Zonal Spherical Functions on Some Symmetric Spaces

by

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§ 0. Introduction

Let $G$ be a real semisimple Lie group with finite center, and $K$ a maximal compact subgroup of $G$. A zonal spherical function on the symmetric space $X=G/K$ is an simultaneous eigenfunction $\varphi(x)$ of all the invariant differential operators on $X$ satisfying $\varphi(kx) = \varphi(x)$ for any $x \in X$, $k \in K$, and $\varphi(eK) = 1$, where $e$ is the identity element in $G$. By the Cartan decomposition $G=KAK$, $\varphi(x)$ is considered as a function on $A$. And by the separation of variables, we obtain differential operators on $A$ from the invariant differential operators, which are called their radial components. In this paper, we investigate the radial components of the invariant differential operators and the zonal spherical functions when $G$ is a real, complex or quaternion unimodular group. The eigenvalues of the zonal spherical functions is parametrized by the element in $\mathfrak{a}^*$. Therefore, the system of differential equations on $A$ satisfied by the zonal spherical function has as many parameters as $\dim \mathfrak{a}$. However, we can construct a new system of differential equations which admits the other parameter $\nu$. It is shown that the zonal spherical function on the real, complex or quaternion unimodular group corresponds to the case in which $\nu = \frac{1}{2}$, 1, 2, respectively.

§ 1. Radial Components of Invariant Differential Operators

Let $\mathfrak{a}$ be a vector space of dimension $n$, and $\mathfrak{a}^*$ its dual space. $\mathfrak{a}^*$ is generated by $e_i$, $i=1$, 2, $\cdots$, where $e_i(H) = t_i$ for $H = (t_1, \cdots, t_n) \in \mathfrak{a}$.

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First we will define $n$ differential operators $A_i^{(v)}, i=1, 2, \cdots, n,$ by the following formula,

$$A_i^{(v)} = \frac{1}{\delta(H)} \sum_{s \in \mathbb{S}_n} (\det s) e^{\frac{s}{2} (aH)} \sum_{i=1}^{n} \langle \zeta + D_{it(i)} + (n+1-2i)\nu \rangle^{e_{s_i}} + A_{i+1}^{(v)} \zeta^{e_{s_{i+1}}} + \cdots + A_n^{(v)}.$$  

Here

$$\delta(H) = \prod_{i<j} (e^{t_i-t_j} - e^{t_j-t_i} \chi_i \chi_j)$$

$$\rho(H) = \frac{1}{2} \sum_{i<j} (t_i-t_j)$$

$$\sigma(H) = (t_1, \cdots, t_n),$$

for $H=(t_1, \cdots, t_n) \in \mathfrak{a},$ $s \in \mathbb{S}_n,$ and $\zeta$ is an indeterminate. For example

$$A_1^{(v)} = D_{t_1} + \cdots + D_{t_n}.$$  

$$A_2^{(v)} = \sum_{i<j} (D_{t_i}D_{t_j} - \nu \cosh(t_i-t_j)(D_{t_i}D_{t_j}) - 2\langle \rho, \rho \rangle \nu).$$

Here $\langle , \rangle$ is the inner product on $\mathfrak{a}^*$ defined by $\langle e_i, e_j \rangle = \delta_{ij}$.

Theorem 1. The operators $A_i^{(v)}, i=1, 2, \cdots, n,$ are commutative with each other. And under the condition $\sum_{i=1}^{n} t_i = 0,$ the radial components of generators of the algebra of the invariant differential operators on symmetric spaces $SL(n, \mathbb{R})/SO(n), SL(n, \mathbb{C})/SU(n),$ $SL(n, \mathbb{H})/Sp(n)$ are given by the above operators, if we substitute $\nu$ for $\frac{1}{2},$ $1,$ $2$ respectively.

Proof. In complex unimodular group case, the radial components of invariant differential operators are known (cf. [1]). And in real unimodular group case, it is easy to compute the radial components of invariant differential operators by using a well-known formula called Cappelli’s identity. In these cases, the operators $A_i^{(v)}, i=1, 2, \cdots, n,$ are commutative, and by this fact, we can prove the commutativity of $A_i^{(v)}$ for any fixed $\nu$. If we know the commutativity, it is easy to check quaternion unimodular group case.

Next, we investigate the system of differential equations.
$\mathcal{H}_{\ell}^{(\nu)}; \Delta(\zeta, \nu) u = \prod_{i=1}^{n} (\zeta + \lambda_i) u$ for any $\zeta$.

Here $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ and we assume $\sum_{i=1}^{n} \lambda_i = 0$. This means that a solution $u$ of this system is a simultaneous eigenfunction of the differential operators $\Delta_{\ell}^{(\nu)}, i = 1, 2, \ldots, n$. Following Harish-Chandra [2], we can construct $n!$ solutions $\Phi_{i_1}^{(\nu)}(H) s \in \mathbb{C}_n$ of this system. $\Phi_{i_1}^{(\nu)}(H)$ is defined as follows.

$$\Phi_{i_1}^{(\nu)}(H) = \sum_{\rho \in L} \Gamma_{\mu}^{(\nu)}(\lambda) e^{(i-2\rho-\rho)(\nu)}$$

where $L = \{m_1 \alpha_1 + \cdots + m_n \alpha_n \mid m_i \in \mathbb{N} \ i = 1, 2, \ldots, n-1\}$, $\alpha_i = e_i - e_{i+1}$, $N = \{0, 1, 2, \ldots\}$. And the coefficients $\Gamma_{\mu}^{(\nu)}(\lambda)$ satisfy the recursion formulas

$$\sum_{s \in \mathbb{C}_n} (\det s) \left( \prod_{i=1}^{n} (\zeta + \tau_i(\lambda - \mu, s, \nu)) - \prod_{i=1}^{n} (\zeta + \lambda_i) \right) \Gamma_{\mu+2i}(\rho - \rho)(\lambda) = 0,$$

for any $\zeta$. Here

$$\tau(\lambda, s, \nu) = \lambda + 2(\nu - 1)(s \rho - \rho)$$

$$= (\tau_1(\lambda, s, \nu), \ldots, \tau_n(\lambda, s, \nu)).$$

$\Phi_{i_1}^{(\nu)}(H)$ is holomorphic in the positive Weyl chamber $C = \{H \in \mathbb{C}; \alpha_i(H) > 0 \ i = 1, \ldots, n-1\}$.

§ 2. An Analogue of Gegenbauer's Function in Two Variables

In case $n = 2$, the system $\mathcal{H}_{\ell}^{(\nu)}$ is well-known Gegenbauer's differential equation by taking a suitable coordinate system. In this section, we will obtain integral representation and recursion formulas for the functions satisfying the system $\mathcal{H}_{\ell}^{(\nu)}$ in case $n = 3$.

We set $a_i = 2e_i - \frac{2}{3} \sum_{j=1}^{3} e_j \ i = 1, 2, 3, \ x_1 = \frac{1}{3} \sum_{i=1}^{3} e_{x_1}, \ x_2 = \frac{1}{3} \sum_{i=1}^{3} e_{x_2}$, and assume that $\sum_{i=1}^{3} e_i(H) = 0$ for $H \in \mathbb{C}$ in this section. We represent the operators $\Delta_{x_1}^{(\nu)}, \Delta_{x_2}^{(\nu)}$ by $x_1, x_2$, then

$$\Delta_{x_1}^{(\nu)} = (x_2 - x_1^2) D_{x_1}^2 + (1 - x_1 x_2) D_1 D_2$$

$$+ (x_1 - x_2) D_2^2 - (3x_1 + 1) (x_1 D_1 + x_2 D_2) \nu^2$$

$$\Delta_{x_2}^{(\nu)} = (1 - 3x_1 x_2 + 2x_1^2) D_1^2 + 3(x_1 - 2x_2^2 + x_2 x_1^2) D_1 D_2$$
First we have the following recursion formulas.

**Theorem 2.** There are two recursion formulas between the functions \( \Phi_{i}^{(\nu)}(H) \), \( \lambda \in \mathbb{A}^{*} \), if we normalize the initial value by \( \Gamma_{0}^{(\nu)}(\lambda) = I(\lambda, \nu)/I(2\nu, \nu) \), \( I(\lambda, \nu) = \prod_{i,j} B\left(\frac{\lambda_{i} - \lambda_{j}}{2}, \nu\right) \) (\( B(x, y) \) is the beta function).

\[
\begin{align*}
3 \prod_{i,j} \langle \lambda, e_{i} - e_{j} \rangle x_{i} \Phi_{i}^{(\nu)} & = \sum_{k=1}^{3} \prod_{i,j} \langle \lambda + \nu \sigma_{k}, e_{i} - e_{j} \rangle \Phi_{k}^{(\nu)}_{i} \sigma_{k} \\
3 \prod_{i,j} \langle \lambda, e_{i} - e_{j} \rangle x_{j} \Phi_{j}^{(\nu)} & = \sum_{k=1}^{3} \prod_{i,j} \langle \lambda - \nu \sigma_{k}, e_{i} - e_{j} \rangle \Phi_{i}^{(\nu)}_{k} \sigma_{k}
\end{align*}
\]

(*)

Now, we consider an integral representation of a solution of the system \( \mathcal{M}_{1}^{(\nu)} \).

**Theorem 3.** Set

\[
\begin{align*}
G_{s}(x_{1}, x_{2}; u_{1}, u_{2}) & = \int_{0}^{\infty} u_{0}^{s-1}(P_{1}P_{2}P_{3})^{-r}du_{0} \text{ for } \text{Re} \nu > 0, \\
\phi_{P_{1}P_{2}}^{(\nu)}(x_{1}, x_{2}) & = c_{s}(P_{1}, P_{2}) \int_{0}^{\infty} \int_{0}^{\infty} u_{1}^{P_{1}-1}u_{2}^{P_{2}-1}G_{s}(x_{1}, x_{2}; u_{1}, u_{2})du_{1}du_{2}
\end{align*}
\]

for \( 0 < \text{Re} p_{i} < 2\nu (i = 1, 2) \), where

\[
P_{i} = u_{i} + (1 + u_{i}e^{iM}) (1 + u_{2}e^{i(-M)})
\]

\[
c_{s}(P_{1}, P_{2}) = \frac{1}{B(P_{1}, 2\nu - p_{1})B(P_{2}, 2\nu - p_{2})B(\nu, 2\nu)}.
\]

Then \( \phi_{P_{1}P_{2}}^{(\nu)}(x) \) has following properties.

(1) \( \phi_{P_{1}P_{2}}^{(\nu)} \) is a solution of the system \( \mathcal{M}_{1}^{(\nu)} \), where

\[
p_{1} = \frac{\lambda_{2} - \lambda_{1} + 2\nu}{2}, \quad p_{2} = \frac{\lambda_{3} - \lambda_{2} + 2\nu}{2}.
\]

(2) \( \phi_{P_{1}P_{2}}^{(\nu)}(1, 1) = 1. \)
(3) (Generating function.)

\[ G_\omega(x, u) = \sum_{m,n=1}^{\infty} \frac{(2\nu, m) (2\nu, n)}{m! n!} (-1)^{m+n} \varphi_{m,n}^{(\omega)}(x) u_1^m u_2^n. \]

(4) (Functional equation.)

\[ \varphi_{\nu-\nu', \rho_1+\rho_2-\tau}^{(\omega)} = \varphi_{\rho_1+\rho_2-\nu-\nu'}^{(\omega)} = \varphi_{\rho_1, \rho_2}^{(\omega)}. \]

(5) There is a relation between \( \varphi_{\rho_1, \rho_2}^{(\omega)} \) and \( \Phi_{\mu}^{(\omega)}, s \in \mathbb{S}_k. \)

\[ \varphi_{\rho_1, \rho_2}^{(\omega)}(x) = \sum_{s \in \mathbb{S}_k} \Phi_{\mu}^{(\omega)}(H) \text{ for } H \in \mathbb{C}. \]

(6) The recursion formulas (*) are also valid if we change \( \Phi_{\mu}^{(\omega)} \) by \( \varphi_{\rho_1, \rho_2}^{(\omega)}. \)

The function \( \varphi_{\rho_1, \rho_2}^{(\omega)}(x) \) is an analogue of Gegenbauer's function in two variables. \( \varphi_{\rho_1, \rho_2}^{(\omega)} \) satisfies many interesting properties including (1) \( \sim (6) \) in Theorem 3. And if we substitute \( \nu \) for \( \frac{1}{2}, 1, 2 \), \( \varphi_{\rho_1, \rho_2}^{(\omega)} \) is a zonal spherical function on \( \text{SL}(3, \mathbb{R})/\text{SO}(3), \text{SL}(3, \mathbb{C})/\text{SU}(3), \text{SL}(3, \mathbb{H})/\text{Sp}(3) \), respectively.

Remark. After the work, I knew that Prof. Koornwinder ([3]) has obtained the differential operators \( \mathcal{A}_2^{(\omega)}, \mathcal{A}_3^{(\omega)} \), and investigated the orthogonal polynomials \( \varphi_{\omega m}^{(\omega)}(x) \) \( (m, n \in \mathbb{N}) \).

References


