the same technique as [8; Corollary 1] or [9; Theorem], we can prove this theorem. For the sake of completeness, we shall give a proof of Theorem 5.

**Proof.** Take a $u$ in $u(M, A)$. Let

$$u = \sum_{s \in \sigma} a(g) \rho(g) \quad a(g) \in \pi_s(A), \text{ (in the } \sigma\text{-strong topology)}$$

be the Fourier expansion of $u([5; Lemma 1])$. By the property that $u\pi_s(A)u^* = \pi_s(A)$, we have that

$$\sum_{s \in \sigma} a(g) \alpha_s(a) \rho(g) = \sum_{s \in \sigma} uau^* a(g) \rho(g), \quad a \in \pi_s(A),$$

so that

$$a(g) \alpha_s(a) = uau^* a(g), \quad a \in \pi_s(A), \quad g \in G.$$ 

If $\alpha_s^{-1}Au$ is an outer automorphism of $\pi_s(A)$, then we have that $a(g) = 0$. Since $u$ is unitary, it follows that there exists a $g \in G$ such that $\alpha_s^{-1}Au$ is inner on $\pi_s(A)$. Let $w$ be a unitary in $\pi_s(A)$ such that $\pi_s(A)\alpha_s^{-1}Au = \text{Ad} w$. Put $w' = \rho(g)^* uw^*$, then $w'$ belongs to $\pi_s(A)' \cap M$. 


Corollary 6. Let $A$, $G$, $\alpha$ and $\{v(g, h) ; g, h \in G\}$ be as in Theorem 5. Each $\beta$ in $\text{Int}(M, A) \cap \text{Aut}(A)$ has a form:

$$\beta = \gamma \alpha, \quad \gamma \in \text{Int}(A), \, g \in G.$$  

Theorem 7. Let $A$, $G$, $\alpha$ and $\{v(g, h) ; g, h \in G\}$ be as in Theorem 5. Assume that $\alpha_g$ is an outer automorphism of $A$ for all $g$ in $G$ except the identity. Then $u(M, A)$ is isomorphic to an extension group of $u(A)$ by $G$ and $\text{Int}(M, A)$ is isomorphic to an extension group of $\text{Int}(A)$ by $G$. If $M$ is the usual crossed product $R(G, A, \alpha)$, then these extensions are a semi-direct product.

Proof. If $\alpha_g$ is an outer automorphism of $A$ for all $g$ in $G$ except the unit, then $\pi_\alpha(A)' \cap M$ is the scalar multiples of the identity ([5; Corollary 3]). Hence, by Theorem 5, each $u$ in $u(M, A)$ has a form:

$$u = wp(g), \quad w \in u(\pi_\alpha(A)), \, g \in G.$$  

If

$$wp(g) = w'p(h), \quad w, \, w' \in u(\pi_\alpha(A)), \, g, \, h \in G,$$

then we have that

$$w^*w = \rho(h)\rho(g)^* = \rho(h)\rho(g^{-1})\pi_\alpha(v(g, g^{-1})^*)$$

$$= \rho(hg^{-1})\pi_\alpha(v(h, g^{-1})v(g, g^{-1})^*).$$

On the other hand, by [4; Theorem 6], there exists a faithful normal expectation $e$ of $M$ onto $\pi_\alpha(A)$ such that $e(\rho(g)) = 0$ for all $g$ in $G$ except the unit. Therefore, if the relation (16) is satisfied for $g$ and $h$ in $G$ such that $g \neq h$, then we have that $w^*w = 0$ by (17), which is a contradiction. Thus the decomposition of $u$ in $u(M, A)$ with the form (15) is unique. We shall define a map $\sigma$ on the set $u(\pi_\alpha(A)) \times G$ by $\sigma(w, g) = wp(g), \, w \in u(\pi_\alpha(A)), \, g \in G$. Define a multiplication on $u(\pi_\alpha(A)) \times G$ by

$$w, \, g) (w', \, h) = (\alpha_g(w')\alpha_g(\pi_\alpha(v(g, h)))), \, gh),$$

then $\sigma$ is an isomorphism of the extension group $E(G, u(\pi_\alpha(A)), \alpha, v)$ of $u(\pi_\alpha(A))$ by $G$ under the multiplication (18) onto $u(M, A)$. If
$M$ is the usual crossed product $R(G, A, \alpha)$, then we may always take $v(g, h) = 1$ for all $g, h$ in $G$, so that mapping $\sigma$ gives an isomorphism of a semi direct product of $u(\pi_\alpha(A))$ by $G$ onto $u(M, A)$.

Similarly, define a multiplication in the set $\text{Int}(A) \times G$ by

$$\langle Adu, g \rangle (\text{Ad}_w, h) = \langle \text{Ad}(u\alpha_\tau(w)\alpha_\tau(v(g, h))), gh \rangle.$$  

The group $\text{Int}(A)$ is isomorphic to the factor group $u(\pi_\alpha(A))/TI$ of $u(\pi_\alpha(A))$ by the normal subgroup $\{\mu I; \mu \in T\}$. The extension group $E(G, \text{Int}(A), \alpha, v)$ of $\text{Int}(A)$ by $G$ under the multiplication (19) is isomorphic to the factor group $E(G, u(\pi_\alpha(A)), \alpha, v)/TI \times \{1\}$ of $E(G, u(\pi_\alpha(A)), \alpha, v)$ by the normal subgroup $TI \times 1 = \{(\mu I, 1); \mu \in T\}$. On the other hand, $E(G, u(\pi_\alpha(A)), \alpha, v)/TI \times 1$ is isomorphic to the factor group $u(M, A)/TI$ of $u(M, A)$ by the normal subgroup $TI$, which is isomorphic to $\text{Int}(M, A)$. Thus $\text{Int}(M, A)$ is isomorphic to the extension group $E(G, \text{Int}(A), \alpha, v)$ of $\text{Int}(A)$ by $G$ under the multiplication (19).

References
