Cohomologies of Lie Algebras of Vector Fields with Coefficients in Adjoint Representations
Foliated Case

By

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Introduction

Let \((M, \mathcal{F})\) be a foliated manifold. We have a natural Lie algebra \(\mathcal{L}(M, \mathcal{F})\) of vector fields locally preserving the foliation \(\mathcal{F}\), and its ideal \(\mathcal{T}(M, \mathcal{F})\) of vector fields tangent to leaves of \(\mathcal{F}\). Here we are interested in the first cohomologies of \(\mathcal{L}(M, \mathcal{F})\) and \(\mathcal{T}(M, \mathcal{F})\) with coefficients in their adjoint representations. This work is in a series of F. Takens' work [7] and the author's [3], [4]. In this paper, we use the latter for the general reference.

Our main result is

**Main Theorem**

(i) \(H^1(\mathcal{L}(M, \mathcal{F}); \mathcal{L}(M, \mathcal{F})) = 0\).

(ii) \(H^1(\mathcal{T}(M, \mathcal{F}); \mathcal{T}(M, \mathcal{F})) \cong \mathcal{L}(M, \mathcal{F})/\mathcal{T}(M, \mathcal{F})\).

If \(M\) is compact, \(\mathcal{L}(M, \mathcal{F})\) is identical with the Lie algebra of vector fields preserving \(\mathcal{F}\). There are compact foliated manifolds \((M, \mathcal{F})\) such that \(H^1(\mathcal{T}(M, \mathcal{F}); \mathcal{T}(M, \mathcal{F}))\) are of dimension \(r\) for any \(r\) \((0 \leq r \leq \infty)\).

The content of this paper is arranged as follows. In §1, we introduce Lie algebras \(\mathcal{L}\) and \(\mathcal{T}\) for a standard foliation on a euclidean space, and study their structures. In §2, we investigate properties of derivations of \(\mathcal{L}\) and \(\mathcal{T}\), and in §3, we prove Main Theorem for \(\mathcal{L}\) and \(\mathcal{T}\)(flat case). In §4, we give the proof of Main Theorem and

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some examples.

All manifolds, foliations, vector fields, etc. are assumed to be of $C^\infty$-class, throughout this paper.

§ 1. Lie Algebras $\mathcal{L}$ and $\mathcal{F}$

1.1. Notations and Definitions. Fix a coordinate system $x_1, \ldots, x_p, y_1, \ldots, y_q$ in a $(p+q)$-dimensional euclidean space $V = \mathbb{R}^{p+q}$. Denote $\frac{\partial}{\partial x_i}$ by $\partial_i (i = 1, \ldots, p)$, and $\frac{\partial}{\partial y_\alpha}$ by $\partial_\alpha (\alpha = 1, \ldots, q)$. Use Latin indices $i, j, k, \ldots$ for $x_1, \ldots, x_p$, and Greek indices $\alpha, \beta, \ldots$ for $y_1, \ldots, y_q$, otherwise stated. Put

$$
\mathcal{F} = \{ \sum_{i=1}^p f_i(x, y) \partial_i ; f_i(x, y) \text{ are } C^\infty \text{-functions of } x_1, \ldots, x_p, y_1, \ldots, y_q \},
$$

$$
\mathcal{L}' = \{ \sum_{\alpha=1}^q g_\alpha(y) \partial_\alpha ; g_\alpha(y) \text{ are } C^\infty \text{-functions of } y_1, \ldots, y_q \},
$$

$$
\mathcal{L} = \mathcal{F} + \mathcal{L}' \quad \text{(as vector spaces)}.
$$

Then they are subalgebras of the Lie algebra $\mathfrak{X}$ of all vector fields on $V$, and $\mathcal{F}$ is an ideal of $\mathcal{L}$.

Let $\mathcal{F}$ be a standard codimension-$q$ foliation, defined by parallel $p$-planes: $y_1 = \text{constant}, \ldots, y_q = \text{constant}$, in $V$. Any vector field $X$ in $\mathcal{F}$ is tangent to leaves of $\mathcal{F}$, and $X$ is called leaf-tangent. Let $\phi_\epsilon$, be the one-parameter group of diffeomorphisms generated by $Y \in \mathcal{F}$, then $\phi_\epsilon$, transforms every leaf to some leaf for each $t$, and $Y$ is called foliation preserving.

Denote by $\mathcal{F}_x$ or $\mathcal{F}_y$, the subalgebra of $\mathcal{F}$ of all vector fields in $\mathcal{F}$ whose coefficient functions depend only on $x_1, \ldots, x_p$, or $y_1, \ldots, y_q$, respectively.

Here we summarize the facts which will be applied later.

\textbf{Lemma 1.1.} \quad (i) Let $X \in \mathfrak{X}$. If $[\partial_i, X] = 0$ for all $i = 1, \ldots, p$, then $X$ is independent of the variables $x_1, \ldots, x_p$.

(ii) $[\mathcal{F}_x, \mathcal{L}'] = 0$, and $[\mathcal{F}_x, \mathcal{L}] \subseteq \mathcal{F}$.

(iii) Let $X \in \mathcal{L}$. If $[\partial_\alpha, X] \in \mathcal{L}'$ for all $\alpha$, then $X$ is independent of the variables $x_1, \ldots, x_p$. 


(iv) Let \( X \in \mathcal{F}_r \). Then \([X, I] = X\), where \( I = \sum_{i=1}^r x_i \partial_i \in \mathcal{F}_r\).

(v) Let \( X \in \mathcal{L}'\). If \([X, y_\alpha \partial_i] = 0\) for all \( i \) and \( \alpha \), then \( X = 0\).

This can be proved by elementary calculations.

1.2. Vector Fields with Polynomial Coefficients. The vector field \( X = \sum_{i=1}^p f_i(x, y) \partial_i + \sum_{a=1}^q g_a(x, y) \partial_a \) on \( V \) is said to be with polynomial coefficients, if all \( f_i(x, y) \) and \( g_a(x, y) \) \((i = 1, \ldots, p, a = 1, \ldots, q)\) are polynomials. Such vector fields form a Lie subalgebra \( \mathfrak{A} \) of \( \mathfrak{A} \). Put \( \mathcal{F} = \mathcal{F} \cap \mathfrak{A}, \mathcal{D} = \mathcal{L} \cap \mathfrak{A} \), and \( \mathcal{D}' = \mathcal{L}' \cap \mathfrak{A} \). Put

\[ \mathcal{F}_r = \left\{ \sum_{i=1}^r f_i(x, y) \partial_i \in \mathcal{F} ; f_i(x, y) \text{ are homogeneous polynomials of degree } n+1 \text{ in } x_1, \ldots, x_r \text{ and of degree } m+1 \text{ in } y_1, \ldots, y_s \right\}. \]

Then

\[ \mathcal{F} = \sum_{n=0}^{n-1} \mathcal{F}_n, \quad \mathcal{D} = \mathcal{F} \cap \mathfrak{A} = \sum_{n=0}^{n-1} \mathcal{F}_n, \quad \mathcal{D}' = \mathcal{F} \cap \mathfrak{A} = \sum_{n=0}^{n-1} \mathcal{F}_n. \]

Moreover, we have easily

**Lemma 1.2.** (cf. [4]) Let \( I \) be defined in Lemma 1.1 (iv), then \( \mathcal{F}_n = \{ X \in \mathcal{F}_r ; [I, X] = nX \} \).

Put \( \mathcal{L}_n = \{ \sum_{a=1}^q g_a(y) \partial_a \in \mathcal{L}' ; g_a(y) \text{ are homogeneous of degree } m+1 \} \).

Then \( \mathcal{D}' = \sum_{n=0}^{n-1} \mathcal{L}_n \), and we have

**Lemma 1.3.** Let \( J = \sum_{s=1}^s y_s \partial_s \in \mathcal{L}' \), then \( \mathcal{L}_n = \{ Y \in \mathcal{L} ; [J, Y] = mY \} \).

1.3. Proposition 1.4. If a vector field \( X \in \mathcal{F} \) satisfies \( j^s(X)(0) = 0 \), then there exist a finite number of vector fields \( X_1, \ldots, X_r \in \mathcal{F} \) such that

\[ X = \sum_{i=1}^r [X_i, X_{i+r}] \text{ and } j^i(X_i)(0) = 0 \text{ (} i = 1, \ldots, 2r \text{).} \]
Proof. Clearly it is enough to show the assertion for the case

\[ X = x_i^{i_1} \ldots x_i^{i_s} y_i^{j_1} \ldots y_i^{j_s} f(x, y) \partial_i \]

for \[ \sum_{i=1}^s i_s + \sum_{a=1}^s j_a \geq 4 \]. Put \( h(x, y) = x_i^{i_1} \ldots x_i^{i_s} y_i^{j_1} \ldots y_i^{j_s} \).

**Case 1.** The case where \( i_s \geq 2 \) for some \( k \).

\[
\begin{align*}
[x_i^a \partial_{\alpha} x_i^{a-1} X] - [x_i^a \partial_{\alpha} x_i^{a^2} X] \\
= (i_s - 1 - 2\delta_{ik}) X - x_i h(x, y) (\partial_i f(x, y)) \partial_i \\
- (i_s - 2 - 3\delta_{ik}) X + x_i h(x, y) (\partial_i f(x, y)) \partial_i \\
= (1 + \delta_{ik}) X.
\end{align*}
\]

Here \( \delta_{ik} \) is Kronecker's delta, so \((1 + \delta_{ik}) \geq 1 > 0 \). And \( j_i(x_1 x^2) (0) = 0 \) is obvious.

In the following, we can assume that \( i_s \leq 1 \) for all \( k \).

**Case 2.** The case where \( \sum_i i_s \geq 2 \). We can assume \( i_1 = i_2 = 1 \). Let \( \phi \) be a coordinate transformation

\[
\phi: \begin{cases} 
  x_1 = x_1 + x_2, & x_2 = x_1 - x_2, \\
  x_i = x_i (i \geq 3), & \gamma_a = \gamma_a \text{ (all } \alpha) 
\end{cases}
\]

then \( \phi(\mathcal{F}) = \mathcal{F} \). So this case is reduced to Case 1.

In the following, we can assume that \( i_s = 0 \) for all \( k \) except at most one \( k \).

**Case 3.** The case where \( j_s \geq 2 \) for some \( \alpha \). We get

\[
[y_0^a \partial_{\gamma_0} x_0^{-1} y_0^{-2} X] - [y_0^a x_0 \partial_{\gamma_0} y_0^{-1} X] = (1 + \delta_{ik}) X.
\]

Obviously \( j_i(Y) (0) = 0 \) for all vector fields \( Y \) in the left hand.

**Case 4.** The case where \( j_s \leq 1 \) for all \( \alpha \). Since we have \( \sum_{a=1}^s j_a \geq 4 - 1 = 3 \), so this case is also reduced to Case 3, similarly as Case 2. Q.E.D.

**Proposition 1.5.** If a vector field \( Y \in \mathcal{L}' \) satisfies \( j_i(Y) (0) = 0 \), then there exist a finite number of vector fields \( Y_1, \ldots, Y_s \in \mathcal{L}' \) such that
\[ Y = \sum_{i=1}^{r} [Y_i, Y_{i+r}] \quad \text{and} \quad j^{i}(Y_i)(0) = 0 \quad (i = 1, \ldots, 2r). \]

**Proof.** Similarly as in Cases 1 and 2 in the proof of the above proposition. Q.E.D.

§ 2. **Derivations of \( \mathcal{I} \) and \( \mathcal{L} \) (I)

2.1. Let \( \mathcal{D} = \mathcal{D}_{\mathcal{I}}(\mathcal{I}; \mathcal{L}) \) be the space of derivations of \( \mathcal{I} \) with values in \( \mathcal{L} \). And let \( \mathcal{D}_{\mathcal{L}} \) or \( \mathcal{D}_{\mathcal{I}} \) be the derivation algebra of \( \mathcal{L} \) or \( \mathcal{I} \) respectively. Remember that a derivation \( D \) satisfies the property \( D[X, Y] = [D(X), Y] + [X, D(Y)] \).

**Proposition 2.1.** If a derivation \( D \) in \( \mathcal{D} \) is zero on \( \mathcal{I}_{n,m} \) for \( n+m \leq -1 \), then \( D \) is zero on \( \mathcal{I} \).

**Proof.** Step 1. To show that \( D \) is zero on \( \mathcal{I} \). We prove this by the induction on \( n \) for the decomposition \( \mathcal{I}_x = \sum_{s=-1}^{z} \mathcal{I}_{x-s} \). When \( n \) is non-positive, the assertion holds by the assumption. Assume that \( D \) is zero on \( \mathcal{I}_{n-1}(k \leq n-1) \). Let \( Z \in \mathcal{I}_{n-1}(n \geq 1) \), and define the vector fields \( X \in \mathcal{I} \) and \( Y \in \mathcal{L} \) as \( D(Z) = X + Y \).

Apply \( D \) to \([\partial_i, Z] \in \mathcal{I}_{n-1} \) \((1 \leq i \leq p)\), then we get \( X \in \mathcal{I} \), by Lemma 1.1 (i) and the hypothesis of the induction.

We get \([I, Z] = nZ\), by Lemma 1.2. Apply \( D \) to the both sides of this equality, then by Lemma 1.1 (iv), we get

\[-X = nX + nY,\]
hence \( X = Y = 0 \), so \( D(Z) = 0 \).

Step 2. To show that \( D \) is zero on \( \mathcal{I}_{0,0} \). Clearly it is enough to show the assertion for the case \( X = x_{i}y_{j} \partial_{i} \in \mathcal{I}_{0,0} \). Apply \( D \) to

\[ X = x_{i}y_{j} \partial_{i} = 2^{-1}[y_{i} \partial_{i}, x_{j} \partial_{j}], \]
then we have \( D(X) = 0 \), because \( y_{i} \partial_{i} \in \mathcal{I}_{-1,0} \) and \( x_{j} \partial_{j} \in \mathcal{I}_{0} \).
Step 3. To show that $D$ is zero on $\mathcal{F}_y$. The proof is carried out by the induction on $m$ for the decomposition $\mathcal{F}_y = \sum_{n \geq -1} \mathcal{F}_{-m}$. When $m$ is non-positive, the assertion holds by the assumption. Assume that $D$ is zero on $\mathcal{F}_{-1,k}(k \leq m-1)$. Clearly it is enough to show that $D(Y) = 0$ for the case

$$Y = y_{i_1} \ldots y_{i_m} \partial_i$$

for $\sum_i j_i = m + 1$. There is an index $\beta$ such that $j_i \geq 1$. Apply $D$ to

$$Y = [y_{i_1}^{-1}Y, y_{\beta}x_i \partial_i],$$

then $D(Y) = 0$, because $y_{i_1}^{-1}Y \in \mathcal{F}_{-1,n-1}$, and $y_{\beta}x_i \partial_i \in \mathcal{F}_{0,0}$.

Last Step. Decompose $\mathcal{F}$ as $\mathcal{F} = \sum_{n \geq -1} (\sum_{m \geq -1} \mathcal{F}_{n,m})$. We prove the assertion of the proposition by the induction on $n$. The assertion for $n = -1$ holds by Step 3. Assume that $D$ is zero on $\sum_{m \geq -1} \mathcal{F}_{n,m}(n \leq n_0 - 1)$. It is enough to show that $D(X) = 0$ for the case

$$X = x_{i_1} \ldots x_{i_s} f(y) \partial_i$$

for $\sum_i i_j = n_0 + 1$, and some polynomial $f(y)$ of $y_1, \ldots, y_q$. Apply $D$ to the equality

$$X = \begin{cases} (i_s + 1)^{-1}[x_{i_s}^{-1}X, x_{i_s}^{i_s} \partial_i] & \text{if } i_s > 0, \\
[x_{i_k}^{-1}X, x_{i_k} x_{i_s} \partial_i] & \text{if } i_s = 0, \text{ and } i_k > 0 \text{ for some } k, \end{cases}$$

we get $D(X) = 0$, because $x_{i_k}^{-1}X, x_{i_k}^{-1}X \in \sum_{n \geq n_0 - 1} (\sum_{m \geq -1} \mathcal{F}_{n,m})$, and $x_{i_k}^{i_s} \partial_i, x_{i_k} x_{i_s} \partial_i \in \mathcal{F}_s$. Q.E.D.

Corollary 2.2. The derivation $D \in \mathcal{B}$ is zero on $\mathcal{F}$, under the same assumption as Proposition 2.1.

Proof. It follows from Propositions 1.3 and 1.4 in [4], and Proposition 1.4. Q.E.D.

2.2. Proposition 2.3. If a derivation $D \in \mathcal{B}$ is zero on $\mathcal{F}$, then $D$ is zero on $\mathcal{F}'$. 
Proof. Step 1. To show that $D(\partial_\alpha) = 0$ ($\alpha = 1, \ldots, q$). Apply $D$ to $[\partial_\alpha, \partial_\alpha] = [I, \partial_\alpha] = 0$, then we get $D(\partial_\alpha) \in \mathcal{L}'$, by Lemma 1.1 (i), (iv).

Define the functions $g(\cdot; y)$ as $D(\partial_\alpha) = \sum_j g_j(\cdot; y) \partial_j \in \mathcal{L}'$. Apply $D$ to $\partial_\alpha \partial_i = [\partial_\alpha, y_i \partial_i]$, then we get

$$0 = \left[ \sum_j g_j(\cdot; y) \partial_j, y_i \partial_i \right] = g_i(\cdot; y) \partial_i,$$

hence $g_i(\cdot; y) = 0$, so $D(\partial_\alpha) = 0$.

Step 2. To show that $D(J) = 0$, where $J = \sum_i y_i \partial_i$. Apply $D$ to $[\partial_i, J] = [I, J] = 0$, then we get $D(J) \in \mathcal{L}'$, by Lemma 1.1 (i), (iv).

Apply $D$ to $[J, y_i \partial_i] = y_i \partial_i \in \mathcal{F}$, then we have $D(J) = 0$, by Lemma 1.1 (v).

Last Step. Since $\mathcal{L}'$ is decomposed as $\tilde{\mathcal{L}}' = \sum_{n+1}^{m} \mathcal{L}'$, (cf. §1.2), then by Lemma 1.3, this step is carried out similarly as Step 1 in the proof of Proposition 2.1. Q. E. D.

Corollary 2.4. If a derivation $D$ of $\mathcal{L}$ is zero on $\mathcal{F}_{n,m}$ for $n + m \leq -1$, then $D$ is zero on $\mathcal{L}$.

Proof. Let $D$ be a derivation of $\mathcal{L}$ such that $D$ is zero on $\mathcal{F}_{n,m}$ for $n + m \leq -1$. Let $D'$ be the restriction of $D$ to $\mathcal{F}$. Then by Corollary 2.2, $D'$ is zero on $\mathcal{F}$, hence by Proposition 2.3, $D$ is zero on $\mathcal{L}'$. The assertion follows from Propositions 1.3 and 1.4 in [4] and Proposition 1.5. Q. E. D.

§ 3. Derivations of $\mathcal{F}$ and $\mathcal{L}$ (II)

3.1. Determination of $\mathcal{D}$. Let $Z$ be a vector field on $V$. We define $\text{ad}Z$ as $\text{ad}Z(X) = [Z, X]$ for $X \in \mathfrak{K}$. Then we have

Lemma 3.1. The map $Z \mapsto \text{ad}Z|_\mathcal{F}$, or $Z \mapsto \text{ad}Z|_\mathcal{L}$ of $\mathcal{L}$ into $\mathcal{D}$ or $\mathcal{D}_\mathcal{L}$ respectively is an into isomorphism.
Proof. It is sufficient to show the injectivity. Let $Z \in \mathcal{L}$. Assume that $\text{ad}Z(\mathcal{F}) = 0$. By Lemma 1.1 (i), we get the vector fields $X \in \mathcal{F}$, and $Y \in \mathcal{L}'$ such that $Z = X + Y$. Then by Lemma 1.1 (ii), (iv), we have $X = [Z, I] = 0$, whence $Y = 0$, by Lemma 1.1 (v).

Q. E. D.

**Theorem 3.2.** Let $D \in \mathcal{D}$. Then there exists a unique vector field $W$ on $V$ such that $D = \text{ad}W|_{\mathcal{F}}$. Moreover, $W$ is in $\mathcal{L}$.

The proof of this theorem will be given in § 3.3.

**Corollary 3.3.** Let $D \in \mathcal{D}_{\mathcal{F}}$ or $\mathcal{D}_{\mathcal{F}}$. Then there exists a unique vector field $W \in \mathfrak{X}$ such that $D = \text{ad}W|_{\mathcal{F}}$ or $= \text{ad}W|_{\mathcal{F}}$. Moreover, $W$ is in $\mathcal{L}$.

Proof. Obvious for the case $D \in \mathcal{D}_{\mathcal{F}}$. Let $D \in \mathcal{D}_{\mathcal{F}}$. The restriction of $D$ to $\mathcal{F}$ belongs to $\mathcal{D}_{\mathcal{F}}$. Then the assertion follows from Theorem 3.2 and Corollary 2.4. Q. E. D.

**Theorem 3.4.** (i) All derivations of $\mathcal{L}$ are inner, that is, $\mathcal{D}_{\mathcal{F}} = \text{ad} \mathcal{L} \cong \mathcal{L}$. Hence

$$H^1(\mathcal{L} ; \mathcal{L}) = 0.$$ 

(ii) The derivation algebra of $\mathcal{F}$ is naturally isomorphic to $\mathcal{L}$, that is, $\mathcal{D}_\mathcal{F} = \{\text{ad}W|_{\mathcal{F}} ; W \in \mathcal{L}\} \cong \mathcal{L}$. Hence

$$H^1(\mathcal{F} ; \mathcal{F}) \cong \mathcal{L}/\mathcal{F} \cong \mathcal{L}'.$$

In particular, the space $H^1(\mathcal{F} ; \mathcal{F})$ is of infinite dimension.

Proof. (ii) By Corollary 3.3, we have $\mathcal{D}_\mathcal{F} \subseteq \{\text{ad}W|_{\mathcal{F}} ; W \in \mathcal{L}\}$. The converse inclusion is obvious, because $\mathcal{F}$ is an ideal of $\mathcal{L}$. For the latter half, remember that $H^1(\mathcal{F} ; \mathcal{F}) \cong \mathcal{D}_\mathcal{F}/\text{ad} \mathcal{F}$ (see § 1 in [3]). Q. E. D.

3.2. To prove Theorem 3.2, we prepare the following four
Lemma 3.5. Let $D \in \mathcal{D}$. Then there exists a vector field $W_i \in \mathcal{F}$ such that $D(\partial_i) \equiv [W_i, \partial_i] \pmod{\mathcal{L}'}$ for $i=1,\ldots,p$.

Proof. Define the functions $f_i^j(x, y)$, and the vector fields $Y_i \in \mathcal{L}'$, as

$$D(\partial_i) = \sum_{j=1}^p f_i^j(x, y)\partial_j + Y_i \quad (1 \leq i \leq p).$$

Apply $D$ to the both sides of $[\partial_i, \partial_i] = 0$, then we have, by Lemma 1.1 (ii),

$$\sum_{j=1}^p [\partial_i(f_i^j(x, y)) - \partial_i(f_i^j(x, y))]\partial_j = 0 \quad (1 \leq i, k \leq p),$$

and so

$$\partial_i(f_i^j(x, y)) = \partial_i(f_i^j(x, y)) \quad (1 \leq i, j, k \leq p).$$

Therefore, there are unique functions $h^i(x, y) \ (1 \leq i \leq p)$ such that

$$\begin{align*}
\partial_i(h^i(x, y)) &= f_i^i(x, y) \quad (1 \leq i \leq p), \\
h^i(0, y) &= 0 \quad (1 \leq i \leq p).
\end{align*}$$

Put $W_i = -\sum_{j=1}^p h^i(x, y)\partial_j$, then we have the assertion of the lemma. Q. E. D.

Lemma 3.6. Let $D \in \mathcal{D}$. Assume that $D(\partial_i) \in \mathcal{L}' \ (1 \leq i \leq p)$. Then

(i) $D(\partial_i) = 0 \quad (1 \leq i \leq p)$,

(ii) there exists a vector field $W_i \in \mathcal{F}$ such that $[\partial_i, W_i] = 0 \ (1 \leq i \leq p)$, and $D(I) \equiv [W_i, I] \pmod{\mathcal{L}'}$.

Proof. Define the vector fields $X \in \mathcal{F}$ and $Y \in \mathcal{L}'$ as $D(I) = X + Y$. Apply $D$ to $[\partial_i, I] = \partial_i$, then by Lemma 1.1 (ii), (iii), we have that $D(\partial_i) = 0 \ (1 \leq i \leq p)$, and $X \in \mathcal{F}'$. Hence, by Lemma 1.1 (iv), we get

$$[X, I] = X \equiv D(I) \pmod{\mathcal{L}'}.$$

Therefore, we can put $W_i = X$. Q. E. D.
Lemma 3.7. Let \( D \in \mathcal{D} \). Assume that \( D(\partial_i) = 0 \) \((1 \leq i \leq p)\), and \( D(I) \in \mathcal{L}' \). Then, \( D(x_i \partial_i) \in \mathcal{L}' \) \((1 \leq i, j \leq p)\).

Proof. Define the vector fields \( X_{ij} \in \mathcal{F} \) and \( Y_{ij} \in \mathcal{L}' \) as \( D(x_i \partial_i) = X_{ij} + Y_{ij} \) \((1 \leq i, j \leq p)\).

Apply \( D \) to \([\partial_i, x_j \partial_j] = \delta_{ij} \partial_i\), then by Lemma 1.1 (i), we have \( X_{ij} \in \mathcal{F} \) \((1 \leq i, j \leq p)\). Apply \( D \) to \([I, x_j \partial_j] = 0\), then by Lemma 1.1 (ii), (iv), we get \( X_{ij} = 0 \) \((1 \leq i, j \leq p)\). Q.E.D.

Lemma 3.8. Let \( D \in \mathcal{D} \). Assume that \( D(\partial_i) = 0 \), and that \( D(I) \in \mathcal{L}' \), \( D(x_i \partial_i) \in \mathcal{L}' \) \((1 \leq i, j \leq p)\). Then,

(i) \( D(I) = 0 \), \( D(x_i \partial_i) = 0 \) \((1 \leq i, j \leq p)\),

(ii) there exists a unique vector field \( W_s \) on \( V \) such that

\[
[W_s, \partial_i] = [W_s, I] = [W_s, x_i \partial_i] = 0,
\]

\[
[W_s, y_i \partial_i] = D(y_i \partial_i) \quad (1 \leq i, j \leq p, 1 \leq \alpha \leq q).
\]

Moreover, \( W_s \) is in \( \mathcal{L}' \).

Proof. Define the vector fields \( X_i \in \mathcal{F} \) and \( Y_i \in \mathcal{L}' \) as \( D(y_i \partial_i) = X_i + Y_i \) \((1 \leq i \leq p, 1 \leq \alpha \leq q)\). Apply \( D \) to \([\partial_i, y_i \partial_i] = 0\), then by Lemma 1.1 (i), we have \( X_i \in \mathcal{F} \), for all \( i \) and \( \alpha \). Apply \( D \) to \( y_i \partial_i = [y_i \partial_i, I] \), then by Lemma 1.1 (ii), (iv), we get that \( D(I) = 0 \) and \( Y_i = 0 \) for all \( i \) and \( \alpha \).

Define the functions \( f_i(y) \) \((1 \leq i, j \leq p, 1 \leq \alpha \leq q)\) as \( X_i = \sum_j f_i(y) \partial_j \). Apply \( D \) to \( y_i \partial_i = [y_i \partial_i, x_i \partial_i] \), then we get

\[
\sum_j f_i(y) \partial_j = f_i(y) \partial_i + [y_i \partial_i, D(x_i \partial_i)],
\]
hence \( D(x_i \partial_i) = 0 \) \((1 \leq i \leq p)\), and \( f_i(y) = 0 \) for all \( i \neq j \) and \( \alpha \).

Apply \( D \) to \( y_i \partial_i = [y_i \partial_i, x_i \partial_i] \) for \( i \neq k \), then we get

\[
f_i(y) \partial_i = f_i(y) \partial_i + [y_i \partial_i, D(x_i \partial_i)],
\]
hence \( D(x_i \partial_i) = 0 \) \((1 \leq i, k \leq p)\), and \( f_i(y) = f_i(y) \) for all \( i \neq k \) and \( \alpha \). Denote \( f_i(y) \) by \( f_i(y) \) \((1 \leq \alpha \leq q)\), then \( D(y_i \partial_i) = f_i(y) \partial_i \).

Let \( W_s \) be a vector field on \( V \) satisfying the equations in (ii). Since \([W_s, \partial_i] = [W_s, I] = 0 \) \((1 \leq i \leq p)\), then we get \( W_s \in \mathcal{L}' \), by Lemma.
1. 1(i), (iv). Write $W_3$ as $W_3 = \sum_i h_i(y) \partial_i$, then

$$[W_3, y_\alpha \partial_i] = h_i(y) \partial_i \quad (1 \leq i \leq p, \ 1 \leq \alpha \leq q).$$

Hence, $h_i(y)$ must be equal to $f_i(y)$ for all $\alpha$.

Q. E. D.

3.3. Proof of Theorem 3.2. Let $D \in \mathcal{D}$. Then, by Lemmata 3.5–3.8, we have a unique vector field $W$ on $V$ such that $D = \text{ad}W$ on $\mathcal{F}_{n,m}$ for $n + m \leq -1$. We can determine $W$ as $W = W_1 + W_2 + W_3$, where $W_i (i=1, 2, 3)$ are given in the above lemmata.

Clearly $W^3 = 0$.

Hence, by Corollary 2.2, we get that $D = \text{ad}W$ on $\mathcal{F}$.

Q. E. D.

3.4. Remarks. (i) Any derivation of $\mathcal{F}$ or $\mathcal{L}$ is continuous, because it is realized as $\text{ad}W$ for some $W \in \mathcal{L}$.

(ii) Let $V'$ be a subspace of $V$, spanned by $y_1, \ldots, y_r$. Then Theorem 3.4 (i) can be rewritten as in the following form in terms of $C^\infty(V')$, which is suggestive for calculations of cohomologies of $\mathcal{F}$ with various coefficients.

**Theorem 3.9.** Let $\mathcal{D}_{\text{er}} (C^\infty(V'))$ be the derivation algebra of the associative algebra $C^\infty(V')$. Then

$$H^1(\mathcal{F}; \mathcal{F}) \cong \mathcal{D}_{\text{er}} (C^\infty(V')).$$

This follows immediately from the following well-known fact.

**Lemma 3.10.** There is an natural Lie algebra isomorphism of $\mathcal{L}'$ onto $\mathcal{D}_{\text{er}} (C^\infty(V'))$.

We give here its elementary proof for completeness. Let $D \in \mathcal{D}_{\text{er}} (C^\infty(V'))$. Define functions $g_\alpha(y) \ (\alpha = 1, \ldots, q)$ as $D(y_\alpha) = g_\alpha(y)$. Let $Y = \sum g_\alpha(y) \partial_\alpha \in \mathcal{L}'$. The vector field $Y$ operates on $C^\infty(V')$ as a first-order partial differential operator, then it defines a derivation
Easily by induction, we can show that $D_y$ coincides with $D$ on the polynomial algebra $R[y_1, \ldots, y_q]$. Hence we obtain Lemma 3.10, because when $f^2(g)(0) = 0$, $g$ is expressed as $g(y) = \sum s_i y_i g_{*i}(y)$ with $g_{*i} \in C^\infty(V')$.

§ 4. Lie Algebras $\mathcal{L}(M, \mathcal{F})$, $\mathcal{F}(M, \mathcal{F})$, and Their Derivations

4.1. Lie Algebras Associated with Foliations. Let $M$ be a $(p + q)$-dimensional manifold and $\mathcal{F}$ a codimension-$q$ foliation on $M$. Around any point of $M$, there is a distinguished coordinate neighborhood $(U; x_1, \ldots, x_p, y_1, \ldots, y_q)$, for which a plate represented as $y_1 = \text{constant}, \ldots, y_q = \text{constant}$ in $U$ is a connected component of $L \cap U$ for some leaf $L$ of $\mathcal{F}$ (see e.g. [6] for definitions).

A vector field $X$ on a foliated manifold $(M, \mathcal{F})$ is called leaf-tangent, if $X$ is tangent to the leaf $L$ through $p$ for any point $p$ of $M$, that is, the vector $X_p$ belongs to the tangent space $T_p L$ of $L$ at $p$. A vector field $X$ is called to be locally foliation preserving (or l.f.p., in short), if $\phi$, maps every plate to some plate, where $\phi_t$ is a one-parameter group of local diffeomorphisms generated by $X$.

Locally for any distinguished coordinates $(x_1, \ldots, x_p, y_1, \ldots, y_q)$, a leaf-tangent vector field is represented as $\sum \alpha f_i(x, y) \partial_i$, and a l.f.p. vector field is represented as $\sum f_i(x, y) \partial_i + \sum g_{*i}(y) \partial_{*i}$, where $f_i(x, y)$ $(i = 1, \ldots, p)$ are $C^\infty$-functions of $x_1, \ldots, x_p, y_1, \ldots, y_q$, and $g_{*i}(y)$ $(\alpha = 1, \ldots, q)$ are $C^\infty$-functions of $y_1, \ldots, y_q$. Here we use the notations $\partial_i$ or $\partial_{*i}$ instead of $\frac{\partial}{\partial x_i}$ or $\frac{\partial}{\partial y_{*i}}$ respectively, and the convention on indices (see § 1.1).

All l.f.p. vector fields on $(M, \mathcal{F})$ form a Lie algebra $\mathcal{L}(M, \mathcal{F})$, and all leaf-tangent vector fields form its ideal $\mathcal{F}(M, \mathcal{F})$.

If a l.f.p. vector field $X$ is complete, then $X$ is foliation preserving, that is, the diffeomorphism $\phi$, maps every leaf of $\mathcal{F}$ to some leaf for each $t$. Similarly, if a leaf-tangent vector field $X$ is complete, $\phi_t$ leaves every leaf of $\mathcal{F}$ stable. Thus, when $M$ is compact, l.f.p. vector fields are foliation preserving.
4.2. Derivations. Let \( \mathcal{D}(M, \mathcal{F}) = \mathcal{D}^e \mathcal{F} (\mathcal{F}(M, \mathcal{F}) ; \mathcal{L}(M, \mathcal{F})) \)
be the space of derivations of \( \mathcal{F}(M, \mathcal{F}) \) with values in \( \mathcal{L}(M, \mathcal{F}) \).
And let \( \mathcal{D}_{\mathcal{G}}(M, \mathcal{F}) \) or \( \mathcal{D}_{\mathcal{F}}(M, \mathcal{F}) \) be the derivation algebra of
\( \mathcal{L}(M, \mathcal{F}) \) or \( \mathcal{F}(M, \mathcal{F}) \) respectively. Sometimes we omit \( \mathcal{F} \) in the
notations \( \mathcal{F}(M, \mathcal{F}), \mathcal{D}(M, \mathcal{F}) \), etc.

**Lemma 4.1.** Let \( U \) be an open subset of \( M \), and \( X \in \mathcal{L}(M, \mathcal{F}) \). Assume that \( [X, Y] = 0 \) on \( U \) for any \( Y \in \mathcal{F}(M, \mathcal{F}) \) with support contained in \( U \). Then, \( X = 0 \) on \( U \).

**Proof.** Let \( p \in U \). Take a sufficiently small neighborhood \( U' \) of \( p \) in \( U \), and distinguished coordinates \( (x_1, \ldots, x_n, y_1, \ldots, y_q) \) in \( U' \). Let a vector field \( Y' \) on \( U' \) be any one of \( \partial_i, x_j \partial_i \), and \( y_a \partial_i (1 \leq i, j \leq p, 1 \leq a \leq q) \). Since \( \mathcal{F}(M) \) is \( C^\infty(M) \)-module, there is a vector field \( Y \in \mathcal{F}(M) \) such that \( Y = Y' \) on \( U' \) and the support of \( Y \) is contained in \( U \). Then we have \( [X, Y] = 0 \) on \( U \) by the assumption.

By the proof of Lemma 3.8, we have that \( X = 0 \) on \( U' \), in particular, at \( p \). Hence we get \( X = 0 \) on \( U \). Q. E. D.

From this lemma, we get the following two lemmata, similarly as
Proposition 2.4 and Corollary 2.5 in [4].

**Lemma 4.2.** Let \( D \in \mathcal{D}(M, \mathcal{F}) \) or \( \mathcal{D}_{\mathcal{G}}(M, \mathcal{F}) \). Then, \( D \) is local.

**Lemma 4.3.** Let \( D \in \mathcal{D}(M, \mathcal{F}) \). Then, \( D \) is localizable (see §1.2

4.3. **Proposition 4.4.** Let \( D \in \mathcal{D}(M, \mathcal{F}) \). Then, there exists a
vector field \( W \) on \( M \) such that \( D = \text{ad}W \rvert_{\mathcal{F}(M, \mathcal{F})} \). Moreover, \( W \) is in
\( \mathcal{L}(M, \mathcal{F}) \).

**Proof.** Take a distinguished coordinate neighborhood system \( \{U_i; (x_i^1, \ldots, x_i^n, y_i^1, \ldots, y_i^q) \}_{i \in A} \) on \( (M, \mathcal{F}) \). Since \( D \) is localizable, the
derivation \( D_{U_i} \in D(U_i, \mathcal{F} \rvert_{U_i}) \) can be defined for all \( \lambda \in \Lambda \) in such a
way that \( D(X) \rvert_{U_i} = D_{U_1}(X \rvert_{U_1}) \) for all \( X \in \mathcal{F}(M) \). Then by Theorem
3.2, there exists a unique vector field \( W_i \) on \( U_i \) such that \( \langle D_{v_i} \rangle = \text{ad} W_i \lvert_{\mathcal{F}(U_i)} \) for any \( \lambda \in \Lambda \). On the other hand, we have \( \langle D_{v_i} \rangle \lvert_{\mathcal{F}(U_i \cap U_i')} = \langle D_{v_i} \rangle \lvert_{\mathcal{F}(U_i \cap U_i')} \) so \( W_i = W_i' \) on \( U_i \cap U_i' \). Hence there is a vector field \( W \) on \( M \) such that \( W = W_i \) on \( U_i \) for all \( \lambda \in \Lambda \) and that \( D = \text{ad} W \lvert_{\mathcal{F}(U_i)} \). Moreover, we have \( W \in \mathcal{L}(M) \), because \( W_i \in \mathcal{L}(U_i) \) for all \( \lambda \in \Lambda \).

\[ \text{Q. E. D.} \]

**Corollary 4.5.** Let \( D \in \mathcal{D}_\mathcal{F}(M, \mathcal{F}) \) or \( \mathcal{D}_\mathcal{F}(M, \mathcal{F}) \). Then there exists a vector field \( W \) on \( M \) such that \( D = \text{ad} W \lvert_{\mathcal{F}(M, \mathcal{F})} \) or \( D = \text{ad} W \lvert_{\mathcal{F}(M, \mathcal{F})} \) respectively. Moreover, \( W \) is in \( \mathcal{L}(M, \mathcal{F}) \).

**Proof.** Obvious for the case \( D \in \mathcal{D}_\mathcal{F}(M) \). Let \( D \in \mathcal{D}_\mathcal{F}(M) \). The restriction of \( D \) to \( \mathcal{F}(M) \) belongs to \( \mathcal{D}(M) \). Then the assertion follows from Proposition 4.4 and Lemma 4.1. \[ \text{Q. E. D.} \]

Then we get Main Theorem similarly as Theorem 3.4.

**Theorem 4.6.** (i) All derivations of \( \mathcal{L}(M, \mathcal{F}) \) are inner, that is, \( \mathcal{D}_\mathcal{F}(M, \mathcal{F}) = \text{ad} \mathcal{L}(M, \mathcal{F}) \simeq \mathcal{L}(M, \mathcal{F}) \). Hence

\[ H^1(\mathcal{L}(M, \mathcal{F}); \mathcal{L}(M, \mathcal{F})) = 0. \]

(ii) The derivation algebra of \( \mathcal{F}(M, \mathcal{F}) \) is naturally isomorphic to \( \mathcal{L}(M, \mathcal{F}) \), that is, \( \mathcal{D}_\mathcal{F}(M, \mathcal{F}) = \{ \text{ad} W \lvert_{\mathcal{F}(M, \mathcal{F})}; \ W \in \mathcal{L}(M, \mathcal{F}) \} \simeq \mathcal{L}(M, \mathcal{F}) \). Hence

\[ H^1(\mathcal{F}(M, \mathcal{F}); \mathcal{F}(M, \mathcal{F})) \simeq \mathcal{L}(M, \mathcal{F})/\mathcal{F}(M, \mathcal{F}). \]

**4.4. Examples.** Let \( H^1 = H^1(\mathcal{F}(M, \mathcal{F}); \mathcal{F}(M, \mathcal{F})) \simeq \mathcal{L}(M, \mathcal{F})/\mathcal{F}(M, \mathcal{F}) \) for a foliated manifold \( (M, \mathcal{F}) \). In many cases, \( H^1 \) are of infinite dimension.

**Proposition 4.7.** Assume that there is a compact leaf \( L \) of \( \mathcal{F} \) such that there is a saturated neighborhood \( U \) of \( L \), which is a product foliation \( D^r \times L \), where \( D^r \) is a q-dimensional disk. Then, \( H^1 \) is of infinite dimension.
Proof. Every leaf in $U$ is represented by a point of $D'$. Let $f$ be a function supported in $D'$. Then $f \cdot \mathcal{L}(M, \mathcal{F}) \subset \mathcal{L}(M, \mathcal{F})$. Hence the assertion follows from Theorem 3. 4. Q. E. D.

However, $H^1$ may be of finite dimension. Assume that $M$ is compact. J. Leslie [5] gives examples of $\dim H^1 = 0$, or 1: (i) an Anosov flow with an integral invariant for $\dim H^1 = 0$, and (ii) irrational flows on a two dimensional torus $T^2$ for $\dim H^1 = 1$. We can modify the latter to get a foliated manifold with $\dim H^1 = n$ (for arbitrary $n < +\infty$), that is, irrational flows on an $(n + 1)$-dimensional torus $T^{n+1}$.

We have also other examples. Fukui and Ushiki [2] shows that $\dim H^1 = 2$ for the Reeb foliation on a 3-sphere $S^3$. Further, Fukui [1] shows that the following: let $(M, \mathcal{F})$ be a Reeb foliated 3-manifold, then $\dim H^1$ is finite, and equals to the number of generalized Reeb components.

References
