G-Homotopy Types of G-Complexes and Representations of G-Cohomology Theories

By

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Dedicated to Professor Ryoji Shizuma on his 60th birthday

Let $G$ be a finite group throughout the present work. Bredon [2] discussed CW-complexes on which $G$ acts nicely, and called them G-complexes. In the present work we discuss first, about $G$-spaces having $G$-homotopy types of G-complexes, parallel properties to Milnor [8] in § 2, where main results are Theorem 2.3 and Corollary 2.6. Then, in § 3 we apply Corollary 2.6 to representations of $G$-equivariant cohomology theories, defined by Segal [9], by $G$-$G$-spectra (Theorems 3.3 and 3.4).

§ 1. G-Complexes

By a $G$-complex $X$ we mean a CW-complex $X$ on which $G$ acts as a group of automorphisms of its cell-structure such that $X^g$, the fixed-point set of $g$, is a subcomplex for each $g \in G$, Bredon [2]. We refer the basic properties of $G$-complexes to [2]. By $G$-maps and $G$-homotopies we mean equivariant maps between $G$-spaces and equivariant homotopies between $G$-maps, respectively, for simplicity.

First we quote two basic properties of $G$-complexes.

$G$-Homotopy Extension Property ([2], Chap. I, § 1). Let $(X, A)$ be a pair of $G$-complex $X$ and its $G$-subcomplex $A$, and $Y$ a $G$-space. Given a $G$-map $f:X \to Y$ and a $G$-homotopy $F:A \times I \to Y$ such that $F|A \times \{0\} = f|A$, then there exists a $G$-homotopy $\tilde{F}:X \times I \to Y$ such that $\tilde{F}|X \times \{0\} = f$ and $\tilde{F}|A \times I = F$. 

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**G-Cellular Approximation Theorem** ([2], Chap. II, Proposition (5.6)). Let \((X, A)\) be a pair of \(G\)-complexes and \(Y\) another \(G\)-complex. Given a \(G\)-map \(f: X \rightarrow Y\) such that \(f|A\) is cellular, then there exists a \(G\)-homotopy \(F: X \times I \rightarrow Y\) such that \(F|A \times \{t\} = f|A, 0 \leq t \leq 1, F|X \times \{0\} = f\) and \(F|X \times \{1\}\) is cellular.

Because of these two properties we can make constructions such as mapping-cylinders, mapping-cones, equalizers, telescopes, \(G\)-cofibration sequences (Puppe sequences) etc., in the category of (pointed) \(G\)-complexes.

Secondly we quote

**Theorem of J.H.C. Whitehead for \(G\)-Complexes** ([2], Chap. II, Corollary (5.5)). Let \(f: X \rightarrow Y\) be a \(G\)-map between two \(G\)-complexes. \(f\) is a \(G\)-homotopy equivalence iff \(f^H: X^H \rightarrow Y^H\) is a weak homotopy equivalence for every subgroup \(H\) of \(G\).

As to the denomination of the above theorem we refer to Matumoto [5].

Now, in our case \(X^H\) is a \(CW\)-complex for each subgroup \(H\) of \(G\) by definition. Thus the above theorem can be restated in the following form:

**Proposition 1.1.** Let \(f: X \rightarrow Y\) be a \(G\)-map between two \(G\)-complexes. \(f\) is a \(G\)-homotopy equivalence iff \(f^H: X^H \rightarrow Y^H\) is a homotopy equivalence for every subgroup \(H\) of \(G\).

The above proposition holds also for pairs of \(G\)-complexes. By \(f \simeq g\) we denote that two \(G\)-maps \(f\) and \(g\) are \(G\)-homotopic.

**Proposition 1.2.** Let \(f: (X, A) \rightarrow (Y, B)\) be a \(G\)-map between two pairs of \(G\)-complexes. \(f\) is a \(G\)-homotopy equivalence iff \(f^H: (X^H, A^H) \rightarrow (Y^H, B^H)\) is a homotopy equivalence for every subgroup \(H\) of \(G\).

**Proof.** The "only if" part is clear. To prove the "if" part we

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need the following

**Lemma.** Let \((K, L)\) be a pair of \(G\)-complexes and \(f: X \to Y\) be a \(G\)-homotopy equivalence of \(G\)-spaces. Let \(g: L \to X\) and \(h: K \to Y\) be \(G\)-maps such that \(h \upharpoonright L = f \circ g\). Then there exists a \(G\)-map \(\tilde{g}: K \to X\) such that \(\tilde{g} \upharpoonright L = g\) and \(f \circ \tilde{g} \simeq \alpha h\) relative to \(L\).

This lemma follows from [2], Chap. II, Lemma (5.2), if we replace \(f\) by an inclusion map making use of the mapping cylinder of \(f\).

Now suppose that \(f'' : (X'', A'') \to (Y'', B'')\) is a homotopy equivalence for every subgroup \(H\) of \(G\). Then \(f'' : X'' \to Y''\) and \((f \upharpoonright A)'' : A'' \to B''\) are homotopy equivalences for all subgroups \(H\) of \(G\). Thus, by Proposition 1.1 we see that \(f : X \to Y\) and \(f \upharpoonright A : A \to B\) are \(G\)-homotopy equivalences.

Let \(g_B : B \to A\) be a \(G\)-homotopy inverse of \(f \upharpoonright A\), and \(H_B : (f \upharpoonright A) \circ g_B \simeq \alpha 1_B\) a \(G\)-homotopy. By \(G\)-homotopy extension property we have a \(G\)-map \(H_1 : Y \times I \to Y\) such that \(H_1 \upharpoonright B \times I = H_B\) and \(H_1 \upharpoonright Y \times 1 = 1_Y\). Put \(h = H_1 \upharpoonright Y \times 0\), then \(h \upharpoonright B = (f \upharpoonright A) \circ g_B\). Apply the above lemma to the pair \((h, g_B)\) of \(G\)-maps, and get a \(G\)-map \(g : Y \to X\) such that \(g \upharpoonright B = g_B\) and \(f \circ g \simeq \alpha h\) relative to \(B\). Let \(H_2 : f \circ g \simeq \alpha h\) be this \(G\)-homotopy relative to \(B\). The sum \(H_1 + H_2 : f \circ g \simeq \alpha 1_{(Y, B)}\) is a \(G\)-homotopy of \(G\)-maps \((Y, B) \to (Y, B)\) of pairs by construction. As is easily seen, \(H_2 + H_1 \upharpoonright B \times I\) can be equivariantly deformed to \(H_B\) relative to \(B \times 0 \cup B \times 1\). Then, by \(G\)-homotopy extension property we can deform \(H_1 + H_2\) to a \(G\)-homotopy \(H : f \circ g \simeq \alpha 1_{(Y, B)}\) such that \(H \upharpoonright B \times I = H_B\).

Take a \(G\)-homotopy \(H_A : (f \upharpoonright A) \circ g \simeq \alpha 1_A\) and apply the same argument as above to \((f \upharpoonright A, g)\), then we get a \(G\)-map \(\tilde{f} : (X, A) \to (Y, B)\) and a \(G\)-homotopy \(H' : g \circ f \simeq \alpha 1_{(X, A)}\) of \(G\)-maps of pairs such that \(\tilde{f} \upharpoonright A = f \upharpoonright A\) and \(H' \upharpoonright A \times I = H_A\). Then

\[
f \simeq f \circ g \circ \tilde{f} \simeq \alpha f
\]

as \(G\)-maps of pairs. Thus

\[
g \circ f \simeq \alpha g \circ \tilde{f} \simeq \alpha 1_{(X, A)}
\]

as \(G\)-maps of pairs. q.e.d.

More generally we obtain
Proposition 1.3. Let \( f: (A; A_1, \cdots, A_{n-1}) \to (B; B_1, \cdots, B_{n-1}) \) be a \( G \)-map between two \( n \)-ads of \( G \)-complexes. \( f \) is a \( G \)-homotopy equivalence iff \( f^H: (A^H; A^H_1, \cdots, A^H_{n-1}) \to (B^H; B^H_1, \cdots, B^H_{n-1}) \) is a homotopy equivalence for every subgroup \( H \) of \( G \).

Proof. Again the “only if” part is clear.

Suppose that \( f^H: (A^H; A^H_1, \cdots, A^H_{n-1}) \to (B^H; B^H_1, \cdots, B^H_{n-1}) \) is a homotopy equivalence of \( n \)-ads for every subgroup \( H \) of \( G \). Then \( f|_{A_{i_1} \cap \cdots \cap A_{i_r}}: A_{i_1} \cap \cdots \cap A_{i_r} \to B_{i_1} \cap \cdots \cap B_{i_r} \) is a \( G \)-homotopy equivalence for every subset \( \{i_1, \cdots, i_r\} \subset \{1, \cdots, n-1\} \). Using the same argument as in the above proof we can construct a right \( G \)-homotopy inverse \( g: (B; B_1, \cdots, B_{n-1}) \to (A; A_1, \cdots, A_{n-1}) \) of \( f \) and \( G \)-homotopy \( H: f \circ g \simeq g \circ f \) of \( G \)-maps of \( n \)-ads by stepwise construction of \( g|_{A_{i_1} \cap \cdots \cap A_{i_r}} \) and \( H|_{A_{i_1} \cap \cdots \cap A_{i_r}} \) so that they extend \( G \)-maps and \( G \)-homotopies already constructed, starting from \( g|_{A_1 \cap \cdots \cap A_{n-1}} \) and \( H|_{A_1 \cap \cdots \cap A_{n-1}} \). Then we construct a right \( G \)-homotopy inverse \( \tilde{f} \) of \( f \) as \( G \)-maps of \( n \)-ads in the same way. Finally we see that \( f \circ g \circ \tilde{f} \simeq g \circ f \) as \( G \)-maps of \( n \)-ads and that \( g \) is a left \( G \)-homotopy inverse of \( f \).

A simplicial \( G \)-complex \( K \) is a simplicial complex \( K \) endowed with a group \( G \) of automorphisms of its simplicial structure. Then, its geometric realization \( K_w \) (or \( K_\ast \)) in the weak (or strong) topology is a \( G \)-space, but not always a \( G \)-complex in our sense. As is easily seen, \( K_w \) is a \( G \)-complex when, for each \( g \in G \) and each simplex \( \sigma \) of \( K \),

\[
g \sigma = \sigma \text{ iff } g \text{ fixes all vertices of } \sigma.
\]

In particular we have

Proposition 1.4. Let \( K \) be a simplicial \( G \)-complex. The barycentric subdivision \( Sd K_w \) of \( K_w \) is a \( G \)-complex.

In virtue of this proposition we regard \( K_w \) as a \( G \)-complex.

A simplicial \( G \)-set \( K \) is a simplicial set \( K \) together with a group \( G \) of automorphisms of its simplicial structure. For each \( g \in G \) its action on \( K \) commutes with all structure maps of \( K \); hence its fixed-point set \( K^g \) is a simplicial subset of \( K \). Let \( |K| \) be the geometric realization of
$K$ in the weak topology (Milnor [7]). $G$-actions on $K$ induce $G$-actions on $|K|$. As is easily seen we have

**Proposition 1.5.** Let $K$ be a simplicial $G$-set. Then $|K|$ is a $G$-complex and $|K|^H = |K^H|$ for any subgroup $H$ of $G$.

Let $X$ be a $G$-space. Its singular complex $S(X)$ is a simplicial $G$-set with induced $G$-actions. Then $S(X^g) = S(X)^g$ for any $g \in G$ as is easily seen. By a routine argument we obtain the following

**Proposition 1.6.** The assignment

$$X \mapsto |S(X)|$$

is functorial on the category of $G$-spaces, and the map

$$\alpha: |S(X)| \to X,$$

defined by $\alpha(\sigma, y) = \sigma(y)$ for $(\sigma, y) \in |S(X)|$, is a natural $G$-map.

Let $X = (X; X_1, \ldots, X_{n-1})$ be an $n$-ad of $G$-spaces. Then $S(X) = (S(X); S(X_1), \ldots, S(X_{n-1}))$ is an $n$-ad of simplicial $G$-sets, and $|S(X)| = (|S(X)|; |S(X_1)|, \ldots, |S(X_{n-1})|)$ is an $n$-ad of $G$-complexes.

**Proposition 1.7.** Let $X = (X; X_1, \ldots, X_{n-1})$ be an $n$-ad of $G$-complexes. Then

$$\alpha: |S(X)| \to X$$

is a $G$-homotopy equivalence of $n$-ads.

**Proof.** Since $|S(X)|^H = (|S(X^H)|; |S(X_1^H)|, \ldots, |S(X_{n-1}^H)|)$ for each subgroup $H$ of $G$, we see that

$$\alpha: |S(X)|^H: |S(X)|^H \to X^H = (X^H; X_1^H, \ldots, X_{n-1}^H)$$

is a homotopy equivalence for each $H$ by Milnor [7], Theorem 4, and [8], Lemma 1. Thus Proposition 1.3 completes the proof.
§ 2. G-Homotopy Types of G-Complexes

In this section we discuss G-spaces having G-homotopy types of G-complexes in a parallel way to Milnor [8]. Roughly to say, since G is finite, averaging procedures over G allow us parallel arguments to [8].

Let X be a G-space. A covering $\mathcal{U} = \{ U_\lambda \}_{\lambda \in \Lambda}$ of X is called a G-covering when $gU_\lambda \in \mathcal{U}$ for each $g \in G$ and $\lambda \in \Lambda$. Then, putting

$$gU_\lambda = U_{g\lambda},$$

G acts on the indexing set $\Lambda$.

Let $\mathcal{U} = \{ U_\lambda \}_{\lambda \in \Lambda}$ be an open G-covering of a G-space X. A partition of unity $\{ p_\lambda \}_{\lambda \in \Lambda}$ subordinate to $\mathcal{U}$ is called a G-partition of unity (subordinate to $\mathcal{U}$) when

$$p_\lambda (g^{-1}x) = p_{g\lambda}(x)$$

for $g \in G$ and $x \in X$.

First we prove an analogue of Milnor [8], Theorem 2. We denote by $\mathcal{W}_G^n$ the category of G-spaces having G-homotopy types of G-complexes and by $\mathcal{W}_n^G$ the category of n-ads of G-spaces having G-homotopy types of n-ads of G-complexes.

**Theorem 2.1.** The following restrictions on an n-ad $A = (A; A_1, \ldots, A_{n-1})$ are equivalent:

(a) $A$ belongs to the category $\mathcal{W}_G^n$,

(b) $A$ is G-dominated by an n-ad of G-complexes,

(c) $A$ has the G-homotopy type of an n-ad of simplicial G-complexes in the weak topology.

(d) $A$ has the G-homotopy type of an n-ad of simplicial G-complexes in the strong topology.

**Proof.** The implications (c) ⇒ (a) ⇒ (b) are clear (by Proposition 1.4). Remark that, for an n-ad $A$ of G-spaces, the barycentric subdivision of $|\Sigma(A)|$ is an n-ad of simplicial G-complexes in the weak topology. Because of Proposition 1.7 we get a proof that (b) ⇒ (c) by the same argument as [8], p.275, (using the same diagram).

Proof that (c) ⇔ (d). Let $K = (K; K_1, \ldots, K_{n-1})$ be an n-ad of simpli-
cial $G$-complexes, and $K_w$ and $K_s$ denote the $n$-ads of geometric realizations of $K$ in the weak and strong topology respectively. Recall that the topology of $K_s$ is given by the standard metric $d$ defined by barycentric coordinates which is $G$-invariant.

Let $\{\beta\}$ be the set of vertices of $K$ and $\mathcal{U} = \{U_\beta\}$ be the locally finite open covering defined as in [8], p.276. $\mathcal{U}$ is a $G$-covering as is easily seen. Let $p_\beta : K_s \to R$ be defined by

$$p_\beta(x) = d(x, K_s - U_\beta) / \sum d(x, K_s - U_\gamma),$$

$x \in K_s$, for each vertex $\beta$ of $K$, where the summation runs over all vertices $\gamma$ of $K$. Then $\{p_\beta\}$ is a $G$-partition of unity subordinate to $\mathcal{U}$. Define $p : K_s \to K_w$ by letting $p(x)$ be the point in $K_w$ with barycentric coordinates $p_\beta(x)$. Now it is clear that $p$ is a continuous $G$-map, and maps each $(K_j)_s$ into $(K_j)_w$.

Let $i : K_w \to K_s$

be the canonical map which is obviously equivariant. The composition $i \circ p : K_s \to K_s$ maps each simplex into itself equivariantly, hence a linear homotopy gives a $G$-homotopy of $i \circ p$ to the identity. Similarly $p \circ i : K_w \to K_w$ is $G$-homotopic to the identity. q.e.d.

Let $X$ be a $G$-space. $X \times X$ is a $G$-space by diagonal actions. $X$ is called to be $G$-ELCX ($G$-equi-locally convex) if there exists a $G$-invariant neighborhood $U$ of the diagonal in $X \times X$ and a $G$-map

$$\lambda : U \times I \to X$$

(which will be called the structure map) satisfying Milnor's conditions (1), (2) and (3) of [8], p.277. Even though we do not assume the open covering $\mathcal{U} = \{V_\beta\}$ of $X$ by convex set (which we call the convex covering of $X$) to be a $G$-covering, we can actually choose $\mathcal{U}$ so as to be a $G$-covering by adding all $gV_\beta$ to $\mathcal{U}$, $g \in G$ and $V_\beta \in \mathcal{U}$, because of equivariancy of the structure map $\lambda$. This will be called the convex $G$-covering of $X$.

An $n$-ad $X = (X; X_1, \cdots, X_{n-1})$ is called a $G$-ELCX $n$-ad when $X$ is $G$-ELCX. $X_i$ is a closed $G$-subspace for each $i$, $1 \leq i \leq n - 1$, and $X$ is an
ELCX $n$-ad in the sense of Milnor [8].

Here we remark the following. Let $X$ be a paracompact $G$-space and $\mathcal{U}$ an open $G$-covering of $X$, then we can choose a locally finite $G$-covering $\mathcal{W}$ of $X$ which refines $\mathcal{U}$. (Choose any locally finite refinement $\mathcal{W}'$ of $\mathcal{U}$, and add all $g$-transforms of elements of $\mathcal{W}'$ to $\mathcal{W}$; the resulting $G$-covering $\mathcal{W}$ is still locally finite since $G$ is finite.) Next, let $\mathcal{W}$ be a locally finite open $G$-covering of a paracompact $X$. We can choose a $G$-partition of unity subordinate to $\mathcal{W}$ (by averaging over $G$ an arbitrary chosen partition of unity subordinate to $\mathcal{W}$).

**Lemma 2.2.** Every $n$-ad of simplicial $G$-complexes in the strong topology is $G$-ELCX.

**Proof.** Let $K = (K; K_1, \ldots, K_{n-1})$ be an $n$-ad of simplicial $G$-complexes in the strong topology. Use the same constructions and notations as [8], p.278, Proof of Lemma 2. It is easy to check that $U$ is $G$-invariant and the maps

$$\mu: U \to K \quad \text{and} \quad \lambda: U \times I \to K$$

are $G$-maps. q.e.d.

**Theorem 2.3.** The following restrictions on an $n$-ad $A = (A; A_1, \ldots, A_{n-1})$ are equivalent:

(i) $A$ belongs to $\mathcal{W}_n$,

(ii) $A$ has a $G$-homotopy type of a metrizable $G$-ELCX $n$-ad,

(iii) $A$ has a $G$-homotopy type of a paracompact $G$-ELCX $n$-ad.

**Proof.** Since simplicial complexes in the strong topology are metrizable, Theorem 2.1 and Lemma 2.2 imply that (i) $\Rightarrow$ (ii). Since metrizable spaces are paracompact, it is obvious that (ii) $\Rightarrow$ (iii).

Proof that (iii) $\Rightarrow$ (i). This part corresponds to Lemma 4 of [8]. Let $A = (A; A_1, \ldots, A_{n-1})$ be a paracompact $G$-ELCX $n$-ad. Because of Theorem 2.1, it is sufficient to prove that $A$ is $G$-dominated by an $n$-ad of $G$-complexes.

Let $\mathcal{V} = \{V_b\}$ be the convex $G$-covering of $A$. Since $A$ is fully
normal, we can find an open covering $\mathcal{W}' = \{W_i'\}$ of $A$ which is sufficiently fine so that the star of any point $a$ of $A$ with respect to $\mathcal{W}'$ is contained in some convex set $V_\varphi$ (as in [8], p.279).

Since $G$ is finite and $A_i$'s are closed, we can choose at every point $a$ of $A$ a sufficiently small open neighborhood $W_a$ of $a$ such that i) $W_a$ is $G_a$-invariant and $W_a \cap gW_a = \phi$ for $g \in G - G_a$, where $G_a$ is the isotropy subgroup of $G$ at $a$, ii) $W_a$ is contained in some $W_i'$, iii) $gW_a = W_{g^a}$ for any $g \in G$, and iv) if $W_a \cap A_i \neq \phi$ then $a \in A_i$. We call each open set $W_a$ an admissible open set centered at $a$. The totality $\mathcal{W} = \{W_a; a \in A\}$ of these admissible centered open sets forms an open $G$-covering which refines $\mathcal{W}'$; hence $\mathcal{W}$ is also sufficiently fine so that the star of any point $a$ of $A$ with respect to $\mathcal{W}$ is contained in some convex set $V_\varphi$.

Practically we need only a $G$-subsystem of $\mathcal{W}$ which covers $A$. So, choosing one representative among $G$-orbits in $\mathcal{W}$ which coincide mutually as families of subsets of $A$, we may assume that $W_a \neq W_b$ if $a \neq b$.

Let $\mathcal{U} = \{U_s\}$ be a locally finite open $G$-covering of $A$ which refines $\mathcal{W}$. Let $N$ denote the nerve of $\mathcal{U}$, considered as a geometric simplicial complex in the weak topology. Define subcomplexes $N_t$ such that the vertices $\delta_0, \ldots, \delta_k$ span a simplex of $N_t$ iff $U_{s_0} \cap \cdots \cap U_{s_k}$ intersects $A_t$. Then we obtain an $n$-ad $N = (N; N_1, \ldots, N_{n-1})$ of simplicial $G$-complexes in the weak topology. Choose a $G$-partition of unity $\{p_s\}$ subordinate to $\mathcal{U}$. Define $p: A \rightarrow N$ by letting $p(a)$ be the point in $N$ with barycentric coordinates $p_s(a)$. $p$ is clearly continuous and determines a $G$-map

$$p: A \rightarrow N$$

of $n$-ads of $G$-spaces.

Next we define a $G$-map

$$q: N \rightarrow A$$

of $n$-ads. Let $SDN$ be the barycentric subdivision which is a $G$-complex. Vertices of $SDN$ corresponds to simplices of $N$ which are mutually identified by an abuse of notations. Order vertices of $SDN$ so that $\sigma < \sigma'$ iff $\sigma \supset \sigma'$ in $N$. Then $G$-actions on $SDN$ preserve this ordering. Set

$$SD\mathcal{U} = \{U'_s = U_{s_0} \cap \cdots \cap U_{s_k}\},$$

where $\sigma = \langle \delta_0, \ldots, \delta_k \rangle$ runs over all simplices of $N$. For each $U'_s \in SD\mathcal{U}$
we choose from \( W \) an admissible open set \( W' \) centered at \( a' \), so that 
\( U' \subset W' \) and \( gW' = W' \) for any \( g \in G \).

Now we define the wanted \( G \)-map \( q \) as follows, by induction on the skeletons of \( S^d N \). For each vertex \( \sigma \) of \( S^d N \) set \( q(\sigma) = a' \). Consider any \( k \)-simplex \( \xi \) in \( S^d N \) with vertices \( \sigma_0 < \cdots < \sigma_k \). Each point \( x \) of \( \xi \) can be written uniquely in the form \( x = (1 - t) \sigma_0 + ty \), \( 0 \leq t \leq 1 \), where \( y \) lies in the \((k-1)\)-face opposite to the leading vertex \( \sigma_0 \). Put 
\[
q(x) = \lambda(a', q(y), t),
\]
assuming \( q \) is defined and a \( G \)-map on the \((k-1)\)-skeleton inductively by the above formula. \( q \) is well defined and continuous on the \( k \)-skeleton. As \( G \)-actions preserve ordering of \( S^d N \), it is easy to see that \( q \) is a \( G \)-map on the \( k \)-skeleton. Suppose \( q \) maps the \((k-1)\)-skeleton of \( S^d N_i \) to \( A_i \) for each \( i \), \( 1 \leq i \leq n-1 \). If \( \xi \) is a \( k \)-simplex of \( S^d N_i \) with vertices \( \sigma_0 < \cdots < \sigma_k \), then \( U_{s_i} \) intersects \( A_i \) and \( a_{s_i} \in A_i \) by our choices; hence \( q(x) \in A_i \) for any point \( x \) of \( \xi \) by definition of \( G \)-ELCX \( n \)-ad. Thus \( q \) maps the \( k \)-skeleton of \( S^d N_i \) to \( A_i \) for each \( i \), \( 1 \leq i \leq n-1 \), completing the induction.

For each point \( a \in A \), let \( V_a \) be a convex set which contains the star of \( a \) with respect to \( W \). Then \( q \circ p(a) \) is a convex combination of points in \( V_a \), whence \( (a, q \circ p(a)) \in V_a \times V_a \subset U \). Therefore the formula 
\[
(a, t) \mapsto \lambda(a, q \circ p(a), t)
\]
defines a \( G \)-homotopy between \( q \circ p \) and the identity of \( A \). q.e.d.

Corresponding to Proposition 3 of [8] we obtain the following

**Proposition 2.4.** If \( A \) belongs to \( \mathcal{W}_n^a \) and \( B \) belongs to \( \mathcal{W}_m^a \), then \( A \times B \) belongs to \( \mathcal{W}_{n+m-1}^a \).

**Proof.** The product \( A \times B \) is an \((n+m-1)\)-ad as defined in [8], p.277. Because of Theorem 2.3 we may suppose \( A \) and \( B \) to be metrizable and \( G \)-ELCX. Then \( A \times B \) is metrizable by product-metric. Using products of convex sets as convex sets, and the product of the structure maps as the structure map, it is routine to check that \( A \times B \) is \( G \)-ELCX. q.e.d.
If $X$ and $Y$ are $G$-spaces then the function space $F(X; Y)$ from $X$ to $Y$, endowed with compact-open topology, is a $G$-space by the formula

$$(g\varphi)(x) = g \varphi(g^{-1}x)$$

for $\varphi \in F(X; Y)$, $x \in X$ and $g \in G$.

The following theorem corresponds to Theorem 3 of [8].

**Theorem 2.5.** If $A = (A; A, \cdots, A_{n-1})$ belongs to $\mathcal{W}_n^g$ and if $C = (C; C_1, \cdots, C_{n-1})$ is an $n$-ad of compact $G$-spaces, then the $n$-ad

$$(F(C; A); F(C, C_1; A, A_1), \cdots, F(C, C_{n-1}; A, A_{n-1}))$$

belongs to $\mathcal{W}_n^g$.

**Proof.** By Theorem 2.3 we may assume that $A$ is metrizable and $G$-ELCX. Since $A$ is metrizable and $C$ is compact, $F(C; A)$ is metrizable; and $F(C, C_i; A, A_i)$ is its closed $G$-subspaces for each $i$, $1 \leq i \leq n-1$.

Define the neighborhood $U'$ of diagonals in $F(C; A) \times F(C; A_1)$, the structure map $\lambda'$ and convex sets of $F(C; A)$ as in [8], Proof of Lemma 3. It is easy to check that $U'$ is $G$-invariant and $\lambda'$ is a $G$-map. Thus $F(C; A)$ is $G$-ELCX, and the $n$-ad mentioned in the theorem is also $G$-ELCX.

Let $V$ be a finite-dimensional $G$-module, and $\Sigma^V$ denote the one-point compactification of $V$. Let $X$ be a pointed $G$-space with base point $x_0$. We put

$$\Omega^V X = F(\Sigma^V, *; X, x_0),$$

which we call the $(\dim V)$-fold loop space of $X$ with $G$-actions of type $V$ in parameters. $\Omega^V X$ is a pointed $G$-space with the constant map $e$ as base point.

**Corollary 2.6.** If a pair $(X, x_0)$ belongs to $\mathcal{W}_2^g$, then the pair $(\Omega^V X, e)$ also belongs to $\mathcal{W}_2^g$.

This corollary corresponds to Corollary 3 of [8], and will be used in the next section.
§ 3. **Representations of G-Cohomology Theories**

Segal [9] proposed to discuss generalized G-equivariant cohomology theories with degrees in the real representation ring $RO(G)$ of $G$. These are called **G-cohomology theories** for the sake of simplicity. Here we discuss to represent $G$-cohomology theories by $\Omega G$-spectra (defined below) in virtue of the method of Brown [3, 4].

A reduced $G$-cohomology theory will be defined as follows. Let $\mathcal{W}_G^\Omega$ and $\mathcal{F}_G^\Omega$ be the categories of pointed $G$-spaces and $G$-maps whose objects have $G$-homotopy types of $G$-complexes and of finite $G$-complexes, respectively; and let $\mathcal{F}_G^\Omega$ and $\mathcal{F}_G^\Omega$ denote the full subcategories of them with pointed $G$-complexes and finite $G$-complexes as objects, respectively. When we are given with an abelian-group-valued contravariant functor $\tilde{h}^\alpha$ for each $\alpha \in RO(G)$ simultaneously on the category $\mathcal{W}_G^\Omega$ or $\mathcal{F}_G^\Omega$, satisfying the following two axioms A1) and A2), then we call the system

$$\tilde{h}_G^\Omega = \{ \tilde{h}^\alpha; \alpha \in RO(G) \}$$

a **reduced $G$-cohomology theory** on $\mathcal{W}_G^\Omega$ or on $\mathcal{F}_G^\Omega$.

A1) Each $\tilde{h}^\alpha$ is a $G$-homotopy functor satisfying wedge axiom and Mayer-Vietoris axiom on $\mathcal{W}_G^\Omega$ or on $\mathcal{F}_G^\Omega$. (Cf., Adams [1] and Brown [4].)

A2) For each finite-dimensional $G$-module $V$, the natural suspension isomorphism

$$\sigma^v: \tilde{h}^\alpha(X) \simeq \tilde{h}^{\alpha + v}(\Sigma^vX)$$

is defined for every $\alpha \in RO(G)$ (where $\Sigma^vX = \Sigma^v \setminus X$).

Take an infinite-dimensional $G$-module $W$ which contains a discrete countable $G$-subset $S$ such that every finite subset of $S$ is linearly independent and, for every subgroup $H$ of $G$, there exists an infinite number of points of $S$ at which the isotropy groups of $G$ are $H$. Let $L$ be the simplicial complex consisting of all simplices spanned by finite subsets of $S$. $L$ is a simplicial $G$-complex and every finite simplicial $G$-complex is isomorphic to a $G$-subcomplex of $L$. Let $\mathcal{F}_G^\Omega$ be the full subcategory
of \( C_\mathcal{L}^g \) having all finite \( G \)-subcomplexes of \( L \) as objects. \( C_\mathcal{L}^g \) is a small category and contains countably-infinite many objects. Now the pairs \((C_\mathcal{W}^g, C_\mathcal{L}^g)\) and \((C_\mathcal{L}^g, C_\mathcal{W}^g)\) are homotopy categories and all functors \( \tilde{h}^g \), restricted to \( C_\mathcal{W}^g \) or \( C_\mathcal{L}^g \), are homotopy functors in the sense of Brown [4]; and we can apply Brown’s theory to our functor \( \tilde{h}^g \).

Here we remark the following. Every finite \( G \)-complex is \( G \)-homotopy equivalent to a finite simplicial \( G \)-complex (by simplicial approximations of attaching maps of cells); hence the set of \( G \)-homotopy types of finite \( G \)-complexes is countable, and we can choose a representative system \( \mathcal{K} = \{ K, K', \ldots \} \) such that all elements of \( K \) belong to \( C_\mathcal{L}^g \). Next, for any two complexes \( K \) and \( K' \) in \( \mathcal{K} \), the set \( [K, K']^g \) is countable (where \( [ , , ]^g \) stands for the set of \( G \)-homotopy classes of pointed \( G \)-maps), because any \( G \)-map \( f: K \to K' \) can be \( G \)-approximated by a simplicial \( G \)-map of some subdivisions of \( K \) and \( K' \) (i.e., take barycentric subdivisions \( Sd K \) and \( Sd K' \) first to make them \( G \)-complexes in our sense, secondly subdivide \( Sd K \) sufficiently fine so that we can apply the usual simplicial approximation to \( f \), then, taking care of \( G \)-equivariancy, we can apply the usual argument of simplicial approximation to get simplicial \( G \)-approximation of \( f \)). These remarks will be used later to apply the device of Adams [1], §3, to our case.

Let \( \mathcal{C} \) be a full subcategory of \( C_\mathcal{W}^g \) and \( h \) a Brown’s homotopy functor on \( \mathcal{C} \) (in the sense of \( G \)-homotopy). Let \( Y \) be an object of \( \mathcal{C} \) and \( u \in h(Y) \). The map

\[
T_u: [X, Y]^g \to h(X),
\]

defined by \( T_u[f] = f^*u \), is a natural transformation of functors on \( \mathcal{C} \), and the correspondence

\[
u \mapsto T_u
\]
gives a bijection

\[h(Y) \approx \text{Nat Trans}([ , , Y]^g, h),\]

[3], Lemma 3.1. When \( T_u \) is an isomorphism for each object \( X \) of \( \mathcal{C} \), \( Y \) is called a representing complex of \( h \) as usual.

Let \( C_\mathcal{W}^g_* \) and \( C_\mathcal{L}^g_* \) be the full subcategories of \( C_\mathcal{W}^g \) and...
\( \mathscr{F}^0 \), respectively, in which objects are \( G \)-complexes \( X \) such that \( X^H \) are arcwise connected for all subgroups \( H \) of \( G \).

As is easily seen
\[
\left( \frac{(G/H)^n \wedge S^n}{Y} \right)^\sigma = \pi_n(Y)
\]
for all \( n \geq 0 \) and all subgroups \( H \) of \( G \) (where \( G \) acts trivially on \( S^n \) and \( Y \) is a pointed \( G \)-complex). Hence, if \( f: Y \rightarrow Y' \) is a map in \( \mathscr{W}^0_\ast \)
such that
\[
f_*: [X, Y]^\sigma = [X, Y']^\sigma
\]
for all \( G \)-complexes \( X \) in \( \mathscr{F}^0_\ast \), then \( f \) is a \( G \)-homotopy equivalence by J.H.C. Whitehead's theorem for \( G \)-complexes. Thus we can apply [4], Theorem 2.8, to a Brown's homotopy functor on \( \mathscr{W}^0_\ast \) and we obtain

**Proposition 3.1.** Let \( h \) be a Brown's homotopy functor defined on \( \mathscr{W}^0_\ast \). There exists a representing couple \( (Y, u) \) of \( h \), where \( Y \) lies in \( \mathscr{W}^0_\ast \) and \( u \in h(Y) \), i.e.,
\[
T_u: [X, Y]^\sigma \approx h(X),
\]

a natural isomorphism of sets for \( X \) in \( \mathscr{W}^0_\ast \). \( Y \) is unique up to \( G \)-homotopy equivalence.

(Let \( K_\ast \) be the subset of \( K \) consisting of all elements which belong to \( \mathscr{F}^0_\ast \). Remark that we can use only elements of \( K_\ast \) as attaching data in the constructions in the proof of Theorem 2.8 of [4], which supplements the proof of the above proposition.)

Before discussing representations of Brown's homotopy functor on \( \mathscr{F}^0_\ast \), we remark the following

**Lemma.** Every \( G \)-complex \( X \) in \( \mathscr{W}^0_\ast \) can be expressed as a union of finite \( G \)-subcomplexes which belong to \( \mathscr{F}^0_\ast \).

**Proof.** It is clear that \( X \) can be expressed as a union of finite \( G \)-subcomplexes. Hence it is sufficient to show that, for arbitrary finite \( G \)-subcomplex \( K' \) of \( X \), we can find a finite \( G \)-subcomplex \( K \) of \( X \) such
that $K \supset K'$ and $K$ belongs to $\mathcal{WF}_q$.

Let $H$ be a subgroup of $G$. We want to find a finite $G$-subcomplex $K_1$ of $X$ satisfying that $K_1 \supset K'$ and, for every vertex $v$ of $K_1$ such that $G_v$ is contained in $H$, $v$ can be joined to the base point by a path in $K_1^q$. Suppose we obtained a finite $G$-subcomplex $K_2$ of $X$ satisfying that $K_2 \supset K'$ and, for every vertex $w$ of $K_2$ such that $G_w$ is a proper subgroup of $H$, $w$ can be joined to the base point by a path in $K_2^q$. Now, for each vertex $v$ of $K_2$ such that $G_v = H$, we can find a path $L_v$, which is a subcomplex of $X^H$ and joins $v$ to the base point. Set

$$K_i = K_1 \cup (\cup_v G_L_v)$$

where $v$ runs over all vertices of $K_1$ such that $G_v = H$. $K_1$ is the wanted $G$-complex.

Now, inductively on inclusions of subgroups $H$ of $G$, after a finite times of the above construction we obtain a finite $G$-subcomplex $K$ of $X$ such that $K \supset K'$ and every vertex $v$ of $K$ can be joined to the base point by a path in $K^q$, which is equivalent to saying that $K$ belongs to $\mathcal{WF}_q$.

q.e.d.

Let $h$ be a group-valued Brown’s homotopy functor on $\mathcal{WF}_q$. Put

$$\hat{h}(X) = \lim_{\leftarrow \gamma} h(X_\gamma)$$

for each $G$-complex $X$ in $\mathcal{WF}_q$, where $X_\gamma$ runs over all finite $G$-subcomplexes of $X$ which belong to $\mathcal{WF}_q$. $\hat{h}$ is a weak $G$-homotopy functor on $\mathcal{WF}_q$ in the parallel sense to “weak homotopy” in [1]. For each object $Y$ in $\mathcal{WF}_q$ and $u \in \hat{h}(Y)$, the maps

$$T_u : [X, Y]^q \to h(X), \quad X \in \mathcal{WF}_q,$$

and

$$\bar{T}_u : [X', Y]^q \to \hat{h}(X'), \quad X' \in \mathcal{WF}_q,$$

defined by $T_u[f] = f^*u$ and $\bar{T}_u[g] = g^*u$, respectively, are natural transformations of functors and the correspondences

$$u \mapsto T_u \quad \text{and} \quad u \mapsto \bar{T}_u$$

give rise to bijections of sets.
\[ \tilde{h}(Y) \approx \text{Nat Trans}(\text{[ }, Y]\mathcal{g}, h) \approx \text{Nat Trans}(\text{[ }, Y]\mathcal{g}, \tilde{h}), \]

where \([ \text{[ } , ]\mathcal{g}\) stands for the set of weak \(G\)-homotopy classes of \(G\)-maps, [3], Lemma 3.3, and [1], Lemma 4.1.

By the earlier remarks and the above lemma we can apply the arguments of [1], § 3, to the present case. In particular, the functor \(\tilde{h}\) on \(\mathcal{C}\mathcal{W}_k\) satisfies the Wedge axiom, the isomorphism with inverse limits and the Mayer-Vietoris axiom in the weak sense, [1], Lemma 3.3, Lemma 3.4 and Proposition 3.5, without any countability assumption on \(\mathcal{G}\).

Now we can do the same arguments and constructions as [1], Lemma 4.2 and Proposition 4.4, by utilizing only elements of \(\mathcal{K}_k\) as attaching data, and we obtain representations of \(h\), that is,

**Proposition 3.2.** Let \(h\) be a group-valued Brown's homotopy functor defined on \(\mathcal{C}\mathcal{D}_k\). There exists a representing couple \((Y, u)\) of \(h\), where \(Y\) lies in \(\mathcal{C}\mathcal{W}_k\) and \(u \in \tilde{h}(Y)\), i.e.,

\[ T_u : [X, Y]\mathcal{g} \approx h(X), \]

a natural isomorphism of sets for \(X\) in \(\mathcal{C}\mathcal{D}_k\). \(Y\) is unique up to \(G\)-homotopy equivalence.

We can also prove an analogue of [1], Theorem 1.9, and introduce a certain Hopf-space-structure to \(Y\) to make \(T_u\) an isomorphism of groups. But we don't need it to represent \(G\)-cohomology theories.

Now we shall discuss representations of \(G\)-cohomology theories. Let \(\tilde{h}_k^\alpha = \{\tilde{h}^\alpha; \alpha \in \text{RO}(G)\}\) be a reduced \(G\)-cohomology theory defined on \(\mathcal{W}_k^\mathcal{g}\) or \(\mathcal{D}_k^\mathcal{g}\). Since discussions of both cases are quite parallel and since the first case is a bit simpler, we shall discuss only the second case, i.e., we suppose \(\tilde{h}_k^\alpha\) is defined on \(\mathcal{D}_k^\mathcal{g}\).

By Proposition 3.2 we have a representing complex \(Y'\) of \(\tilde{h}_k^\alpha|\mathcal{C}\mathcal{D}_k^\mathcal{g}\) for each \(\alpha \in \text{RO}(G)\), i.e., we have a natural isomorphism

\[ [X, Y']_\mathcal{g} \approx \tilde{h}_k^\alpha(X), \quad X \in \mathcal{C}\mathcal{D}_k^\mathcal{g}, \]

for each \(\alpha \in \text{RO}(G)\).
By $\Sigma Y$ and $\Omega Y$ we denote the suspension and the loop space of a pointed $G$-space $Y$ with trivial $G$-actions on parameters. Put

$$Y_\alpha = \Omega Y'_{\alpha+1},$$

where 1 denotes the real 1-dimensional trivial $G$-module. $Y_\alpha$ is a Hopf-space ($H$-space) with the multiplication defined by usual loop compositions. Moreover, this multiplication in $Y_\alpha$ commutes with every $g$-action, $g \in G$. In this sense we call $Y_\alpha$ a $Hopf$-$G$-$space$. By Corollary 2.6 $Y_\alpha$ belongs to $\mathcal{W}_G^G$; hence we may assume that $Y_\alpha$ is a $Hopf$-$G$-$complex$ (replacing by a $G$-homotopy equivalent one if necessary). Then $Y_\alpha^G$ is a Hopf-subcomplex of $Y_\alpha$ for any subgroup $H$ of $G$.

$\Sigma X$ belongs to $\mathcal{G} \mathcal{W}_G^G$ for any $G$-complex $X$. Thus we have isomorphisms

$$\tilde{h}_\alpha(X) \cong \tilde{h}_{\alpha+1}(\Sigma X) \cong [\Sigma X, Y'_{\alpha+1}]^G \cong [X, Y_\alpha]^G$$

for each $X$ in $\mathcal{G} \mathcal{W}_G^G$ and $\alpha \in RO(G)$, where $\sigma$ is the suspension isomorphism. Moreover, the above isomorphisms are group isomorphisms by a usual argument, endowing $[X, Y_\alpha]^G$ a group structure induced by the Hopf-$G$-structure of $Y_\alpha$. Thus $Y_\alpha$ represents $\tilde{h}_\alpha$ on $\mathcal{G} \mathcal{W}_G^G$ as a group-valued functor.

Let $\tilde{H}_\alpha$ be the associated functor to $\tilde{h}_\alpha$, i.e.,

$$\tilde{H}_\alpha(X) = \lim_{\leftarrow \tau} \tilde{H}_\alpha(X_\tau)$$

for $X$ in $\mathcal{G} \mathcal{W}_G^G$, where $X_\tau$ runs over all finite $G$-subcomplexes of $X$. Since $[X, Y_\alpha]^G = \lim_{\leftarrow \tau} [X_\tau, Y_\alpha]^G$ as is easily seen, we have a natural isomorphism

$$[X, Y_\alpha]^G \cong \tilde{H}_\alpha(X)$$

of groups for each $X$ in $\mathcal{G} \mathcal{W}_G^G$, i.e., $Y_\alpha$ represents $\tilde{h}_\alpha$.

Let $V$ be a finite-dimensional $G$-module. Passing to the inverse limit of suspension isomorphisms $\sigma^V: \tilde{h}_\alpha(X_\tau) \approx \tilde{h}_{\alpha+V}(\Sigma^V X_\tau)$, we obtain a natural isomorphism

$$\sigma^V: \tilde{h}_\alpha(X) \approx \tilde{h}_{\alpha+V}(\Sigma^V X), \quad X \in \mathcal{G} \mathcal{W}_G^G.$$

Again, passing to the inverse limit of the canonical natural isomorphism $[\Sigma^V X_\tau, Y_{\alpha-V}]^G \cong [X_\tau, \mathcal{G} \mathcal{W}_G^G]$, we have a natural isomorphism
Combining the above three natural isomorphisms, we obtain a natural isomorphism

$$[X, Y] \cong [X, \mathcal{Q}^v Y_{a+r}]$$

of groups, where $\mathcal{Q}^v Y_{a+r}$ is a Hopf-G-space with structures induced from those of $Y_{a+r}$, and the group structure of the right hand side of the above isomorphism is induced from Hopf-G-structures of $Y_{a+r}$.

By Corollary 2.6 $\mathcal{Q}^v Y_{a+r}$ belongs to $\mathcal{Q}^v_Y$. And we may suppose that $\mathcal{Q}^v Y_{a+r}$ itself is a Hopf-G-complex. Putting $X = Y_a$ in the above isomorphism, we obtain a $G$-map

$$f(a, v) : Y_a \rightarrow \mathcal{Q}^v Y_{a+r}$$

such that $[f(a, v)]$ corresponds to the class of the identity map of $Y_a$. Next, putting $X = \mathcal{Q}^v Y_{a+r}$ in the same isomorphism, we obtain a $G$-map

$$g(a, v) : \mathcal{Q}^v Y_{a+r} \rightarrow Y_a$$

which corresponds to the class of the identity map of $\mathcal{Q}^v Y_{a+r}$. By the above choices we see easily that $(f(a, v))^{-1} = (g(a, v))^{-1}$ which is the same as the above natural isomorphism.

This shows, on one hand, that $g(a, v) f(a, v)$ and $f(a, v) g(a, v)$ are weakly homotopic to the identity maps; and, on the other hand, the fact that $f(a, v)$ and $g(a, v)$ induce group isomorphisms implies that $f(a, v)$ and $g(a, v)$ are weak morphisms of Hopf-G-complexes (i.e., they commute with Hopf-structure maps up to weak homotopy).

Then, for each subgroup $H$ of $G$, we see easily that $(f(a, v))^H$ is a weak morphism of Hopf-complexes, and

$$(g(a, v))^H (f(a, v))^H \simeq a 1 \quad \text{and} \quad (f(a, v))^H (g(a, v))^H \simeq a 1,$$

where $\simeq$ denotes "weak homotopy", which implies isomorphisms

$$(f(a, v))^H \tilde{\pi}_n(Y_a^H) \cong \tilde{\pi}_n((\mathcal{Q}^v Y_{a+r})^H)$$

for all $n \geq 0$. Hence, $(f(a, v))^H$ is a weak morphism of Hopf-complexes, induces one-one correspondence of path-components, and gives a weak homotopy equivalence of $\epsilon$-components. Thus $(f(a, v))^H$ is a weak homotopy equivalence by a classically well-used argument. Finally, J.H.C. Whitehead's theorem for $G$-complexes concludes that $f(a, v)$ is a $G$-homotopy equivalence.
Summarizing the above arguments we obtain

**Theorem 3.3.** Let \( \overline{h}^a = \{ \overline{h}^a; \alpha \in \text{RO}(G) \} \) be a reduced G-cohomology theory defined on \( \mathcal{W}_0^a \) or \( \mathcal{Q}_0^a \). There exists, for each \( \alpha \in \text{RO}(G) \), a G-complex \( Y_\alpha \) in \( \& \mathcal{W}_\alpha^0 \) which is a Hopf-G-complex and represents \( \overline{h}^a \) as a group-valued functor. Furthermore, for each finite-dimensional G-module \( V \), there exists a G-homotopy equivalence

\[
\int_\alpha Y_\alpha \cong \alpha \mathcal{Q} Y_\alpha.
\]

which is a morphism or weak morphism of Hopf-G-spaces (depending on the categories) and induces the suspension isomorphism \( \sigma^\nu \) for each \( \alpha \in \text{RO}(G) \).

Let \( \omega \) be a G-module containing exactly one copy of each irreducible G-module (including a trivial one) as a direct summand. A **G-spectrum** \( E^\omega \) consists of a G-space \( E_n \) in \( \mathcal{W}_\omega^a \) and a G-map \( \varepsilon_n; \Sigma^n E_n \to E_{n+1} \) for each \( n \in \mathbb{Z} \). Let \( \varepsilon'_n; E_n \to \mathcal{Q}^\omega E_n \) be the adjoint G-map of \( \varepsilon_n \) for each \( n \in \mathbb{Z} \). \( E \) is called an \( \mathcal{Q}\text{-G-spectrum} \) if \( \varepsilon'_n \) is a G-homotopy equivalence for every \( n \in \mathbb{Z} \). Since \( \omega \) contains a 1-dimensional trivial representation as a direct factor, \( \mathcal{Q}^\omega Y \) is a Hopf-G-space for any G-space \( Y \) by compositions along the parameter on which \( G \) acts trivially. Thus, if \( E \) is an \( \mathcal{Q}\text{-G-spectrum}, \) each term of it can be regarded as a Hopf-G-space.

In Theorem 3.3, putting

\[
E_n = Y_{n\alpha},
\]

and

\[
\varepsilon'_n = f_{n\alpha}; E_n \cong \mathcal{Q}^\omega E_n;
\]

for each \( n \in \mathbb{Z} \), we obtain an \( \mathcal{Q}\text{-G-spectrum} \) \( E = \{ E_n; \varepsilon_n; n \in \mathbb{Z} \} \). And we obtain

**Theorem 3.4.** Every reduced G-cohomology theory \( \overline{h}^a = \{ \overline{h}^a; \alpha \in \text{RO}(G) \} \) can be represented by an \( \mathcal{Q}\text{-G-spectrum} \) \( E = \{ E_n; n \in \mathbb{Z} \} \), i.e., we have a natural isomorphism

\[
\overline{h}^a(X) \cong [X, \mathcal{Q}^\omega E_n]_G
\]

\[\text{\footnote{The referee remarked the authors that this notion was defined in somewhat wide sense by C. Kosniowski, \textit{Math. Ann.}, 210 (1974), 83–104.}}\]
for each $\alpha \in RO(G)$, where $V$ is a finite $G$-module such that $\alpha + V = \emptyset$.

Remark 1. A similar representation theory was discussed by Matumoto [6], Theorem 6.1, for certain equivariant cohomology theories defined on the category of his $G$-CW-complexes, where he obtained representations of his cohomology theories by weak $\Omega$-spectra.

Remark 2. As observed by Segal [9], stable $G$-cohomotopy $\pi^G_\ast$ is universal for $G$-cohomology theories, or equivalently, we can say that every reduced $G$-cohomology theory is an $\pi^G_\ast$-module. Then a result of Segal [9], Corollary to Proposition 1, suggests that every $h^a$ should be treated as an $A(G)$-module-valued functor and the suspension $\sigma^a$ as an $A(G)$-module isomorphism, where $A(G)$ denotes the Burnside ring of $G$. Such an $A(G)$-module structure would be important if we want to discuss further structures of $G$-cohomologies such as multiplicative structures, in which units of $A(G)$ might play an important role in sign conventions. Even though it seems to be difficult to discuss the general case, we will discuss the case of $G = \mathbb{Z}/2\mathbb{Z}$, i.e., spaces-with-involutions, in a subsequent paper in details.

References