The Structure of Local Solutions of Partial Differential Equations of the Fuchsian Type

by

Hidetoshi TAHARA*

Linear partial differential equations with regular singularity along a hypersurface were studied by several authors, say, Hasegawa [10][11], Baouendi-Goulaouic [3][4], Alinhac [1][2], Froim [7][8][9], Delache-Leray [6], Kashiwara-Oshima [13], Tsuno [15], etc... in various problems. In this note, we consider the hyperfunction solutions of certain type equations with regular singularity. The details of this note will be published in [14] anywhere else.

Let \((t, z) \in \mathbb{C} \times \mathbb{C}^n\) and let

\[ P(t, z, D_t, D_z) = t^k D_t^m + P_1(t, z, D_t) t^{k-1} D_t^{m-1} + \ldots + P_k(t, z, D_t) D_t^{m-k} + \ldots + P_m(t, z, D_t) \]

be a linear differential operator whose coefficients are holomorphic functions defined in a neighbourhood of the origin such that

(A-i) \quad 0 \leq k \leq m

(A-ii) \quad \text{order of } P_j(t, z, D_t) \leq j \text{ for } 1 \leq j \leq m

(A-iii) \quad \text{order of } P_j(0, z, D_t) \leq 0 \text{ for } 1 \leq j \leq k.

Then \(P\) is said of the Fuchsian type with weight \(m-k\) with respect to \(t\) (by [3]). By the condition (A-iii), \(P_j(0, z, D_t)\) is a function. We set \(P_j(0, z, D_t) = a_j(z)\) for \(1 \leq j \leq k\). Then the indicial equation associated with \(P\) is defined by

\[ \mathcal{E}(\lambda, z) = \lambda(\lambda-1) \cdots (\lambda-m+1) + a_1(z)\lambda(\lambda-1) \cdots (\lambda-m+2) \]

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* Department of Mathematics, Faculty of Science and Technology, Sophia University, 7 Kioicho, Chiyoda-ku, Tokyo, Japan.
The roots, that we call the characteristic exponents of $P$, are denoted by $\lambda = 0, 1, \ldots, m-k-1$, $\rho_1(z), \ldots, \rho_k(z)$. They are functions of $z$.

We set

$$\mathcal{O} = \text{the set of all the germs of multivalued holomorphic functions on } \mathbb{C} \times \mathbb{C}^n \setminus \{t = 0\} \text{ at the origin.}$$

Then we have the next theorem.

**Theorem 1.** Assume that $\rho_i(0), \rho_i(0) - \rho_j(0) \in \mathbb{Z}$ holds for $1 \leq i \neq j \leq k$. Then the equation $Pu = f$ is always solvable in $\mathcal{O}$. Moreover there exist holomorphic functions $K_i(t, z, w)$ $(0 \leq i \leq m-k-1)$, $L_j(t, z, w)$ $(1 \leq j \leq k)$ on

$$s = \min(m, k+1), M = \text{constant}$$

which satisfy the following conditions:

1. For any holomorphic functions $\varphi_i(w), \psi_j(w)$ at the origin, we set

$$u(t, z) = \sum_{i=0}^{m-k-1} \int K_i(t, z, w) t^i \varphi_i(w) \, dw$$

$$+ \sum_{j=1}^{k} \int L_j(t, z, w) t^{j(m)} \psi_j(w) \, dw$$

Then $u(t, z)$ is a solution of the equation $Pu = 0$ in $\mathcal{O}$.

2. If $u(t, z) \in \mathcal{O}$ and $Pu = 0$ holds, then $u(t, z)$ is uniquely expressed in the form (1).

Next, we consider the equation in the real domain and investigate the structure of hyperfunction solutions. Let $(t, x) \in \mathbb{R} \times \mathbb{R}^n$ and let $P(t, x, D_t, D_x)$ be of the Fuchsian type with weight $m-k$ with respect to $t$. Moreover we assume the following conditions on $P$:

(A-iv) $\sigma_m(P)$ has the form: $\sigma_m(P)(t, x, \tau, \xi) = t^p m(t, x, \tau, \xi)$

(A-v) All the roots $\tau(t, x, \xi)$ of the equation $\rho_m(t, x, \tau, \xi) = 0$ are real, when $t, x, \xi$ are real (near the origin).
Then we say that $P$ is a Fuchsian hyperbolic operator with respect to $t$. Note that if $k=0$, then $P$ is nothing but a weakly hyperbolic operator in the direction $dt$ ([5]).

Under these assumptions, we can give the meaning as hyperfunctions to the above $K_i(t,z,w)$, $L_j(t,z,w)$ in Theorem 1. We also denote these hyperfunctions by $K_i(t,x,y)$, $L_j(t,x,y)$ respectively. Then $K_i$, $L_j$ satisfy the following conditions:

**Theorem 2.** Assume that $\rho_i(0)$, $\rho_i(0)-\rho_j(0) \notin \mathbb{Z}$ holds for $1 \leq i \neq j \leq k$. Then the equation $Pu=f$ is always solvable in $\mathcal{B}(\text{where } \mathcal{B} \text{ is the stalk of the sheaf of hyperfunctions at the origin})$. Moreover the above $K_i(t,x,y)$ $(0\leq i \leq m-k)$, $L_j(t,x,y)$ $(1 \leq j \leq k)$ satisfy the following conditions:

(1) For any hyperfunctions $\varphi_i(y)$, $\psi_j^\pm(y)$ at the origin, we set

$$u(t,x) = \sum_{i=0}^{m-k-1} \int K_i(t,x,y) t^i \varphi_i(y) \, dy + \sum_{j=1}^{k} \sum_{\pm} \int L_j(t,x,y) (t \pm i0)^{\rho_j(y)} \psi_j^\pm(y) \, dy$$

or

$$u(t,x) = \sum_{i=0}^{m-k-1} \int K_i(t,x,y) t^i \varphi_i(y) \, dy + \sum_{j=1}^{k} \sum_{\pm} \int L_j(t,x,y) t^{\rho_j(y)} \psi_j^\pm(y) \, dy$$

Then $u(t,x)$ is a solution of the equation $Pu=0$ in $\mathcal{B}$.

(2) If $u(t,x) \in \mathcal{B}$ and $Pu=0$ holds, then $u(t,x)$ is uniquely expressed in the form (1).

**References**