Some Results on Formal Power Series
and Differentiable Functions

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§ 1. Introduction

In [2] we see that any formal power series in two variables with coefficients in $\mathbb{R}$ or $\mathbb{C}$ (in this paper only the real case will be considered,) can be transformed to a polynomial by some automorphism change of the variables. In [3] Whitney shows an example which is a convergent series in three variables but which cannot be transformed to a polynomial. In this paper we give a formal power series example in three variables that is never transformed to be convergent (§ 2).

A formal power series is the Taylor expansion of some $C^\infty$ function at the origin by E. Borel theorem. The followings refine it.

**Theorem 1.** Let $f$ be a formal power series in the variables $x=(x_1, \ldots, x_n)$. Let $K$ be a positive real. There exists a $C^\infty$ function $g$ defined on $|x|<K$ with the Taylor expansion at 0 $Tg=f$ and which is analytic except when $x=0$.

**Theorem 2.** There exists a homomorphism $S$ from the $\mathbb{R}$-algebra $F$ of formal power series in one variable $x$ to the $\mathbb{R}$-algebra $E$ of germs of $C^\infty$ function in one variable $x$ at 0 such that the composition $T\circ S$ is the identity homomorphism of $F$.

There is a question in Malgrange [1] whether any homomorphism between the $\mathbb{R}$-algebras of $C^\infty$ function germs is a morphism (see § 4). Theorem 2 gives a counter-example to it (Corollary).

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§ 2. An Example

The example of Whitney is an analytic function $f$ in three variables of the form $xy(y-x)(y-(3+x)x)(y-\nu(x)x)$ where $\nu$ is a transcendental function with $\nu(0)=4$. If we replace the transcendental function above by a non-convergent formal power series, then $f$ cannot be transformed to a convergent one by any automorphism of the algebra of formal power series.

Proof. Suppose it is not so, then there exist formal power series in $(X,Y,Z)$-variables $x(X,Y,Z)$, $y(X,Y,Z)$, $z(X,Y,Z)$ such that $f(x(X,Y,Z), y(X,Y,Z), z(X,Y,Z))$ is analytic and that the determinant of Jacobian $D(x,y,z)/D(X,Y,Z)$ does not vanish at 0. Moreover we can assume $D(x,y,z)/D(X,Y,Z)$ at $(X,Y,Z) = (0,0,0)$. We know Zariski-Nagata Theorem and the fact that the formal power series ring and the convergent power series ring are unique factorization rings. Therefore there exist formal power series $g_1, \cdots, g_5$ in $(X,Y,Z)$-variables such that $g_i(0)=1$ for each $i$, $g_1\cdots g_5=1$, and $g_1x$, $g_2y, \cdots , g_5(y-\nu(x)x)$ are convergent. Let $G_i$ $i=1,\cdots ,5$ be $C^\infty$ functions in $(X,Y,Z)$-variables such that $TG_i=g_i$ and $G_1\cdots G_5=1$. We assume $g_i, x, \cdots$ converge in a neighbourhood $U$ of $(X,Y,Z) = (0,0,0)$. Let $\phi_i=g_i x/G_i$, $\cdots$, $\phi_5=g_5(y-\nu(x)x)/G_5$ in $U$. Clearly $T\phi_1=x, \cdots , T\phi_5=y-\nu(x)x$, and the Taylor expansions of $\phi_1-(\phi_2-\phi_1)$, $\phi_4-(\phi_3-(3+S)\phi_1)$ and $\phi_5-(\phi_2-R(S)\phi_1)$ are zeros at 0. Here $S, R$ are $C^\infty$ functions in $(X,Y,Z)$-variables respectively such that $TS=x$, $TR=\nu$. From the assumption, $(g_1x, g_3y, Z)$ is an analytic local coordinates system around 0. Hence $\phi_1^{-1}(0) \cap \phi_2^{-1}(0)$ is an analytic curve. These imply that the functions $\phi_2-(\phi_2-\phi_1), \cdots$ are zero identically on the curve $\phi_i^{-1}(0) \cap \phi_j^{-1}(0)$. On the other hand $(\phi_1, \phi_3, Z)$ also is a local coordinates system around 0. Thus, we find $C^\infty$ functions $\psi_{ij}$ $i=3,4,5$ $j=1,2$ flat at 0 such that

$$\phi_3-(\phi_2-\phi_1)=\psi_{31}\phi_1+\psi_{32}\phi_2;$$
$$\phi_4-(\phi_3-(3+S)\phi_1)=\psi_{41}\phi_1+\psi_{42}\phi_2;$$
$$\phi_5-(\phi_2-R(S)\phi_1)=\psi_{51}\phi_1+\psi_{52}\phi_2.$$

Hence the intersection of any two $\phi_i^{-1}(0)$ are the same one $\phi_i^{-1}(0)$.
\( \cap \phi_t^{-1}(0), \) and the intersection is described as \( (X, Y, Z) = (X(Z), Y(Z), Z) \) where \( X(Z) \) and \( Y(Z) \) are analytic in \( Z \)-variable. We see easily that the Taylor expansions at 0 of the cross ratios of \( (\phi_1^{-1}(0), \phi_2^{-1}(0), \phi_3^{-1}(0), \phi_4^{-1}(0)) \) and \( (\phi_1^{-1}(0), \phi_2^{-1}(0), \phi_3^{-1}(0), \phi_4^{-1}(0)) \) are \( 1/(3+\varepsilon(X(Z), Y(Z), Z)) \) and \( 1/\nu(z(X(Z), Y(Z), Z)) \), respectively.

In the same way we find analytic functions \( \chi_t \) such that \( \chi_t(0) \neq 0 \) and

\[
G_t \phi_i = \chi_t G_i \phi_i - \chi_t G_2 \phi_2 \quad i = 3, 4, 5, \]

and we see that the cross ratios above are \( \chi_t \chi_4 / \chi_t \chi_2 \) and \( \chi_3 \chi_4 / \chi_3 \chi_2 \), respectively. Hence they are analytic, but both \( z(X(Z), Y(Z), Z) \) and \( \nu(z(X(Z), Y(Z), Z)) \) are not convergent by the assumption. That is a contradiction.

§ 3. Proof of Theorem 1

We prove only the case \( n = K = 1 \). In the general case there is nothing to prove moreover.

Let \( f \) be a formal power series \( \sum a_n x^n \) where \( a_n \) are reals. It is enough to find sufficiently large reals \( m_n \) such that \( \sum a_n (1 - \exp(-1/m_n x^n)) x^n \) converges on

(1) the real interval \([-1, 1]\) with its each derivatives; and

(2) any compact subset of the complex domain \( 0 < |x| < 1 \).

Proof of (1). For \( n \geq 2 \) and \( k \leq n/3 \), we have

the \( k \)-th derivative of \( (1 - \exp(-1/m x^k)) x^n \)

\[
= n \cdots (n-k+1) (1 - \exp(-1/m x^k)) x^{n-k} + P(x, m) \exp(-1/m x^k).
\]

Here \( P(x, m) \) is a polynomial in \( x \) and uniformly converges to 0 when \( m \to +\infty \). We can see that \( (1 - \exp(-1/m x^k)) x^{n-k} \) and \( \exp(-1/m x^k) \) are monotonous in the intervals \([-1, 0]\) and \([0, 1]\). Hence these functions take the maximal values at \( x = -1 \) or 1. Now, it follows that the \( k \)-th derivative of \( (1 - \exp(-1/m x^k)) x^n \) uniformly converges to 0 when \( m \to +\infty \) for \( n \geq 2 \) and \( k \leq n/3 \). This proves (1).

Proof of (2) is also easy.

§ 4. Homomorphism

Proof of Theorem 2. Let \( X \) be the ordered set consisting of the
pairs \((A, \phi)\). Here \(A\) is a subring of \(\mathcal{E}\) containing \(\mathbb{R}\) and \(\phi\) is a homomorphism from \(A\) to \(\mathcal{E}\) such that the composition \(T \circ \phi\) is the identity of \(A\). Order two elements \((A, \phi), (B, \psi)\) of \(X\) as follows

\[(A, \phi) \leq (B, \psi) \text{ if } A \subset B \text{ and } \psi|_A = \phi .\]

Apply Zorn's lemma, and \(X\) has a maximal element \((A, \phi)\).

Now, we prove that \(A\) of the maximal is itself \(\mathcal{E}\). Assume that \(A\) is a proper subset of \(\mathcal{E}\), and that \(\zeta\) is an element in \(\mathcal{E}\) but not in \(A\). There are two cases,

1. \(\zeta\) is algebraic over \(A\);
2. \(\zeta\) is not so.

The case (1). Let \(A[\zeta]\) and \(A[t]\) be the ring generated by \(\zeta\) over \(A\) and the polynomial ring in \(t\)-variable with coefficients in \(A\) respectively, and let \(\theta\) be the homomorphism from \(A[t]\) to \(A[\zeta]\) naturally defined by \(\theta(t) = \zeta\). Let \(P(t)\) be an element of \(\ker \theta\) whose degree as a \(t\)-polynomial takes the minimal in \(\ker \theta\). For any element \(Q\) of \(\ker \theta\), dividing \(Q\) by \(P\) we have \(QQ' = PP' + R\) with \(Q' \in A, P', R \in A[t]\). Since \(R \in \ker \theta\) and degree \(R \leq \text{degree } P\), we see \(R = 0\). Hence we have the equality \((a)\) \(QQ' = PP'\) for some \(Q' \in A - \{0\}\) and \(P' \in A[t]\). Let \(P(t) = a_1t^n + \cdots + a_{n+1}\). We may assume \(\zeta = x^s\) for an integer \(s\) through some change of the variable \(x\). Let \(\phi_* P(t)\) denote \(\phi(a_1)t^n + \cdots + \phi(a_{n+1})\). We shall define an extension homomorphism \(\phi\) of \(\phi\) from \(A[t]\) to \(\mathcal{E}\) such that \(T \circ \phi(t) = x^s\) and \(\phi(\ker \theta) = 0\). This follows from \((a)\) if we choose a germ \(g(x)\) flat at 0 such that \(\phi_* P(x^s + g) = 0\). Let \(y\) be a variable. Then \(\phi_* P(x^s + y)\) is a polynomial \(b_1y^n + \cdots + b_{n+1}\) in \(y\) with coefficients in \(\mathcal{E}\). We see that \(b_{n+1}\) is flat at 0 and that \(b_n = (\partial \phi_* P/\partial t)(x^s)\) is not flat. Because we have degree \(\partial P/\partial t\) at 0 and therefore \(\partial \phi_* P/\partial t \notin \ker \theta\). Put \(y = x^Nz\) for a sufficiently large \(N\) and a new variable \(z\). We can divide \(b_1y^n + \cdots + b_{n+1}\) by \(x^N b_n\). The quotient is \(c_1z^n + \cdots + c_{n+1}\), here \(c_i\) are in \(\mathcal{E}\) and \(c_{n+1}\) is flat at 0. Applying the implicit function theorem, we give a germ \(z(x)\) flat at 0 such that \(c_1z^n(x) + \cdots + z(x) + c_{n+1} = 0\). The germ \(g(x) = x^Nz(x)\) is what we want. Now we have defined a homomorphism \(\phi\). It is clear that \(\phi\) induces a homomorphism \(\rho\) from \(A[\zeta]\) to \(\mathcal{E}\) such that the composition \(T \circ \rho\) is the identity of \(A[\zeta]\). This contradicts the maximality of \((A, \phi)\). Hence \(A\) is \(\mathcal{E}\).
The proof of the case (2) is trivial from the proof of (1). Theorem 2 follows.

**Remark.** Even if we treat only the homomorphisms where the image of a convergent power series is naturally defined, there are infinitely many homomorphisms. We can prove this from the fact that any non-convergent formal power series is algebraically independent over the convergent series ring.

**Definition** [1]. An endomorphism \( u \) of \( E \) is called a *morphism* if there exists a germ \( \phi \) with \( \phi(0) = 0 \) such that for any \( f \in E \), we have \( u(f) = f \circ \phi \).

The following answers the question in [1].

**Corollary.** *The composed homomorphism \( S \circ T \) is not a morphism.* Here \( S \) is defined in Theorem 2.

**Proof.** Suppose it is a morphism induced by some \( \phi \). The first derivative of \( \phi \) takes a non-zero value at 0. Hence \( S \circ T \) is an automorphism, on the other hand we have \( S \circ T(f) = 0 \) for \( f \) flat at 0.

**Remark.** The general preparation theorem in [1] does not hold in the homomorphism case. That is, this \( S \circ T \) is quasifinite but not finite.

**References**


*Added in proof:* The author was informed that Theorem 2 and its corollary were also obtained independently by K. Reichard (Manuscripta Math. 15 (1975), 243-250) and M. van der Put (to appear in Compositio Math.).