On a Non-Linear Semi-Group Attached to Stochastic Optimal Control

By

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§ 1. Introduction

In [6] we introduced a non-linear semi-group attached to the stochastic control of diffusion type, by the following way. Let $\Gamma$ be a $\sigma$-compact subset of $\mathbb{R}^k$, called by a control region. Let a triple $(\mathcal{Q}, B, U)$ be an admissible system where $\mathcal{Q}$ is a probability space, $B$ is an $n$-dimensional Brownian motion on $\mathcal{Q}$ and $U$ is a $\Gamma$-valued $\mathcal{B}$-non-anticipative process on $\mathcal{Q}$. For an admissible system $(\mathcal{Q}, B, U)$ we consider the following $n$-dimensional stochastic differential equation

$$dX(t) = \alpha(X(t), U(t)) dB(t) + \gamma(X(t), U(t)) dt$$

where $\alpha(x, u)$ is a symmetric $n \times n$-matrix and $\gamma(x, u)$ an $n$-vector. Under the condition of smoothness and boundness of the coefficients $\alpha$ and $\gamma$, there exists a unique solution $X$, which is called the response for $U$.

By $C$ we denote the Banach lattice of all bounded and uniformly continuous functions on $\mathbb{R}^n$ endowed with the usual supremum norm and the usual order. Let $c(x, u)$ be non-negative and $f(x, u)$ real. We assume that both $c$ and $f$ are smooth and bounded. For any $\phi \in C$ we define $Q_t$ by

$$Q_t \phi(x) = \sup_{\text{admiss. syst.}} E_x \left[ \exp \left\{ - \int_0^t c(X(\theta), U(\theta)) d\theta \right\} \times f(X(s), U(s)) ds + \exp \left\{ - \int_0^t c(X(\theta), U(\theta)) d\theta \right\} \phi(X(t)) \right],$$

where $X$ is the response for $U$, starting at $X(0) = x$. Then $Q_t$ is a strongly continuous non-linear semi-group on $C$, which is contractive and


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monotone. Moreover the generator $G$ of $Q_t$ is given by

\begin{align}
G\phi &= \sup_{u \in \Gamma} [A_u\phi + f^u] \\
&= \sup_{u \in \Gamma} [A_u\phi + f^u] \\
&= \frac{1}{2} \sum_{ij} \alpha^2(x, u) \frac{\partial^2 \phi}{\partial x_i \partial x_j}(x) \\
&+ \sum_i r_i(x, u) \frac{\partial \phi}{\partial x_i}(x) - c(x, u) \phi(x)
\end{align}

for $\phi$ whose first and second derivatives are in $C$. The right side of (3) can be found in the famous Bellman equation, [2], [4]. Furthermore the least $Q_t$-excessive majorant has a close relation to the optimal stopping problem, [3], [4].

In this note we shall discuss a similar problem in a more general set-up. Let $A^u$ be the generator of a Markov process. We seek a semi-group of operators acting on $L^0(\mathbb{R}^n, \mu)$ whose generator is an extension of $G\phi = \sup_u (A^u\phi + f^u)$. Such a semi-group (with generator $G$) will be obtained as the envelope of the semi-groups

$$T^u_t \phi = P^u_t \phi + \int_0^t P^u_s f^u \, ds \quad u \in \Gamma$$

whose generators are

$$G^u\phi = A^u\phi + f^u, \quad u \in \Gamma$$

respectively, as we can image from the fact that $G$ is the envelope of $G^u, u \in \Gamma$. In fact we will prove the following theorem in §3.

**Theorem 1.** Let $A^u$ be the generator of positive contractive and strongly continuous linear semi-group $P^u_t$ on $L_\infty(\mathbb{R}^n, \mu)$. We assume the following conditions (A1)~(A3).

(A1) If $\phi_n \in L_\infty(\mathbb{R}^n, \mu)$ is an increasing sequence tending to $\phi \in L_\infty(\mathbb{R}^n, \mu)$ $\mu$-a.e., then $P^u_t \phi_n$ increases and tends to $P^u_t \phi$ $\mu$-a.e. for every $u \in \Gamma$ and every $t \geq 0$.

(A2) Let $D(A^u)$ denote the domain of the generator $A^u$. The subset $D$ of $L_\infty(\mathbb{R}^n, \mu)$ defined by

$$D = \{ \phi \in \cap_u D(A^u) ; \sup_u \| A^u \phi \| < \infty \}$$
is strongly dense in \( L_\infty(\mathbb{R}^n, \mu) \).

\[(A3) \quad \sup_u \|f^u\| < \infty.\]

Then there exists a unique non-linear semi-group \( S_t \) on \( L_\infty(\mathbb{R}^n, \mu) \) satisfying the following conditions \((0) \sim (vi)\):

\begin{enumerate}
  \item \textbf{semi-group property:} \( S_0 = \text{identity}, \ S_{t+s}\phi = S_t(S_s\phi) = S_s(S_t\phi) \),
  \item \textbf{monotone:} \( S_t\phi \leq S_{t'}\phi \), whenever \( \phi \leq \psi \),
  \item \textbf{contractive:} \( \|S_t\phi - S_{t'}\phi\| \leq \|\phi - \psi\| \),
  \item \textbf{strongly continuous:} \( \|S_t\phi - S_{t'}\phi\| \to 0 \), as \( t \to 0 \),
  \item \( P_t\phi + \int_0^t P_s f^u d\theta \leq S_t\phi \), for \( \forall t \) and \( u \), where the integral stands for the Bochner integral,
  \item \textbf{the generator} \( G \) of \( S_t \) is expressed by
    \[(5) \quad G\phi = \sup_u [A^u \phi + f^u] \quad \text{for} \quad \phi \in D(G) \cap D , \]
  \item \textbf{minimum:} if \( \tilde{S}_t \) is a non-linear semi-group with \((i) \sim (iv)\), then \( S_t\phi \leq \tilde{S}_t\phi \).
\end{enumerate}

In § 4, we shall show the existence of the least \( S_t \)-excessive function.

\textbf{Theorem 2.} Suppose that there exists a positive \( c \) such that \( |P_t u| \leq e^{-ct} \) for any \( u \). Then, for any \( g \in L_\infty(\mathbb{R}^n, \mu) \), there exists a unique \( v \in L_\infty(\mathbb{R}^n, \mu) \) such that

\begin{enumerate}
  \item \( S_t \)-excessive majorant of \( g \): \( g \leq v \) and \( S_t v \leq v \ \forall t \geq 0 \)
  \item least: if \( V \) is an \( S_t \)-excessive majorant of \( g \), then \( v \leq V \).
\end{enumerate}

In § 5 we will mention two simple examples as applications of our results. Since we formulate control problems in terms of non-linear semi-groups on \( L_\infty(\mathbb{R}^n, \mu) \) in this note, the stochastic control of diffusion type does not lie in our framework, but some optimal controls can be treated in our way, as we shall see in § 5.

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§ 2. Preliminaries

Let \( \mu \) be a \( \sigma \)-finite measure on \( \mathbb{R}^n \). Let \( L(=L_\infty(\mathbb{R}^n, \mu)) \) denote the set of all Borel measurable, essential bounded functions, defined \( \mu - a.e. \) on \( \mathbb{R}^n \). \( L \) becomes a complete Banach lattice by the usual norm and partial order, [cf. 7], i.e.

\[
\| \phi \| = \text{ess} \sup_{x \in \mathbb{R}^n} |\phi(x)|
\]

and \( \phi \leq \psi \) is defined by \( \phi(x) \leq \psi(x), \mu-a.e. \). A subset \( \{ \phi_a \} \) of \( L \) is said to be \( \Omega \)-bounded, if there exist \( \psi \) and \( \phi \) in \( L \) such that

\[
\psi \leq \phi_a \leq \phi, \quad \forall \alpha.
\]

Hence a subset \( \{ \phi_a \} \) of \( L \) is \( \Omega \)-bounded, if and only if \( \sup \|\phi_a\| < \infty \). When \( \psi_n \in L \) increasingly tends to \( \psi \in L \), we say \( \psi = \Omega \lim \psi_n. \) Hence, if \( \psi = \Omega \lim \phi_n \), then \( \sup \|\phi_n\| < \infty \). In this note we often use the following well-known facts,

**Proposition 1.** For any \( \Omega \)-bounded set \( \{ \phi_a \} \) of \( L \) there uniquely exist \( \psi^+ \) and \( \psi^- \) in \( L \) such that

(i) \( \psi_a \leq \psi^+, \quad \forall \alpha \)

(ii) if \( \psi \) satisfies \( \psi_a \leq \psi, \quad \forall \alpha \), then \( \psi \leq \psi^+ \),

and

(i)' \( \psi^+ \leq \psi_a, \quad \forall \alpha \)

(ii)' if \( \psi \) satisfies \( \psi \leq \psi_a, \quad \forall \alpha \), then \( \psi \leq \psi^- \),

\( \sup \phi_a \) and \( \inf \phi_a \) are denoted by \( \psi^+ \) and \( \psi^- \) respectively.

Moreover,

\[
\inf(\phi_a - \psi_a) \leq \sup \phi_a - \sup \psi_a \leq \sup(\phi_a - \psi_a).
\]

\[
\| \sup \phi_a - \sup \psi_a \| \leq \sup \| \phi_a - \psi_a \|.
\]

Let \( T_t \phi \) be strongly continuous in \( t \). Then \( T_t \phi \) has a \((t, x)\)-Borel measurable version which is continuous in \( t \).

**Proof:** Let \( \{ r_i \} \) be countable and dense in \([0, \infty)\) and \( \Phi(r_i, \cdot) \) a Borel measurable version of \( T_{r_i} \phi \). Then the set \( \Sigma \) of \( \{ x \in \mathbb{R}^n; \| \Phi(r_i, x) - \Phi(r_j, x) \| \leq \| T_{r_i} \phi - T_{r_j} \phi \| \ \forall i, j \} \) is \( \mu \)-full. On the other hand, for any positives \( \varepsilon \) and \( l \), there exists a positive \( \delta \) such that
\[ \|T_t \phi - T_\theta \phi\| < \varepsilon \quad \text{whenever} \quad |t-\theta| < \delta \quad \text{and} \quad 0 \leq t, \theta \leq l. \]

Hence, for \( x \in \Sigma, \Phi(r_t, x) \) is uniformly continuous on \( \{r_t\} \subset [0, l] \). Thus, \( \Phi(\cdot, x) \) can be extended to a continuous function \( \bar{\Phi}(\cdot, x) \) on \([0, l]\). Letting \( l \) tend to \( \infty \), we get our wanted version \( \bar{\Phi} \).

The Bochner integral \( \int_0^t T_\phi d\theta \) can understood as the usual Riemann integral \( \int_0^t \bar{\Phi}(\theta, x) d\theta \).

Let \( P_t \) be a positive, contractive and strongly continuous linear semi-group on \( L \). Define \( T_t \) for \( f \in L \) by

\[
(1) \quad T_t \phi = P_t \phi + \int_0^t P_s f d\theta, \quad \phi \in L.
\]

Then \( T_t \) is a mapping from \( L \) into \( L \) and has the following properties

\( (T0) \) semi-group property: \( T_t \phi = \phi, \quad T_{t+s} \phi = T_t (T_s \phi) = T_s (T_t \phi) \),

\( (T1) \) monotone: \( T_t \phi \leq T_s \phi \) whenever \( \phi \leq \psi \),

\( (T2) \) contractive: \( \|T_t \phi - T_s \phi\| \leq \|\phi - \psi\| \),

\( (T3) \) strongly continuous: \( \|T_t \phi - T_s \phi\| \to 0 \) as \( t \to \theta \)

\( (T4) \) the generator \( G \) of \( T_t \): Let \( A \) be the generator of \( P_t \). Then \( D(G) = D(A) \) and

\[
(2) \quad G \phi = A \phi + f
\]

\( (T5) \)

\[
T_t \phi - \phi = \int_0^t P_s G \phi d\theta \quad \forall \phi \in D(G).
\]

**Proof.** Since \( (T1), (T2) \) and \( (T3) \) are obvious, we shall only show \( (T0), (T4) \) and \( (T5) \).

\( (T0) \) \( T_{t+s} \phi = P_{t+s} \phi + \int_0^{t+s} P_s f d\theta = P_{t+s} (P_s \phi) + \int_0^t P_s f d\theta + \int_t^{t+s} P_s f d\theta \\
= P_{s} (P_t \phi + \int_0^t P_s f d\theta) + \int_0^t P_s f d\theta = P_{t} (T_s \phi) + \int_0^t P_s f d\theta = T_s (T_t \phi). \)

\( (T4) \) For \( \varepsilon > 0 \), there exists a positive \( \delta \) such that \( \|P_{s} f - f\| < \varepsilon \) for \( \theta < \delta \). Hence

\[
\left| \frac{1}{t} \int_0^t P_s f d\theta - f \right| = \left| \frac{1}{t} \int_0^t (P_s f - f) d\theta \right| \leq \frac{1}{t} \int_0^t \|P_s f - f\| d\theta < \varepsilon \quad \text{for} \quad t < \delta.
\]
Therefore \( \lim_{t \to 0} \frac{1}{t} (T_t \phi - \phi) \) exists if and only if \( \lim_{t \to 0} \frac{1}{t} (P_t \phi - \phi) \) exists. Moreover (2) is valid.

\[(T5) \quad \text{For any } \phi \in D(A), \text{ we have}
\]

\[
T_t \phi - \phi = P_t \phi - \phi + \int_0^t P_s f d\theta
= \int_0^t P_s A \phi d\theta + \int_0^t P_s f d\theta = \int_0^t P_s (A \phi + f) d\theta.
\]

**Proposition 2.** Suppose (A1) and (A3). If \( \phi = O_t - \lim \phi_n \) then

\[
(3) \quad \sup_u T_t^u \phi = O_t - \limsup_n T_t^u \phi_n.
\]

**Proof.** Since \( T_t^u \) satisfies (T1) and (T2), we have \( T_t^u \phi_n \leq T_t^u \phi_{n+1} \) and

\[
(4) \quad \|T_t^u \phi_n\| \leq \|T_t^u \phi_n - T_t^u O\| + \|T_t^u O\| \leq \|\phi_n\| + \sup \|f\| t.
\]

Thus \( \sup_n T_t^u \phi_n \) is increasing as \( n \to \infty \) and the set \( \{\sup_n T_t^u \phi_n, n = 1, 2, \ldots\} \) is \( O \)-bounded. Therefore

\[
(5) \quad O_t - \limsup_n T_t^u \phi_n \leq \sup_n T_t^u \phi.
\]

On the other hand, from (A1) we can derive, for any \( u \)

\[
(6) \quad T_t^u \phi = O_t - \lim_n T_t^u \phi_n \leq O_t - \limsup_n T_t^u \phi_n.
\]

By (5) and (6) we conclude Proposition 2.

**§ 3. Proof of Theorem 1**

We shall construct our required semi-group \( S_t \). Define \( J = J(N) \) by

\[
(1) \quad J \phi = \sup_u T_t^u \phi, \quad \phi \in L.
\]

Then \( J \) is a mapping from \( L \) into \( L \). Define \( J^k \) by

\[
J^{k+1} \phi = J(J^k \phi) \quad \text{and} \quad J^1 \phi = \phi.
\]

**Lemma 1.** \( J^k \) has the following properties,

\[
(\text{J0}) \quad J^{k+1} \phi = J^k (J^1 \phi) = J^k (J^1 \phi),
(\text{J1}) \quad \text{monotone: } J^k \phi \leq J^k \psi \text{ whenever } \phi \leq \psi,
\]
Proof. Since $T_t^u$ is monotone, we have
\[ J\phi \leq J^\phi \] whenever $\phi \leq \psi$.

Hence we can show (J1) by induction.

Put $A = \frac{1}{2^N}$. The following evaluation is clear,
\[ \|J^\phi - J^\psi\| = \sup_u T^u_\phi - \sup_u T^u_\psi \leq \sup_u \|T^u_\phi - T^u_\psi\| \leq \|\phi - \psi\|. \]

Thus if we assume that (J2) holds for $k$, then
\[ \|J^{k+1}\phi - J^{k+1}\psi\| = \|J(J^k\phi) - J(J^k\psi)\| \leq \|J^k\phi - J^k\psi\| \leq \|\phi - \psi\| \]

namely (J2) holds for $k+1$.

Put $K(\phi) = \sup_u \|A^\phi\| + \sup_u \|f^\phi\|$. Recalling (T5) we have, for $\phi \in D$
\[ T^u_\phi - \phi = \int_0^d P^u_\phi A^\phi d\theta + \int_0^d P^u_\phi f^\phi d\theta. \]

So
\[ \|J^\phi - J^\psi\| \leq \sup_u \|T^u_\phi - \phi\| \leq \Delta K(\phi). \]

Therefore by (J2) we see
\[ \|J^k\phi - \phi\| = \sum_{j=1}^k \|J^j\phi - J^{j-1}\phi\| = \sum_{j=1}^k \|J^{j-1}(J^\phi) - J^{j-1}\phi\| \leq k \|J^\phi - \phi\| \leq kA \cdot K(\phi). \]

This completes the proof of (J3).

By the definition of $J$ we get
\[ T^u_\psi \leq J^\psi \quad \forall \psi \in L. \]

Hence, if we assume that (J4) holds for $k$, then
\[ T^u_{(k+1)}\phi = T^u_{(k+1)}(T^u_k\phi) \leq T^u_k(J^\phi) \leq J(J^\phi) = J^{k+1}\phi, \]
namely (J4) holds for \( k + 1 \).

For \( k = 1 \), (J5) is Proposition 2 in § 2. If (J5) holds for \( k \), then

\[
J^{k+1} \phi = J(J^k \phi) = J(O_l - \operatorname{lim} J^k \phi_n) = O_l - \operatorname{lim} J(J^k \phi_n) = O_l - \operatorname{lim} J^{k+1} \phi_n.
\]

Therefore we get (J5).

Put \( S^{(N)}_t \phi = J^k (N) \phi \) for \( t = \frac{k}{2^n} \), \( k = 0, 1, 2, \ldots \).

**Lemma 2.** \( S^{(N)}_t \phi \) is increasing as \( N \to \infty \), i.e.

\[
(2) \quad S^{(N)}_t \phi \leq S^{(N+1)}_t \phi \quad \text{for} \quad t = \frac{k}{2^n}.
\]

**Proof.** Put \( \Delta = 1/2^{n+1} \). Recalling (T0) and (T1), we have

\[
(3) \quad T^u_d \phi = T^u_d (T^u_d \phi) \leq T^u_d (S^{(N+1)}_d \phi).
\]

Taking the supremum of both sides, we get

\[
(4) \quad S^{(N)}_d \phi \leq S^{(N+1)}_d (S^{(N+1)}_d \phi) = S^{(N+1)}_d \phi,
\]

namely (2) is valid for \( k = 1 \). If (2) holds for \( k \), then

\[
(5) \quad S^{(N)}_{d(k+1)} \phi = S^{(N)}_d (S^{(N)}_d \phi) \leq S^{(N)}_d (S^{(N+1)}_d \phi)
\]

\[
\leq S^{(N+1)}_d (S^{(N+1)}_d \phi) = S^{(N+1)}_d \phi.
\]

This completes the proof of Lemma 2.

Hereafter we put \( h = \sup\|f^u\| \). By virtue of (J2), putting \( \Delta = \frac{1}{2^n} \) and \( t = k \Delta \) we have

\[
(6) \quad \| S^{(N)}_t \phi \| \leq \| S^{(N)}_t \phi - S^{(N)}_t O \| + \| S^{(N)}_t O \| \leq \| \phi \| + \| S^{(N)}_t O \|
\]

and

\[
\| S^{(N)}_d O \| \leq \sup\| T^u_d (S^{(N)}_d O) \| \leq \Delta h.
\]

Suppose \( \| S^{(N)}_d O \| \leq k \Delta h \). Then

\[
(7) \quad \| S^{(N+1)}_d O \| = \| S^{(N)}_d (S^{(N)}_d O) \| \leq \sup\| T^u_d (S^{(N)}_d O) \|
\]

\[
\leq \| S^{(N)}_d O \| + \Delta h \leq (k + 1) \Delta h.
\]

Hence we have
This implies that, for any fixed binary \( t = \frac{j}{2^i} \), the set \( \{ S_i^{(N)} \phi, N \geq 1 \} \) is \( O \)-bounded. So we can define \( S_t \) by

\[
S_t \phi = O_t - \lim_{n \to \infty} S_i^{(n)} \phi \quad \text{for binary } t.
\]

\( S_t \) has the following properties:

**Lemma 3.** For binary \( t \) and \( \theta \),

(S0) \( S_0 \phi = \phi \),

(S1) monotone: \( S_t \phi \leq S_t \psi \), whenever \( \phi \leq \psi \),

(S2) contractive: \( \| S_t \phi - S_t \psi \| \leq \| \phi - \psi \| \)

(S3) \( \| S_t \phi - S_t \psi \| \leq \| t - \theta \| K(\phi) \) for \( \phi \in D \),

(S4) \( T_i^n \phi \leq S_t \phi \).

**Proof:** From the definition of \( S_t \) and Lemma 1, these properties are clear. We shall only show (S3). Put \( t = \frac{i}{2^j} \) and \( \theta = \frac{j}{2^j}, \ (j \leq i) \). For any \( N \geq 1 \), we have

\[
\| S_i^{(N)} \phi - S_i^{(N)} \psi \| = \| S_i^{(N)} (S_i^{(N)} \psi) - S_i^{(N)} \phi \| \leq \| S_i^{(N)} \psi - \phi \| \leq |t - \theta| K(\phi).
\]

Since \( S_i^{(N)} \phi - S_i^{(N)} \psi \) converges to \( S_t \phi - S_t \psi \) \( \mu \)-a.e. as \( N \to \infty \), we get

\[
\| S_t \phi - S_t \psi \| \leq \lim_{N \to \infty} \| S_i^{(N)} \phi - S_i^{(N)} \psi \| \leq |t - \theta| K(\phi).
\]

Using (S3) we can define \( S_t \phi, t \geq 0 \), by

\[
S_t \phi = \lim_{n \to \infty} S_i^{(n)} \phi, \ \phi \in D,
\]

where \( \{ t_i \} \) is a sequence of binary times approximating \( t \). (S3) implies that the left side of (10) does not depend on the special choice of \( \{ t_i \} \). Moreover (S1) \( \sim \) (S4) hold.

**Lemma 3'.** For \( \theta, t \geq 0 \) and \( \phi, \psi \in D \),

(S1)' monotone: \( S_t \phi \leq S_t \psi \) whenever \( \phi \leq \psi \),

(S2)' contractive: \( \| S_t \phi - S_t \psi \| \leq \| \phi - \psi \| \),

(S3)' \( \| S_t \phi - S_t \psi \| \leq |t - \theta| K(\phi) \),

(S4)' \( T_i^n \phi \leq S_t \phi \).
Recalling (A2) and (S2)', we can extend $S_t$ on $L$ by

(11) \[ S_t \phi = \lim S_t \phi_n, \quad \phi \in L, \]

where \( \{ \phi_n \} \) is a sequence of functions in $D$ approximating $\phi$.

**Proposition 3.** $S_t$ has the following properties

(i) monotone: $S_t \phi \leq S_t \psi$ whenever $\phi \leq \psi$,

(ii) contractive: $\| S_t \phi - S_t \psi \| \leq \| \phi - \psi \|$, 

(iii) strongly continuous: $\| S_t \phi - S_t \psi \| \to 0$ as $t \to 0$,

(iv) $T_t^n \phi \leq S_t \phi$.

**Proof:** First we shall show (ii). Take $\phi_n \in D$ and $\psi_n \in D$ approximating $\phi$ and $\psi$ respectively. Hence

$$
\| S_t \phi - S_t \psi \| \leq \lim_n \| S_t \phi_n - S_t \psi_n \| \leq \lim_n \| \phi_n - \psi_n \| = \| \phi - \psi \|.
$$

(i). For $\varepsilon > 0$, we take an approximation $\phi_n(\varepsilon) \in D$ to $\phi - \varepsilon$. Let $\psi_n \in D$ approximate $\psi$. Then, for large $n$,

$$
\phi_n(\varepsilon) \leq \psi_n.
$$

Hence, by (S1)',

$$
S_t \phi_n(\varepsilon) \leq S_t \psi_n \quad \text{for large } n.
$$

Therefore tending $n$ to $\infty$ we have

$$
S_t (\phi - \varepsilon) \leq S_t \phi.
$$

On the other hand $\phi - \varepsilon$ converges to $\phi$, so (ii) implies $S_t \phi \equiv \lim_{t \to 0} S_t (\phi - \varepsilon)$. Hence

$$
S_t \phi \leq S_t \psi.
$$

(iii). For $\varepsilon > 0$, we take $\psi \in D$ such that $\| \phi - \psi \| < \varepsilon$. Then we have

$$
\| S_t \phi - S_t \psi \| \leq \| S_t \phi - S_t \psi \| + \| S_t \phi - S_t \psi \| + \| S_t \psi - S_t \phi \|
$$

$$
< 2\varepsilon + \| S_t \psi - S_t \phi \| \leq 2\varepsilon + |t - \theta| K(\psi).
$$

Hence there exists a small positive $\delta = \delta(\phi, \varepsilon)$ such that $\| S_t \phi - S_t \psi \| < 3\varepsilon$ whenever $|t - \theta| < \delta$.

(iv). By (S4)' we have $T_t^n \phi_n \leq S_t \phi_n$ where $\phi_n \in D$ tends to $\phi$. Let-
Proposition 4. \( S_t \) is a semi-group on \( L \).

Proof. Let \( t \) and \( \theta \) be binary, say \( t = \frac{i}{2^i} \) and \( \theta = \frac{j}{2^j} \). For \( N \geq l \), we have

\[
S_{t,N}^N \phi = S_{t}^{(N)}(S_{t}^{(N)} \phi) \leq S_{t}^{(N)}(S_{t} \phi),
\]
(12) \[
S_{\theta}(S_{t} \phi) = O_{t} - \lim_{N \to \infty} S_{\theta,N}^{N}(S_{t} \phi),
\]
(13) and

\[
S_{\theta,t} \phi = O_{t} - \lim_{N \to \infty} S_{\theta,N}^{N} \phi.
\]
(14) Hence

\[
S_{\theta,t} \phi \leq O_{t} - \lim_{N \to \infty} S_{\theta,N}^{N}(S_{t} \phi) = S_{\theta}(S_{t} \phi).
\]
(15) On the other hand, for \( 1 \leq n \leq N \), we see

\[
S_{\theta}^{(n)}(S_{t}^{(n)} \phi) \leq S_{\theta}^{(n)}(S_{t}^{(n)} \phi) = S_{\theta,t}^{(n)} \phi \leq S_{\theta,t} \phi
\]
and recalling (J5) of Lemma 1 we have

\[
S_{\theta}^{(n)}(S_{t} \phi) = O_{t} - \lim_{N \to \infty} S_{\theta}^{(n)}(S_{t}^{(n)} \phi).
\]
Therefore, for \( n \geq l \),

\[
S_{\theta}^{(n)}(S_{t} \phi) \leq S_{\theta,t} \phi.
\]
(16) Tending \( n \) to \( \infty \), we get

\[
S_{\theta}(S_{t} \phi) \leq S_{\theta,t} \phi.
\]
(17) From (15) and (16) we have

\[
S_{\theta}(S_{t} \phi) = S_{\theta,t} \phi \quad \text{for binary} \ t \text{ and} \ \theta.
\]

Let \( t_n \) be a binary approximation to \( t \). Then for any binary \( \theta \),

\[
S_{\theta}(S_{t_n} \phi) = S_{\theta,t_n} \phi.
\]
So appealing to (ii) and (iii) we get

\[
S_{\theta}(S_{t} \phi) = S_{\theta,t} \phi \quad \text{for binary} \ \theta.
\]
(18)
Again by virtue of (iii) we obtain the semi-group property of $S_t$.

Let $G$ be the generator of $S_t$, namely

$$G\phi = \lim_{t \to 0} \frac{1}{t} (S_t\phi - \phi)$$

and

$$D(G) = \left\{ \phi \in L, \lim_{t \to 0} \frac{1}{t} (S_t\phi - \phi) \text{ exists} \right\}.$$

**Proposition 5.**

(19) \(G\phi = \sup_u (A^n\phi + f^n) \quad \text{for} \quad \phi \in D(G) \cap D.\)

Moreover, if $f^n \in D(A^n)$ and $\sup_u \|A^n f^n\| < \infty$, then

(20) \(D(G) \supseteq \{ \phi \in D, A^n\phi \in D(A^n) \quad \text{for} \quad \forall u \) and $\sup_u \|A^n(A^n\phi)\| < \infty$, (say $\Theta$).

**Proof.** In the case $f^n = 0$ for any $u$, we denote $S_t$ by $A_t$. Put

$$A\phi = \sup_u G^n\phi = \sup_u (A^n\phi + f^n) \quad \text{and} \quad A = \frac{1}{2^n}. \quad \text{Recalling (T5) we have for} \quad \phi \in D$$

(21) \(S_t A^n \phi - \phi = \sup_u (T_{d} A^n \phi - \phi) = \sup_u \int_0^1 P_{s} A^n \phi d\theta \)

\[\leq \sup_u \int_0^1 P_{s} A^n \phi d\theta \leq \int_0^1 A_{n} A^n \phi d\theta.\]

Moreover

(22) \(S_t A^n \phi - S_{d} A^n \phi = \sup_u T_{d} (S_t A^n \phi) - \sup_u T_{d} A^n \phi \)

\[\leq \sup_u [T_{d} A^n (S_d A^n \phi) - T_{d} A^n \phi] = \sup_u [P_{d} A^n (S_t A^n \phi) - P_{d} A^n \phi] \]

\[= \sup_u [P_{d} A^n (S_t A^n \phi - \phi)] = A_{d} (S_t A^n \phi - \phi) \]

\[\leq A_{d} \left( \int_0^1 A_{d} A^n \phi d\theta \right) = \int_0^1 A_{d} A_{d} A^n \phi d\theta = \int_0^1 A_{d} A^n \phi d\theta.\]

Suppose $S_{k-1} A^n \phi - S_{k-1} A^n \phi \leq \int_0^1 A_{d} A^n \phi d\theta$. Then, by the similar calcula-
tion, we see

\[ S^{(N)}_{(k+1)\delta} \phi - S^{(N)}_{k\delta} \phi \leq A_{d} \left( S^{(N)}_{k\delta} \phi - S^{(N)}_{(k-1)\delta} \phi \right) \leq \int_{k\delta}^{(k+1)\delta} A_{\theta} A_{\phi} d\theta. \]

Hence taking the summation for \( k \) we get

\[ S^{(N)}_{t} \phi - \phi \leq \int_{0}^{t} A_{\theta} A_{\phi} d\theta \quad \text{for} \quad t = \frac{i}{2^{N}}. \]

Tending \( N \) to \( \infty \) we have

\[ S_{t}\phi - \phi \leq \int_{0}^{t} A_{\theta} A_{\phi} d\theta \quad \text{for binary} \ t \ \text{and} \ \phi \in D. \]

Since the both sides of (24) are continuous in \( t \), (24) holds for any \( t \geq 0 \). Furthermore

\[ \frac{1}{t} (S_{t}\phi - \phi) \leq \frac{1}{t} \int_{0}^{t} A_{\theta} A_{\phi} d\theta \leq \|A_{\phi}\|1, \]

where 1 is the unit in \( L \). On the other hand, by virtue of (T5) and (iv) of Proposition 3, we have

\[ \frac{1}{t} (S_{t}\phi - \phi) \geq \frac{1}{t} (T_{t}^{*} \phi - \phi) = \frac{1}{t} \int_{0}^{t} P_{\theta}^{*} G^{*} \phi d\theta \geq -\|G^{*}\|1. \]

Therefore the set \( \left\{ \frac{1}{t} (S_{t}\phi - \phi), t > 0 \right\} \) is \( O \)-bounded. Hence \( \inf_{\theta > 0} \sup_{t > 0} \frac{1}{t} (S_{t}\phi - \phi) \) exists, and \( \sup_{\theta > 0} \inf_{t > 0} \frac{1}{t} (S_{t}\phi - \phi) \), i.e.

\[ O \lim_{t \downarrow 0} \frac{1}{t} (S_{t}\phi - \phi) \), exists. Since

\[ \lim_{t \downarrow 0} \frac{1}{t} \int_{0}^{t} A_{\theta} A_{\phi} d\theta = A_{\phi}, \]

and

\[ \lim_{t \downarrow 0} \frac{1}{t} \int_{0}^{t} P_{\theta}^{*} G^{*} \phi d\theta = G^{*} \phi, \]

we have by (25), (26), (27) and (28),

\[ O \lim_{t \downarrow 0} \frac{1}{t} (S_{t}\phi - \phi) \leq A_{\phi} \]

and

\[ O \lim_{t \downarrow 0} \frac{1}{t} (S_{t}\phi - \phi) \geq G^{*} \phi \quad \forall \mu. \]
Hence

\[
O - \lim_{t \downarrow 0} \frac{1}{t} (S_t \phi - \phi) \geq \sup_u G^u \phi = A \phi.
\]

From (29) and (31) we have

\[
O - \lim_{t \downarrow 0} \frac{1}{t} (S_t \phi - \phi) = O - \lim_{t \downarrow 0} \frac{1}{t} (S_t \phi - \phi) = A \phi.
\]

Thus, for \( \phi \in D(G) \cap D \), we have

\[
G \phi = \lim_{t \downarrow 0} \frac{1}{t} (S_t \phi - \phi) = O - \lim_{t \downarrow 0} \frac{1}{t} (S_t \phi - \phi) = A \phi.
\]

Next we shall show (20). From (25)

\[
\frac{1}{t} (S_t \phi - \phi) - A \phi \leq \frac{1}{t} \int_0^t A \phi d \theta - A \phi.
\]

By (27) the right side converges to 0 as \( t \to 0 \). Hence, for \( \varepsilon > 0 \), there exists a positive \( \delta = \delta(\varepsilon) \), such that

\[
\text{ess. sup.}_x \left[ \frac{1}{t} (S_t \phi - \phi) (x) - A \phi(x) \right] < \varepsilon \quad \text{for} \quad t \in (0, \delta).
\]

On the other hand, by (26) we have

\[
\frac{1}{t} (S_t \phi - \phi) - A \phi \geq \sup_u \frac{1}{t} \int_0^t P_u^* G^u \phi d \theta - A \phi
\]

\[
= \sup_u \frac{1}{t} \int_0^t P_u^* G^u \phi d \theta - \sup_u G^u \phi \geq \inf_u \left[ \frac{1}{t} \int_0^t P_u^* G^u \phi d \theta - G^u \phi \right].
\]

For \( \phi \in \Theta \), we have \( G^u \phi \in D(A^u) \) and

\[
P_u^* G^u \phi - G^u \phi = \int_0^u P_s^* A^u G^u \phi ds.
\]

Thus

\[
\frac{1}{t} \int_0^t P_u^* G^u \phi d \theta - G^u \phi = \frac{1}{t} \int_0^t \left( \int_0^u P_s^* A^u G^u \phi ds \right) d \theta.
\]

So we have

\[
\frac{1}{t} \int_0^t P_u^* G^u \phi d \theta - G^u \phi \leq \|A^u G^u \phi\| t \leq \|A^u (A^u \phi) + A^u f^u\| t.
\]

Therefore by (33) and (34) we have
(35) \[ \text{ess.inf.} \left( \frac{1}{t} (S_t \phi - \phi) - A \phi \right) \geq - \sup_u A^u (A^u \phi) + A^u f^u t. \]

Hence (32) and (35) complete the proof of (20).

Remark 1. If \( S_t \phi \) is differentiable in \( t > 0 \) and \( S_t \phi \) belongs to \( D \), then
\[
\begin{align*}
\frac{d}{dt} S_t \phi &= \sup_u (A^u S_t \phi + f^u), \quad t > 0, \\
S_t \phi &= \phi.
\end{align*}
\]
This is the so-called Bellman equation. So \( S_t \) is called a Bellman semigroup.

Remark 2. If each \( A^u \) is a bounded operator on \( L \) and
\[ \text{sup} \| A^u \| < \infty, \]
then \( \text{sup} \| A^u f^u \| < \infty \) and \( \Theta = L \). Moreover \( S_t \phi \) is differentiable in \( t \) and satisfies the Bellman equation.

Proof. Since \( A^u \) is a bounded linear operator on \( L \),
\[ P_t^u = \sum_{k=0}^\infty \frac{1}{k!} (t A^u)^k = \exp t A^u \]
and \( D(A^u) = L \). Hence \( f^u \in D(A^u) \) and \( \text{sup} \| A^u f^u \| \leq \text{sup} \| A^u \| h < \infty \). Moreover \( \text{sup} \| A^u \phi \| < \infty \), for any \( \phi \in L \). Thus \( D = L \). Since \( \text{sup} \| A^u (A^u \phi) \| \leq (\text{sup} \| A^u \|)^2 \| \phi \| \), we have \( \Theta = L \).

For the proof of the latter half, we apply the same method as for linear semi-groups. Since \( D(G) \ni \Theta = L \), the right derivative of \( S_t \phi \),
\[ \frac{d^+}{dt} S_t \phi = \lim_{t \to 0} \frac{1}{t} (S_{t + \theta} \phi - S_t \phi) \]
exists and, by \( \Theta = L \),
\[ \frac{d^+}{dt} S_t \phi = \sup_u (A^u S_t \phi + f^u) = A S_t \phi. \]
Hence, for any \( F \in L' \), we have
F(AS_\phi) = F(\frac{d^+}{dt} S_t \phi) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (F(S_{t+\varepsilon} \phi) - F(S_t \phi)) = \frac{d^+}{dt} F(S_t \phi).

On the other hand by (36) we get
\[ \|AS_\phi - AS_t \phi\| \leq \sup_u \|A^u S_t \phi - A^u S_\phi\| = \sup_u \|A^u (S_t \phi - S_\phi)\| \leq (\sup_u \|A^u\|) \|S_t \phi - S_\phi\|. \]

Therefore $AS_\phi$ is continuous in $t$. So $F(AS_t \phi)$ is a real continuous function of $t$, namely the right derivative of $F(S_t \phi)$ is continuous. Therefore $F(S_t \phi)$ is differentiable and its derivative $\frac{dF(S_t \phi)}{dt}$ is continuous. Therefore

\begin{equation}
F(S_t \phi - \phi) = F(S_t \phi) - F(\phi) = \int_0^t \frac{d}{d\theta} F(S_t \phi) d\theta = \int_0^t F(AS_\theta \phi) d\theta = F\left( \int_0^t AS_\theta d\theta \right).
\end{equation}

Since $F$ is arbitrary, (37) implies

\begin{equation}
S_t \phi - \phi = \int_0^t AS_\theta \phi d\theta.
\end{equation}

By the continuity of $AS_\phi$, (38) implies the differentiability of $S_t \phi$. Therefore by Remark 1 $S_t \phi$ satisfies the Bellman equation. In fact the operator $S_t$ thus obtained is identical with $e^{tA}$ in the sense of [1].

**Proposition 6.** If $\bar{S}_t$ is a semi-group on $L$ satisfying the condition (i) \sim (iv), then for any $t \geq 0$ and $\phi \in L$,

\[ S_t \phi \leq \bar{S}_t \phi. \]

**Proof.** Putting $A = \frac{1}{2^\gamma}$, we have

\begin{equation}
S^{(\gamma)}_t \phi = \sup_u T^{u\phi} \leq \bar{S}_t \phi, \quad \forall \phi \in L.
\end{equation}

Suppose

\begin{equation}
S^{(\gamma)}_t \phi \leq \bar{S}_t \phi.
\end{equation}

Then

\[ S^{(\gamma)}_{t+k} \phi = S^{(\gamma)}_t (S^{(\gamma)}_k \phi) \leq S^{(\gamma)}_t (\bar{S}_k \phi) \leq \bar{S}_t (\bar{S}_k \phi) = \bar{S}_{t+k} \phi. \]
Hence, for any \( k \), we have (40).

This implies for any binary \( t \)
\[
S_t^{(n)} \phi \leq S_t \phi \quad \text{for large } n.
\]
Therefore for binary \( t \)
\[
S_t \phi = O_t - \lim_{n \to \infty} S_t^{(n)} \phi \leq S_t \phi.
\]

Since the both sides are continuous in \( t \), we complete the proof of Proposition 6.

For any constant \( c \geq 0 \), we replace \( P_t \) by \( e^{-ct} P_t \). Then we can easily show the following,

**Corollary.** *Theorem 1 is still valid, when we replace (iv) and (5) respectively by*

(iii)
\[
e^{-ct} P_t + \int_0^t e^{-c\theta} P_{\theta} \phi \, d\theta \leq S_t \phi,
\]

and

(v)
\[
Gf = \sup_u (A^u \phi - c \phi + f^u), \quad \text{for } \phi \in D(G) \cap D.
\]

For positive \( c \), we denote the semi-group of Corollary by \( \bar{S}_t \).

**Proposition 7.** *There exists a unique \( v \in L \) such that*

\[
\lim_{t \to \infty} \bar{S}_t \phi = v \quad \text{for any } \phi \in L.
\]

**Proof.** Using \( e^{-ct} P_t \) instead of \( P_t \), we define \( \bar{J}(N) \) and \( \bar{S}^{(N)} \) by the similar way. Then, putting \( \bar{J} = \bar{J}(N) \) and \( \bar{J} = \frac{1}{2N} \), we have

\[
\| \bar{J} \phi - \bar{J} \psi \| \leq \sup_u \| e^{-ct} P_u \phi - e^{-ct} P_u \psi \| \leq e^{-ct \| \phi - \psi \|}.
\]

Moreover we can show (41) by the induction,

\[
\| \bar{J}^k \phi - \bar{J}^k \psi \| \leq e^{-ck \| \phi - \psi \|}.
\]

On the other hand we can easily see the following inequality
\[ \| \tilde{J}\phi \| \leq e^{-\varepsilon t} \| \phi \| + \frac{h}{c} (1 - e^{-\varepsilon t}) \]

and moreover, we have (42) by the induction,

\[ \| \tilde{J}^k \phi \| \leq e^{-\varepsilon t_k} \| \phi \| + \frac{h}{c} (1 - e^{-\varepsilon t_k}) . \]

(41) and (42) mean, for \( t = \frac{k}{2^n} \),

\[ \| \tilde{S}_t^{(\varepsilon)} \phi - \tilde{S}_t^{(\varepsilon')} \phi \| \leq e^{-\varepsilon t} \| \phi - \phi \| \]

and

\[ \| \tilde{S}_t^{(\varepsilon)} \phi \| \leq e^{-\varepsilon t} \| \phi \| + \frac{h}{c} (1 - e^{-\varepsilon t}) . \]

Therefore, for binary \( t \), we have (43) and (44),

\[ \| \tilde{S}_t \phi - \tilde{S}_t \phi \| \leq \lim_{N \to \infty} \| \tilde{S}_t^{(\varepsilon_N)} \phi - \tilde{S}_t^{(\varepsilon_N)} \phi \| \leq e^{-\varepsilon t} \| \phi - \phi \| \]

and

\[ \| \tilde{S}_t \phi \| \leq e^{-\varepsilon t} \| \phi \| + \frac{h}{c} (1 - e^{-\varepsilon t}) \leq \| \phi \| + \frac{h}{c} . \]

Since the both sides of the above inequalities (43) and (44) are continuous in \( t \), we have

\[ \| \tilde{S}_{t+} \phi - \tilde{S}_t \phi \| = \| \tilde{S}_t (\tilde{S}_{t+} \phi) - \tilde{S}_t \phi \| \leq e^{-\varepsilon t} \left( \| \phi \| + \frac{h}{c} \right) . \]

Hence there exists \( \lim_{t \to 0} \tilde{S}_t \phi \), say \( \nu_\phi \). By virtue of (43), we can see that \( \nu_\phi \) does not depend on \( \phi \).

**Corollary.** \( \tilde{S}_t \nu = \nu \) for any \( t \geq 0 \), and if \( \nu \) belongs to \( D \), then

\[ \sup_u (A^u \nu - c \nu + f^u) = 0 . \]

**§ 4. Proof of Theorem 2**

For any \( \lambda \geq 0 \) and \( g \in L \) we define

\[ T_t^{\lambda g} \phi = e^{-i \lambda t} P_t^\phi \phi + \int_0^t e^{-i \lambda \theta} P_t^f (f^u + \lambda g) \, d\theta . \]

Then we have

\[ \| T_t^{\lambda g} \phi \| \leq e^{-(1 + \varepsilon t)} \| \phi \| + \frac{h}{c} (1 - e^{-\varepsilon t}) + (1 - e^{-\varepsilon t}) \| g \| . \]
and its generator $G^{u_tg}$ is as follows

$$G^{u_tg}\phi = A^x\phi - \lambda\phi + f^u + \lambda g.$$  

For simplicity we omit $g$ in $T^{u_tg}$ and $G^{u_tg}$ for the moment, if any confusion does not occur. In order to prove Theorem 2, we apply the same method as [4], namely we take $I' \times [0, \infty)$ for the control region. Appealing to (2), we can define $J = J(N)$ by

$$J\phi = \sup_{u_t} T^{u_tg}_{T_t\phi}, \quad \phi \in L$$

and

$$J^{x+1}\phi = J(J^x\phi), \quad J^0\phi = \phi.$$  

Then Lemma 1 is easy.

**Lemma 1.** Putting $A = \frac{1}{2^n}$, we have

(J0) $J^{x+1}\phi = J^x(J^x\phi) = J^x(J^x\phi)$,

(J1) $J^x\phi \leq J^x\psi$ whenever $\phi \leq \psi$,

(J2) $\|J^x\phi - J^x\psi\| \leq e^{-ckx}\|\phi - \psi\|$,  

(J3) $\|J^x\phi\| \leq e^{-ckx}\|\phi\| + \frac{h}{c} (1 - e^{-ckx}) + \|g\|$,  

(J4) $\phi = O_t - \lim_n \phi_n$ implies $J^x\phi = O_t - \lim_n J^x\phi_n$.  

(J5) $g \leq J^x\phi$.

**Proof.** We show (J3) by the induction. For $k = 1$, (J3) comes from (2). Suppose (J3) holds for $k$. Then we have

(3) $\|J^{x+1}\phi\| = \|J(J^x\phi)\| \leq \sup_{u_t} \|T^{u_tg}_t (J^x\phi)\|$.  

Recalling (2) we see

(4) $\|T^{u_tg}_t (J^x\phi)\| \leq e^{-(k+1)T\phi} + \frac{h}{c} (1 - e^{-\epsilon T}) + (1 - e^{-\epsilon T}) \|g\|$  

$$\leq e^{-\epsilon(k+1)T\phi} + \frac{h}{c} (1 - e^{-\epsilon T}) + \|g\| + (1 - e^{-\epsilon T})\|g\|$$

$$\leq e^{-\epsilon(k+1)T\phi} + \frac{h}{c} (e^{-\epsilon T} - e^{-\epsilon(k+1)T} + 1 - e^{-\epsilon T}) + \|g\|.$$  

From (3) and (4) we have (J3) for $k + 1$.

We have, for any $u \in I$ and $t > 0,$
\[ g = \lim_{i \to \infty} T_i u_i \phi. \]

Hence (J5) is valid.

Define \( S_t^{(N)} \) by \( S_t^{(N)} \phi = J_k^k (N) \phi \) for \( t = \frac{k}{2^N} \). Then \( S_t^{(N)} \phi \) is increasing as \( N \to \infty \). Moreover we have

**Lemma 2.** If \( \phi \leq g \), then \( S_t^{(N)} \phi \) is increasing as \( t \to \infty \).

**Proof.** Putting \( A = 1/2^N \), we get by (J5)
\[
\phi \leq g \leq S_d^{(N)} \phi.
\]
Hence, by (J1),
\[
\phi \leq S_1^{(N)} \phi \leq S_2^{(N)} \phi \leq \cdots \leq S_k^{(N)} \phi \leq S_{k+1}^{(N)} \phi.
\]
(J3) means the following (7).
\[
\|S_t^{(N)} \phi\| \leq e^{-ct} \|\phi\| + \frac{h}{c} (1 - e^{-ct}) + \|g\|.
\]
Therefore, for binary \( t \), the set \( \{S_t^{(N)} \phi, N \text{ large}\} \) is \( O \)-bounded. Hence we can define \( S_t \) by
\[
S_t \phi = O_t - \lim_{N} S_t^{(N)} \phi \quad \text{for binary } t.
\]

From (J4) we can again see, for binary \( t \),
\[
S_t \phi = O_t - \lim_{n} S_t \phi_n \quad \text{if } \phi = O_t - \lim_{n} \phi_n.
\]
Therefore we can derive the semi-group property on binary parameter.
\[
S_{t+\theta} \phi = S_t (S_\theta \phi) = S_\theta (S_t \phi) \quad \text{for binary } t \text{ and } \theta.
\]
Again, by (7), we have
\[
\|S_t \phi\| \leq e^{-ct} \|\phi\| + \frac{h}{c} (1 - e^{-ct}) + \|g\|.
\]
Hence the set \( \{S_t \phi, \text{binary } t\} \) is also \( O \)-bounded.

**Lemma 3.** If \( \phi \leq g \), then \( S_t \phi \) is increasing in \( t \) and \( O_t - \lim_t S_t \phi \) exists, say \( u_\phi \). Moreover
Proof. By Lemma 2 we have for $t<0$,

$$S_t \phi = O_t - \lim_{y \to x} S_t^{(y)} \phi \leq O_t - \lim_{y \to x} S_{t+y} \phi = S_t \phi.$$ 

Hence $S_t \phi$ is increasing as binary $t \to \infty$. (10) is clear by (J5).

For simplicity we put $v = v_0$ if any confusion does not occur.

**Lemma 4.** $v$ is $S_t$-invariance, i.e.

$$S_t v = v \text{ for binary } t.$$ 

**Proof.** By the definition of $v$ and (8),

$$S_t v = S_t (O_t - \lim_{\theta} S_{t+\theta} \phi) = O_t - \lim_{\theta} S_{t+\theta} \phi = v.$$ 

**Proposition 8.** $v$ is an $S_t$-excessive majorant of $g$, i.e. $v \geq g$ and

$$S_t v \leq v, \forall t \geq 0.$$ 

**Proof.** By the definitions of $S_t$ and $S$, we have

$$S_t \phi \leq S_t \phi \quad \forall \text{ binary } t \text{ and } \phi \in L.$$ 

Hence by Lemma 4

$$S_t v \leq S_t v = v.$$ 

Namely we get (12) for binary $t$. Since $S_t v$ is continuous in $t$, (12) is valid for any $t$. Recalling (10) we complete the proof.

**Proposition 9.** For any $\phi \leq g$, $v_\phi$ is the least $S_t$-excessive majorant of $g$.

**Proof.** Let $V$ be an $S_t$-excessive majorant of $g$. Recalling the definitions of $T_t^{uV}$ and $T_t^u$, we have

$$T_t^{uV} \phi = e^{-t} P_t^{u} \psi + \int_0^t e^{-t\theta} P_t^u (f^u + \lambda V) \, d\theta$$
and

\[(14) \quad T_t^{u_0} \psi = P_t^{u_0} \psi + \int_0^t P_t^u f^u \, d\theta = T_t^u \psi.\]

Hence

\[(15) \quad T_t^{u_t} V = e^{-\mu t} T_t^u V + \int_0^t e^{-\mu s} P_t^u f^u \, d\theta + \lambda \int_0^t e^{-\mu s} P_t^u g \, d\theta - e^{-\mu t} \int_0^t P_t^u f^u \, d\theta,\]

and, from (14), we see

\[(16) \quad \lambda \int_0^t e^{-\mu s} P_t^u V \, d\theta = \lambda \int_0^t e^{-\mu s} \left( T_t^u V - \int_0^s P_t^u f^u \, ds \right) \, d\theta = \lambda \int_0^t e^{-\mu s} T_t^u V \, d\theta - \int_0^t (e^{-\mu s} - e^{-\mu t}) P_t^u f^u \, ds.\]

Therefore, by (15) and (16) we have

\[(17) \quad T_t^{u_t} V = e^{-\mu t} T_t^u V + \lambda \int_0^t e^{-\mu s} T_t^u V \, d\theta.\]

Since \( "T_t^{u_0} \psi \leq S_t \psi" \) and \( V \) is \( S_t \)-excessive, we have

\[(18) \quad e^{-\mu t} T_t^u V \leq e^{-\mu t} S_t V \leq e^{-\mu t} V.\]

Combining (18) with (17) we can see

\[(19) \quad T_t^{u_t} V \leq e^{-\mu t} V + \lambda \int_0^t e^{-\mu s} V \, d\theta = V.\]

Hence we have, denoting \( J(N) \) for \( T_t^{u_t} \) by \( \bar{J}(N) \),

\[(20) \quad \bar{J}(N) V \leq V \quad \text{and} \quad \bar{J}^k(N) V \leq V.\]

This tells us the following inequality,

\[(21) \quad \bar{S}_t V \leq V \quad \text{for binary } t.\]

Appealing to \( "g \leq V" \) and the definition of \( T_t^{u_0} \), we have

\[T_t^{u_0} \psi \leq T_t^{u_t} \psi \quad \forall \psi \in L.\]

Hence

\[J(N) \psi \leq \bar{J}(N) \psi \quad \text{and} \quad S_t \psi \leq \bar{S}_t \psi.\]

So, by (21), we have for binary \( t \),

...
\[ S_t \phi \leq S_t g \leq S_t V \leq \bar{S}_t V \leq V. \]

Tending \( t \) to \( \infty \), we can derive
\[ v \leq V. \]

**Corollary.** \( v_\phi = v_\theta \ \forall \phi \leq g \).

**Proof.** Since the least \( S_t \)-excessive majorant of \( g \) is unique, \( v_\phi = v_\theta \).

§ 5. Examples

We will show two simple examples of control problems related Markov processes with exponential holding times, [cf. 5].

**Example 1.** Let \( A^u = (a^u(i, j)) \) be an \( l \times l \)-matrix. Suppose \( a^u(i, j) \geq 0 \) for \( i \neq j \) and \( \sum_{j=1}^{l} a^u(i, j) = 0 \). Then \( A^u \) is the generator of the transition semi-group \( P_t^u = (P_t^u(i, j)) = e^{tA^u} \).

Put \( \mu_i = 1, \ i = 1, \ldots, l \) and \( \mu(R^1 - \{1, 2, \ldots, l\}) = 0 \). Then \( A^u \) becomes a bounded linear operator on \( L = L_0^0(R^1, \mu) \) and \( P_t^u \) a positive contractive and continuous semi-group on \( L \). Assume
\[ \sup_{u} |a^u(i, j)| < \infty, \ \forall i, j = 1, \ldots, l. \]

Thus \( \sup_{u} ||A^u|| < \infty \). Let \( \sup_{u} |f^u(i)| < \infty \) for \( i = 1, \ldots, l \). Then we can construct Bellman semi-group \( S_t \) for \( \{A^u, f^u\} \). Moreover, for \( \phi \in L \), \( S_t \phi \) is a solution of the following Bellman equation,
\[
\begin{align*}
\frac{dS_t \phi(i)}{dt} &= \sup_{u} \left[ \sum_{j=1}^{l} a^u(i, j) S_t \phi(j) + f^u(i) \right], \ i = 1, \ldots, l, \\
S_0 \phi(i) &= \phi(i).
\end{align*}
\]

**Example 2.** Let \( X^u \) be a 1-dimensional Lévy process of pure jump type with finite Lévy measure \( n^u \)
\[ X(t) = x + \int_0^t zN^u(dsz) \]
and \( E N^u(dsz) = dsn^u(dz) \). Thus every point of \( R^1 \) is an exponential
holding point.

Suppose that $n^u$ has the density, say $n^u(dz) = n^u(z)dz$. Put $Y^u(t) = \int_R zN^u(ds)dz$. We denote its $i$-th jump time by $\tau_i^u, \tau_i^u = 0$, and $Y^u(\tau_i^u) - Y^u(\tau_{i-1}^u)$ by $\zeta_i^u$. For simplicity we skip the suffix $u$ if any confusion does not occur. We have the following well-known facts,

(i) $\tau_i - \tau_{i-1}, i = 1, 2, \ldots, \zeta_i, i = 1, 2, \ldots$ are independent.

(ii) $P(\tau_i - \tau_{i-1}>t) = e^{-\lambda t}$ where $\lambda = n(R^\prime)$.

(iii) $P(\zeta_i \in A) = \frac{n(A)}{\lambda} = \frac{1}{\lambda} \int_R n(z)dz$.

Hence

$$P(Y(t) \in A) = \chi_A(O)P(\tau_1 < t) + \sum_{i=1}^{\infty} P(Y(\tau_i) \in A)P(\tau_i \leq t < \tau_{i+1})$$

$$= \chi_A(O)e^{-\lambda t} + \sum_{i=1}^{\infty} P(\zeta_i + \cdots + \zeta_i \in A)P(\tau_{i+1} \leq t < \tau_{i+1})$$

$$= \chi_A(O)e^{-\lambda t} + \mu(A, t).$$

By virtue of (i) and (iii) the measure $m(\cdot, t)$ is absolutely continuous w.r. to the Lebesgue measure $\mu$. Suppose $\phi = \psi - \mu - a.e.$ Then, for any $x$ where $\psi(x) = \phi(x)$ holds, we see

$$P_t \phi(x) = E_\phi(X(t)) = E\phi(x + Y(t))$$

$$= \phi(x)e^{-\lambda t} + \int \phi(x + y)m'(y, t)dy$$

$$= \phi(x)e^{-\lambda t} + \int \phi(y)m'(y-x, t)dy$$

$$= \phi(x)e^{-\lambda t} + \int \phi(y)m'(y-x, t)dy = P_t\psi(x).$$

Hence the transition semi-group $P_t$ can act on $L = L_m(R^\prime, \mu)$. On the other hand we have

$$|P_t \phi(x) - \phi(x)| \leq |\phi(x)| (1 - e^{-\lambda t}) + \|\phi\|P(\tau_1 \leq t)$$

$$\leq 2\|\phi\|(1 - e^{-\lambda t}) \to 0 \quad \text{as} \quad t \downarrow 0.$$ 

So $P_t$ is strongly continuous.

Thus $P_t^n$ is a positive contractive and strongly continuous linear semi-group on $L$ whose generator $A^n$ is
\[ A^u\phi(x) = \int_{\mathbb{R}^1} (\phi(x+y) - \phi(x)) n^u(y) \, dy, \quad \phi \in L. \]

Since \( \|A^u\phi\| \leq 2\|\phi\|\lambda^u \), this example 2 satisfies the condition (36) of Remark 2, if

\[ \sup_u \lambda^u < \infty. \]

Therefore, for \( f^u \in L \) with (A3), we have a solution, \( V(t, x) = S_t \phi(x) \), of Bellman equation

\[
\begin{cases}
\frac{\partial V(t, x)}{\partial t} = \sup_u \left[ \int_{\mathbb{R}^1} (V(t, x+y) - V(t, x)) n^u(y) \, dy + f^u(x) \right] \text{a.e. } \forall t > 0, \\
V(0, x) = \phi(x).
\end{cases}
\]

References
