Notes on Minimality and Ergodicity of Compact Abelian Group Extensions of Dynamics

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§ 0. Introduction and Definitions

W. Parry [5] introduced the notion of a $G$-extension of a topological dynamics, where $G$ is a compact abelian group, and gave necessary and sufficient conditions for a $G$-extension of a minimal (respectively uniquely ergodic) topological dynamics to be minimal (uniquely ergodic). In the first part of this paper a proof of the Minimality Theorem of W. Parry without his "simple free" condition is given. In the purely measure-theoretic case W. Parry [6] introduced the notion of $G$-extension of type $\sigma$, where $\sigma$ is an automorphism of $G$, and spectrally analysed it. In the second part of this paper a necessary and sufficient condition for a $G$-extension of an ergodic measure-preserving dynamics to be ergodic is shown. As particular cases of this result we have well-known necessary and sufficient conditions for a translation, a group-automorphism and an affine transformation on a compact group to be ergodic.

Throughout, $G$ and $\hat{G}$ will respectively denote a compact abelian metric group and its character-group. An element $\gamma$ of $\hat{G}$ is called $n$-periodic with respect to an automorphism $\sigma$ of $G$ if $\gamma \sigma \neq \gamma, \ldots, \gamma \sigma^{n-1} \neq \gamma$ and $\gamma \sigma^n = \gamma$ ($n \geq 1$). A topological dynamics $(X, S)$ is a compact metric space $X$, together with a homeomorphism $S$. A topological dynamics $(X, S)$ is conjugate to $(X, S)$ if there is a homeomorphism $\tau$ of $X$ onto $X$, such that the diagram

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A set $F$ is $S$-invariant if $SF = F$. An $S$-invariant closed set $F$ is $S$-minimal if the only $S$-invariant closed subsets of $F$ are $F$ and $\emptyset$. $(X, S)$ is minimal if $X$ is $S$-minimal. Denote respectively by $C(X)$ and $C(X, K)$, the set of all continuous complex-valued functions defined on $X$ and the set of all functions in $C(X)$ with absolute value 1.

A continuous $G$-action on $X$ is a continuous map $\chi$ of $G \times X$ onto $X$ such that $\chi(g, \chi(h, x)) = \chi(gh, x)$ for $x$ in $X$ and $g, h$ in $G$ and $\chi(e, x) = x$ for $x$ in $X$ where $e$ is the identity element of $G$. If the map $\chi$ is understood we shall write $gx$ for $\chi(g, x)$. If $(X, S)$ is a topological dynamics such that $S$ commutes with the $G$-action (i.e. $Sgx = gSx$ for $x$ in $X$ and $g$ in $G$) then $S$ induces the homeomorphism $S'$ on the $G$-orbit space $X/G$ defined by $S'G(x) = G(Sx)$ where $G(x) = \{gx; g \in G\}$. If a topological dynamics $(X_i, S_i)$ is conjugate to the topological dynamics $(X/G, S')$ we shall say that $(X, S)$ is a $G$-extension of $(X_i, S_i)$. (W. Parry [5]).

A measure-preserving dynamics $(\mathcal{Q}, \mu, \varphi)$ (in this paper) is a Lebesgue measure space $(\mathcal{Q}, \mu)$, $\mu(\mathcal{Q}) = 1$, together with a bimeasurable bijection $\varphi$ such that $\mu(\varphi A) = \mu(A)$ for any measurable set $A$. For simplicity of notation, expressions involving sets or functions will be stated disregarding sets of measure zero. A measure-preserving dynamics $(\mathcal{Q}, \mu, \varphi)$ is conjugate to $(\mathcal{Q}, \mu, \varphi_1)$ if there is a bimeasurable, measure-preserving bijection $\tau$ from $(\mathcal{Q}, \mu)$ onto $(\mathcal{Q}_1, \mu_1)$ such that the diagram

\[
\begin{array}{ccc}
\mathcal{Q} & \xrightarrow{\varphi} & \mathcal{Q} \\
\tau \downarrow & & \downarrow \tau \\
\mathcal{Q}_1 & \xrightarrow{\varphi_1} & \mathcal{Q}_1
\end{array}
\]

commutes. $(\mathcal{Q}, \mu, \varphi)$ is ergodic if every measurable function $f$ with $f(\varphi \omega) = f(\omega)$ is constant. Denote by $L^1(\mathcal{Q}, \mu)$ the set of all square-integrable functions on $\mathcal{Q}$. A measurable $G$-action on $(\mathcal{Q}, \mu)$ is a measurable map
\[ \chi \] of \( G \times \Omega \) onto \( \Omega \) such that \( \chi(g, \chi(h, \omega)) = \chi(gh, \omega) \) for \( \omega \) in \( \Omega \) and \( g, h \) in \( G \), \( \chi(e, x) = x \) for \( x \) in \( \Omega \), and \( \mu(\chi(g, A)) = \mu(A) \) for any measurable set \( A \) and any \( g \) in \( G \). If the map \( \chi \) is understood we shall write \( g\omega \) for \( \chi(g, \omega) \). If \( (\Omega, \mu, \varphi) \) is a measure-preserving dynamics such that \( \varphi g\omega = \sigma(g)\varphi\omega \) for \( \omega \) in \( \Omega \) and \( g \) in \( G \) for some automorphism \( \sigma \) of \( G \), then \( \varphi \) induces the measure-preserving transformation \( \varphi' \) on the \( G \) orbit space \( \Omega/G \). If a measure-preserving dynamics \((\Omega, \mu_1, \varphi_1)\) is conjugate to the \((\Omega/G, \mu_{\varphi_0}, \varphi')\) we shall say that \((\Omega, \mu, \varphi)\) is a \( G \)-extension of type \( \sigma \) of \((\Omega, \mu_1, \varphi_1)\). (W. Parry [6]).

§ 1. Minimality of a \( G \)-extension

**Lemma 1.** Let \((X, S)\) be a \( G \)-extension of a minimal topological dynamics. Then for any \( S \)-minimal set \( C \), \( gC \) is \( S \)-minimal for any \( g \) in \( G \) and \( X = \bigcup_{\varphi \in \Theta} gC \).

**Proof.** It is easy to see that \( gC \) is \( S \)-minimal for any \( g \) in \( G \). We denote by \( \pi \) the map from \((X, S)\) to the minimal topological dynamics \((X, S_\lambda)\) defined by \( \pi x = \tau^{-1}(G(x)) \). The set \( \bigcup_{\varphi \in \Theta} gC \) is closed and \( S \)-invariant and the set \( \pi(\bigcup_{\varphi \in \Theta} gC) \) is closed and \( S_\lambda \)-invariant. From the minimality of \((X, S_\lambda)\) we have \( \pi(\bigcup_{\varphi \in \Theta} gC) = X \). Therefore we have \( \bigcup_{\varphi \in \Theta} gC = X \).

q.e.d.

**Lemma 2.** Let \( Y \) be a compact topological space on which there is a continuous \( G \)-action such that \( Y = \{gy; g \in G\} \) for some (any) point \( y \) in \( Y \). And let \( \Gamma \) be the set of all \( \gamma \) in \( \widehat{G} \) such that there exists an \( f_\gamma \) in \( C(Y, K) \) with \( f_\gamma(gy) = \gamma(g)f_\gamma(y) \) for \( y \) in \( Y \) and \( g \) in \( G \). Then for \( h \) in \( G \), \( \varphi(h) = 1 \) for any \( \gamma \) in \( \Gamma \) if and only if \( hy = y \) for any \( y \) in \( Y \). In particular, \( \Gamma = \{1\} \) if and only if \( Y \) is one point space.

**Proof.** For \( f \) in \( C(Y) \) and \( \gamma \) in \( \widehat{G} \), put \( f_\gamma(y) = \int \gamma(g)f(g^{-1}y) \, dg \) where \( dg \) is the Haar measure on \( G \), then \( f_\gamma(gy) = \gamma(g)f_\gamma(y) \) and \( f_\gamma = 0 \) for \( \gamma \) not in \( \Gamma \). Now \( \varphi(h) = 1 \) for any \( \gamma \) in \( \Gamma \), iff \( f_\gamma(hy) = f_\gamma(y) \) for \( y \) in \( Y \), any \( f \) in \( C(Y) \) and any \( \gamma \) in \( \widehat{G} \), iff \( \int \gamma(g)\{f(g^{-1}hy) - f(g^{-1}y)\} \, dg = 0 \).
for $y$ in $Y$, any $f$ in $C(Y)$ and any $\gamma$ in $\tilde{G}$, iff $f(hy) = f(y)$ for $y$ in $Y$, any $f$ in $C(Y)$. All these hold iff $hy = y$ for $y$ in $Y$. q.e.d.

**Theorem 1.** Let $(X, S)$ be a $G$-extension of a minimal topological dynamics. Then $(X, S)$ is not minimal if and only if there exists a $\gamma$ in $\tilde{G}$, $\gamma \neq 1$ and $f$ in $C(X, K)$ such that $f(gx) = \gamma(g)f(x)$ and $f(Sx) = f(x)$ for any $x$ in $X$ and any $g$ in $G$.

**Proof.** Note that the quotient space $X/C$ is Hausdorff (and compact) and apply Lemma 1 and Lemma 2. q.e.d.

**Corollary 1.** (H. Fürstenberg [2], W. Parry [5]). Let $(X, S)$ be a minimal topological dynamics and $\alpha$ be a continuous $G$-valued function defined on $X$. $\tilde{S}$ is a homeomorphism of the product space $X \times G$ defined by

$$\tilde{S}(x, g) = (Sx, \alpha(x)g), \quad (x, g) \text{ in } X \times G.$$ 

Then the topological dynamics $(X \times G, \tilde{S})$ is not minimal if and only if there exists a $\gamma$ in $\tilde{G}$, $\gamma \neq 1$ and an $f$ in $C(X, K)$ such that $\gamma(\alpha(x))f(Sx) = f(x)$ for all $x$ in $X$.

**Proof.** Consider the $G$-action $g(x, h) = (x, gh)$ on $X \times G$. $(X \times G, \tilde{S})$ is a $G$-extension of $(X, S)$. Corollary 1 follows from Theorem 1. q.e.d.

§ 2. Ergodicity of a $G$-extension

**Lemma 3.** (W. Parry [6]). Let $(\Omega, \mu, \varphi)$ be a measure-preserving dynamics such that $\varphi(g\omega) = \sigma(g)\varphi(\omega)$ for $\omega$ in $\Omega$ and $g$ in $G$ for some automorphism $\sigma$ of $G$.

Let $V_\gamma$ ($\gamma \in \tilde{G}$) be the set of all $f_\gamma$ in $L^2(\Omega, \mu)$ such that $f_\gamma(g\omega) = \gamma(g)f_\gamma(\omega)$ with $\omega$ in $\Omega$ and $g$ in $G$. Then

1. $L^2(\Omega, \mu) = \sum_{\gamma \in \tilde{G}} V_\gamma$ (orthogonal sum) and
2. if $f_\gamma$ is in $V_\gamma$, then $f_\gamma \varphi$ is in $V_{\gamma \sigma}$. 
Proof. (1) For an \( f_r \) in \( V_r \) and an \( f_r' \) in \( V_r' \) we have

\[
\int f_r(\omega)\overline{f_r'(\omega)} \, d\mu(\omega) = \int f_r(g\omega)\overline{f_r'(g\omega)} \, d\mu(\omega)
\]

\[
= \gamma(g)\overline{\gamma'(g')} \int f_r(\omega)\overline{f_r'(\omega)} \, d\mu(\omega).
\]

If \( \gamma \neq \gamma' \), \( \gamma(g)\overline{\gamma'(g')} \neq 1 \) for some \( g \) in \( G \), and so \( f_r \) is orthogonal to \( f_r' \). Suppose that \( f \) in \( L^2(\Omega, \mu) \) is orthogonal to any function in \( \cup_r V_r \). Put

\[
f_r(\omega) = \int \gamma(g) f(\gamma^{-1}(\omega)) \, dg
\]

for \( \gamma \) in \( G \), then \( f_r \) is in \( V_r \). We have

\[
\int f_r(\omega)\overline{f_r'(\omega)} \, d\mu(\omega) = \int f_r(\omega) \int \gamma(g)f(\gamma^{-1}(\omega)) \, dg \, d\mu(\omega)
\]

\[
= \int \int f_r(\gamma^{-1}(\omega))\overline{f(\gamma^{-1}(\omega))} \, d\mu(\omega) \, dg
\]

\[
= \int f_r(\omega)f(\omega) \, d\mu(\omega) = 0.
\]

Hence \( f_r(\omega) = 0 \) for \( \omega \) in \( \Omega \) and \( \gamma \) in \( G \), and thus \( f(\omega) = 0 \) for \( \omega \) in \( \Omega \). Assertion (2) follows from the equation

\[
f_r(\phi\omega) = f_r(\phi)(\phi\omega) = \gamma(\phi(\omega)) f_r(\omega).
\]

q.e.d.

Theorem 2. Let \( (\Omega, \mu, \phi) \) be a \( G \)-extension of type \( \sigma \) of an ergodic measure-preserving dynamics. Then \( \phi \) is not ergodic if and only if there exists a positive integer \( n \) and a \( \gamma \) in \( G \), \( n \)-periodic with respect to \( \sigma \) and not equal to \( 1 \), and an \( f_r \) in \( L^2(\Omega, \mu) \), \( f_r \neq 0 \), such that \( f_r(\phi^n) = f_r(\omega) \) and \( f_r(g\omega) = \gamma(g)f_r(\omega) \), for \( \omega \) in \( \Omega \) and \( g \) in \( G \).

Proof. Let \( f_r \) be a function which satisfies the conditions of Theorem 2. Put \( f(\omega) = f_r(\omega) + f_r(\phi\omega) + \cdots + f_r(\phi^{n-1}\omega) \). Then \( f \) is in \( V_r \oplus V_{\sigma r} \oplus \cdots \oplus V_{\sigma^{n-1} r} \) and \( f(\phi\omega) = f(\omega) \) for \( \omega \) in \( \Omega \). That is, \( f \) is not constant and \( \phi \)-invariant. Hence \( \phi \) is not ergodic. Conversely, let \( f \) be a not constant function drawn from \( L^1(\Omega, \mu) \) such that \( f\phi = f \), and let \( f = \sum_{r \in G} f_r \) with \( f_r \) in \( V_r \) be the direct sum decomposition of \( f \). Then \( f\phi = \sum_{r \in G} f_r \phi \) where \( f_r\phi \) is in \( V_{\sigma r} \). From \( f\phi = f \) we have \( f_r\phi = f_{\sigma r} \) and \( \|f_r\|_\Lambda = \|f_{\sigma r}\|_\Lambda \) for \( \gamma \) in \( G \). From the orthogonality of \( f_r \)'s we have \( f_r = 0 \) if \( \gamma \) is not periodic w.r.t. \( \sigma \). Since any \( \phi \)-invariant, \( G \)-invariant function is constant
from the ergodic assumption, there exists a positive integer $n$ and an $n$-periodic $\gamma$ in $\widehat{G}$, $\gamma \neq 1$ such that $f_\gamma \neq 0$. We have $f_\gamma (\varphi^\omega) = f_{\gamma^*} (\omega) = f_\gamma (\omega)$ for $\omega$ in $\mathcal{O}$.

Corollary 2. Let $(\mathcal{O}, \mu, \varphi)$ be an ergodic measure-preserving dynamics, $\alpha (\omega)$ be a measurable $G$-valued function and $\sigma$ be an automorphism of $G$. $\varphi$ is a measure-preserving transformation of the product $\mathcal{O} \times G$ defined by

$$
\varphi (\omega, g) = (\varphi(\omega), \alpha (\omega) \sigma (g)), \quad (\omega, g) \text{ in } \mathcal{O} \times G.
$$

Then $(\mathcal{O} \times G, \mu \times dg, \varphi)$ is not ergodic if and only if there exists a positive integer $n$ and a $\gamma$ in $\widehat{G}$, $n$-periodic with respect to $\sigma$ and not equal to 1, and an $f$ in $L^1(\mathcal{O}, \mu), f \neq 0$ such that $\gamma (\alpha (\varphi^{n-1} \omega) \sigma (\alpha (\varphi^{n-2} \omega)) \cdots \sigma^{-1} (\alpha (\omega))) f (\varphi^\omega) = f (\omega)$ for $\omega$ in $\mathcal{O}$.

Proof: Consider the $G$-action $g (\omega, h) = (\omega, gh)$ on $\mathcal{O} \times G$. $(\mathcal{O} \times G, \mu \times dg, \varphi)$ is a $G$-extension of type $\sigma$ of $(\mathcal{O}, \mu, \varphi)$. Corollary 2 follows from Theorem 2.

Corollary 3. (1) When $\sigma$ of Corollary 2 is the identity, $\varphi$ is not ergodic if and only if there exists a $\gamma$ in $\widehat{G}$, $\gamma \neq 1$, and a measurable function $f$ such that $|f (\omega)| = 1$, and $\gamma (\alpha (\omega)) f (\varphi^\omega) = f (\omega)$ for $\omega$ in $\mathcal{O}$. (H. Anzai [1]).

(2) When $\alpha (\omega) = h$ for $\omega$ in $\mathcal{O}$ and $G$ is connected, $\varphi$ of Corollary 2 is not ergodic if and only if (i) there exists an $n \geq 2$ and an $n$-periodic $\gamma$ in $G$, or (ii) there exists a $1$-periodic $\gamma$ in $G$, $\gamma \neq 1$, and a measurable function $f$, such that $|f (\omega)| = 1$ and $\gamma(h) f (\varphi^\omega) = f (\omega)$ for $\omega$ in $\mathcal{O}$, that is, $\gamma(h)^{-1}$ is in the point spectrum of $\varphi$.

Proof. (1) Clear from Corollary 2.

(2) If $n \geq 2$ and $\gamma$ be $n$-periodic, put $\gamma_1 = \gamma \sigma^k$. Then $\gamma_1$ is in $\widehat{G}$, $\gamma_1 \neq 1$ and $\gamma_1 \sigma^n = \gamma_1$. Let $n_1$ be the period of $\gamma_1$; we may represent $n$ as $n_1 p$ where $p$ is a positive integer. If $\left(\frac{\gamma_1 \sigma^k}{\gamma_1}\right)^p = 1$ for a positive integer $k$, we have $\gamma_1 \sigma^k = 1$ from the connectedness of $G$. This means that $\gamma_1^p$
is also $n_t$-periodic. Since $\gamma_1^n(h \sigma h \ldots \sigma^{n_t-1}h) = \gamma_1(h \sigma h \ldots \sigma^{n_t-1}h) = 1$, $\gamma_1^n(h \sigma h \ldots \sigma^{n_t-1}h)f(\varphi^{n_t-1}\omega) = f(\omega)$ for any constant function $f$. The rest of the proof is obvious.

q.e.d.

**Corollary 4.** (1) The affine transformation $g \mapsto h \sigma(g)$ on connected $G$ is not ergodic if and only if there exists an $n$-periodic $\gamma$ in $\widehat{G}$ with $n \geq 2$ or there exists a $1$-periodic $\gamma$ in $\widehat{G}$, $\gamma \neq 1$ with $\gamma(h) = 1$ (F. Hahn [3]).

(2) The group automorphism $g \mapsto \sigma(g)$ on $G$ is not ergodic iff there exists an $n$-periodic $\gamma$ in $\widehat{G}$, $\gamma \neq 1$ for some $n \geq 1$. (P.R. Halmos [4]).

(3) The translation $g \mapsto hg$ on $G$ is not ergodic iff there exists a $\gamma$ in $\widehat{G}$, $\gamma \neq 1$ with $\gamma(h) = 1$.

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**References**


